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A NOTE ON THE EXCHANGEABILITY CONDITION IN STEIN'S METHOD

ADRIAN RÖLLIN

ABSTRACT. We show by a surprisingly simple argument that the exchangeability condition, which is key to the *exchangeable pair* approach in Stein's method for distributional approximation, can be omitted in many standard settings. This is achieved by replacing the usual antisymmetric function by a simpler one, for which only equality in distribution is required. In the case of normal approximation we also slightly improve the constants appearing in previous results. For Poisson approximation, a different antisymmetric function is used, and additional error terms are needed if the bound is to be extended beyond the exchangeable setting.

1. INTRODUCTION

In the context of normal approximation, a variant of Stein's method that is often used is the *exchangeable pair coupling* introduced in Diaconis (1977) and Stein (1986). There are many applications based on this coupling, see e.g. Rinott and Rotar (1997), Fulman (2004b), Meckes (2006) and others, but also in the context of non-normal approximation such as Chatterjee et al. (2005), Chatterjee and Fulman (2006) and Röllin (2007).

Assume that W is a random variable which we want to approximate. The key concept introduced by Stein (1986) is that of *auxiliary randomization*. In the context of exchangeable pairs, this means that we construct a random variable W' on the same probability space such that (W, W') is exchangeable, that is $\mathcal{L}(W, W') = \mathcal{L}(W', W)$ and, in general, the pair should be constructed such that $|W' - W|$ is small. One can then prove for instance a bound as in Theorem 2.1 below (often under additional conditions on the

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exchangeable pair such as Equation (2.1)). If W and W' are two consecutive steps of a reversible Markov chain in equilibrium, Rinott and Rotar (1997) note that exchangeability automatically follows. However, the pairs constructed in some of the examples of Rinott and Rotar (1997) and Fulman (2004a) are based on non-reversible Markov chains and therefore some effort has to be put into showing that the pairs satisfy the exchangeability condition.

The key fact to prove a bound as in Theorem 2.1 is that, for any antisymmetric function F , exchangeability of (W, W') implies the identity

$$(1.1) \quad \mathbb{E}F(W, W') = 0$$

(see (Stein, 1986, p. 10)). In fact, this is often the only place where exchangeability is used. For example, in the case of the normal distribution, the standard choice for F is

$$(1.2) \quad F(w, w') = (w' - w)(f(w') + f(w)),$$

where f is the solution to the Stein equation

$$(1.3) \quad f'(x) - xf(x) = h(x) - \mathbb{E}h(Z)$$

and Z has the standard normal distribution. We show in this paper that instead of the choice (1.2) we can take the simpler function

$$(1.4) \quad F(w, w') = \int_0^{w'} f(x)dx - \int_0^w f(x)dx,$$

which is, in particular, antisymmetric, but for which (1.1) of course holds without exchangeability as long as $\mathcal{L}(W') = \mathcal{L}(W)$. If exchangeable pairs are used in a discrete setting, the integrals in (1.4) of course have to be replaced by corresponding sums.

In the following section we restate and prove results obtained by the exchangeable pairs approach for normal and Poisson approximation. The main purpose is to demonstrate in detail how the exchangeability condition can be

omitted in some standard settings, sometimes also yielding better constants. With the approach in this paper, however, it is possible to remove the exchangeability condition in many other situations, such as in Stein (1995) and Meckes (2006), Chatterjee and Fulman (2006), and Röllin (2007).

Most ingredients in the following proofs are taken directly from the proofs of the corresponding papers; hence many details have been omitted.

2. MAIN RESULTS

2.1. Normal approximation. For normal approximation we need some more assumptions. Assume that W is a random variable with $\mathbb{E}W = 0$ and $\text{Var } W = 1$. Construct an exchangeable pair such that

$$(2.1) \quad \mathbb{E}^W W' = (1 - \lambda)W + R$$

for some constant $0 < \lambda < 1$ and some random variable R , where \mathbb{E}^W denotes the conditional expectation with respect to W .

