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ON NEUMANN TYPE PROBLEMS FOR NON-LOCAL EQUATIONS SET IN A HALF SPACE

GUY BARLES, EMMANUEL CHASSEIGNE, CHRISTINE GEORGELIN, AND ESPEN R. JAKOBSEN

Abstract. We study Neumann type boundary value problems for nonlocal equations related to Lévy processes. Since these equations are nonlocal, Neumann type problems can be obtained in many ways, depending on the kind of "reflection" we impose on the outside jumps. To focus on the new phenomenas and ideas, we consider different models of reflection and rather general non-symmetric Lévy measures, but only simple linear equations in half-space domains. We derive the Neumann/reflection problems through a truncation procedure on the Lévy measure, and then we develop a viscosity solution theory which includes comparison, existence, and some regularity results. For problems involving fractional Laplacian type operators like e.g. $(-\Delta)^{\alpha/2}$, we prove that solutions of all our nonlocal Neumann problems converge as $\alpha \to 2^-$ to the solution of a classical Neumann problem. The reflection models we consider include cases where the underlying Lévy processes are reflected, projected, and/or censored upon exiting the domain.

1. Introduction

In the classical probabilistic approach to elliptic and parabolic partial differential equations via Feynman-Kac formulas, it is well-known that Neumann type boundary conditions are associated to stochastic processes having a reflection on the boundary. We refer the reader to the book of Freidlin [9] for an introduction and to Lions and Sznitman [17] for general results. A key result in this direction is roughly speaking the following: for a PDE with Neumann or oblique boundary conditions, there is a unique underlying reflection process, and any consistent approximation will converge to it in the limit (see [17] and Barles & Lions [5]). At least in the case of normal reflections, this result is strongly connected to the study of the Skorohod problem and relies on the underlying stochastic processes being continuous.

The starting point of this article was to address the same question, but now for jump diffusion processes related to partial integro-differential equations (PIDEs in short). What is a reflection for such processes, and is a PIDE with Neumann boundary conditions naturally connected to a reflection process? It turns out that the situation is more complicated in this setting, at least the questions have to be reformulated in a slightly different way. In this article we address these questions

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through an analytical PIDE approach where we keep in mind the idea of having a reflection process but without defining it precisely or even proving its existence.

For jump processes which are discontinuous and may exit a domain without touching its boundary, it turns out that there are many ways to define a "reflection" or a "process with a reflection". This remains true even if we restrict ourselves to a mechanism which is connected to a Neumann boundary condition (see below). But because of the way the PIDE and the process are related, defining a reflection on the boundary will change the equation inside the domain. This is a new nonlocal phenomenon which is not encountered in the case of continuous processes and PDEs.

PIDE WITH NEUMANN-TYPE BOUNDARY CONDITION – In order to simplify the presentation of paper and focus on the main new ideas and phenomenas, we consider different models of reflections and rather general non-symmetric Lévy measures, but only for problems involving linear equations set in simple domains. The cases we will consider already have interesting features and difficulties. To be precise, we consider half space domains $\Omega := \left\{(x_1, \dots, x_N) = (x', x_N) \in \mathbb{R}^N : x_N > 0\right\}$ and simple linear Neumann type problems that we write as

$$\begin{cases} u(x) - I[u](x) - f(x) = 0 & \text{in} \quad \Omega, \\ -\frac{\partial u}{\partial x_N} = 0 & \text{in} \quad \partial\Omega, \end{cases} \tag{1.1}$$

or sometimes as

$$\begin{cases} F(x, u, I[u]) = 0 & \text{in} \quad \Omega, \\ -\frac{\partial u}{\partial x_N} = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where $F(x, r, l) = r - l - f(x)$ and

$$I[u](x) = \lim_{b \to 0^+} \int_{b<|z|} \left[u(x + \eta(x, z)) - u(x)\right] \, d\mu(z).$$

We will assume that $f \in C_b(\overline{\Omega})$, i.e. f is bounded and continuous, that μ is a nonnegative Radon measure satisfying

$$\int |z|^2 \wedge 1 \, d\mu(z) < \infty, \tag{1.2}$$

and that

$$x + \eta(x, z) \in \overline{\Omega} \text{ for all } x \in \overline{\Omega}\,, \ \eta(x, z) = z \text{ if } x + z \in \overline{\Omega}. \tag{1.3}$$

Note that $I[u]$ is a principal value (P.V.) integral, and that (1.2) is the most general integrability assumption satisfied by the Lévy measure associated to any Lévy process [1]. When $\eta(x, z) \equiv z$, then $I[u]$ is the generator of a stochastic process which can jump from $x \in \overline{\Omega}$ to $x + z$ with a certain probability, see e.g. [1, 8, 10]. Assumption (1.3) is a type of reflection condition preventing the jump-process from leaving the domain: nothing happens and $\eta(x, z) = z$ if $x + z \in \overline{\Omega}$, while if $x + z \notin \overline{\Omega}$, then a "reflection" is performed in order to move the particle back to a point $P(x, z) = x + \eta(x, z)$ inside $\overline{\Omega}$. Note that we have to check at some point that the reflection is consistent with a Neumann boundary condition.

The main examples of η are the following model cases, where we use the notation $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+$, $\eta(x,z) = (\eta(x,z)', \eta(x,z)_N)$, etc.:

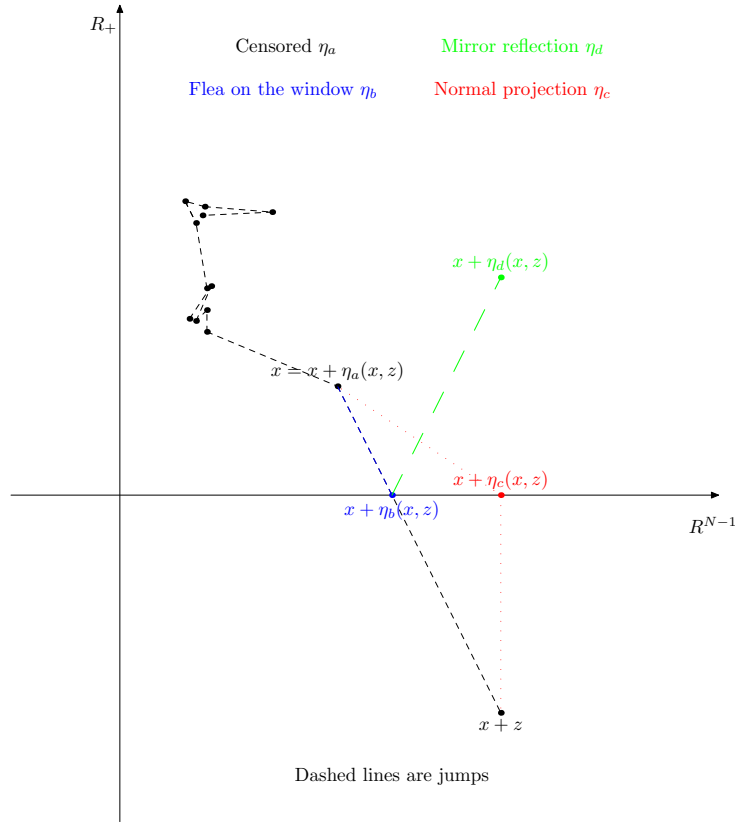
$$\text{(a)} \quad \eta(x,z) = \begin{cases} z & \text{if } x_N + z_N \geq 0 \\ 0 & \text{if not} \end{cases} \qquad \text{[censored]}$$

$$\text{(b)} \quad \eta(x,z) = \begin{cases} z & \text{if } x_N + z_N \geq 0 \\ z\frac{x_N}{|z_N|} & \text{if not} \end{cases} \qquad \text{[fleas on the window]}$$

$$\text{(c)} \quad \eta(x,z) = \begin{cases} z & \text{if } x_N + z_N \geq 0 \\ (z', -x_N) & \text{if not} \end{cases} \qquad \text{[normal projection]}$$

$$\text{(d)} \quad \eta(x,z) = \begin{cases} z & \text{if } x_N + z_N \geq 0 \\ (z', -2x_N - z_N) & \text{if not} \end{cases} \qquad \text{[mirror reflection]}$$

for all $x \in \Omega$ and $z \neq 0$. The different reflections are depicted in the figure below.



We will discuss later whether the naively proposed "reflections" are realized by a concrete Markov process, *i.e.* if they correspond to the generator of such a process.

Main results – From an analytical (PIDE) point of view, we first have to give a sense to problem (1.1) and relate it to an homogeneous Neumann boundary value problem. This is done in Sections 2 and 3. The first part is classical: to take into account singular Lévy measures, we split the integral operator in two and write the

equation in a more convenient way. Here classical arguments in viscosity solution theory are used, see e.g. [4] and references therein. Viscosity solution theory is also used to give a good definition of the Neumann boundary conditions.

If μ is a nice bounded measure, then the problem (1.1) can be solved easily without caring much about the Neumann boundary condition. Moreover, the solutions will be uniformly bounded by $||f||_\infty$. Intuitively (1.1) carries the information that the particles remain in $\overline{\Omega}$ since they can only jump inside $\overline{\Omega}$. This mass conservation is an other way to understand that we are dealing with a (homogeneous) Neumann type of boundary condition.

When μ then is a singular measure, we can approximate it by a sequence of bounded measures $(\mu_n)_n$, consider the associated (uniformly bounded!) solutions $(u_n)_n$, and wonder what the limiting problem is if μ_n converges to μ in a suitable sense. This is the way we choose to make sense of both the definition of problem (1.1) and the associated notion of (viscosity) solutions. We point out here that the "real" Neumann boundary condition arises only if the measure is singular enough. In the other cases, either the process will never reach the boundary as in the censored case for α-stable process with $\alpha < 1$ (see e.g. [6]), or the equation will hold up to the boundary.

The natural next step is then to prove uniqueness results for all the above models and equations. In this paper different types of proofs are given depending on the singularity of the measure and the structure of the "reflection" mechanism at the boundary. These results are given in Sections 4 – 6. The first case we treat is when the jump function η enjoys a contraction property in the normal direction. This covers all the non-censored cases listed above – models (b)–(d). Had we had a contraction in all directions, then the usual viscosity solution doubling of variables argument would work. Here we have to modify that argument to take into account the special role of the normal direction.

The second case we consider is the censored case (a) when the singularity of the measure is not too strong. By this we mean typically a stable process with Lévy measure with density $\frac{d\mu}{dz} \sim \frac{1}{|z|^{N+\alpha}}$ for $\alpha \in (0,1)$. We construct an approximate subsolution which blows up at the boundary and this allows us to derive a comparison result by a penalization procedure. Such a construction is related to the fact that the process does not reach the boundary in this case, see e.g. [6].

The last case is the censored case (a) when the singularity is strong, *i.e.* when $\alpha \in [1,2)$. This case requires much more work because no blow-up subsolutions exist here. In fact, when $\alpha \in [1,2)$, the censored process does reach the boundary (see e.g. [6]). We first prove that the Neumann boundary condition is already encoded in the equation under the additional assumption that the solution is β-Hölder continuous at the boundary for some $\beta > \alpha - 1$. Then we prove a comparison result for sub/super solutions with this Hölder regularity at the boundary. The proof is then similar to the proof in the $\alpha < 1$ case, except that the special subsolutions in this case are bounded and only penalize the boundary when the sub/super solutions are Hölder continuous there. Finally, we construct solutions in this class. In dimension $N = 1$, we use and prove that any bounded uniformly continuous solution is Hölder continuous provided μ satisfies some additional integrability condition. In higher dimensions, we use and prove a similar result under additional regularity assumptions on f in the tangential directions.

Finally, in Section 7 we show that all the proposed nonlocal models converge to the same local Neumann problem when the Lévy measure approches the "local" case $\alpha = 2$. More precisely, we consider Lévy measures μ_α with densities $(2-\alpha)\frac{g(z)}{|z|^{N+\alpha}}$, where g is a nonnegative bounded function which is C^1 in a neighborhood of 0 and satisfies $g(0) > 0$. In this case we prove that whatever nonlocal Neumann model we use, the solutions u_α converge as $\alpha \to 2$ to the unique viscosity solution of the same limit problem, namely

$$\begin{cases} -a\Delta u - b \cdot Du + u = f & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{1.4}$$

where $a := g(0)\frac{|S^{N-1}|}{N}$ and $b := Dg(0)\frac{|S^{N-1}|}{N}$.

Related work – One of the first papers on the subject was Menaldi and Robin [15]. In that paper stochastic differential equations for reflection problems are solved in the case of diffusion processes with inside jumps, i.e. when there are no jumps outside the domain. They use the method of "penalization on the domain" inspired by Lions, Menaldi and Sznitman [18].

In model (a) (the censored case), any outside jump of the underlying process is cancelled (censored) and the process is restarted (resurrected) at the origin of that jump. We refer to e.g. [6, 10, 13, 16] for more details on such processes. The process can be constructed using the Ikeda-Nagasawa-Watanabe piecing together procedure [6, 10, 16], as Hunt processes associated to some Dirichlet forms [6, 13], or via the Feynman-Kac transform involving the killing measure [6]. In particular, the underlying processes in this paper are related to the censored stable processes of Bogdan et al. [6] and the reflected α-stable process of Guan and Ma [13]. But note that we essentially only construct the generators and not yet the processes themselves. On the technical side, we use viscosity solution methods, while [6, 13] use the theory of Dirichlet forms and potential theory. Our assumptions are also different, e.g. with our arguments we treat more general measures and we have the potential to treat non-linear problems, while [6, 13] e.g. treat much more general domains.

Let us also mention that the "natural" Neumann boundary condition for the reflected α-stable process of Guan and Ma [13] is slighly different from the one we consider here. They claim that the boundary condition arising through the variational formulation and Green type of formulas is

$$\lim_{t\to 0} t^{2-\alpha}\frac{\partial u}{\partial x_N}(x + te_N) = 0.$$

This formula allows the normal derivative $\frac{\partial u}{\partial x_N}$ to explode less rapidly than $|x_N|^{\alpha-2}$.

In model (b) outside jumps are stopped where the jump path hits the boundary, and then the process is restarted there. Model (c) is close to the approach of Lions-Sznitman in [17], and here outside jumps are immediately projected to the boundary along the normal direction. This type of models will be thoroughly investigated in the forthcoming paper [3] by three of the authors, but this time in the setting of fully non-linear equations set in general domains. Note that model (b) and (c) coincide in one dimension, i.e. when $N = 1$. Finally, in model (d), outside jumps are mirror reflected at the boundary. This is intuitively the natural way of understanding a

"reflection", but the model may be problematic to handle in general domains due to the possibility of multiple reflections. E.g. it is not clear to us if it correspond to an underlying Markov process in a general domain.

To the best of our knowledge, processes with generators of the form (b)–(d) have not been considered before. E.g. the works of Stroock [19] and Taira [20, 21] seem not to cover our cases because their integrodifferential operators involve measures and jump vectors η that are more regular than ours. Moreover in these works, it is the measure and not η that prevents the process to jump outside Ω.

In the case of symmetric α-stable processes (a subordinated Brownian motion), our formulation follows after a "reflection" on the boundary. So such processes can be constructed from a Brownian motion by first subordinating the process and then reflecting it. Another possible way to construct a "reflected" process in this case would be to reflect the Brownian motion first and then subordinate the reflected process. A related approach is described e.g. in Hsu [14] where pure jump processes like Lévy processes are connected via the Dirichlet-Neumann operator to the trace at the boundary of a Reflected Brownian Motion in one extra space dimension $\Omega \times \mathbb{R}_+$. An analytic PIDE version of this approach is introduced by Caffarelli and Silvestre in [7] in order to obtain Harnack inequalities for solutions of integrodifferential equations, and then these ideas have been used by many authors since.

Finally we mention the more classical work of Garroni and Menaldi [11], where a large class of uniformly elliptic integro differential equations are considered. There are two main differences with our work: (i) In [11] the principle part of all equations is a local non-degenerate 2nd order term. This allows the non-local terms to be controlled by local terms (the solution and its 1st and 2nd derivatives) via interpolation inequalities, and the local $W^{2,p}$ and $C^{2,\alpha}$ theories can therefore be extended to this nonlocal case. In our paper, it is the non-local terms that are the principal terms, and interpolation is not available. In addition, most of our results can be extended to degenerate problems. (ii) In [11] Dirichlet type problems are considered, and the authors have to assume extra decay properties of the jump vector η near the boundary, conditions that are not satisfied in our Neumann models.

2. Assumptions and Definition of solutions

In this section we state the assumptions on the problem (1.1) that we will use in the rest of the paper, give the definition of solutions for (1.1), and show that the quantities in this defintion is well-defined under our assumptions.

In this paper we let $Du(x)$ and $D^2u(x)$ denote the gradient and hessian matrix of a function u at x. We also define $P(x,z) = x + \eta(x,z)$, and then we can state our assumptions as follows:

(H_f) $f \in C_b(\overline{\Omega})$.

(H_μ) The measure μ is the sum of two nonnegative Radon measures μ_* and $\mu_\#$,

$$\mu = c\mu_* + \mu_\#,$$

where c is either 0 or 1, μ_* is a symmetric measure satisfying (1.2) and

$$\int_{|z|<1} |z|\, d\mu_* = \infty, \quad \text{and} \quad \int_{\mathbb{R}^N} (1 \wedge |z|)\, d\mu_\# < \infty.$$

(H^0_η) Neuman problem: $P(x,z)_N = x_N + \eta(x,z)_N > 0$ for any x, z and

$$\eta(x,z) = z \quad \text{for any } x_N + z_N > 0.$$

(H^1_η) At most linear growth of the jumps: there exists $c_\eta > 0$ such that

$$|\eta(x,z)| \le c_\eta |z| \text{ for any } x, z.$$

(H^2_η) Antisymmetry with respect to the z'-variables: for any $i = 1, \dots, (N-1)$,

$$\eta(x, \sigma_i z)_i = -\eta(x,z)_i \text{ where } \sigma_i z = (z_1, \dots, -z_i, \dots, z_N)\,.$$

(H^3_η) Weak continuity condition:

$$\eta(y,z) \to \eta(x,z) \quad \mu\text{-a.e.} \quad \text{as } y \to x.$$

(H^4_η) Continuity in the x'-variable:

$$|\eta(x', s, z)' - \eta(y', s, z)'| \le C|z||x' - y'| \quad \text{for any } x', y', z \text{ and any } s > 0.$$

(H^5_η) Non-censored cases: Contraction in the N-th direction:

$$|P(x,z)_N - P(y,z)_N| \le |x_N - y_N|.$$

(H^6_η) Censored case: For all $z \neq 0$ and $x \in \Omega$,

$$\eta(x,z) = \begin{cases} z & \text{if } x_N + z_N \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1. Assumption (H_μ) means that we can decompose μ into a sum of a singular *symmetric* Lévy measure and a not so singular Lévy measure. Symmetric here means that $\int_{\mathbb{R}^N} \chi(z)\, d\mu = 0$ for any odd μ-integrable function χ. This assumption covers the stable, the tempered stable, and the larger class of self-decomposable processes in $\mathbb{R}^N$, cf. chapter 1.2 in [1]. In all these cases the Lévy measures satisfy

$$\frac{d\mu}{dz} = \frac{g(z)}{|z|^{N+\alpha}} \quad \text{for} \quad \alpha \in (0,2),$$

and (H_μ) holds with $c = 0$ if $\alpha \in (0,1)$, while if $\alpha \in [1,2)$ and g is Lipschitz in $B_1(0)$ and bounded, then (H_μ) holds with $c = 1$. In the last case we may take

$$\frac{d\mu_*}{dz} = \frac{h(z)}{|z|^{N+\alpha}} \quad \text{and} \quad \frac{d\mu_\#}{dz} = \frac{g(z) - h(z)}{|z|^{N+\alpha}} \quad \text{for} \quad h(z) := \min_{|y|=|z|} g(y),$$

and note that h is symmetric and $g - h$ is nonnegative. More generally, we can consider measures where $\frac{d\mu}{dz} = g(z)\frac{d\mu_*}{dz}$ and μ^* is symmetric.