The following calculations are essential for the proof of Theorem 1.2 of Rinott and Rotar (1997), but we now assume for the sake of clarity that $R = 0$. Let f be the solution to (1.3) for a Lipschitz-continuous test function h . Now, with the standard choice of F as in (1.2) and exchangeability, we have

$$(2.2) \quad \begin{aligned} 0 &= \mathbb{E}F(W, W') \\ &= \mathbb{E}\{(W' - W)(f(W') + f(W))\} \\ &= \mathbb{E}\{(W' - W)(2f(W) + f(W') - f(W))\} \\ &= -2\lambda\mathbb{E}\{Wf(W)\} + \mathbb{E}\{(W' - W)(f(W') - f(W))\}, \end{aligned}$$

where for the last equality we used (2.1). Let now τ be a random variable uniformly distributed on $[0, 1]$, independent of all other random variables, and put $V = W' - W$. Noting that the second derivative f'' exists almost everywhere because h is Lipschitz continuous, Taylor's expansion yields

$$f(W') = f(W) + Vf'(W) + V^2\mathbb{E}^{W, W'}\{(1 - \tau)f''(W + \tau V)\}.$$

Thus, from (2.2),

$$(2.3) \quad \mathbb{E}\{Wf(W)\} = \frac{1}{2\lambda}\mathbb{E}\{V^2f'(W)\} + \frac{1}{2\lambda}\mathbb{E}\{V^3(1-\tau)f''(W+\tau V)\}.$$

In the proof of Theorem 2.1 we show that an equation similar to (2.3) can be deduced without exchangeability. To state the theorem, we need some notation. Define for a given function h and $\varepsilon > 0$,

$$h_\varepsilon^+(x) = \sup\{h(x+y) : |y| \leq \varepsilon\}, \quad h_\varepsilon^-(x) = \inf\{h(x+y) : |y| \leq \varepsilon\}.$$

Let \mathcal{H} be a class of measurable functions on the real line such that for all $h \in \mathcal{H}$, we have $\|h\| \leq 1$, where $\|\cdot\|$ denotes the supremum norm; for any real numbers c and d , $h \in \mathcal{H}$ implies $h(c \cdot + d) \in \mathcal{H}$; for any $\varepsilon > 0$, $h \in \mathcal{H}$ implies $h_\varepsilon^+, h_\varepsilon^- \in \mathcal{H}$ and there is a constant a (depending only on the class \mathcal{H}) such that $\mathbb{E}\{h_\varepsilon^+(Z) - h_\varepsilon^-(Z)\} \leq a\varepsilon$ where Z has standard normal distribution. As in Rinott and Rotar (1997) we assume without loss of generality that $a \geq \sqrt{2/\pi}$.

Theorem 2.1 (cf. Theorem 1.2 of Rinott and Rotar (1997)). *Assume that W and W' are random variables on the same probability space such that $\mathcal{L}(W') = \mathcal{L}(W)$, $\mathbb{E}W = 0$, $\text{Var} W = 1$. Assume that (2.1) holds for some λ and R . Then, for $\delta := \sup_{h \in \mathcal{H}} |\mathbb{E}h(W) - \mathbb{E}h(Z)|$,*

$$(2.4) \quad \delta \leq \frac{6}{\lambda} \sqrt{\text{Var} \mathbb{E}^W(W' - W)^2} + \frac{19\sqrt{\mathbb{E}R^2}}{\lambda} + 4\sqrt{\frac{a\mathbb{E}|W' - W|^3}{\lambda}}.$$

If, in addition, there is a constant A such that $|W' - W| \leq A$ almost surely, we have

$$(2.5) \quad \delta \leq \frac{12}{\lambda} \sqrt{\text{Var} \mathbb{E}^W(W' - W)^2} + \frac{37\sqrt{\mathbb{E}R^2}}{\lambda} + 32\frac{A^3}{\lambda} + 6\frac{A^2}{\sqrt{\lambda}}$$