Remark 2.2. The cases (a), (b), (c), and (d) mentioned in the introduction satisfy assumptions (H^i_η) for $i = 0, 1, 2, 3, 4$, where in fact (H^4_η) holds with $C = 0$. Assumption (H^5_η) holds except in case (a), and case (a) is equivalent to (H^6_η). Note that η is continuous in x for $z \neq 0$ in (b), (c), and (d), while in (a), η is continuous except on the codimension 1 hypersurface $\{z_N = x_N\}$.

Now we will define generalized solutions in the viscosity sense, and to do that we need the following notation:

$$I[\phi] = I_\delta[\phi] + I^\delta[\phi]\,,$$

where

$$I^\delta[\phi] = \int_{|z| \ge \delta} \phi(x + \eta(x,z)) - \phi(x)\, d\mu(z)\,.$$

The I^δ-term is well-defined for any bounded function ϕ. For the I_δ-term there are two cases, depending on whether $c = 0$ or 1 in (H_μ). If $c = 0$, a Taylor expansion shows that $I_\delta[\phi](x)$ is well-defined for $\phi \in C^1$ and $x \in \Omega$. If $c = 1$, and the measure μ is very singular, we add and subtract a compensator and write

$$I_\delta[\phi](x) = \tilde{I}_\delta[\phi](x) + \text{P.V.} \int_{|z|<\delta} D\phi(x) \cdot \eta(x,z)\, d\mu(z),$$

for

$$\tilde{I}_\delta[\phi](x) := \int_{|z|<\delta} \phi(x + \eta(x,z)) - \phi(x) - D\phi(x) \cdot \eta(x,z)\, d\mu(z).$$

By the C^2-regularity of ϕ, these two terms will be well-defined – see Lemma 2.1 below. Note that this results is non-trivial because the compensator is not well defined in general!

Definition 2.1. Assume that (H_μ), (H^i_η) for $i = 0, 1, 2$ hold.
(i) A bounded usc function u is a viscosity *subsolution* to (1.1) if, for any test-function $\phi \in C^2(\mathbb{R}^N)$ and maximum point x of $u - \phi$ in $B(x, c_\eta\delta) \cap \overline{\Omega}$,

$$\begin{aligned} & F(x, u(x), I_\delta[\phi] + I^\delta[u]) \le 0 && \text{if } x \in \Omega \quad \text{and} \\ \text{either} \quad & F(x, u(x), I_\delta[\phi] + I^\delta[u]) \le 0 && \text{if } x \in \partial\Omega \text{ and } c = 0, \\ \text{or} \quad & -\frac{\partial \phi}{\partial x_N}(x) \le 0 && \text{if } x \in \partial\Omega \text{ and } c = 1. \end{aligned}$$

(ii) A bounded lsc function v is a viscosity *supersolution* to (1.1) if, for any test-function $\phi \in C^2(\mathbb{R}^N)$ and minimum point x of $v - \phi$ in $B(x, c_\eta\delta) \cap \overline{\Omega}$,

$$\begin{aligned} & F(x, v(x), I_\delta[\phi] + I^\delta[v]) \ge 0 && \text{if } x \in \Omega \quad \text{and} \\ \text{either} \quad & F(x, v(x), I_\delta[\phi] + I^\delta[v]) \ge 0 && \text{if } x \in \partial\Omega \text{ and } c = 0, \\ \text{or} \quad & -\frac{\partial \phi}{\partial x_N}(x) \ge 0 && \text{if } x \in \partial\Omega \text{ and } c = 1. \end{aligned}$$

(iii) A viscosity *solution* is both a sub- and a supersolution.

Remark 2.3. The constant c_η is defined in (H^1_η). If u and ϕ are smooth and x is a maximum point of $u - \phi$ over $B(x, c_\eta\delta) \cap \overline{\Omega}$, then by ($\mathrm{H}^1_\eta$),

$$u(x) - \phi(x) \ge u(x + \eta(x,z)) - \phi(x + \eta(x,z)) \quad \text{for all} \quad |z| < \delta.$$

If we rewrite this inequality and integrate, we find formally that $I_\delta[u](x) \le I_\delta[\phi](x)$. Lemma 2.1 below makes this computation rigorous. From this inequality it is easy to prove that classical (sub)solutions of (1.1) are viscosity (sub)solutions. Moreover, smooth viscosity (sub)solutions are classical (sub)solutions (simply take $\phi = u$).

Remark 2.4. In general to pose boundary value problems in the viscosity sense, one requires that either the minimum of the equation and the boundary condition is nonpositive or the maximum of the equation and the boundary condition is nonnegative. Here this is not the case and for a natural reason. If the measure is very singular (and $c = 1$) then the equation cannot hold on the boundary and the inequality holds just for the boundary condition. In the $c = 0$ case, on the contrary, the equation will hold up to the boundary and the boundary condition can not be imposed in general. In other words, we only end up with a Neumann boundary condition if $c = 1$, i.e. the measure has a "strong" singular part μ_*. In this case

the intensity of small jumps is so strong that the jump-reflection mechanisms, e.g. as in (a) – (d), are not enough to prevent the process from "diffusing" onto the boundary, and we need to add a reflection process at the boundary to keep the process inside (just as in the case of Brownian motion). We also note that the symmetry of μ_* is a natural condition in order to obtain Neumann and not oblique derivative boundary conditions, cf. Lemma 3.3 and proof.

We now prove that $I_\delta[\phi]$ is well-defined for $\phi \in C^2$.

Lemma 2.1. *Assume* (H_μ) *and* (H^i_η) *for* $i = 0, 1, 2$, *and let* $x \in \Omega$, $\phi \in C^2$, *and* $\delta > 0$. *Then* $I_\delta[\phi](x)$ *is well-defined since*

$$I_\delta[\phi](x) = \tilde{I}_\delta[\phi](x) + \text{P.V.} \int_{|z|<\delta} D\phi(x) \cdot \eta(x,z)\, d\mu(z),$$

and the compensator term satisfies

$$\begin{aligned} &\text{P.V.} \int_{|z|<\delta} D\phi(x) \cdot \eta(x,z)\, d\mu(z) \\ &= \int_{|z|<\delta} D\phi(x) \cdot \eta(x,z)\, d\mu_\#(z) + c \int_{x_N<|z|<\delta} D\phi(x) \cdot \eta(x,z)\, d\mu_*(z). \end{aligned}$$

Moreover, there is $R = R(x,\eta) > 0$ *such that*

$$I_\delta[\phi](x) = o_\delta(1)\|\phi\|_{C^2(B_R(x))} \quad \text{as} \quad \delta \to 0.$$

In the following, we often drop the P.V. notation for such integrals since they may be expressed in terms of converging integrals. Note that the integral over $\{x_N < |z| < \delta\}$ need not vanish since this regions exceeds the boundary and hence $\eta(x,z)$ will not be odd there.

To prove Lemma 2.1, we need the following result.

Lemma 2.2. *Assume* (H_μ) *and* (H^i_η) *for* $i = 0, 1, 2$, *and let* $x_N > 0$, $\rho \in (0, x_N)$, *and* $v \in \mathbb{R}^N$ *be a fixed vector.*

(i) For $r \in (0,\rho)$, $\int_{r<|z|<\rho} v \cdot \eta(x,z)\, d\mu(z) = \int_{r<|z|<\rho} v \cdot \eta(x,z)\, d\mu_\#(z)$, *and*

$$\text{P.V.} \int_{|z|<\rho} v \cdot \eta(x,z)\, d\mu(z) = \int_{|z|<\rho} v \cdot \eta(x,z)\, d\mu_\#(z)\,.$$

(ii) For $r \in (0,1)$, $\int_{r<|z|<\delta} v' \cdot \eta(x,z)'\, d\mu(z) = \int_{r<|z|<\delta} v' \cdot \eta(x,z)'\, d\mu_\#(z)$, *and*

$$\text{P.V.} \int_{|z|<\delta} v' \cdot \eta(x,z)'\, d\mu(z) = \int_{|z|<\delta} v' \cdot \eta(x,z)'\, d\mu_\#(z)\,,$$

(iii) For $r \in (0,1)$,

$$\int_{r<|z|<\delta} \eta(x,z)_N\, d\mu_*(z) \geq \int_{\substack{r<|z|<\delta \\ z_N > x_N}} (z_N - x_N)\, d\mu_*(z) \geq 0\,.$$

Proof. (i) If $|z| < \rho < x_N$, then $\eta(x,z) = z$ by (H^0_η). Hence η is odd with respect to the z variable, and the integral with respect to the symmetric part $c\mu_*$ is zero. Passing to the limit as $r \to 0$ and using the integrability of $\mu_\#$ along with (H^1_η) finishes the proof of (i).

(ii) Let σ be the rotation $\sigma(z', z_N) = (-z', z_N)$. Since μ_* is symmetric,

$$\int_{r<|z|<\delta} v' \cdot \eta(x, \sigma z)' \, d\mu_*(z) = \int_{r<|z|<\delta} v' \cdot \eta(x, z)' \, d\mu_*(z).$$

Other hand, since $\eta(x, -z', z_N)' = -\eta(x, z', z_N)'$ by (H^2_η), the above integral is zero. The result on the principal value is obtained as in the first case, after letting $r \to 0$.

(iii) Let us decompose

$$\begin{aligned}\int_{r<|z|<\delta} \eta(x,z)_N \, d\mu_*(z) &= \int_{\substack{r<|z|<\delta \\ -x_N \le z_N \le x_N}} \eta(x,z)_N \, d\mu_*(z) \\ &+ \int_{\substack{r<|z|<\delta \\ z_N > x_N}} \eta(x,z)_N \, d\mu_*(z) + \int_{\substack{r<|z|<\delta \\ z_N < -x_N}} \eta(x,z)_N \, d\mu_*(z)\,.\end{aligned}$$

The integral over $-x_N \le z_N \le x_N$ vanishes since $\eta(y, z) = z$ in this region and μ_* is symmetric. By (H^0_η) we have that $\eta(x,z)_N \ge -x_N$ if $z_N < -x_N$ and $\eta(x,z) = z_N > x_N$ if $z_N > x_N$. Hence by symmetry of μ_*,

$$\int_{r<|z|<\delta} \eta(x,z)_N \, d\mu_*(z) \ge \int_{\substack{r<|z|<\delta \\ z_N > x_N}} (z_N - x_N) \, d\mu_*(z) \ge 0,$$

and the proof is complete. □

Proof of Lemma 2.1. The expression for I_δ is obtained by adding and substracting the compensator term. The first integral in this expression is well-defined since the integrand is smooth and bounded by the function $\frac{1}{2}|z|^2 \max_{B(x,R)} |D^2\phi|$, for $R = \max_{y \in B(0,\delta)} |\eta(x,y)|$, which is an μ-integrable function over $B(0,\delta)$. Moreover, $\int_{0<|z|<\delta} |z|^2 \, d\mu(z) = o_\delta(1)$ as $\delta \to 0$ since $|z|^2$ is μ-integrable near 0.

In the compensator term, the integral with respect to $\mu_\#$ exists by (H^1_η), while the integral with respect to μ_* over $B(0, x_N)$ vanishes by Lemma 2.2-(i). Since $|z|$ is integrable near the origin for $\mu_\#$, this term is $|D\phi(x)| o_\delta(1)$ as $\delta \to 0$. □

3. Derivation of the boundary value problem - PIDE approach

In this section we derive the boundary value problems from approximate problems involving a sequence of bounded positive Radon measures $\mu^k = 1_{\{|z|>1/k\}}\mu$ converging to μ. Assume (H_μ) and let $\mu^k_\# = 1_{\{|z|>1/k\}}\mu_\#$ and $\mu^k_* = 1_{\{|z|>1/k\}}\mu_*$, it then easily follows that

$$\begin{aligned}&(\mathrm{H}^1_\mu) \quad \lim_{k\to+\infty} \int |z| \wedge 1 \; d\mu^k_\#(z) = \int |z| \wedge 1 \; d\mu_\#(z),\\ &(\mathrm{H}^2_\mu) \quad \lim_{k\to+\infty} \int |z|^2 \wedge 1 \; d\mu^k_*(z) = \int |z|^2 \wedge 1 \; d\mu_*(z),\\ &(\mathrm{H}^3_\mu) \quad \lim_{k\to+\infty} \int |z| \wedge 1 \; d\mu^k_*(z) = \infty.\end{aligned}$$

The approximation problem we consider is then given by

$$u(x) - I_{\mu_k}[u](x) = f(x) \quad \text{in} \quad \overline{\Omega}, \tag{3.1}$$

where, for $\phi \in C_b(\overline{\Omega})$,

$$I_{\mu_k}[\phi](x) = \int_{|z|>0} \phi(x + \eta(x,z)) - \phi(x) \; d\mu^k(z).$$

Since the measures μ^k are bounded, this equation holds in a classical, pointwise sense. Moreover, it is well-posed in $C_b(\overline{\Omega})$ and the solutions u_k are uniformly bounded in k:

Lemma 3.1. *Assume* (H_f), (H_μ), (H^0_η), *and* (H^3_η).
(a) For every k, there is a unique pointwise solution u_k of (3.1) *in $C_b(\overline{\Omega})$.*
(b) If u_k and v_k are pointwise sub- and supersolutions of (3.1), *then $u_k \le v_k$ in $\overline{\Omega}$.*
(c) If u_k is a pointwise solution of (3.1), *then $\|u_k\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}$.*

Proof. (a) Let $T : C_b(\overline{\Omega}) \to C_b(\overline{\Omega})$ be the operator defined by

$$Tu := u - \varepsilon\big(u - I_{\mu_k}[u] - f\big)\,,$$

where $\varepsilon < (1 + 2\|\mu^k\|_1)^{-1}$ and $\|\mu^k\|_1$ is the total (finite!) mass of the measure μ^k. Then T is a contraction in the Banach space $C_b(\overline{\Omega})$ since

$$\begin{aligned}\|Tu - Tv\|_\infty &\le (1-\varepsilon)\|u-v\|_\infty + 2\varepsilon\|\mu^k\|_1\|u-v\|_\infty \\ &\le \big(1 - \varepsilon(1+2\|\mu^k\|_1)\big)\|u-v\|_\infty \\ &\le C(k)\|u-v\|_\infty\,,\end{aligned}$$

and $C(k) < 1$. Hence there exists a unique $u_k \in C_b(\overline{\Omega})$ such that $Tu_k = u_k$, which is equivalent to (3.1).

(b) If $\sup_\Omega(u-v)$ is attained at a point $x \in \overline{\Omega}$, then by the equation and the easy fact that $I_{\mu_k}[\phi] \le 0$ at a maximum point of ϕ,

$$\sup_\Omega(u-v) = u(x) - v(x) \le I_{\mu_k}[u-v](x) \le 0.$$

The general case follows after a standard penalization argument.

(c) Follows from (b) since $\pm\|f\|_{L^\infty(\Omega)}$ are sub- and supersolutions of (3.1). □

The limiting problem can be identified through the half relaxed limit method:

Theorem 3.2. *Assume* (H_f), (H_μ), *and* (H^i_η) *for* $i = 0,1,2,3$ *hold. Then the half-relaxed limit functions*

$$\overline{u}(x) = \limsup_{k\to+\infty,\, y\to x} u_k(y) \quad \textit{and} \quad \underline{u}(x) = \liminf_{k\to+\infty,\, y\to x} u_k(y)$$

are respectively sub- and supersolutions of the Neumann boundary problem in the sense of Definition 2.1.

In the proof we will need the following result whose proof is given at the end of this section.

Lemma 3.3. *Assume* (H^i_η) *holds for* $i = 0,1,2$, (H_μ) *holds with* $c = 1$, *and let* $\delta > 0$ *and* $\gamma_{\mu_k,r}(x) := \int_{|z|<r} \eta(x,z)\, d\mu^k(z)$. *If* $y_k \to x \in \partial\Omega$ *as* $k \to \infty$, *then*

$$|\gamma_{\mu_k,\delta}(y_k)| \to \infty \quad \textit{and} \quad \frac{\gamma_{\mu_k,\delta}(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|} \to -\mathbf{n},$$

where $\mathbf{n} = (0,0,\dots,0,-1)$ *is an outward normal vector of* $\partial\Omega$.

Proof of Theorem 3.2. Since the proofs are similar for $\overline{u}$ and $\underline{u}$, we only do the one for $\overline{u}$. Let $\delta > 0$ and $\phi \in C^2$, and assume that $\overline{u} - \phi$ has a maximum point x in $B(x, c_\eta\delta) \cap \overline{\Omega}$. Let us first consider the case when $x \in \Omega$, i.e. when $x_N > 0$. By modifying the test-function, we may always assume that the maximum is strict.

By standard arguments, $u_k - \phi$ has a maximum point y_k in $B(x, c_\eta\delta)$, and when $k \to +\infty$,

$$y_k \to x \qquad \text{and} \qquad u_k(y_k) \to \overline{u}(x).$$

Let $\delta_k = \delta - |x - y_k|$ and $0 < r \leq \delta_k$, and note that $B(y_k, c_\eta r) \subset B(x, c_\eta\delta)$. Since the max of $(u_k - \phi)$ in $B(y_k, c_\eta r)$ is attained at y_k, we find that

$$\begin{aligned}(I_{\mu_k})_r[u_k](y_k) &:= \int_{|z|<r} u_k(y_k + \eta(y_k, z)) - u_k(y_k)\, d\mu^k \\ &\leq \int_{|z|<r} \phi(y_k + \eta(y_k, z)) - \phi(y_k)\, d\mu^k = (I_{\mu_k})_r[\phi](y_k).\end{aligned}$$

Hence, since u_k is a pointwise solution of (3.1), we find for all $0 < r \leq \delta_k$,

$$u_k(y_k) - (I_{\mu_k})_r[\phi](y_k) - (I_{\mu_k})^r[u_k](y_k) \leq f(y_k),$$

where $(I_{\mu_k})^r[u_k](x) := \int_{|z|\geq r} u_k(x + \eta(x, z)) - u_k(x)\, d\mu^k(z)$.

We want to pass to the limit in this equation and consider first the $(I_{\mu_k})^r$-term. By the definition of $\overline{u}$ and (H^3_η),

$$\limsup_{k\to+\infty} u_k(y_k + \eta(y_k, z)) \leq \overline{u}(x + \eta(x, z)) \quad \text{for a.e. } z.$$

Hence, since we integrate away from the singularity of μ, we can use Fatou's lemma and (H^1_μ) and (H^2_μ) to show that

$$\limsup_{k\to\infty} (I_{\mu_k})^r[u_k](y_k) \leq \int_{|z|>r} \overline{u}(x + \eta(x, z)) - \overline{u}(x)\, d\mu(z) = I^r[\overline{u}](x).$$

To pass to the limit in the $(I_{\mu_k})_r$-term, we have to write it as

$$(I_{\mu_k})_r[\phi](y_k) = \underbrace{\int_{|z|<r} \phi(x + \eta(y_k, z)) - \phi(y_k) - D\phi(y_k)\cdot\eta(y_k, z)\, d\mu^k(z) + \gamma_{\mu_k,r}(y_k)\cdot D\phi(y_k)}_{=(\tilde{I}_{\mu_k})_r[\phi](y_k)},$$

where $\gamma_{\mu_k,r}(x) := \int_{|z|<r} \eta(x, z)\, d\mu^k(z)$. For $|z| < r$, a Taylor expansion then yields

$$\left|\phi(y_k + \eta(y_k, z)) - \phi(y_k) - D\phi(y_k)\cdot\eta(y_k, z)\right| \leq \|D^2\phi\|_{L^\infty(B(x,c_\eta r))}|z|^2.$$

Hence by (H^1_η), (H^3_η), (H^1_μ) and (H^2_μ), we can use the Dominated Convergence Theorem to show that

$$(\tilde{I}_{\mu_k})_r[\phi](y_k) \to \int_{|z|<r} \phi(x + \eta(x, z)) - \phi(x) - \eta(x, z)D\phi(x)\ d\mu(z) = \tilde{I}_r[\phi](x).$$

Next, by Lemma 2.1,

$$\gamma_{\mu_k,r}(y_k) = \int_{|z|<r} \eta(y_k, z)\, d\mu^k_\#(z) + c\int_{y_{k,N}\leq|z|<r} \eta(y_k, z)\, d\mu^k_*(z),$$

where the last integral is understood to be zero if $y_{k,N} > r$. Note that since $y_{k,N} \to x_N > 0$, the domain of integration of the μ_*-integral is always bounded away from $z = 0$ when k is big. Along with (H^3_η) and (H^2_μ), this allows us to pass to

the limit in the μ_*-integral using the Dominated Convergence Theorem. Similarly, we may pass to the limit in the $\mu_\#$-integral by (H^1_η), (H^3_η) and (H^1_μ). We find that

$$\begin{aligned}\gamma_{\mu_k,r}(y_{k,N}) \to \gamma_r(x) &:= \int_{|z|<r} z\, d\mu_\#(x) + c\int_{x_N\le|z|<r} \eta(x,z)\, d\mu_*(x)\\ &= \text{P.V.}\int_{|z|<r} \eta(x,z)\, d\mu(x) =: \gamma_r(x),\end{aligned}$$

where we used Lemma 2.1 again. Hence we can conclude that

$$\lim_{k\to\infty} (I_{\mu_k})_r[\phi](y_k) = \tilde{I}_r[\phi](y_k) + \gamma_r(x)\cdot D\phi(x) = I_r[\phi](x).$$

Since $\delta_k \to \delta$, we end up with the following limit equation,

$$\overline{u}(x) - I_r[\phi](x) - I^r[\overline{u}](x) \le f(x)$$

for every $0 < r < \delta$. Using the Dominated Convergence Theorem again, we send $r \to \delta$ and obtain the subsolution condition for (1.1) at the point $x \in \Omega$.

The second part of the proof is to consider the case of when $x \in \partial\Omega$, i.e the case when $x_N = 0$. We first do it in the case $c = 1$. By adding, subtracting, and divinding by terms, we may rewrite the subsolution condition as

$$\frac{u_k(y_k) - (\tilde{I}_{\mu_k})_\delta[\phi](y_k) - (I_{\mu_k})^\delta[u_k](y_k) - f(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|} - \frac{\gamma_{\mu_k,\delta}(y_k)\cdot D\phi(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|} \le 0.$$

By Lemma 3.3, $|\gamma_{\mu_k,\delta}(y_k)| \to \infty$, and since u_k and f are uniformly bounded,

$$\frac{u_k(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|}, \quad \frac{f(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|}, \quad \frac{(I_{\mu_k})^\delta[u_k](y_k)}{|\gamma_{\mu_k,\delta}(y_k)|},$$

all converge to zero. The same is true for

$$\frac{(\tilde{I}_{\mu_k})_\delta[\phi](y_k)}{|\gamma_{\mu_k,\delta}(y_k)|}$$

since the integrand of the numerator is controlled by $\|D^2\phi\|_\infty |z|^2 1_{\{|z|<\delta\}}$ and μ^k satisfies (H^1_μ). Using Lemma 3.3 again, we have $\gamma_{\mu_k,\delta}(y_k)/|\gamma_{\mu_k,\delta}(y_k)| \to -\mathbf{n}$, so that we may go to the limit in the above inquality to find that

$$-\frac{\partial\phi}{\partial x_N}(x) = \frac{\partial\phi}{\partial\mathbf{n}}(x) \le 0\,.$$

In the case when $c = 0$, the measure $\mu = \mu_\#$ which less singular than μ_*. The same line of arguments as in the proof for $x \in \Omega$ (much easier this time) now shows that the equation holds at $x \in \partial\Omega$. □

Proof of Lemma 3.3. First note that by Lemma 2.2 with y_k instead of x and μ^k instead of μ,

$$\gamma_{\mu_k,\delta}(y_k)' = \int_{|z|<\delta} \eta(y_k,z)'\, d\mu^k(z) = \int_{|z|<\delta} \eta(y_k,z)'\, d\mu^k_\#(z)\,,$$

which remains uniformly bounded in k because of (H^1_η) and our assumption on $\mu_\#$. Since $y_{k,N} \to x_N = 0$, we can assume that $0 \le y_{k,N} < \delta$, and by Lemma 2.2,

$$(\gamma_{\mu_k,\delta})_N(y_k) = \int_{|z|<\delta} \eta(y_k,z)_N\, d\mu^k_\#(z) + \int_{y_{k,N}<|z|<\delta} \eta(y_k,z)_N\, d\mu^k_*(z)\,.$$

As above, the first integral is uniformly bounded as $k \to \infty$. For the second one, we send $r \to 0$ in Lemma 2.2-(iii) to find that

$$\int_{|z|<\delta} \eta(y_k, z)_N \, d\mu_*^k(z) \geq \int_{\substack{|z|<\delta \\ y_{k,N}<z_N}} (z_N - y_{k,N}) \, d\mu_*^k(z) \geq 0, \tag{3.2}$$

and, since $y_{k,N} \to 0$, we can then use Fatou's lemma to show that

$$\int_{\substack{|z|<\delta \\ z_N>0}} z_N \, d\mu_*(z) \leq \liminf_{k\to\infty} \int_{\substack{|z|<\delta \\ y_{k,N}<z_N}} (z_N - y_{k,N}) \, d\mu_*^k(z) \, .$$

Applying symmetry of the measure μ_* twice, we are lead to

$$\int_{\substack{|z|<\delta \\ z_N>0}} z_N \, d\mu_*(z) = \frac{1}{2} \int_{|z|<\delta} |z_N| \, d\mu_*(z) = \frac{1}{2N} \int_{|z|<\delta} \sum_{i=1}^{N} |z_i| \, d\mu_*(z),$$

so by taking (H^3_μ) into account, there is a constant $C = C(N) > 0$ such that

$$\int_{\substack{|z|<\delta \\ z_N>0}} z_N \, d\mu_*(z) \geq \frac{C}{2N} \int_{|z|<\delta} |z| \, d\mu_*(z) = \infty.$$

Hence we have proved that $(\gamma_{\mu_k,\delta})_N(y_k) \to \infty$ as $k \to \infty$, and if we use that $\big(\gamma_{\mu_k,\delta}\big)'(y_k)$ is uniformly bounded, we see that

$$\frac{\gamma_{\mu_k,\delta}(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|} = \Big(\frac{(\gamma_{\mu_k,\delta})'(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|} \, , \, \frac{(\gamma_{\mu_k,\delta})_N(y_k)}{|\gamma_{\mu_k,\delta}(y_k)|} \Big) \longrightarrow (0, 0, \cdots, 0, 1) = -\mathbf{n}.$$

□

4. Comparison in non-censored cases

In this section we prove a comparison result for the non-censored cases, i.e. under assumption (H^i_η) for $i = 0, \ldots, 5$. These assumptions covers all the examples given in the introduction, except example (a) – the censored case. As a conseqence of the comparison result and the results of the previous sections, we also obtain well-posedness for (1.1). The comparison result is the following:

Theorem 4.1. *Assume* (H_μ), (H_f), *and* (H^i_η) *hold for* $i = 0, 1, 2, 3, 4, 5$. *Let* u *be a bounded usc subsolution of (1.1) with data* $f \in C_b(\mathbb{R}^N)$, v *be a bounded lsc supersolution of (1.1) with data* $g \in C_b(\mathbb{R}^N)$ *such that* $f \leq g$ *in* $\overline{\Omega}$. *Then* $u \leq v$ *on* Ω.

From this result it follows that the half-relaxed limits in Theorem 3.2 satisfy $\overline{u} \leq \underline{u}$ in $\overline{\Omega}$. Since the opposite inequality is always satisfied, this means that $u := \overline{u} = \underline{u}$ is solution of (1.1) according to Definition 2.1. Uniqueness and continuous dependence (on f) follows from Theorem 4.1 by standard arguments and we have the following result:

Corollary 4.2. *Under the assumptions of Theorem 4.1, there exits a unique viscosity solution of* (1.1) *depending continuously on* f.

Proof of Theorem 4.1. We argue by contradiction assuming that $M := \sup_{\overline{\Omega}}(u - v) > 0$. We provide the full details only when $c = 1$. The case $c = 0$ is far simpler since the equation then holds even on the boundary.

To get a contradiction, we first introduce the function

$$\Psi_R(x) := u(x) - v(x) - \psi_R(x, x) \, ,$$

where ψ_R is a localisation term which makes the max attained:

$$\psi_R(x,y) = \psi(|x_N|/R) + \psi(|y_N|/R) + \psi(|x'|/R) + \psi(|y'|/R)\,,$$

with ψ a smooth function such that

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \le s < 1/2\,, \\ \text{increasing} & \text{for } 1/2 \le s < 1\,, \\ 2(\|u\|_\infty + \|v\|_\infty + 1) & \text{for } s \ge 1\,. \end{cases}$$

Of course, $M_R := \max \Psi_R \to M$ as $R \to \infty$ so that for R big, $\max \Psi_R > 0$. Then there are two cases for such $R > 0$:

(a) either there exists a maximum point $\bar{x}_R$ of Ψ_R which is not located on the boundary. In this case the proof is quite classical: we use the doubling of variables plus a localisation term around $\bar{x}_R$ by considering the max of

$$u(x) - v(y) - \frac{|x'-y'|^2}{\varepsilon'^2} - \frac{|x_N - y_N|^2}{\varepsilon_N^2} - \psi_R(x,y) - \sigma|x - \bar{x}_R|^4\,,$$

where $\sigma > 0$ is small. The localization term $\sigma|x-\bar{x}_R|^4$ is chosen so that $\bar{x}_R$ is the unique maximum point of $x \mapsto \Psi_R(x) - \sigma|x-\bar{x}_R|^4$, and by choosing σ small enough, the contribution of this function in the integral term is small[1].

Hence the maximum points $(\bar{x},\bar{y})$ of the above test-function converges necessarely to $(\bar{x}_R,\bar{x}_R)$ as $\varepsilon', \varepsilon_N \to 0$ and they are also bounded away from the boundary if $\varepsilon', \varepsilon_N$ are small enough. This property implies that we can use directly the equation and obtain $\max \Psi_R \le 0$, which is a contradiction.

(b) or any maximum point $\bar{x}_R$ is located on the boundary. In this case we use the doubling of variables plus some extra term to push the points inside (see below)[1]:

$$\Phi_{\varepsilon',\varepsilon_N,\nu,R}(x,y) := u(x) - v(y) - \frac{|x'-y'|^2}{\varepsilon'^2} - \frac{|x_N-y_N|^2}{\varepsilon_N^2} - \psi_R(x,y) + \mathrm{d}_\nu(x_N) + \mathrm{d}_\nu(y_N)\,.$$

In this case we can assume without loss of generality that the maximum points $\bar{x}, \bar{y}$ are always such that $0 < \bar{x}_N, \bar{y}_N < 1$, whatever $\varepsilon_N, \varepsilon', \nu > 0$ are.

Note that in both cases we take two distinct real parameters $\varepsilon', \varepsilon_N > 0$ in order to take into account the special contraction property in the N-th direction, see (H^5_η). Now, since case (a) is rather standard, we only provide a proof of case (b) which is more involved.

The term d_ν plays the role of a distance to the boundary of the domain; such term is usual in classical Neumann proofs in order to prevent the maximum points to be on the boundary. More precisely, for $\nu > 0$, we take $\mathrm{d}_\nu(\cdot) = \nu \mathrm{d}(\cdot)$ where d is a smooth function such that

$$\mathrm{d}(s) = \begin{cases} s & \text{for } 0 \le s < 1/2\,, \\ \text{increasing} & \text{for } 1/2 \le s < 1\,, \\ 1 & \text{for } s \ge 1\,. \end{cases}$$

Let us note that if $0 < \nu < 1$ and $R \gg 1$ are fixed, then $\Phi := \Phi_{\varepsilon',\varepsilon_N,\nu,R} \le 0$ for $|x|, |y|$ large enough, while, by choosing $x = y$ in a suitable way by taking into account the fact that $M > 0$, we have $\Phi_{\varepsilon',\varepsilon_N,\nu,R}(x,x) > 0$ for ν small; hence the

[1] In the viscosity inequalities, the various penalization terms are only integrated near the origin, e.g. in $\{|z| < \delta\}$. Therefore we do not need to worry about the integrability at infinity of $|x|^2$ and $|x|^4$ with respect to the measure μ.

maximum of Φ is attained at some point $(\bar{x},\bar{y}) \in \bar{\Omega}^2$, that we still denote by (x,y) for simplicity.

After proving that the points $\bar{x},\bar{y}$ are inside Ω, we are going to first let $\varepsilon_N \to 0$, then $\varepsilon' \to 0$, then $\nu \to 0$ and finally $R \to \infty$. Because of this use of parameters, we have

$$M_{\varepsilon_N,\varepsilon',\nu,R} := \max \Phi_{\varepsilon_N,\varepsilon',\nu,R} \to M > 0\,.$$

In particular, this implies that

$$x_N - y_N = O(\varepsilon_N)\,,\ x' - y' = O(\varepsilon')\,,\ \frac{|x'-y'|^2}{\varepsilon'^2} = o_{\varepsilon_N,\varepsilon'}(1)\,,$$

where the $O(\varepsilon_N), O(\varepsilon')$ are uniform with respect to all the parameters, and the $o_{\varepsilon_N,\varepsilon'}(1)$ means precisely that after passing to the limit as $\varepsilon_N \to 0$, we are left with a quantity which is an $o_{\varepsilon'}(1)$. Also note that

$$u(x) - v(y) = M + o_{\varepsilon_N,\varepsilon',\nu,R}(1)\,,$$

where the order of the parameters is important as explained above.

Step 1 – *Pushing the points inside.*
We denote by

$$\varphi(x,y) := \frac{|x'-y'|^2}{\varepsilon'^2} + \frac{|x_N-y_N|^2}{\varepsilon_N^2} + \psi_R(x,y) - \mathrm{d}_\nu(x_N) - \mathrm{d}_\nu(y_N)\,,$$

where we have dropped the parameters for the sake of simplicity of notations.

In this step, we prove that the F-viscosity inequalities hold for u and v. According to Definition 2.1, this is clearly the case if $c = 0$ since these viscosity inequalities hold even if the maximum or minimum points are on the boundary.

In the $c = 1$ case, let us assume that the maximum point (x,y) is such that $x_N = 0$, then x is a (global) maximum point of the function $z \mapsto u(z) - v(y) - \varphi(z,y)$ and, thanks to Definition 2.1, we should have $-\frac{\partial \varphi}{\partial x_N}(x,y) \le 0$. But, recalling that $\frac{\partial \psi_R}{\partial x_N}$ is zero in a neighborhood of the boundary, we have

$$-\frac{\partial \varphi}{\partial x_N}(x,y) = -\frac{2(x_N-y_N)}{\varepsilon_N^2} - \frac{\partial \psi_R}{\partial x_N}(x,y) + \frac{d}{ds}\mathrm{d}_\nu(0) = \frac{2y_N}{\varepsilon_N^2} + \nu > 0\,,$$

which is a contradiction. Therefore x_N cannot be zero and a similar argument shows that $y_N > 0$ as well, hence both x and y are inside Ω.

Step 2 – *Writing the viscosity inequalities and sending δ to zero.*
We introduce a (small) fixed parameter $0 < \delta < 1$ which is the parameter appearing in Definition 2.1 in order to give sense to different terms in the equation. We write the definition of viscosity sub and super solutions, using the test-function in the ball B_δ for a $\delta < \rho := \min(x_N, y_N, 1)$, and the functions u and v outside this ball. Since u is a viscosity subsolution and the function $u(\cdot) - v(y) - \varphi(\cdot, y)$ reaches a maximum at x, then we have the viscosity subsolution condition that we write as follows, thanks to Lemma 2.1:

$$\begin{aligned} &u(x) - \int_{|z|<\delta} [\varphi(x+\eta(x,z),y) - \varphi(x,y) - D_x\varphi(x,y)\cdot\eta(x,z)]\,d\mu(z) \\ &- \text{P.V.} \int_{|z|<\delta} D_x\varphi(x,y)\eta(x,z)\,d\mu(z) - \int_{|z|\ge\delta} [u(x+\eta(x,z)) - u(x)]\,d\mu(z) \le f(x)\,. \end{aligned}$$

For simplicity of notations, we leave out the P.V. notation since the integral can be expressed as converging integrals and we use the notation $P(x,z) := x+\eta(x,z)$. Next we use Lemma 2.2-(i) and the similar super solution condition on v to get

$$\begin{aligned}
&-\int_{|z|<\delta}[\varphi(P(x,z),y)-\varphi(x,y)-D_x\varphi(x,y)\cdot\eta(x,z)]\,d\mu(z)\\
&-D_x\varphi(x,y)\cdot\int_{|z|<\delta}\eta(x,z)\,d\mu_{\#}(z)\\
&-\int_{|z|<\delta}[\varphi(x,P(y,z))-\varphi(x,y)+D_y\varphi(x,y)\cdot\eta(y,z)]\,d\mu(z)\\
&+D_y\varphi(x,y)\cdot\int_{|z|<\delta}\eta(y,z)\,d\mu_{\#}(z)\\
&-\int_{|z|\geq\delta}[u(P(x,z))-v(P(y,z))-u(x)+v(y)]\,d\mu(z)\\
&+u(x)-v(y)\leq f(x)-f(y)\,.
\end{aligned}$$

In order to pass to the limit as $\delta\to 0$ to get rid of the test-function φ, we use Lemma 2.1 for all the terms which are smooth functions: the integrals over $B(0,\delta)$ all vanish as $\delta\to 0$ and we are left with limit of the integral over $\{|z|>\delta\}$. To this end, we split this integral into two integrals, one over $\{|z|\geq 1\}$ (which is independent of δ of course) and the other over $\{\delta\leq|z|<1\}$ that we have to deal with.