Proof. From Lemma 4.1 of Rinott and Rotar (1997) we have that, for any $0 < t < 1$,

$$(2.6) \quad \delta \leq 2.8 \sup_{h \in \mathcal{H}} |\mathbb{E}h_t(W) - \mathbb{E}h_t(Z)| + 4.7at,$$

where $h_t(x) = \mathbb{E}h(x + tZ)$. Let f be the solution to the Stein equation

$$(2.7) \quad f'(x) - xf(x) = h_t(x) - \mathbb{E}h_t(Z).$$

Then, f satisfies

$$(2.8) \quad \|f\| \leq 2.6, \quad \|f'\| \leq 4, \quad \|f''\| \leq 2\|h'_t\| \leq 1.6t^{-1};$$

recalling that $\|h\| \leq 1$, the first two bounds and the first part of the third one follow from Lemma 3 of Stein (1986) and, as noted by Rinott and Rotar (1997), the second inequality of the third bound can be deduced using the equality $h'_t(x) = -t^{-1} \int h(x + ty)\varphi'(y)dy$, where φ is the standard normal density, so that $\|h'_t\| \leq t^{-1} \int |\varphi'(x)|dx = t^{-1}\sqrt{2/\pi}$.

Define the function

$$(2.9) \quad G(w) = \int_0^w f(x) dx$$

and note that $|G(w)| \leq |w|\|f\|$, so that $\mathbb{E}G(W)$ exists. By Taylor's expansion, we have

$$(2.10) \quad \begin{aligned} G(W') &= G(W) + Vf(W) + \frac{1}{2}V^2f'(W) \\ &\quad + \frac{1}{2}V^3\mathbb{E}^{W,W'}\{(1-\tau)^2f''(W+\tau V)\}, \end{aligned}$$

where, again, $V = W' - W$. Thus, together with (2.1), we obtain

$$\begin{aligned} 0 &= \mathbb{E}G(W') - \mathbb{E}G(W) \\ &= -\lambda\mathbb{E}\{Wf(W)\} + \mathbb{E}\{Rf(W)\} \\ &\quad + \frac{1}{2}\mathbb{E}\{V^2f'(W)\} + \frac{1}{2}\mathbb{E}\{V^3(1-\tau)^2f''(W+\tau V)\}, \end{aligned}$$

which can be rearranged to obtain the following analogue of (2.3)

$$(2.11) \quad \begin{aligned} \lambda\mathbb{E}\{Wf(W)\} &= \frac{1}{2}\mathbb{E}\{V^2f'(W)\} + \frac{1}{2}\mathbb{E}\{V^3(1-\tau)^2f''(W+\tau V)\} \\ &\quad + \mathbb{E}\{Rf(W)\}. \end{aligned}$$

With $\alpha := \mathbb{E}\{RW\}$ and noting that $\mathbb{E}V^2 = 2(\lambda - \alpha)$ from (2.1), we thus have from (2.7) and (2.11)

$$\begin{aligned} \lambda(\mathbb{E}h_t(W) - \mathbb{E}h_t(Z)) &= \mathbb{E}\left\{\left((\lambda - \alpha) - \frac{1}{2}V^2\right)f'(W)\right\} \\ &\quad + \mathbb{E}\{\alpha f'(W) - Rf(W)\} \\ &\quad - \frac{1}{2}\mathbb{E}\{V^3(1 - \tau)^2 f''(W + \tau V)\} \\ &=: J_1 + J_2 - \frac{1}{2}J_3. \end{aligned}$$

Using (2.8), we obtain the estimates

$$|J_1| \leq 2\sqrt{\text{Var } \mathbb{E}^W V^2}, \quad |J_2| \leq 6.6\sqrt{\mathbb{E}R^2}, \quad |J_3| \leq \frac{1.6}{3t}\mathbb{E}|V|^3;$$

for details see Rinott and Rotar (1997). Choosing $t = 0.4(\mathbb{E}|V|^3/(a\lambda))^{1/2}$ and with (2.6), this proves (2.1).

Assume now that $|V| \leq A$. Note that because of (2.7), $f''(x) = f(x) + xf'(x) + h'_t(x)$. Following the proof of Rinott and Rotar (1997), but recalling that in our remainder J_3 the term $(1 - \tau)$ is squared, we have

$$\begin{aligned} |J_3| &= \mathbb{E}\{V^3(1 - \tau)^2(f(W + \tau V) + (W + \tau V)f'(W + \tau V) + h'_t(W + \tau V))\} \\ &\leq 0.9A^3 + 1.4\mathbb{E}\{V^3(|W| + |W'|)\} + \mathbb{E}\{V^3(1 - \tau)^2 h'_t(W + \tau V)\} \\ &\leq 3.7A^3 + \mathbb{E}\{V^3(1 - \tau)^2 h'_t(W + \tau V)\}, \end{aligned}$$