Using the definition of the maximum point for Φ, we have that for any z:

$$u(P(x,z))-v(P(y,z))-\varphi(P(x,z),P(y,z))\leq u(x)-v(y)-\varphi(x,y)\,.$$

Hence, it follows that

$$\begin{aligned}
&u(P(x,z))-v(P(y,z))-(u(x)-v(y))\\
&\leq\frac{|P(x,z)_N-P(y,z)_N|^2}{\varepsilon_N^2}-\frac{|x_N-y_N|^2}{\varepsilon_N^2}+\frac{|P(x,z)'-P(y,z)'|^2}{\varepsilon'^2}-\frac{|x'-y'|^2}{\varepsilon'^2}\\
&\quad+\psi_R(P(x,z),P(y,z))-\psi_R(x,y)\\
&\quad-\mathrm{d}_\nu(P(x,z)_N)+\mathrm{d}_\nu(x_N)-\mathrm{d}_\nu(P(y,z)_N)+\mathrm{d}_\nu(y_N)\,,
\end{aligned}$$

and we put this inequality into the integral over $\{\delta\leq|z|<1\}$ which gives rise to several terms denoted by (with obvious notation):

$$\int_{\delta\leq|z|<1}\big\{u(P(x,z))-v(P(y,z))-u(x)+v(y)\big\}\,d\mu(z)\leq T^\delta_{\varepsilon_N}+T^\delta_{\varepsilon'}+T^\delta_{\psi_R}+T^\delta_{\mathrm{d}_\nu}\,.$$

As for the ε_N-terms, we get rid of them by (H^5_η) which implies $T^\delta_{\varepsilon_N}\leq 0$. Then for the ε'-terms we write

$$\begin{aligned}
T^\delta_{\varepsilon'}&=\int_{\delta\leq|z|<1}\Big(\frac{|P(x,z)'-P(y,z)'|^2}{\varepsilon'^2}-\frac{|x'-y'|^2}{\varepsilon'^2}\Big)d\mu\\
&\leq\frac{1}{\varepsilon'^2}\int_{\delta\leq|z|<1}|\eta(x,z)'-\eta(y,z)'|^2\,d\mu(z)\\
&\quad+\frac{2}{\varepsilon'^2}\int_{\delta\leq|z|<1}(x'-y')\cdot(\eta(x,z)'-\eta(y,z)')\,d\mu(z)\,.
\end{aligned}$$

For the first term of $T^\delta_{\varepsilon'}$, we use the domination of the integrand by $c|z|^2$ to pass to the limit as $\delta \to 0$. For the second one, we use Lemma 2.2-(iii) which allows to wipe out the symmetric μ_*-contribution, so that we get in the limit

$$\limsup_{\delta\to 0} T^\delta_{\varepsilon'} \le \frac{1}{\varepsilon'^2}\int_{0<|z|<1} |\eta(x,z)' - \eta(y,z)'|^2\, d\mu(z) + \frac{2}{\varepsilon'^2}\int_{0<|z|<1} (x'-y')\cdot(\eta(x,z)' - \eta(y,z)')\, d\mu_\#(z)\,.$$

We concentrate now on the penalisation terms which are given by integrals of smooth functions. Note first that we are in the case when $0 < x_N, y_N < 1$, so that the $\psi(x_N/R)$ and $\psi(y_N/R)$-terms vanish (we assumed that $R \gg 1$). Hence, using Lemma 2.1 we get as $\delta \to 0$ the following two contributions:

$$\lim_{\delta\to 0} T^\delta_{\mathrm{d}_\nu} = -\tilde I_1[\mathrm{d}_\nu](x) - \frac{d}{ds}\mathrm{d}_\nu(x_N)\,\mathrm{P.V.}\int_{0<|z|<1}\eta(x,z)_N\, d\mu(z) + (\dots)(y),$$

$$\lim_{\delta\to 0} T^\delta_{\psi_R} \le -\tilde I_1[\tilde\psi_R](x) - D\tilde\psi_R(x)'\cdot \mathrm{P.V.}\int_{0<|z|<1}\eta(x,z)_N\, d\mu(z) + (\dots)(y).$$

where $\tilde\psi_R(x) := \psi(|x'|/R)$ and the $(\dots)(y)$ notation stands for the same terms but calculated at y instead of x. Now, note that $\frac{d}{ds}\mathrm{d}(x_N) > 0$ and use Lemma 2.2-(iii) (with $r = x_N > 0$). This gives that in the principal value for T_{d_ν}, the μ_*-term which is multiplied by $(-\nu)$ has a nonpositive contribution. So we find that

$$\lim_{\delta\to 0} T^\delta_{\mathrm{d}_\nu} \le -\nu\,\tilde I_1[\mathrm{d}](x) - \nu\,\frac{d}{ds}\mathrm{d}(x_N)\int_{0<|z|<1}\eta(x,z)_N d\mu_\#(z) + (\dots)(y) = o_\nu(1)\,.$$

As for the T_{ψ_R}-term, this time we use Lemma 2.2-(ii), which implies that the symmetric μ_*-part of the principal value vanishes:

$$\begin{aligned}\lim_{\delta\to 0} T^\delta_{\psi_R} &= -\tilde I_1[\tilde\psi_R](x) - \frac{1}{R}\left[\frac{d\psi}{ds}(|x'|/R)\right]'\cdot\int_{0<|z|<1}\eta(x,z)'\, d\mu_\#(z) + (\dots)(y)\\ &\le C(\mu)\Big(\frac{1}{R^2}\|\psi\|_{C^2} + \frac{1}{R}\|\psi\|_{C^1}\Big) = o_R(1)\,.\end{aligned}$$

Thus, the parameters $\varepsilon', \varepsilon_N, \nu, R > 0$ are still fixed for the moment and after sending $\delta \to 0$ we have obtained:

$$\begin{aligned}u(x) - v(y) \le\ & f(x) - f(y) + o_\nu(1) + o_R(1)\\ &+ \frac{1}{\varepsilon'^2}\int_{|z|<1}|\eta(x,z)' - \eta(y,z)'|^2\, d\mu(z)\\ &+ \frac{2}{\varepsilon'^2}\int_{|z|<1}(x'-y')\cdot(\eta(x,z)' - \eta(y,z)')\, d\mu_\#(z)\\ &+ \int_{|z|\ge 1}\big\{u(P(x,z)) - v(P(y,z)) - u(x) + v(y)\big\}\, d\mu(z)\\ &= f(x) - f(y) + o_\nu(1) + o_R(1) + \mathrm{Int}_1 + \mathrm{Int}_2 + \mathrm{Int}_3\,.\end{aligned}$$

Step 3 – *Sending the parameters to their limits*
We let first $\varepsilon_N \to 0$, the other parameters remaining fixed for the moment and we recall that $|x_N - y_N| = O(\varepsilon_N)$. Moreover, as long as R is fixed, the points x, y remain in a compact subset of $\overline{\Omega}$; therefore we can assume without loss of generality that x, y are converging to points (still denoted by x, y) such that $x_N = y_N$.

Combining (H^3_η) and (H^4_η), we obtain in particular that

$$\lim_{\varepsilon_N\to 0} |\eta(x,z)'-\eta(y,z)'| \le C|z||x'-y'| \quad \text{for } \mu\text{-a.e. } z\,.$$

Then, (H^1_η) and the integrability condition on $\mu_\#$ justify that we can use dominated convergence in Int_2. The argument is similar for Int_1, using the domination

$$|\eta(x,z)'-\eta(y,z)'|^2 \le (2c_\eta)^2|z|^2\,.$$

So we find that $\lim_{\varepsilon_N\to 0}\mathrm{Int}_2 = 0$ while

$$\lim_{\varepsilon_N\to 0}\mathrm{Int}_1 \le C\frac{|x'-y'|^2}{\varepsilon'^2}\int_{|z|<1}|z|^2 d\mu(z) = o_{\varepsilon'}(1).$$

The $o_R(1)$ and $o_\nu(1)$ terms are uniform with respect to the other parameters, so there is no problem to send $\varepsilon_N,\varepsilon'\to 0$ here. Next, since $|x-y|\to 0$ as $\varepsilon_N,\varepsilon'\to 0$ here, we may assume that $x,y\to\bar x\in\Omega$ by considering a subsequence is necessary. By continuity of f, it then follows that $\big(f(x)-f(y)\big)\to 0$.

We then pass also to the limit as $\nu\to 0$ and get:

$$u(\bar x)-v(\bar x) \le \limsup_{\nu\to 0}\ \limsup_{\varepsilon'\to 0}\ \limsup_{\varepsilon_N\to 0}\ [\mathrm{Int}_3] + o_R(1)\,. \tag{4.1}$$

Passage to the limit in the Int_3 term is possible because the domain of integration does not meet the singularity of the integral: we need only use the u.s.c. and l.s.c. properties of u and v, together with Fatou's Lemma (because the integrand is bounded and μ is finite on $\{|z|\ge 1\}$). After passing to the limit in $\varepsilon_N,\varepsilon'$ and ν, we have by definition

$$\lim_{\nu\to 0}\lim_{\varepsilon'\to 0}\lim_{\varepsilon_N\to 0}\big(u(x)-v(y)\big) = M + o_R(1)$$

so that

$$\begin{aligned}&\limsup_{\nu\to 0}\ \limsup_{\varepsilon'\to 0}\ \limsup_{\varepsilon_N\to 0}\ [\mathrm{Int}_3]\\ &\le \int_{|z|\ge 1}\big\{u(P(\bar x,z))-v(P(\bar x,z))-\big(M+o_R(1)\big)\big\}\,d\mu(z)\,.\end{aligned}$$

Now since and $u(P(\bar x,z))-v(P(\bar x,z))\le \sup_{\overline\Omega}(u-v) = M$,

$$\limsup_{\nu\to 0}\ \limsup_{\varepsilon'\to 0}\ \limsup_{\varepsilon_N\to 0}\,[\mathrm{Int}_3] \le \int_{|z|\ge 1} o_R(1)\,d\mu(z) = o_R(1).$$

When $R\to\infty$ in (4.1), we get $M\le 0$ and the proof is complete. □

5. Comparison in the censored case I.

In this section we give comparison and well-posedness results for the initial value problem (1.1) in the censored case (under assumption (H^6_η)) when the measure μ is not too singular:

(H'_μ) The measure μ is a nonnegative Radon measure satisfying

$$(i)\quad \int_{\mathbb{R}^N}|z|\wedge 1\,d\mu<\infty \quad\text{and}\quad (ii)\quad \int_{\{z_N=a\}} d\mu = 0 \quad \text{for any } a<0\,.$$

In addition, we assume the existence of a "blow-up supersolution"

(*U*) There exists $R_0 > 0$ such that, for any $R > R_0$, there is a positive function $\mathcal{U}_R \in C^2(\Omega)$ such that

$$-I[\mathcal{U}_R](x) \geq -K_R \quad \text{in} \quad \{x : 0 < x_N \leq R\},$$

for some $K_R \geq 0$, and

$$\mathcal{U}_R(x) \geq \frac{1}{\omega_R(x_N)} \quad \text{in} \quad \Omega,$$

for some function ω_R which is nonnegative, continuous, strictly increasing in a neighbourhood of 0, and satisfies $\omega(0) = 0$.

Remark 5.1. See Appendix A for a discussion of this assumption. E.g. in Remark A.1 we prove that (U) holds if

$$\mu = \bar{\mu} + \sum_{i=1}^{M} c_i \delta_{x^i}$$

where $c_i \in \mathbb{R}$, δ_{x^i} are delta measures supported at $\{x^i\}$ for $x^i_N > 0$, and

$$\frac{d\bar{\mu}}{dz} = \frac{g(z)}{|z|^{N+\alpha}} \qquad \text{where} \quad \alpha \in (0,1),\ 0 \leq g \in L^\infty(\mathbb{R}),\ \lim_{z\to 0} g(z) = g(0) > 0.$$

This class of measures include the Lévy measures of the stable, tempered stable, and self-decomposable Lévy processes. Much more general examples are presented in Appendix A.

Theorem 5.1. *Assume* (H'_μ), (H_f), (H^6_η) *and (U) hold. Let u be a bounded usc subsolution of* (1.1) *and v be a bounded lsc supersolution of* (1.1). *Then $u \leq v$ in $\overline{\Omega}$.*

As in the previous section, we immediatly get a well-posedness result for (1.1) by Theorems 5.1 and 3.2.

Corollary 5.2. *Under the assumptions of Theorem 5.1, there exists a unique viscosity solution of* (1.1) *depending continuously on f.*

Proof of Theorem 5.1. We argue by contradiction assuming that $M := \sup_{\overline{\Omega}}(u - v) > 0$. Take $R > R_0$ and $0 < \kappa \ll 1$. Using $0 < \varepsilon \ll 1$, we double the variables by introducing the quantities

$$\phi(x,y) = \frac{|x-y|^2}{\varepsilon^2} + \kappa[\mathcal{U}_R(x) + \mathcal{U}_R(y)] + \psi_R(x) + \psi_R(y),$$
$$\Phi(x,y) = u(x) - v(y) - \phi(x,y),$$

where $\mathcal{U}_R$ is given by (U) and $\psi_R(x) = 2(\|u\|_\infty + \|v\|_\infty)\psi(\frac{|x|}{R})$ for an increasing function $\psi(s) \in C^\infty(0,\infty)$ which is 0 in $(0, \frac{1}{2})$ and 1 in $(1, \infty)$.

For any R, κ and ε, the function Φ achieves its maximum at $(\bar{x}, \bar{y}) = (\bar{x}_{R,\kappa,\varepsilon}, \bar{y}_{R,\kappa,\varepsilon})$ and, by the definition of $\mathcal{U}_R$ and ψ_R, we have

$$\bar{x}_N, \bar{y}_N \geq \delta_0 = \omega_R^{-1}\Big(\frac{\kappa}{2(\|u\|_\infty + \|v\|_\infty)}\Big) \qquad \text{and} \qquad |\bar{x}|, |\bar{y}| \leq R. \tag{5.1}$$

These estimates will hold in most of the proof since we are going to keep R and κ fixed untill the end, sending $\varepsilon \to 0$ first. A standard argument also shows that

$$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By the estimates on $\bar{x}, \bar{y}$ and extracting a subsequence if necessary, we can assume without loss of generality that $\bar{x}, \bar{y} \to X$, $u(\bar{x}) \to u(X)$, and $v(\bar{y}) \to v(X)$ where X is a maximum point of $\Phi(x,x) = u(x) - v(x) - \phi(x,x)$. Finally, when we first send $\kappa \to 0$ and then $R \to +\infty$, we have

$$u(X) - v(X) \to M \quad \text{and} \quad \kappa \mathcal{U}_R(X) + \psi_R(X) \to 0 \, .$$

Now we write down the viscosity inequalities. Since $u - \phi(\cdot, \bar{y})$ has a global maximum at $\bar{x}$ and $v - (-\phi(\bar{x}, \cdot))$ has a global minimum at $\bar{y}$, we have that

$$u(\bar{x}) - I^\delta[u](\bar{x}) - I_\delta[\phi(\cdot, \bar{y})](\bar{x}) \leq f(\bar{x}),$$
$$v(\bar{y}) - I^\delta[v](\bar{y}) - I_\delta[-\phi(\bar{x}, \cdot)](\bar{y}) \geq f(\bar{y}).$$

With this in mind we see that

$$\begin{aligned} M + o(1) &= u(\bar{x}) - v(\bar{y}) - \phi(\bar{x}, \bar{y}) \\ &\leq I^\delta[u](\bar{x}) - I^\delta[v](\bar{y}) + I_\delta[\phi(\cdot, \bar{y})](\bar{x}) - I_\delta[\phi(\bar{x}, \cdot)](\bar{y}) + f(\bar{x}) - f(\bar{y}). \end{aligned} \tag{5.2}$$

In this inequality, we aim at first sending $\delta \to 0$ in order to get rid of the ε-depending $I_\delta[\phi]$-terms. In fact $I_\delta[\varphi] \to 0$ as $\delta \to 0$ by the Dominated Convergence Theorem since $|\eta(x,z)| \leq c_\eta |z|$, and hence for any C^1-function φ,

$$\int_{\mathbb{R}^N} |\varphi(x + \eta(x,z)) - \varphi(x)| \; 1_{|z|<\delta} d\mu(z) \leq c_\eta \|D\varphi\|_{L^\infty(B_{c_\eta \delta})} \int_{\mathbb{R}^N} 1_{|z|<\delta} \; |z| \; d\mu(z).$$

Next we consider the I^δ-terms. We restrict ourselves to a subsequence such that $\bar{x}_N \geq \bar{y}_N$ (if $\bar{x}_N \leq \bar{y}_N$ the argument is similar). Then

$$\begin{aligned} I^\delta[u](\bar{x}) - I^\delta[v](\bar{y}) &= \int_{-\bar{x}_N < z_N < -\bar{y}_N} [u(\bar{x} + z) - u(\bar{x})] \; 1_{|z|>\delta} d\mu(z) \\ &\quad + \int_{-\bar{y}_N < z_N} [u(\bar{x} + z) - v(\bar{y} + z) - (u(\bar{x}) - v(\bar{y}))] \; 1_{|z|>\delta} d\mu(z) \\ &=: I_1 + I_2. \end{aligned}$$

For I_1, we have

$$|I_1| \leq 2\|u\|_\infty \int_{|z|>\delta} 1_{\{-\bar{x}_N < z_N < -\bar{y}_N\}}(z) \; d\mu(z) \, .$$

Keeping κ and R fixed and recalling (5.1), we see that this integral is independent of δ as soon as $\delta < \delta_0$. Furthermore, because of (H'_μ) (ii), the Dominated Convergence Theorem implies that

$$I_1 \to 0 \quad \text{as} \quad \varepsilon \to 0$$

since $|\bar{x} - \bar{y}| \to 0$ as $\varepsilon \to 0$.

For I_2, we use the maximum point property for $\bar{x}, \bar{y}$,

$$\big(u(\bar{x} + z) - v(\bar{y} + z)\big) - \big(u(\bar{x}) - v(\bar{y})\big) \leq \phi(\bar{x} + z, \bar{y} + z) - \phi(\bar{x}, \bar{y}) \, ,$$

which after cancellation of the ε-terms leads to

$$I_2 \leq \kappa \Big(I^\delta[\mathcal{U}_R](\bar{x}) + I^\delta[\mathcal{U}_R](\bar{y}) \Big) + \Big(I^\delta[\psi_R](\bar{x}) + I^\delta[\psi_R](\bar{y}) \Big) \, .$$

Recalling again (5.1) and using the regularity of $\mathcal{U}_R$ and ϕ, we can send $\delta \to 0$ and obtain

$$\limsup_\delta \; I_2 \leq \kappa \Big(I[\mathcal{U}_R](\bar{x}) + I[\mathcal{U}_R](\bar{y}) \Big) + \Big(I[\psi_R](\bar{x}) + I[\psi_R](\bar{y}) \Big) \, ,$$

where each term on the right-hand side have a sense.

Consider equation (5.2) again. Using all the previous estimates, we can send $\delta \to 0$ first and obtain using (U) for the $\mathcal{U}_R$-terms that

$$M + o(1) \leq 2K_R\kappa + (I[\psi_R](\bar{x}) + I[\psi_R](\bar{y})) + (f(\bar{x}) - f(\bar{y}))\ .$$

In this inequality, we can first send $\varepsilon \to 0$, keeping R and κ fixed. Then $f(\bar{x}) - f(\bar{y}) \to 0$ as $\varepsilon \to 0$ since f is uniformly continuous in $\overline{B}_R$, and we find that

$$M + o(1) \leq 2K_R\kappa + 2I[\psi_R](X)\ .$$

We conclude by first sending $\kappa \to 0$ and then $R \to +\infty$. □

6. Comparison results in the censored case II.

In this section we give comparison and well-posedness results for the initial value problem (1.1) in the censored case (under assumption (H^6_η)) when the measure μ is very singular

(H''_μ) Hypothesis (H_μ) holds with

$$\mu_*(dz) = \frac{dz}{|z|^{N+\alpha}}\ ,\ \int_{\mathbb{R}^N} (1 \wedge |z|^{\overline{\beta}})\, \mu_\#(dz) < \infty\ ,\ \int_{\{z_N = a\}} \mu_\#(dz) = 0 \text{ for any } a < 0\ ,$$

for $\alpha \in (1,2)$ and $\overline{\beta} := \alpha - 1$.

This assumption is much more restrictive than (H_μ), and the results of this section are not completely satisfactory. We had lot of difficulties to obtain comparison results because on one hand, it is not possible to get rid of the boundary and the boundary condition in such a general way as we did in the less singular case I. On the other hand a lot of technical difficulties come from the the way the x-depending domain of integration in I interferes with the singularity of the measure and the boundary.

Our first result is the following

Theorem 6.1. *Assume* (H_f), (H^6_η), *and* (H''_μ) *hold.*
(a) Let u and v be respectively a bounded usc subsolution and a bounded lsc supersolution of

$$w(x) - I[w](x) = f(x) \quad \text{in} \quad \Omega\ , \tag{6.1}$$

and let us also denote by u and v respectively their usc or lsc extensions to $\overline{\Omega}$[2]. If there exists $C > 0$ and $\beta > \overline{\beta}$ such that

$$u(x', x_N) \geq u(x', 0) - Cx_N^\beta \quad \text{and} \quad v(x', x_N) \leq v(x', 0) + Cx_N^\beta \tag{6.2}$$

then u and v are respectively a bounded usc subsolution and a bounded lsc supersolution of (1.1).
(b) If u and v are respectively a bounded usc subsolution and a bounded lsc supersolution of (1.1) *satisfying* (6.2), *then*

$$u \leq v \quad \text{in} \quad \overline{\Omega}.$$

In particular, there exists at most one solution of (1.1) *in $C^{0,\beta}(\Omega)$ for $\beta > \overline{\beta}$.*

[2]For any $x' \in \mathbb{R}^{N-1}$, $u(x',0) := \limsup_{(y',y_N)\to(x',0)} u(y',y_N)$ and $v(x',0) := \liminf_{(y',y_N)\to(x',0)} v(y',y_N)$

Several comments have to be made on the different statements in Theorem 6.1. Part (a) means that, for sub and supersolutions having a suitable regularity at the boundary, the Neumann boundary condition is already encoded in the equation inside. This might be expected from the proof of Theorem 3.2 or from the intuition coming from the censored process. But the result is not true in general since we need anyway (6.2) to prove it.

Unfortunately part (b) does not provide the full comparison result for semi-continuous solutions, and we do not know if this result is optimal or not. Of course, in view of Theorem 6.1 (b), it is clear that we need a companion existence result providing the existence of solutions satisfying (6.2) or belonging to $C^{0,\beta}(\Omega)$ for $\beta > \overline{\beta}$. We address this question after the proof of Theorem 6.1.

Proof. We prove (a) only in the subsolution case since the supersolution case is analogous. Let ϕ be a smooth function which is bounded and has bounded first and second-order derivatives and assume that $u-\phi$ has a maximum point $(x',0) \in \partial\Omega$ in $B((x',0), c_\eta\delta) \cap \overline{\Omega}$. We may assume that the maximum is strict and global without any loss of generality.

We set $\theta(t) = t^{\overline{\beta}} \wedge 1$ for $t \geq 0$ and, for $0 < \kappa \ll 1$, we consider the function $u(x) - \phi(x) + \kappa\theta(x_N)$. By standard arguments, using the properties of ϕ, this function achieves a global maximum at a point nearby $(x',0)$, and we claim that this point cannot be on $\partial\Omega = \{x : \ x_N = 0\}$. Indeed, otherwise it would have to be $(x',0)$, the strict global maximum point of $u - \phi$ on $\partial\Omega$. But then by (6.2),

$$u(x',0) - \phi(x',0) \geq u(x) - \phi(x) + \kappa\theta(x_N) \geq u(x',0) - \phi(x',0) - 2Cx_N^\beta + \kappa\theta(x_N),$$

and we have a contradiction since $\beta > \overline{\beta}$ and hence $-2Cx_N^\beta + \kappa\theta(x_N) > 0$ for x_N small enough.

Therefore the function $x \mapsto u(x) - \phi(x) + \kappa\theta(x_N)$ has a maximum point at x_κ with $(x_\kappa)_N > 0$. We may write the viscosity inequality at x_κ as

$$u(x_\kappa) - \tilde{I}_\delta[\phi](x_\kappa) - \gamma(x_\kappa) \cdot D\phi(x_\kappa) + \kappa I_\delta[\theta](x_\kappa) - I^\delta[u](x_\kappa) \leq f(x_\kappa),$$

for (say) $0 < \delta < 1$, where $\gamma(x_\kappa) = P.V. \int_{|z|<\delta} \eta(x_\kappa, z)\mu(dz)$.

We first consider the term $\kappa I_\delta[\theta](x_\kappa)$. On one hand, the $\mu_\#$-part is $O(\kappa)$ since θ is in $C^{0,\overline{\beta}}$ and (H''_μ) holds. On the other hand, the singular part (the μ_* part) is nothing but

$$\kappa \text{ P.V.} \int_{\substack{|z|\leq\delta \\ x_N+z_N\geq 0}} \theta(x_N + z_N) - \theta(x_N) \ \frac{dz}{|z|^{N+\alpha}} \ ,$$

where we have dropped the subscript κ to simplify the notation. Since $\delta < 1$ and $x_N \to 0$ as $\kappa \to 0$, we may assume that $0 \leq x_N + z_N < 1$ for $|z| \leq \delta$, and hence that the principal value reduces to

$$\kappa \text{ P.V.} \int_{\substack{|z|\leq\delta \\ x_N+z_N\geq 0}} |x_N + z_N|^{\overline{\beta}} - |x_N|^{\overline{\beta}} \ \frac{dz}{|z|^{N+\alpha}} \ .$$

By the computations of Lemma B.1 in the Appendix,

$$-\text{P.V.} \int_{x_N+z_N\geq 0} |x_N + z_N|^{\overline{\beta}} - |x_N|^{\overline{\beta}} \ \frac{dz}{|z|^{N+\alpha}} = 0$$

for $x_N > 0$. Writing

$$\kappa \text{ P.V.} \int_{\substack{|z|\le\delta \\ x_N+z_N\ge 0}} (\cdots) = \kappa \text{ P.V.} \int_{x_N+z_N\ge 0} (\cdots) \; - \kappa \int_{\substack{|z|>\delta \\ x_N+z_N\ge 0}} (\cdots) \;\; ,$$

we conclude that for fixed δ,

$$\kappa \text{ P.V.} \int_{\substack{|z|\le\delta \\ x_N+z_N\ge 0}} \theta(x_N+z_N) - \theta(x_N) \; \frac{dz}{|z|^{N+\alpha}} = O(\kappa) \; .$$

Finally, the u, $\tilde{I}_\delta$, and I^δ terms are uniformly bounded in κ while $\gamma(x_\kappa) \to \infty$ since $(x_\kappa)_N \to 0$. We divide the above inequality by $|\gamma(x_\kappa)|$ and send $\kappa \to 0$. As in the proof of Theorem 3.2 – the second part, when $x \in \partial\Omega$ and $c = 1$ – the result is that all terms vanish except the γ-term and we are left with the boundary condition

$$\frac{\partial \phi}{\partial \mathbf{n}}(x) \le 0 \, .$$

Now we prove part (b). By linearity of the problem and part (a), the function $w = u - v$ is a subsolution of (1.1) with $f \equiv 0$, and we are done if we can prove that $w \le 0$. To prove this, we consider the function

$$\chi_{R,\nu}(x) := \psi(|x_N|/R) + \psi(|x'|/R) - \nu \mathrm{d}(x_N) \; ,$$

where ψ and d are defined as in the proof of Theorem 4.1, replacing, in the case of ψ, $2(\|u\|_\infty + \|v\|_\infty + 1)$ by $2\|w\|_\infty + 1$. The function $\chi_{R,\nu}$ is smooth and easy computations show that $\chi_{R,\nu}$ is a supersolution of (1.1) with an $f \ge \varpi(R,\nu)$ where $\varpi(R,\nu)$ converges uniformly to 0 as $R \to \infty$ and $\nu \to 0$. At the boundary $\partial\Omega$,

$$-\frac{\partial \chi_{R,\nu}}{\partial x_N} = 0 + \nu \cdot 1 > 0.$$

Because of the behavior of $\chi_{R,\nu}$ at infinity, the function $w - \chi_{R,\nu}$ achieves its maximum at some point x, and because of the behaviour of $\chi_{R,\nu}$ at the boundary, $x_N > 0$. Writing the viscosity subsolution inequality then yields that

$$w(x) - \chi_{R,\nu}(x) \le -\chi_{R,\nu}(x) + I[\chi_{R,\nu}](x) + I^\delta[u - \chi_{R,\nu}](x) \le -\varpi(R,\nu) + 0,$$

where we have used that $I^\delta[\psi](x) \le 0$ at any maximum point x of ψ. Hence, for any $y \in \overline{\Omega}$,

$$w(y) - \chi_{R,\nu}(y) \le -\varpi(R,\nu),$$

and part (b) follows from sending $R \to \infty$ and then $\nu \to 0$. □

Now we turn to the existence of Hölder continuous solutions and we begin with a result in 1-d.

Theorem 6.2. *Assume $N = 1$ and that* (H_f), (H^6_η), *and* (H''_μ) *hold.*
(a) Any bounded, uniformly continuous solution of (1.1) *is in $C^{0,\beta}(\overline{\Omega})$ for some $\beta > \overline{\beta}$.*
(b) There exists a solution of (1.1) *in $C^{0,\beta}(\overline{\Omega})$ for some $\beta > \overline{\beta}$.*

Proof. (a) To prove the Hölder reglarity we consider

$$M = \sup_{[0,+\infty)\times[0,+\infty)} (u(x) - u(y) - C|x-y|^\beta) \; , \tag{6.3}$$

and argue by contradiction assuming that $M > 0$. The aim is to show that this is impossible for $C > 0$ large enough. A rigorous proof would consists in introducing localization terms like the ψ-terms in the proof of Theorem 4.1 and d_ν-terms in

order to take care of the Neumann boundary condition. We drop these terms for the sake of simplicity in order to emphasize the main idea and not loose the reader in technicalities.

Therefore we assume that the above supremum is achieved at (x, y) with $x, y > 0$. Since $M > 0$ we have $x \neq y$, and we assume below $x < y$. The other case can be treated analogously. To simplify the notation, we introduce the function $\phi(z) := C|x-y+z|^\beta$. Note that this function is concave in the intervals $(-\infty, y-x)$ and $(y-x, +\infty)$, and that it is smooth in $(-\delta, \delta)$ for $\delta \leq y-x$ so that it can be used as a test function. By the maximum point property for (x, y),

$$u(x+z_1) - u(y+z_2) - C|x-y+(z_1-z_2)|^\beta \leq u(x) - u(y) - C|x-y|^\beta \ ,$$

for $z_1 \geq -x$ and $z_2 > -y$, and hence

$$u(x+z) - u(y+z) - [u(x) - u(y)] \leq 0 \qquad \text{for} \qquad z \geq -x \ (> -y), \tag{6.4}$$

$$u(x+z) - u(x) \leq [\phi(z) - \phi(0)] \quad \text{for} \quad z \geq -x, \tag{6.5}$$

$$u(y+z) - u(y) \geq -[\phi(-z) - \phi(0)] \quad \text{for} \quad z \geq -y. \tag{6.6}$$

Using the definition of viscosity solution and the symmetry of the measure μ_*, for $\delta, \delta' > 0$ small enough, we have the inequalities

$$-(I_\delta[\phi] + I^\delta[u])(x) + u(x) \leq f(x) \quad \text{and} \quad -(I_{\delta'}[\phi] + I^{\delta'}[u])(y) + u(y) \geq f(y),$$

which reduce here to

$$\begin{aligned} -\int_{-x}^{-\delta} (u(x+z) - u(x)) d\mu(z) - \int_{-\delta}^{\delta} [\phi(z) - \phi(0) - \phi'(0)z] d\mu(z) \\ - \int_{-\delta}^{\delta} \phi'(0) z d\mu(z) - \int_{\delta}^{+\infty} (u(x+z) - u(x)) d\mu(z) + u(x) \leq f(x) \ , \end{aligned} \tag{6.7}$$

$$\begin{aligned} -\int_{-y}^{-\delta'} (u(y+z) - u(y)) d\mu(z) + \int_{-\delta'}^{\delta'} [\phi(-z) - \phi(0) + \phi'(0)z] d\mu(z) \\ - \int_{-\delta'}^{\delta'} \phi'(0) z d\mu(z) - \int_{\delta'}^{+\infty} (u(y+z) - u(y)) d\mu(z) + u(y) \geq f(y) \ . \end{aligned} \tag{6.8}$$

In the proof below we will subtract these inequalities and the main difficulty of the proof will come from the term

$$J := -\int_{-y}^{-x} (u(y+z) - u(y)) d\mu(z)$$

which is *not* a difference of terms from (6.7) and (6.8). Indeed the domain of integration $z \in (-y, -x)$ appears in inequality (6.8) but not in (6.7). Because of the singularity of μ, if x is close to 0 it is not obvious how to get an estimate for J which is independent of C, or how to control this "bad term" by a good term. Therefore we have problems with this term if $x \to 0$ when $C \to +\infty$. For the $\mu_\#$-part of J there is no problem, we can use (6.6) to see that

$$-\int_{-y}^{-x} (u(y+z) - u(y)) d\mu_\#(z) \leq \int_{-y}^{-x} [\phi(-z) - \phi(0)] d\mu_\#(z) \leq C \int_{-y}^{-x} |z|^\beta d\mu_\#(z) \ ,$$

and we will see later that this term can be controlled since $|z|^\beta$ is $\mu_\#$-integrable.

First case – We first consider the case when $x \le y - x$, or equivalently, $2x \le y$. In this case $J \ge 0$ and can be dropped from inequality (6.8). To see this we note that for $-y \le z \le -x$,

$$2x - y \le x - y - z \le x$$

with $x \le y - x$ and $2x - y = -(y - x) + x \ge -(y - x)$, and hence by (6.6)

$$u(y+z) - u(y) \ge -[\phi(-z) - \phi(0)] = |x-y|^\beta - |x-y-z|^\beta \ge 0 \,. \tag{6.9}$$

In this first case, we choose $\delta = x$ and $\delta' = y - x$ and subtract the viscosity inequalities (6.7) and (6.8). After some computations using (6.5), (6.6), and (6.9), and dropping the J term, we are lead to the inequality

$$\begin{aligned} &-\int_{-x}^{y-x} [\phi(z) + \phi(-z) - 2\phi(0)] d\mu(z) \\ &-\int_{y-x}^{+\infty} ((u(x+z) - u(y+z)) - (u(x) - u(y))) d\mu(z) + u(x) - u(y) \le f(x) - f(y) \,. \end{aligned}$$

Some easy computations then shows that the first integral equals

$$-C(y-x)^{\beta-\alpha} \int_{-\frac{x}{y-x}}^{1} [|1+z|^\beta + |1-z|^\beta - 2] \frac{dz}{|z|^{1+\alpha}} + O(C) \,,$$

where the $O(C)$-term comes from the $\mu_\#$ part of the measure since the integrand can be estimated by $2|z|^\beta$ which is integrable on, say, $(-1, 1)$. The second integral is nonpositive by (6.4) and can be dropped because of the "$-$" in front.

Finally, since f is bounded and $u(x) - u(y) \ge 0$ (by assumption), we obtain

$$-C(y-x)^{\beta-\alpha} \int_{-\frac{x}{y-x}}^{1} [|1+z|^\beta + |1-z|^\beta - 2] \frac{dz}{|z|^{1+\alpha}} \le 2\|f\|_\infty + O(C) \,. \tag{6.10}$$

In order to conclude, we use that $M = u(x) - u(y) - C|x-y|^\beta > 0$ (by assumption) and $\beta \le 1 \le \alpha$ to find that

$$|x-y| \le \Big(\frac{2\|u\|_\infty}{C}\Big)^{1/\beta} \quad \text{and} \quad C(y-x)^{\beta-\alpha} \ge K C^\zeta \,,$$

where $\zeta := 1 + (\alpha - \beta)\beta^{-1} > 1$ and $K = (2||u||_\infty)^{\frac{\beta-\alpha}{\beta}}$. Then we note that

$$-\int_{-\frac{x}{y-x}}^{1} [|1+z|^\beta + |1-z|^\beta - 2] \frac{dz}{|z|^{1+\alpha}} \ge -\int_0^1 [|1+z|^\beta + |1-z|^\beta - 2] \frac{dz}{|z|^{1+\alpha}} > 0 \,,$$

since $z \mapsto |1+z|^\beta$ is strictly concave on $(-1, 1)$. From inequality (6.10) we then find that

$$\tilde{K} C^\zeta \le 2\|f\|_\infty + O(C) \,,$$

which cannot hold for C large enough and we have a contradiction in the first case.