where the latter expectation can be bounded by

$$|\mathbb{E}\{V^3(1 - \tau)^2 h'_t(W + \tau V)\}| \leq \frac{aA^3}{3} + \frac{A^3}{3t}(2\delta + aA);$$

see again Rinott and Rotar (1997). Collecting all the bounds on the J_i we obtain

$$(2.12) \quad \begin{aligned} |\mathbb{E}h_t(W) - \mathbb{E}h_t(Z)| &\leq \frac{2}{\lambda}\sqrt{\text{Var } \mathbb{E}^W V^2} + \frac{6.6}{\lambda}\sqrt{\mathbb{E}R^2} \\ &\quad + \frac{3.7A^3}{2\lambda} + \frac{aA^3}{6\lambda} + \frac{A^3(2\delta + aA)}{6\lambda t}. \end{aligned}$$

Recalling that $a \geq \sqrt{2/\pi}$, putting (2.12) into (2.6) and with the choice $t = 0.32A(\frac{A(2\delta+aA)}{a\lambda})^{1/2}$, we finally have

$$\delta \leq 5.6\sqrt{\text{Var } \mathbb{E}^W V^2} + 18.5\sqrt{\mathbb{E}R^2} + 7\frac{aA^3}{\lambda} + 3\frac{aA^2}{\sqrt{\lambda}} + 4.2\sqrt{\frac{a\delta A^3}{\lambda}}.$$

This inequality is of the form $\delta \leq a + b\sqrt{\delta}$, for which we can show that $\delta \leq 2a + b^2$ and which hence proves (2.5). \square

Note that if $R = 0$ almost surely, equation (2.11) is the same as (2.3), except that the factor $(1 - \tau)$ is squared; this is the only reason for the improved constants. If (W, W') is exchangeable, both equalities (2.3) and (2.11) hold. At first glance this may seem to be a contradiction, but the remainders with the second derivatives are in fact equal. To see this, write

$$\begin{aligned} & \mathbb{E}\{V^3(1 - \tau)^2 f''(W + \tau V)\} \\ &= \mathbb{E}\{V^3(1 - \tau)f''(W + \tau V)\} - \mathbb{E}\{V^3\tau(1 - \tau)f''(W + \tau V)\}. \end{aligned}$$

We need only show that the second term on the right hand side is equal to zero. Note to this end the simple fact that $\mathcal{L}(\tau) = \mathcal{L}(1 - \tau)$, thus, using this in the following calculations to obtain the first equality and the exchangeability of (W, W') for the third equality,

$$\begin{aligned} & \mathbb{E}\{(W' - W)^3\tau(1 - \tau)f''(W + \tau(W' - W))\} \\ &= \mathbb{E}\{(W' - W)^3\tau(1 - \tau)f''(W + (1 - \tau)(W' - W))\} \\ &= \mathbb{E}\{(W' - W)^3\tau(1 - \tau)f''(W' + \tau(W - W'))\} \\ &= \mathbb{E}\{(W - W')^3\tau(1 - \tau)f''(W + \tau(W' - W))\} \\ &= -\mathbb{E}\{(W' - W)^3\tau(1 - \tau)f''(W + \tau(W' - W))\}, \end{aligned}$$

which proves the claim.

2.2. Poisson approximation. The situation in the discrete setting, in which W takes values only in \mathbb{Z} , is more delicate. Assume that f is the solution

to a Stein equation of a discrete distribution. Instead of (2.9), define the function G as

$$(2.13) \quad G(w) = \sum_{k=1}^w f(k) - \sum_{k=0}^{-w-1} f(-k),$$

where here and in what follows $\sum_{k=a}^b$ is defined to be zero if $b < a$. For many standard distributions, f will be bounded, thus $|G(w)| \leq |w| \|f\|$, so that $\mathbb{E}G(W)$ exists if $\mathbb{E}|W| < \infty$.