Second case – When $x > y - x$, or equivalently, $2x > y$. In this case we choose $\delta = \delta' = y - x$, subtract viscosity inequalities (6.7) and (6.8), and use (6.6) to see that

$$\begin{aligned} &-\int_{-y}^{-x} [\phi(-z) - \phi(0)] d\mu(z) - \int_{-(y-x)}^{y-x} [\phi(z) + \phi(-z) - 2\phi(0)] d\mu(z) \\ &\quad -\int_{y-x}^{+\infty} ((u(x+z) - u(y+z)) - (u(x) - u(y))) d\mu(z) + u(x) - u(y) \le f(x) - f(y) \,. \end{aligned}$$

Arguing as in the first case, we can drop all u-terms and are lead to an inequality of the form

$$-C(y-x)^{\beta-\alpha}(B(a)+G) \le 2\|f\|_\infty + O(C)\,,$$

where

$$B(a) = \int_{-a-1}^{-a} (|1+z|^\beta - 1)\ \frac{dz}{|z|^{N+\alpha}},$$
$$G = \int_{-1}^{1} (|1+z|^\beta + |1-z|^\beta - 2)\ \frac{dz}{|z|^{N+\alpha}},$$

with $a = x/(y-x) > 1$. A technical computation (Corollary B.3 in the appendix) then shows that $B(a) + G \le -\kappa < 0$ for some $\beta > \overline{\beta}$ and we can conclude the argument as in the first case. The proof of (a) is complete.

Note the important estimate, valid in both cases: there exist $k_1, k_2 > 0$ such that

$$I[u](x) - I[u](y) \le -k_1 C|x-y|^{\beta-\alpha} + k_2(1+C)\,, \tag{6.11}$$

where the 1 comes from the localization terms. This formal estimate should be interpreted in the viscosity sense and with the above choice(s) of test function and parameters δ and δ', cf. e.g. (6.10).

(b) To show the existence of solutions with a suitable regularity property, we follow the so-called "Sirtaki method in 4 steps". We just give a formal sketch the proof which is an easy adaptation of the above arguments.

We start by building a suitable approximate problem. We approximate the Lévy measure μ by bounded measures $\mu_n = \mu\, 1_{|z|>1/n}$ for $n \ge 1$ and denote the associated nonlocal term by I_n. Then we introduce a truncation of the nonlocal term and add an additional viscosity term. The results is the approximate equation

$$-\epsilon u_{xx} - T_R(I_n[u]) + u = f \quad \text{in } \Omega\,,$$

where $T_R(s) := \max(\min(s,R), -R)$, $R, \epsilon > 0$.

1. For fixed ϵ, n, R, since the T_R-term and the measure μ_n are bounded, this equation can easily be solved by classical viscosity solutions' methods (Perron's method and comparison result). This provides us with a continuous solution which is bounded and we even have $||u||_\infty \le ||f||_\infty$.

Moreover, in order to obtain the $C^{0,\beta}$-regularity and $C^{0,\beta}$-bounds, we consider (6.3) and follow the arguments in the first part of this proof. After subtracting the viscosity sub- and supersolution inequalities, we formally obtain

$$\begin{aligned} -\epsilon\,[u_{xx}(x) - u_{xx}(y)] - [T_R(I_n[u])(x) - T_R(I_n[u])(y)] \\ +u(x) - u(y) \le f(x) - f(y)\,. \end{aligned} \tag{6.12}$$

For the second-derivatives, we have an analogue estimate to (6.11), namely there exists $k_1', k_2' > 0$ such that

$$u_{xx}(x) - u_{xx}(y) \le -k_1' C|x-y|^{\beta-2} + k_2'\,. \tag{6.13}$$

Note that to give meaning to this formal estimate, we must consider instead of u_{xx} the sub- and super jets of the theorem of sums, cf. e.g. [4]. Now consider (6.12) with fixed $R, \varepsilon > 0$. Since the T_R-terms are bounded, we can rewrite it as

$$-\epsilon\,[u_{xx}(x) - u_{xx}(y)] \le 2R + 2(||u||_\infty + ||f||_\infty)\,,$$

and use (6.13) to find that the inequality cannot hold for C large enough. This implies that the solution $\{u^{n,R,\epsilon}\}$ is at least $C^{0,\beta}$ by the arguments of the regularity proof above.

2. The above argument also shows that, for fixed ϵ, the $C^{0,\beta}$-bounds for the $\{u^{n,R,\epsilon}\}$ are uniform in n since they depend only on R through the T_R-term. This allows us to pass to the limit $n \to +\infty$ and get a solution $u^{R,\epsilon} := \lim_{n\to+\infty} u^{n,R,\epsilon}$ of the limiting equation enjoying the same $C^{0,\beta}$-bound. This solution satisfies the truncated viscous equation with μ_n replaced by the singular measure μ.

3. Next, we repeat the proof of the $C^{0,\beta}$-bound for the truncated viscous equation : Estimate (6.11) together with the fact that T_R is an increasing and a 1-Lipschitz continuous function, implies that

$$T_R(I[u](x)) - T_R(I[u](y)) \le k_2\,.$$

at least for C big enough. Rewriting the analogue of (6.12) as

$$-\epsilon\,[u_{xx}(x) - u_{xx}(y)] \le [T_R(I[u])(x) - T_R(I[u])(y)] + 2(||u||_\infty + ||f||_\infty)\,,$$

this new estimates on the difference of the truncated terms shows that the $C^{0,\beta}$-bound which is obtained in Step 1, is independent of R and we can let $R \to +\infty$. The result is that the limit $u^\varepsilon := \lim_{R\to\infty} u^{R,\varepsilon}$ is a $C^{0,\beta}$-solution of the non-truncated viscous equation

$$-I[u] - \epsilon u_{xx} + u = f \quad \text{in } \Omega\,.$$

4. Finally we come back again to the proof of the $C^{0,\beta}$-bound but, this time, the main role is played by the non-local term via estimate (6.11). Indeed we rewrite the analogue of (6.12) as

$$-\,[I[u](x) - I[u](y)] \le \epsilon\,[u_{xx}(x) - u_{xx}(y)] + 2(||u||_\infty + ||f||_\infty)\,,$$

and remark that, since the u_{xx}-terms satisfy (6.13), the ϵ-term in (6.12) can be estimated by $\epsilon k_2'$. Using (6.11), we obtain again a contradiction for large enough C. The argument is the same as in Step 3 with the roles of the local and nonlocal terms exchanged. This also explains the terminology "Sirtaki's method", since Sirtaki is a danse where we exchange the roles of the two feet as we exchange here the role of the ϵu_{xx} and $I[u]$ terms. To conclude the argument, we have found that the $C^{0,\beta}$-bound is independent of ϵ, and we pass to the limit as $\epsilon \to 0$. We get a solution u of the original problem belonging to $C^{0,\beta}$. Since this solution is unique, it is the solution we are looking for. □

Now we turn to the case when $N \ge 2$. Unfortunately we require far more retrictive assumptions on f.

Theorem 6.3. *Assume $N \ge 2$, that (H_f), (H^6_η), and (H''_μ) hold, and that $f(\dots, x_N)$ is in $W^{2,\infty}(\mathbb{R}^{N-1})$ for any $x_N > 0$ with uniformly bounded $W^{2,\infty}$-norms.*
(a) Any bounded, uniformly continuous solution of (1.1) is in $C^{0,\beta}(\overline{\Omega})$ for some $\beta > \overline{\beta}$.
(b) There exists a solution of (1.1) in $C^{0,\beta}(\overline{\Omega})$ for some $\beta > \overline{\beta}$.

Proof. We are not going to provide the full proof since it is rather long and tedious and is mostly based on two ingredients which we have already seen. But we remark that an easy consequence of the the comparison result and linearity of the problem,

is that u inherits the regularity of f. I.e. there exists a constant $K > 0$ such that, for any $x', z' \in \mathbb{R}^{N-1}$ and $x_N > 0$,

$$-K|z'|^2 \le u(x'+z', x_N) + u(x'-z', x_N) - 2u(x'+z', x_N) \le K|z'|^2 \,. \tag{6.14}$$

Then we repeat the 1-d proof essentially considering

$$\sup_{[0,+\infty)\times[0,+\infty)} (u(x', x_N) - u(x', y_N) - C|x_N - y_N|^\beta) \,.$$

Of course, a doubling of variables in x' is necessary to take care of the singularity of the measure, but using the $W^{2,\infty}$ property in x', we can go back to the 1-d computations without any difficulty. Let us just mention the key decomposition we use here. We rewrite the integrals with respect to μ_*, first replacing the integrands by

$$u(x'+z', x_N+z_N) + u(x'-z', x_N-z_N) - 2u(x', x_N),$$

and then by

$$\Delta^2_{z'} u(x', x_N+z_N) + \Delta^2_{z'} u(x', x_N-z_N) + 2\Delta^2_{z_N} u(x', x_N),$$

where

$$\Delta^2_{z'} u(x', x_N) := \frac{1}{2}\Big(u(x'+z', x_N) + u(x'+z', x_N) - 2u(x', x_N)\Big) \,,$$
$$\Delta^2_{z_N} u(x', x_N) := \frac{1}{2}\Big(u(x', x_N+z_N) + u(x', x_N-z_N) - 2u(x', x_N)\Big) \,.$$

These expressions are not equal pointwise of course, but they give the same integrals because of the symmetry of μ_*. We deal with the $\Delta^2_{z'}$-terms using (6.14), and the $\Delta^2_{z_N}$-term is treated as in the one dimensional case. Also note that we use a decomposition of Ω into sets like $\mathbb{R}^{N-1} \times \{z_N : \ a \le z_N \le b\}$, for $a, b > 0$, following the 1-d proof.

Finally, concerning the nonsymetric part $\mu_\#$, we use as usual the fact that it is a controlled term since it is less singular.

The existence is proved as in the proof of Theorem 6.2. □

Remark 6.1. The regularity results of the $N = 1$ and $N \ge 2$ cases are different. In the first case, the results is purely elliptic and we gain regularity. In the second case, the result is elliptic in the x_N-direction while in the other directions we just use a preservation of regularity argument. It is an open problem to find an elliptic argument also in the x'-directions.

7. The limit as $\alpha \to 2^-$

In this section we prove that all the Neumann models we consider converge to the same local Neumann problem as $\alpha \to 2^-$, provided that the nonlocal operators include the normalisation constant $(2-\alpha)$. To be more precise, we consider the following problem

$$\begin{cases} -(2-\alpha)\int_{\mathbb{R}^N} u_\alpha(x+\eta(x,z)) - u_\alpha(x)\, d\mu_\alpha + u_\alpha(x) = f(x) & \text{in } \Omega \,, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{in } \partial\Omega \,, \end{cases} \tag{7.1}$$

where $\alpha \in (0,2)$, η depends on the Neumann model we consider, and

$$\frac{d\mu_\alpha}{dz} = \frac{g(z)}{|z|^{N+\alpha}}$$

where g is nonnegative, continuous and bounded in $\mathbb{R}^N$, $g(0) > 0$ and $g \in C^1(B)$ for some ball B around 0.

We prove below that the solution of (7.1) converge to the solution of the following local problem,

$$\begin{cases} -a\Delta u - b \cdot Du + u = f & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{7.2}$$

where

$$a := g(0)\frac{|S^{N-1}|}{N} \quad \text{and} \quad b := Dg(0)\frac{|S^{N-1}|}{N}.$$

In this section $|S^{N-1}|$ denotes the measure of the unit sphere in $\mathbb{R}^N$ and Id_N the $N \times N$ identity matrix.

Theorem 7.1. *Assume* (H^i_η), $i = 0 \dots 4$ *hold and let* u_α *be the solutions of* (7.1) *for* $\alpha \in (0,2)$. *Then, as* $\alpha \to 2^-$, u_α *converges locally uniformly to the unique solution* u *of* (7.2).

Before providing the proof, we introduce the following sequences of measures:

$$\begin{aligned} (d\nu^1_\alpha)_{i,j} &= (2-\alpha) z_i z_j \frac{g(z)}{|z|^{N+\alpha}}\, dz\,, \\ d\nu^2_\alpha &= (2-\alpha) z \frac{g(z)-g(0)}{|z|^{N+\alpha}} dz\,, \\ (d\nu^3_{\alpha,y})_{i,j} &= (2-\alpha)\eta(y,z)_i \eta(y,z)_j \frac{g(z)}{|z|^{N+\alpha}}\, dz\,, \\ d\nu^4_{\alpha,y} &= (2-\alpha)\eta(y,z)\frac{g(z)-g(0)}{|z|^{N+\alpha}} dz\,, \end{aligned}$$

where $\eta(y,z)_i$ denotes the i-th component of the vector $\eta(y,z)$. Note that ν^1_α and $\nu^3_{\alpha,y}$ are matrix measures while ν^2_α and $\nu^4_{\alpha,y}$ are vector measures. The localization phenomenon occuring as $\alpha \to 2$ is reflected in the following lemma:

Lemma 7.2.
(a) As $\alpha \to 2^-$, $\nu^1_\alpha \rightharpoonup a\delta_0 \mathrm{Id}_N$ *and* $\nu^2_\alpha \rightharpoonup b\delta_0$ *in the sense of measures.*
(b) For any sequence $\alpha_k \to 2$ *and* $y_k \to x$, *there exist two vector functions* $\bar a(x), \bar b(x) \in \mathbb{R}^N$ *satisfying*

$$\frac{1}{2}a \le \bar a_i(x) \le \Lambda \quad \textit{and} \quad |\bar b_i(x)| \le \Lambda \quad \textit{for some } \Lambda = \Lambda(g,\eta) < \infty\,,$$

such that, at least along a subsequence,

$$\nu^3_{\alpha_k, y_k} \rightharpoonup \mathrm{diag}(\bar a(x))\delta_0\,, \quad \nu^4_{\alpha_k,y_k} \rightharpoonup \bar b(x)\delta_0\,,$$

where $\mathrm{diag}(\bar a(x))$ *is the diagonal matrix with diagonal coefficients* $\bar a_i(x)$.

Proof. If $\delta \in (0,1)$ is fixed, we notice first that, for any $K > 1$,

$$0 \le (2-\alpha)\int_{\delta<|z|<K} |z|^2 \frac{g(z)\,dz}{|z|^{N+\alpha}} \le \|g\|_\infty(\delta^{2-\alpha} - K^{2-\alpha}) \to 0 \text{ as } \alpha \to 2^-\,,$$

so that the only possible limit in the sense of measure is supported in $\{0\}$. Similar calculations show that the same is true for all the measures ν^i, $i = 2 \dots 4$.

Coming back to ν^1, we compute the inner integral as follows,

$$\begin{aligned}&(2-\alpha)\int_{|z|<\delta} z_i z_j \frac{g(z)}{|z|^{N+\alpha}}\,dz\\&= g(0)(2-\alpha)\int_{|z|<\delta} z_i z_j \frac{dz}{|z|^{N+\alpha}} + (2-\alpha)\int_{|z|<\delta} z_i z_j \frac{g(z)-g(0)}{|z|^{N+\alpha}}\,dz\,.\end{aligned}$$

The second integral vanishes as $\alpha \to 2$ since

$$\begin{aligned}&\Big|(2-\alpha)\int_{|z|<\delta} z_i z_j \frac{g(z)-g(0)}{|z|^{N+\alpha}}\,dz\Big|\\&\le C_g(2-\alpha)\int_{|z|<\delta}\frac{|z|^3}{|z|^{N+\alpha}}\,dz \le C_g(2-\alpha)\frac{\delta^{3-\alpha}}{3-\alpha}\to 0 \quad \text{as } \alpha\to 2^-,\end{aligned}$$

for $C_g = \|Dg\|_{L^\infty(B_\delta)}$. By symmetry, the first integral is zero for $i \neq j$, while for $i = j$,

$$\begin{aligned}g(0)(2-\alpha)\int_{|z|<\delta} z_i^2 \frac{dz}{|z|^{N+\alpha}} &= g(0)\frac{|S^{N-1}|}{N}(2-\alpha)\int_{r=0}^{\delta}\frac{r^{2+N-1}}{r^{N+\alpha}}\,dr\\&= g(0)\frac{|S^{N-1}|}{N}\delta^{2-\alpha} \longrightarrow a \quad \text{as } \alpha\to 2^-\,.\end{aligned}$$

This means that the measures $\{\nu^1_\alpha\}$ concentrate to a delta mass δ_0 multiplied by the diagonal matrix $a\mathrm{Id}_N$.

Let us now consider the inner integral for each component of the measures ν^2_α : using similar arguments, we have

$$\begin{aligned}&(2-\alpha)\int_{|z|<\delta} z_i \frac{g(z)-g(0)}{|z|^{N+\alpha}}\,dz\\&= (2-\alpha)\int_{|z|<\delta} z_i \frac{(z,Dg(0))+(z,Dg(z)-Dg(0))}{|z|^{N+\alpha}}\,dz\\&= \sum_{j=1}^N \frac{\partial g}{\partial x_j}(0)(2-\alpha)\int_{|z|<\delta} z_i z_j \frac{dz}{|z|^{N+\alpha}} + o_\delta(1)\\&= \frac{\partial g}{\partial x_i}(0)(2-\alpha)\int_{|z|<\delta} z_i^2 \frac{dz}{|z|^{N+\alpha}} + o_\delta(1)\\&\longrightarrow \frac{\partial g}{\partial x_i}(0)\frac{|S^{N-1}|}{N} + o_\delta(1) \quad \text{as} \quad \alpha\to 2^-\,.\end{aligned}$$

Hence, by the definition of b, ν^2_α concentrates to $b\delta_0$.

We now come to the measures ν^3 which is more complex to analyse due to the presence of the perturbation $\eta(y_k, z)$. We first notice that by using (H^2_η), it follows that for $i \neq j$,

$$\int_{|z|<\delta} \eta(y_k,z)_i \eta(y_k,z)_j \frac{g(z)dz}{|z|^{N+\alpha}} = 0\,.$$

Then by (H^1_η) $|\eta(y_k,z)| \le c_\eta|z|$, and we have

$$\begin{aligned}0 &\le g(0)(2-\alpha)\int_{|z|<\delta}\eta(y_k,z)_i^2\frac{dz}{|z|^{N+\alpha}}\,dz\\&\le g(0)c_\eta^2(2-\alpha)\int_{|z|<\delta}|z|^2\frac{dz}{|z|^{N+\alpha}}\,dz \le g(0)c_\eta^2|S^{N-1}|\,.\end{aligned}$$

So, the total mass of ν^3 is bounded and, by the same arguments as above, it is clear that the support of ν^3 shrinks to $\{0\}$ (or the empty set).

Then, we split the integral over $\{|z| < \delta\}$ as follows

$$(2-\alpha)\int_{|z|<\delta} \eta(y_k,z)_i^2 \frac{g(z)dz}{|z|^{N+\alpha}} = \int_{\substack{|z|<\delta \\ z_N > -y_{k,N}}} (\cdots) + \int_{\substack{|z|<\delta \\ z_N \le -y_{k,N}}} (\cdots) = (A_i) + (B_i)\,.$$

The first integral is easy to handle since $\eta_i(y_k,z) = z$ when $z_N > -y_{k,N}$,

$$\begin{aligned}(A_i) &= (2-\alpha)\int_{\substack{|z|<\delta \\ z_N > -y_{k,N}}} z_i^2 \frac{g(z)dz}{|z|^{N+\alpha}} \\ &= (2-\alpha)\int_{\substack{|z|<\delta \\ z_N > 0}} z_i^2 \frac{g(z)dz}{|z|^{N+\alpha}} + o(y_{k,N}) \to \frac{1}{2}a\,.\end{aligned}$$

The other integral has a sign and can take different values according to the structure of the jumps, but in all cases we see that the weak limit of ν^3 can be written as $\bar{a}(x)\delta_0$ where $\bar{a}(x)$ satisfies $a/2 \le \bar{a}_i(x) \le \Lambda$.

The measure ν^4 is treated similarly: the total mass can be bounded by

$$\begin{aligned}&(2-\alpha)\int_{|z|<\delta} |\eta(y,z)| \frac{|g(z)-g(0)|}{|z|^{N+\alpha}} dz \\ &\le c_\eta C_g (2-\alpha)\int_{|z|<\delta} |z|^2 \frac{dz}{|z|^{N+\alpha}} = c_\eta C_g |S^{N-1}| \delta^{2-\alpha}\,,\end{aligned}$$

so that, up to a subsequence, there exists indeed a vector function $\bar{b}$ such that $\nu^4_{\alpha_n,y_n} \to \bar{b}\delta_0$ in the sense of measures, with $\|\bar{b}\|_\infty \le c_\eta C_g |S^{N-1}|$. The result then holds with $\Lambda := |S^{N-1}| c_\eta \max\{C_g, g(0)\}$. □

Remark 7.1. Note that in the censored case, $\bar{a}(x) \equiv a/2$ since the jumps below level $-y_N$ are censored, while $\bar{a}(x) = a$ by symmetry when the jumps are mirror reflected. Under our general hypotheses, different structures of the jumps (i.e. different η's) lead to different $\bar{a}$'s which could in principle depend on x and the sequences α_k, y_k. We will overcome this difficulty by using the extremal Pucci operator associated to $\bar{a}(x)$: for any symmetric $N \times N$ matrix A with eigenvalues (λ_i) we define

$$\mathcal{M}^+(A) := \frac{a}{2}\sum_{\lambda_i<0} \lambda_i + \Lambda \sum_{\lambda_i>0} \lambda_i\,. \tag{7.3}$$

Proposition 7.3. *Let us define the half relaxed limits as $\alpha \to 2^-$,*

$$\bar{u}(x) := \limsup_{\alpha\to 2, y\to x} u_\alpha(y) \qquad \text{and} \qquad \underline{u}(x) := \liminf_{\alpha\to 2, y\to x} u_\alpha(y)\,.$$

Under the assumptions of Theorem 7.1, $\bar{u}$ is a viscosity subsolution of (7.2)*, and $\underline{u}$ is a viscosity supersolution of* (7.2)*.*

Proof. The proofs for $\bar{u}$ and $\underline{u}$ are similar, therefore we only provide it for $\bar{u}$. We have to check that $\bar{u}$ satisfy the viscosity subsolution inequalities for the Neumann problem (7.2) at any point $x \in \overline{\Omega}$. There are two separate cases to check, (i) when $x \in \Omega$ and (ii) when $x \in \partial\Omega$.

Step 1. Case (i) where $x \in \Omega$, that is $x_N > 0$. Let ϕ be a smooth function and assume that x is a strict local maximum point of $u - \phi$. By standard arguments there exists a sequence $(y_\alpha)_\alpha$ of local maximum points of $u_\alpha - \phi$ such that $y_\alpha \to x$

as $\alpha \to 2^-$. Moreover, since $x_N > 0$, by taking α close to 2, we can assume that $y_{\alpha,N} > \delta$ for some small $\delta > 0$. By the subsolution inequality for u_α at y_α,

$$-(2-\alpha)\int_{|z|<\delta} \phi(y_\alpha+z)-\phi(y_\alpha)-D\phi(y_\alpha)\cdot z\,d\mu_\alpha - (2-\alpha)\int_{|z|<\delta} D\phi(y_\alpha)\cdot z\,d\mu_\alpha$$
$$-(2-\alpha)\int_{|z|\geq\delta} u_\alpha(P(y_\alpha,z))-u_\alpha(y_\alpha)\,d\mu_\alpha + u_\alpha(y_\alpha) \leq f(y_\alpha).$$

We recall that the second integral of the left-hand side is well-defined : see the remark after Lemma 2.1.

We denote the three integral terms by I_1, I_2, and I_3. Then

$$\begin{aligned} I_1 &= -(2-\alpha)\int_{|z|<\delta} ((D^2\phi(y_\alpha)+o_\delta(1))z,z)\,d\mu_\alpha \\ &= -(2-\alpha)\int_{|z|<\delta} (D^2\phi(y_\alpha)z,z)\,d\mu_\alpha + o_\delta(1). \end{aligned}$$

Note that the $o_\delta(1)$-term is independent of α because the measure $(2-\alpha)|z|^2\mu_\alpha$ has bounded mass. The symmetry of μ_α implies that $\int_{|z|<\delta} z_i z_j\,d\mu_\alpha = 0$ and then, by Lemma 7.2, we get

$$I_1 = -(2-\alpha)\mathrm{Tr}(D^2\phi(y_\alpha))\int_{|z|<\delta}|z|^2\,d\mu_\alpha + o_\delta(1) = a\Delta\phi(x)+o_\alpha(1)+o_\delta(1)\,.$$

Similarly we have

$$I_2 = -(2-\alpha)\int_{|z|<\delta} D\phi(y_\alpha)\cdot z\,d\mu_\alpha = -(2-\alpha)D\phi(y_\alpha)\int_{|z|<\delta} z\,d\mu_\alpha,$$

and by symmetry of μ_α and Lemma 7.2 we see that

$$I_2 = -(2-\alpha)D\phi(y_\alpha)\int_{|z|<\delta} z\frac{g(z)-g(0)}{|z|^{N+\alpha}}\,dz = b\cdot D\phi(x)+o_\alpha(1)+o_\delta(1)\,.$$

The $o_\delta(1)$-terms are independent of α since the measures ν_α^2 of Lemma 7.2 have unformly bounded mass. For the last integral I_3, we use the boundedness of $(u_\alpha)_\alpha$ with respect to α to see that

$$|I_3| \leq C(2-\alpha)\int_{|z|\geq\delta}\frac{dz}{|z|^{N+\alpha}} \leq C'\frac{2-\alpha}{\alpha\delta^\alpha} \to 0 \quad \text{as} \quad \alpha\to 2\,. \tag{7.4}$$

So we keep $\delta > 0$ fixed and pass to the limit as $\alpha \to 2^-$ (and $y_\alpha \to x$) to get

$$-a\Delta\phi(x) - b\cdot D\phi(x) + \bar{u}(x) \leq f(x) + o_\delta(1)\,.$$

Then, since $\delta < x_N$ could be arbitrarily small, we pass to the limit as $\delta \to 0$ and get the viscosity subsolution condition for $\bar{u}$ at x.

Step 2. Case (ii) where $x \in \partial\Omega$, that is $x_N = 0$. We again consider a smooth function ϕ such that $\bar{u}-\phi$ has a strict local maximum point at x and, as above, we have a sequence $(y_\alpha)_\alpha$ of maximum points of $u_\alpha - \phi$ such that $y_\alpha \to x$ as $\alpha \to 2^-$.

In this step we are going to prove that

$$\min\Big(-\mathcal{M}^+(D^2u(x)) - \Lambda|Du(x)| + \bar{u}(x) - f(x)\ ;\ \frac{\partial\phi}{\partial\mathbf{n}}(x)\Big) \leq 0\,, \tag{7.5}$$

where $\mathcal{M}^+$ is defined in (7.3). We may assume that $\frac{\partial\phi}{\partial\mathbf{n}}(x) = -\frac{\partial\phi}{\partial x_N}(x) > 0$ since otherwise (7.5) is already satisfied. Then for α close to 2, $-\frac{\partial\phi}{\partial x_N}(y_\alpha) > 0$ by the

continuity of $D\phi$. We can also assume $y_\alpha \in \Omega$, since otherwise $y_\alpha \in \partial\Omega$ and then $\frac{\partial\phi}{\partial \mathbf{n}}(y_\alpha) = -\frac{\partial\phi}{\partial x_N}(y_\alpha) \leq 0$ for α close to 2 by Definition 2.1, and this would contradict our assumption.

Therefore $0 < y_{\alpha,N} \to 0$ as $\alpha \to 2$, and the subsolution inequality for u_α takes the form

$$\begin{aligned}
&-(2-\alpha)\int_{|z|<\delta} \phi(y_\alpha + \eta(y_\alpha, z)) - \phi(y_\alpha) - D\phi(y_\alpha)\cdot\eta(y_\alpha, z)\, d\mu_\alpha \\
&-(2-\alpha)\int_{|z|<\delta} D\phi(y_\alpha)\cdot\eta(y_\alpha, z)\, d\mu_\alpha \\
&-(2-\alpha)\int_{|z|\geq\delta} u_\alpha(P(y_\alpha) - u_\alpha(y_\alpha)\, d\mu_\alpha + u_\alpha(y_\alpha) \leq f(y_\alpha).
\end{aligned}$$

We denote as before the three integral terms by I_1, I_2, I_3. The compensator term I_2 can be written as

$$\begin{aligned}
I_2 &= -g(0)D\phi(y_\alpha)\cdot(2-\alpha)\int_{|z|<\delta}\eta(y_\alpha, z)\frac{dz}{|z|^{N+\alpha}} \\
&\quad - D\phi(y_\alpha)\cdot(2-\alpha)\int_{|z|<\delta}\eta(y_\alpha, z)\frac{g(z)-g(0)}{|z|^{N+\alpha}}dz \\
&= I_{2,1} + I_{2,2}\,.
\end{aligned}$$

For symmetry reasons of both η and the measure, $I_{2,1}$ reduces to the scalar product of the N-th components, and it has a sign,

$$I_{2,1} = -g(0)\frac{\partial\phi}{\partial x_N}(y_\alpha)(2-\alpha)\int_{|z|<\delta}\eta(y_\alpha, z)_N\frac{dz}{|z|^{N+\alpha}} \geq 0\,,$$

since $g(0)$, $-\frac{\partial\phi}{\partial x_N}(y_\alpha)$, and the η-integral are nonnegative (see Lemma 2.2 (iii)). Thus we may drop the $I_{2,1}$ term from the inequality above and get that

$$I_1 + I_{2,2} + I_3 + u_\alpha(y_\alpha) \leq f(y_\alpha)\,.$$

We now pass to the limit in this inequality as $\alpha \to 2$ and hence $y_{\alpha,N} \to 0$. The difference with Step 1 above, is that now y_α converge to the boundary so that we cannot take a fixed $0 < \delta < y_{\alpha,N}$ as $\alpha \to 2$. For the first integral, Lemma 7.2 enables us to take subsequences $\alpha_k \to 2$ and $y_\alpha \to 0$ such that (dropping the subscript k for simplicity)

$$\begin{aligned}
I_1 &= -(2-\alpha)\int_{|z|<\delta}\phi(y_\alpha + \eta(y_\alpha, z)) - \phi(y_\alpha) - D\phi(y_\alpha)\cdot\eta(y_\alpha, z)\frac{g(z)dz}{|z|^{N+\alpha}} \\
&= -\sum_{i,j}\int_{|z|<\delta}(\partial^2_{i,j}\phi(y_\alpha) + o_\delta(1))\, d(\nu^3_{\alpha,y_\alpha})_{i,j}(z) \\
&= -\sum_{i,j}\partial^2_{i,j}\phi(x)\int_{|z|<\delta} d(\nu^3_{\alpha,y_\alpha})_{i,j}(z) + o_\alpha(1) + o_\delta(1) \\
&= -\sum_i \bar{a}_i(x)\partial^2_{i,i}\phi(x) + o_\alpha(1) + o_\delta(1) \\
&\geq -\mathcal{M}^+(D^2\phi(x)) + o_\alpha(1) + o_\delta(1)\,.
\end{aligned}$$

The last term I_3 can be treated as in Step 1 and vanishes as $\alpha \to 2$. We are left with the $I_{2,2}$ term and use again Lemma 7.2, this time for the measure ν^4. The

result is the existence of a vector $\bar{b}(x)$ such that along subsequences we have

$$I_{2,2} = D\phi(y_\alpha)\cdot(2-\alpha)\int_{|z|<\delta}\eta(y_\alpha,z)\frac{g(z)-g(0)}{|z|^{N+\alpha}}dz$$
$$= D\phi(x)\cdot\bar{b}(x) + o_\alpha(1) \geq -\Lambda|D\phi(x)| + o_\alpha(1)\,.$$

Hence, passing to the limit $\alpha \to 2$ in the above inequality, leads to

$$-\mathcal{M}^+(D^2\phi(x)) - \Lambda|D\phi(x)| + \bar{u}(x) - f(x) \leq 0,$$

and (7.5) still holds.

Step 3. We shall prove now that boundary condition (7.5) reduces to the condition $\frac{\partial\phi}{\partial\mathbf{n}} \leq 0$. Let us assume on the contrary that $\frac{\partial\phi}{\partial\mathbf{n}}(x) > 0$ for some point x at the boundary $\{x_N = 0\}$ and some smooth function ϕ such that $u - \phi$ has a maximum point at x. For any $\tau, \varepsilon > 0$, we take a smooth, bounded function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\psi(t) = \tau\Big(t - \frac{t^2}{\varepsilon^2}\Big) \quad \text{for} \quad 0 \leq t \leq \varepsilon^2/2\,.$$

Since $\psi(0) = 0$ and $0 \leq \psi$ for $0 \leq t \leq \varepsilon^2/2$, it follows that $u(x) - \phi(x) - \psi(x_N)$ has again a local maximum point at x. Hence (7.5) holds with $\phi(x) + \psi(x_N)$ replacing $\phi(x)$, i.e.

$$\min\Big(E(\phi) + \frac{a}{2}\frac{2\tau}{\varepsilon^2} - \Lambda\tau;\ \frac{\partial\phi}{\partial\mathbf{n}}(x) - \tau\Big) \leq 0\,, \tag{7.6}$$

where

$$E(\phi) := -\mathcal{M}^+(D^2\phi(x)) - \Lambda|D\phi(x)| + \bar{u}(x) - f(x)\,.$$

Since we assumed that $\frac{\partial\phi}{\partial\mathbf{n}}(x) > 0$, we first fix $\tau > 0$ small enough so that the inequality $\frac{\partial\phi}{\partial\mathbf{n}}(x) - \tau > 0$ still holds. Then we can choose $\varepsilon > 0$ small enough to ensure that also

$$E(\phi) + \frac{a}{2}\frac{2\tau}{\varepsilon^2} - \Lambda\tau > 0\,.$$

But then we contradict (7.6), and hence the boundary condition for $\bar{u}$ reduces to $\partial\phi/\partial\mathbf{n} \leq 0$ everywhere on the boundary. This concludes the proof of Proposition 7.3. □

Proof of Theorem 7.1. We have seen that $\bar{u}$ is a subsolution of (7.2) while $\underline{u}$ is a supersolution of the same problem. Since $\underline{u} \leq \bar{u}$ on $\overline{\Omega}$ by definition and $\underline{u} \geq \bar{u}$ on $\overline{\Omega}$ by the comparison principle for (7.2), we see that $\underline{u} = \bar{u}$ on $\overline{\Omega}$. Setting $u := \underline{u} = \bar{u}$ on $\overline{\Omega}$, it immediately follows that u is a continuous (since $\underline{u}$ is lsc and $\bar{u}$ is usc) and the unique viscosity solution of (7.2). By classical arguments in the half-relaxed limit method, the sequence $(u_\alpha)_\alpha$ also converge locally uniformly to u. □

Appendix A. Blowup supersolution in censored case I.

In this section we assume (H^6_η) and (H'_μ) as in Section 5. Remeber that $\Omega := \big\{(x_1,\dots,x_N) = (x', x_N) : x_N \geq 0\big\}$. First we show that in the censored fractional Laplace case (i.e. the censored alpha stable case), we can essentially take

$$\mathcal{U}(x) = -\ln x_N$$

as our blowup supersolution in assumption (U) in Section 5.

Lemma A.1. *If* $d\mu(z) = \frac{dz}{|z|^{N+\alpha}}$ *for* $\alpha \in (0,1)$ *and* $\mathcal{U}(x) = -\ln(x_N)$*, then*

$$-I[\mathcal{U}](x) = -\int_{x_N+z_N\geq 0} \mathcal{U}(x+z) - \mathcal{U}(x)\ \frac{dz}{|z|^{N+\alpha}} > 0 \qquad \textit{for} \qquad x \in \Omega.$$

Proof. We first change variables, $\bar{z} = \frac{z}{x_N}$, to find that

$$-I[\mathcal{U}](x) = \int_{x_N+z_N\geq 0} \ln\Big(1+\frac{z}{x}\Big)\ \frac{dz}{|z|^{N+\alpha}} = \frac{1}{x_N^\alpha}\int_{\bar{z}_N\geq -1} \ln(1+\bar{z})\ \frac{d\bar{z}}{|\bar{z}|^{N+\alpha}}.$$

Now we are done if we can prove that

$$J = \int_{\bar{z}_N\geq -1} \ln(1+\bar{z})\ \frac{d\bar{z}}{|\bar{z}|^{N+\alpha}} > 0.$$

When $N = 1$, we take $1+\bar{z} = e^y$ and note that simple computations lead to

$$J = \int_{-\infty}^{\infty} y\frac{e^y dy}{|e^y-1|^{1+\alpha}} = \int_{-\infty}^{\infty} F(y)e^{\frac{y}{2}(1-\alpha)}dy \quad \text{where} \quad F(y) = \frac{y}{|2\sinh\frac{y}{2}|^{1+\alpha}}.$$

Since $F(y)$ is odd and $1-\alpha > 0$,

$$0 < -F(-y)e^{-\frac{y}{2}(1-\alpha)} < F(y)e^{\frac{y}{2}(1-\alpha)} \qquad \text{for} \qquad y > 0,$$

and hence by symmetry $J > 0$.