With such G , one verifies that $G(w) - G(w-1) = f(w)$ for all $w \in \mathbb{Z}$. Define $I_i = I[W' - W = i]$ and $\Delta_i G(w) := G(w+i) - G(w)$ for all $i \in \mathbb{Z}$; then,

$$(2.14) \quad G(W') - G(W) = \sum_{i \in \mathbb{Z}} I_i \Delta_i G(W)$$

and thus, if $\mathcal{L}(W') = \mathcal{L}(W)$,

$$(2.15) \quad 0 = \mathbb{E}G(W') - \mathbb{E}G(W) = \sum_{i \in \mathbb{Z}} \mathbb{E}\{I_i \Delta_i G(W)\} = \sum_{i \in \mathbb{Z}} \mathbb{E}\{P_i(W) \Delta_i G(W)\}$$

where $P_i(W) := \mathbb{P}^W[W' - W = i]$.

As mentioned in the introduction, Rinott and Rotar (1997) suggest constructing W and W' via an underlying stationary Markov process, which implies $\mathcal{L}(W) = \mathcal{L}(W')$; if the chain is reversible, exchangeability follows immediately. However, the Markov chain that they use for the anti-voter model is not reversible, and so exchangeability has to be proved separately. One way of doing this is to assume that

$$(2.16) \quad W' - W \in \{-1, 0, 1\},$$

almost surely, which is the main assumption in Röllin (2007).

Chatterjee et al. (2005) applied the method of exchangeable pairs to the Poisson distribution. Assume that (W', W) is an exchangeable pair taking

values only on the non-negative integers. Define the antisymmetric function

$$(2.17) \quad F(w, w') = f(W')I[W' - W = 1] - f(W)I[W' - W = -1].$$

This yields

$$(2.18) \quad \begin{aligned} 0 &= \mathbb{E}F(W, W') \\ &= \mathbb{E}\{f(W')I[W' - W = 1] - f(W)I[W' - W = -1]\} \\ &= \mathbb{E}\{\mathbb{E}^W\{f(W')I[W' - W = 1]\} - \mathbb{E}^W\{f(W)I[W' - W = -1]\}\} \\ &= \mathbb{E}\{f(W+1)\mathbb{P}[W' - W = 1|W] - f(W)\mathbb{P}[W' - W = -1|W]\} \\ &= \mathbb{E}\{f(W+1)P_1(W) - f(W)P_{-1}(W)\}. \end{aligned}$$

We can use now the standard argument for Poisson approximation by Stein's method (see (Barbour et al., 1992, p. 6)). Denoting by $\text{Po}(\lambda)$ the Poisson distribution with mean λ , it follows that

$$(2.19) \quad \begin{aligned} d_{\text{TV}}(\mathcal{L}(W), \text{Po}(\lambda)) &\leq \sup_f |\mathbb{E}\{\lambda f(W+1) - Wf(W)\}| \\ &\leq \sup_f |\mathbb{E}\{(cP_1(W) - \lambda)f(W+1) - (cP_{-1}(W) - W)f(W)\}| =: \kappa_c \end{aligned}$$

for any $c > 0$ and where the supremum ranges over all solutions $f = f_A$ to the Stein-equation

$$(2.20) \quad \lambda f(j+1) - jf(j) = I[j \in A] - \text{Po}(\lambda)\{A\},$$

for subsets A of the non-negative integers. For many applications investigated by Chatterjee et al. (2005), the following further bound on κ_c is used:

$$\kappa_c \leq \lambda^{-1/2}(\mathbb{E}|cP_1(W) - \lambda| + \mathbb{E}|cP_{-1}(W) - W|),$$

which results from the well known bound $\|f\| \leq \lambda^{-1/2}$ (see (Barbour et al., 1992, Lemma 1.1.1)). However, it is at times beneficial to work directly with the expression in (2.19) in the hope of re-arranging things so that the better bound $\|\Delta_1 f\| \leq (1 - e^{-\lambda})\lambda^{-1}$ may be applied.

It may seem surprising that, although in the bound (2.19) only the jump probabilities of size 1 appear, no assumptions concerning the sizes of other jumps were made in the above calculations; we did not, for instance, assume condition (2.16). However, with the choice $f(w) = I[w = k]$ for $k \in \mathbb{Z}$, we obtain from (2.17) the detailed balance equation for reversible Markov chains. Even if the chain makes jumps of size larger than 1, the stationary distribution is determined by the jump probabilities of size 1, so that in (2.19) the full information about the distribution under consideration is actually used, just by starting from (2.17). If exchangeability is not assumed, the effects of jumps of size larger than 1 have also to enter, and this is reflected in the bounds of the next theorem. The choice of F in (2.17) is fundamentally different from the standard choice in the continuous setting, where we obtained the same bounds as before, but under weaker assumptions.