In the case $N > 1$ we introduce polar coordinates $z = ry$ where $r \geq 0$ and $|y| = 1$, and we let $dS(y)$ be the surface measure of the sphere $|y| = 1$ in $\mathbb{R}^N$. We then find that

$$J = \Big(\int_{|y|=1,y_N>0}\int_0^{\infty} + \int_{|y|=1,y_N<0}\int_0^{-\frac{1}{y_N}}\Big)\ln(1+ry_N)\frac{r^{N-1}dr\,dS(y)}{r^{N+\alpha}}.$$

The change of variables $s = y_N r$ then leads to

$$\begin{aligned} J &= \Big(\int_{|y|=1,y_N>0}\int_0^{\infty} + \int_{|y|=1,y_N<0}\int_0^{-1}\Big)\operatorname{sgn}(y_N)|y_N|^\alpha \ln(1+s)\frac{ds}{|s|^{1+\alpha}}\,dS(y) \\ &= \int_{|y|=1,y_N>0}|y_N|^\alpha\ dS(y)\ \int_{-1}^{\infty}\ln(1+s)\frac{ds}{|s|^{1+\alpha}}. \end{aligned}$$

The lemma now follows from the computations for we did for $N = 1$. □

We now generalize to a much larger class of integral operators with Lévy measures μ such that $d\mu(z) \sim \frac{dz}{|z|^{N+\alpha}}$ near $|z| = 0$. In this case the blowup supersoution will be the modified log-function $\mathcal{U}_R$ defined as

$$\mathcal{U}_R(x) = \bar{\mathcal{U}}_R(x_N) \qquad \text{for} \qquad x \in \Omega, \quad R > 1,$$

where $\bar{\mathcal{U}}_R$ is a (nonnegative) monotone decreasing $C^\infty(0,\infty)$ function such that

$$\bar{\mathcal{U}}_R(s) = \begin{cases} -\ln(s) + \frac{3}{2}\ln R & \text{if } 0 < s \leq R, \\ 0 & \text{if } s \geq 2R. \end{cases}$$

The main result in this appendix says that $\mathcal{U}_R$ will be the blowup "supersolution" of assumption (U) provided the Lévy measure μ also satisfies:

$(U)'$ For all $R, \varepsilon > 0$ there are $r, c, K > 0$ and $\alpha \in (0,1)$ such that

$$(a) \int_{-1<z_N\le R} \ln(1+z_N)\Big(s^\alpha \mu(sdz) - \frac{c\,dz}{|z|^{N+\alpha}}\Big) > -\varepsilon \qquad \text{for} \quad s \in (0,r),$$

$$(b) \int_{-1<z_N\le -\frac{1}{2}} \ln(1+z_N)\ \mu(sdz) \ge -K \qquad \text{for} \quad s \in (r,R).$$

Theorem A.2. *Assume* (H^6_η), $(\mathrm{H}_\mu)'$, *and* $(U)'$ *hold. Then the function* $\mathcal{U}_R$ *defined above satisfy the assumptions in (U). In particular, there is* $R_0 > 0$ *such that for any* $R > R_0$ *there is* $K_R \ge 0$ *such that*

$$-I[\mathcal{U}_R](x) \ge -K_R \quad \textit{in} \quad \{x : 0 < x_N \le R\}.$$

Before we prove this result, we show how assumption $(U)'$ can be checked when μ is Lévy measure whos restriction to $\{z : |z_N| \le r\}$ has a density

$$\frac{d\mu}{dz} = \frac{g(z)}{|z|^{N+\alpha}} \quad \text{where} \quad \begin{cases} \alpha \in (0,1), \\ 0 \le g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N; \frac{dz}{1+|z|^{N+\alpha}}), \\ \lim_{z\to 0} g(z) = g(0) > 0. \end{cases} \tag{A.1}$$

Note that the L^1 assumption makes $\frac{d\mu}{dz}$ integrable near infinity and that $L^\infty(\mathbb{R}^N) \subset L^1(\mathbb{R}^N; \frac{dz}{1+|z|^{N+\alpha}})$ for $\alpha > 0$.

Corollary A.3. *If* μ *has a density satisfying* (A.1)*, then the function* $\mathcal{U}_R$ *defined above satisfy the assumptions in (U).*

Proof. By Theorem A.2 we have to check that $(U)'$ holds. Part (b) follows from Hölder's inequality since $\ln(1+s) \in L^1(-1,0)$. Now we check part (a). Note that

$$s^\alpha \mu(sdz) - \frac{c\,dz}{|z|^{N+\alpha}} = \frac{g(sz)-c}{|z|^{N+\alpha}} dz.$$

Now choose $c = g(0)$ and write

$$\begin{aligned} &\int_{-1<z_N\le R} \ln(1+z_N)\Big(s^\alpha \mu(sdz) - \frac{c\,dz}{|z|^{N+\alpha}}\Big) \\ &\ge -\sup_{-s<r<Rs} |g(r)-g(0)| \int_{-1<z_N\le R} |\ln(1+z_N)| \frac{dz}{|z|^{N+\alpha}}. \end{aligned}$$

Part (a) now follows since the last integral is finite for any $R > 0$, while the sup-term goes to zero as $s \to 0$ by continuity of g at $z = 0$. □

Remark A.1. Assumption (A.1) also includes measures like

$$\mu = \sum_{i=1}^{M_1} \mu_i,$$

where μ_i have densities satifying (A.1) for different g_i and α_i. To see this, simply take $\alpha = \max_i \alpha_i$ and $g(z) = \sum_{i=1}^M g_i(z)|z|^{\alpha-\alpha_i}$ and note that $g \in L^1(\mathbb{R}^N; \frac{dz}{1+|z|^{N+\alpha}})$. We can even relax this assumption to include measures with zero or arbitrary negative α_i provided that $\max_i \alpha_i$ remains in $(0,1)$. Finally we mention that we need some assumption to insure that μ does not give to much mass to the negative part of the integral $-I[\mathcal{U}_R]$. In (A.1) we do this by requiring continuity at 0 of g, but a carefull reader can extend this assumption to allow some discontinuities at 0.

Remark A.2. In assumption $(U)'$ it is only the restriction of μ to the set

$$\{z : -r < z_N < Rr\} \cap \{z : -1 < z_N < -\frac{1}{2}\}$$

that plays any role. Hence if μ satisfies $(U)'$, by taking $\bar{r}$ small enough, so will $\mu + \bar{\mu}$ for any measure $\bar{\mu}$ satisfying

$$\int_{|z|>0} d\bar{\mu} < \infty \qquad \text{and} \qquad \operatorname{supp}\bar{\mu} \cap \{z : -1 < z_N < \bar{r}\} = \emptyset \quad \text{for some } \bar{r} > 0.$$

E.g. the delta-measure $\bar{\mu} = \sum_{i=1}^{M} \delta_{x^i}$ is ok if $x_N^i > 0$.

Proof of Theorem A.2. First note that there is an $R_0 > 0$ such that

$$J_{R_0} := \int_{-1<z_N\le R_0} \ln(1+z_N)\frac{dz}{|z|^{N+\alpha}} > 0.$$

Indeed, in the proof of Lemma A.1, we showed that $J_\infty = J > 0$. The result then follows by the Dominated Convergence Theorem since the integrand is positive for $z_N > 0$ and integrable.

For any $R > R_0$, we note immediatly that $\mathcal{U}_R$ is a nonnegative decreasing function which trivially satisfies the second part of (U) with $\omega_R(s) = \frac{1}{\mathcal{U}_R(s)}$. We will now check that $\mathcal{U}_R$ has the appropriate supersolution properties and hence complete the proof that $\mathcal{U}_R$ satisfies (U) under $(U)'$. By the definition of $\mathcal{U}_R$, we can write

$$\begin{aligned} -I[\mathcal{U}_R](x) &= \int_{-x_N<z_N\le Rx_N} \ln\Big(1+\frac{z_N}{x_N}\Big)\ \mu(dz) + I_R \\ &= \int_{-1<y_N\le R} \ln(1+y_N)\ \mu(x_N\,dy) + I_R, \end{aligned}$$

where $I_R = -\int_{z_N>Rx_N} \mathcal{U}_R(x+z) - \mathcal{U}_R(x)\ \mu(dz) > 0$ since $\mathcal{U}_R$ is decreasing. By assumption $(U)'$ we then find a $r > 0$ such that for $x_N \in (0, r)$,

$$-x_N^\alpha I[\mathcal{U}_R](x) \ge J_R + \int_{-1<y_N\le R} \ln(1+y)\ \Big(x_N^\alpha \mu(x_N\,dy) - \frac{dy}{|y|^{1+\alpha}}\Big) \ge \frac{1}{2}J_R > 0.$$

When $x_N \in (r, R)$, another application of $(U)'$ along with $(\mathrm{H}_\mu)'$ leads to

$$\begin{aligned} &-I[\mathcal{U}_R](x) \\ &\ge \Big(\int_{-x_N<z_N<-\frac{x_N}{2}} + \int_{-\frac{x_N}{2}<z_N<R\cap|z|<1} + \int_{-\frac{x_N}{2}<z_N<R\cap|z|>1}\Big)\ln\Big(1+\frac{z}{x_N}\Big)d\mu(dz) \\ &\ge -K - \max_{s\in(-\frac{1}{2},\frac{R}{r})}\frac{|\partial_s\ln(1+s)|}{|x_N|}\int_{|z|<1}|z|d\mu(z) - \max_{s\in(-\frac{1}{2},\frac{R}{r})}|\ln(1+s)|\int_{|z|>1}d\mu(z). \end{aligned}$$

Since this last expression is bounded for $x_N \in (r, R)$, this completes the proof. □

Appendix B. Estimates for the censored case II.

Lemma B.1. *Let $\mu(dz) = \frac{dz}{|z|^{N+\alpha}}$, $\alpha \in (1,2)$, and define $\tilde{\theta}(x) = |x_N|^\beta$. If $\beta \in (0,1)$ and $x \in \Omega$, then*

$$I[\tilde{\theta}](x) = P.V.\int_{x_N+z_N\ge 0} \tilde{\theta}(x+z) - \tilde{\theta}(x)\ \mu(dz) \begin{cases} > 0 & \text{if} \quad \beta > \alpha - 1, \\ = 0 & \text{if} \quad \beta = \alpha - 1, \\ < 0 & \text{if} \quad \beta < \alpha - 1. \end{cases}$$

Proof. First let $\beta \in (0,1)$ and $N = 1$, and define $\tilde{\theta}(x) = |x|^\beta$. Note that the change of variables $z = x\bar{z}$ followed by $1 + \bar{z} = e^s$ reveals that

$$\begin{aligned} I[\tilde{\theta}](x) &= P.V. \int_{x+z\geq 0} |x+z|^\beta - |x|^\beta \ \frac{dz}{|z|^{1+\alpha}} \\ &= |x|^{\beta-\alpha} \ P.V. \int_{\bar{z}\geq -1} |1+\bar{z}|^\beta - 1 \ \frac{d\bar{z}}{|\bar{z}|^{1+\alpha}} \\ &= |x|^{\beta-\alpha} \ P.V. \int_{-\infty}^{\infty} \frac{2\sinh\frac{\beta s}{2}}{|2\sinh\frac{s}{2}|^{1+\alpha}} e^{\frac{s}{2}(1+\beta-\alpha)} dx. \end{aligned}$$

When $\beta = \alpha - 1$, the integrand is odd and hence the integral is zero. For $\beta > \alpha - 1$ ($\beta < \alpha - 1$) the exponential factor makes the integral positive (negative). Hence when $\beta + 1 - \alpha = 0, > 0$ or < 0, then $I[\tilde{\theta}] = 0, > 0$, or < 0 respectively.

When $N > 1$, a similar result holds for $\tilde{\theta}(x) = |x_N|^\beta$. The idea is to work in polar coordinates. We set $x = ry$ for $r \geq 0$ and $|y| = 1$ and let $dS(y)$ denote the surface area element of the N-sphere $|y| = 1$. We also use the change of variables $ry_N = \bar{r}x_N$.

$$\begin{aligned} &I[\tilde{\theta}](x) \\ &= \int_{x_N+z_N\geq 0} |x_N+z_N|^\beta - |x_N|^\beta \ \frac{dz}{|z|^{N+\alpha}} \\ &= \int_{|y|=1} \int_{x_N+ry_N>0} |x_N+ry_N|^\beta - |x_N|^\beta \ \frac{r^{N-1} dr \, dS(y)}{r^{N+\alpha}} \\ &= \left(\int_{|y|=1, y_N>0} \int_0^\infty + \int_{|y|=1, y_N<0} \int_0^{-\frac{x_N}{y_N}} \right) (\cdots) \frac{dr \, dS(y)}{r^{1+\alpha}} \\ &= \left(\int_{|y|=1, y_N>0} \int_0^\infty - \int_{|y|=1, y_N<0} \int_0^{-1} \right) |x_N|^{\beta-\alpha} |y_N|^\alpha \Big(|1+\bar{r}|^\beta - 1 \Big) \frac{d\bar{r} \, dS(y)}{|\bar{r}|^{1+\alpha}} \\ &= |x_N|^{\beta-\alpha} \int_{|y|=1, y_N>0} |y_N|^\alpha \ dS(y) \int_{-1}^\infty |1+\bar{r}|^\beta - 1 \ \frac{d\bar{r}}{|\bar{r}|^{1+\alpha}}. \end{aligned}$$

Here the first integral is just a positive constant while the second integral is the same we found in the $N = 1$ case. The conclusion is therefore as in that case: When $\beta + 1 - \alpha = 0, > 0$ or < 0, then $I[\tilde{\theta}] = 0, > 0$, or < 0 respectively.

□

Next we consider the two integrals

$$\begin{aligned} B(a) &= \int_{-a-1}^{-a} |1+z|^\beta - 1 \ d\mu(z), \\ G &= \int_{-1}^{1} |1+z|^\beta + |1-z|^\beta - 2 \ d\mu(z), \end{aligned}$$

where $a > 1$, $\beta \in (0,1)$, and $d\mu(z) = \frac{dz}{|z|^{N+\alpha}}$ for $\alpha \in [1,2)$.

Proposition B.2. *If $\beta = \alpha - 1$ then there is a $\kappa > 0$ such that*

$$B(a) + G \leq -\kappa < 0$$

for any $a > 1$.

By continuity of the integrals in β we have the following corollary:

Corollary B.3. *There is $\kappa > 0$ and $\beta > \alpha - 1$ such that*

$$B(a) + G \leq -\kappa \leq 0$$

for any $a > 1$.

To prove Proposition B.2, note that $z + 1 \leq 0$ for $z \in (-a-1, -a)$ $(a > 1)$ and that the change of variable $1 + z = -e^x$ in $B(a)$ leads to

$$B(a) = \int_{\ln(a-1)}^{\ln a} \frac{2\sinh\frac{\beta x}{2}}{|2\cosh\frac{x}{2}|^{1+\alpha}} e^{\frac{x}{2}(1+\beta-\alpha)}\, dx \overset{\beta=\alpha-1}{=} \int_{\ln(a-1)}^{\ln a} \frac{2\sinh\frac{\beta x}{2}}{|2\cosh\frac{x}{2}|^{1+\alpha}}\, dx.$$

For the G integral we have the following result.

Lemma B.4.

$$G = 2\ P.V. \int_{-\ln 2}^{\ln 2} \frac{2\sinh\frac{\beta x}{2}}{|2\sinh\frac{x}{2}|^{1+\alpha}} e^{\frac{x}{2}(1+\beta-\alpha)} dx \ - \ 2\int_{\ln 2}^{\infty} \frac{2\sinh\frac{\beta x}{2}}{|2\sinh\frac{x}{2}|^{1+\alpha}} e^{-\frac{x}{2}(1+\beta-\alpha)} dx,$$

and if $\beta = \alpha - 1$,

$$G = -\ 2\int_{\ln 2}^{\infty} \frac{2\sinh\frac{\beta x}{2}}{|2\sinh\frac{x}{2}|^{1+\alpha}} dx.$$

Proof. First note that by symmetry

$$G = 2\lim_{b\to 0^+} \int_{(-1,1)\setminus(-b,b)} |1+z|^\beta - 1\ d\mu(z).$$

Then, since $1 + z > 0$ for $z \in (-1, 1)$, the change of variable $1 + z = e^x$ leads to

$$G = 2\lim_{b\to 0^+} \int_{(-\infty,\ln 2)\setminus(\ln(1-b),\ln(1+b))} \frac{2\sinh\frac{\beta x}{2}}{|2\sinh\frac{x}{2}|^{1+\alpha}} e^{\frac{x}{2}(1+\beta-\alpha)}\, dx.$$

Note that $\ln(1 \pm b) = \pm b + O(b^2)$ and decompose the above integral as follows,

$$\begin{aligned}
&\int_{(-\infty,\ln 2)\setminus(\ln(1-b),\ln(1+b))} (\cdots)\, dx \\
&= \Big(\int_{(-\infty,\ln 2)\setminus(-b,+b)} + \int_{(-\infty,\ln 2)\setminus(\ln(1-b),-b)} - \int_{(-\infty,\ln 2)\setminus(\ln(1+b),b)} \Big)(\cdots)\, dx
\end{aligned}$$

Now since $\sinh x = x + O(x^3)$, the last two integrals are bounded by $Cb^2\frac{b}{b^{1+\alpha}} = Cb^{2-\alpha}$ for $b \ll 1$, and we have

$$G = 2\lim_{b\to 0^+} \int_{(-\infty,\ln 2)\setminus(-b,b)} (\cdots)\, dx = 2\left(P.V. \int_{(-\ln 2,\ln 2)} + \int_{(-\infty,-\ln 2)} \right)(\cdots)\, dx.$$

A change of variables in the last integral then gives the first statement of the Lemma. The last part of the lemma follows since the integrand is odd when $\beta = \alpha - 1$, and hence the integral over $(-\ln 2, \ln 2)$ vanishes. □

We also need the next lemma.

Lemma B.5. *If $\beta = \alpha - 1$, then $B(2) < -\frac{G}{2}$.*

Proof. We will show that

$$B(2) = \int_0^{\ln 2} \frac{2\sinh\frac{\beta x}{2}}{|2\cosh\frac{x}{2}|^{1+\alpha}}\,dx \le \int_0^{\ln 2} \frac{2\sinh\frac{\beta(x+\ln 2)}{2}}{|2\sinh\frac{x+\ln 2}{2}|^{1+\alpha}}\,dx < \int_0^{\infty}(\cdots)\,dx = -\frac{G}{2}.$$

The last inequality is trivial, and since sinh is an increasing function, the first inequaliy follows if we can show that

$$\cosh\frac{x}{2} \ge \sinh\frac{x+\ln 2}{2} \quad \text{for all} \quad x \in (0, \ln 2).$$

But this easily follows since $f(x) = \cosh\frac{x}{2} - \sinh\frac{x+\ln 2}{2}$ satisfy

$$f'(x) = \frac{1}{2}\sinh\frac{x}{2} - \frac{1}{2}\cosh\frac{x+\ln 2}{2} \le 0 \quad \text{for all } x,$$
$$f(\ln 2) = \frac{\sqrt{2}-1}{4} \ge 0.$$

□

Proof of Proposition B.2. Divide the integral $B(a)$ into three parts

$$\Big(\int_{\ln(a-1)\wedge 0}^{0} + \int_{\ln(a-1)\vee 0}^{\ln 2\wedge \ln a} + \int_{\ln 2}^{\ln 2\vee \ln a}\Big)(\cdots)\,dx.$$

Now we conclude since the first integral is negative, the second one is less than $-\frac{G}{2}$ by Lemma B.5, and the last one is less than $-\frac{G}{2}$ by definition of G. □

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(Guy Barles) Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Fédération Denis Poisson, Université François Rabelais, Parc de Grandmont, 37200 Tours, France
E-mail address: barles@lmpt.univ-tours.fr

(Emmanuel Chasseigne) Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Fédération Denis Poisson, Université François Rabelais, Parc de Grandmont, 37200 Tours, France
E-mail address: Emmanuel.Chasseigne@lmpt.univ-tours.fr

(Christine Georgelin) Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Fédération Denis Poisson, Université François Rabelais, Parc de Grandmont, 37200 Tours, France
E-mail address: christine.georgelin@lmpt.univ-tours.fr

(Espen R. Jakobsen) Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway
E-mail address: erj@math.ntnu.no