Theorem 2.2 (cf. Proposition 3 of Chatterjee et al. (2005)). *Let W and W' be non-negative random variables such that $\mathcal{L}(W') = \mathcal{L}(W)$. Then, for any constant $c > 0$,*

$$(2.21) \quad d_{\text{TV}}(\mathcal{L}(W), \text{Po}(\lambda)) \leq \kappa_c + c\rho$$

where ρ satisfies the bounds

$$(2.22) \quad \rho \leq \lambda^{-1/2} \sum_{i \geq 2} i \sum_{k \in \mathbb{Z}} |p_{k, k+i} - p_{k+i, k}| \leq \lambda^{-1/2} \sum_{|i| \geq 2} |i| \mathbb{E}P_i(W),$$

for $p_{k,j} = \mathbb{P}[W = k, W' = j]$ and the P_i are as before.

Proof. Taking expectation over (2.20) with respect to W , using (2.15) and noting that $\Delta_1 G(W) = f(W+1)$ and $\Delta_{-1} G(W) = -f(W)$, we obtain

$$\begin{aligned} & \mathbb{E}\{\lambda f(W+1) - W f(W)\} \\ &= \mathbb{E}\{(\lambda - cP_1(W))f(W+1) - (W - cP_{-1}(W))f(W)\} \\ &+ c \sum_{i \geq 2} \mathbb{E}\{I_i \Delta_i G(W) + I_{-i} \Delta_{-i} G(W)\}, \end{aligned}$$

where, as before, $I_i = I[W' - W = i]$. Now, it is easy to see that

$$(2.23) \quad \mathbb{E}\{I_i \Delta_i G(W) + I_{-i} \Delta_{-i} G(W)\} = \sum_{k \in \mathbb{Z}} (p_{k, k+i} - p_{k+i, k}) \Delta_i G(k)$$

Recalling the bound $\|f\| \leq \lambda^{-1/2}$ for all solutions f of (2.20), and hence $|\Delta_i G(W)| \leq |i| \lambda^{-1/2}$ for all $i \in \mathbb{Z}$, (2.23) yields the first bound of (2.22).

Now, as $p_{k, k+i} = \mathbb{P}[W = k] P_i(k)$, the second bound follows from the first. \square

Under condition (2.16), Theorem 2.2 and estimate (2.19) clearly yield the same bound. If (2.16) does not hold, exchangeability is not automatically implied. Then the first bound of (2.22) is a measure of the non-exchangeability of (W, W') ; if the pair is in fact exchangeable, this term vanishes and we regain (2.16). The second bound in (2.22) is particularly useful if the jump probabilities of larger jumps are small.

3. DISCUSSION

The key to understand the present approach is the generator interpretation introduced in Barbour (1988). Recall that

$$(\mathcal{A}G)(x) = G''(x) - xG'(x) = f'(x) - xf(x)$$

is the generator of the Ornstein-Uhlenbeck diffusion, applied to the function G . Now assume that a Markov chain $\{X(n); n \in \mathbb{Z}_+\}$ with stationary distribution $\mathcal{L}(W)$ is given and let $(W, W') = (X(0), X(1))$, assuming that the chain starts in its equilibrium. Construct a Markov jump process $Z = \{Z(t); t \geq 0\}$ by randomising the fixed steps of size 1 of the Markov chain X and wait instead an exponentially distributed amount of time with rate $1/\lambda$. It is easy to see that, under the assumption of (2.1), the infinitesimal operator of Z is

$$(\mathcal{B}G)(x) = \lim_{h \rightarrow 0} \frac{\mathbb{E}[G(Z(h)) \mid Z(0) = x] - G(x)}{h} = \frac{\mathbb{E}^x G(W') - G(x)}{\lambda}$$

$$\begin{aligned}
&= -xG'(x) + \frac{\mathbb{E}^x V^2}{2\lambda} G''(x) \\
&\quad + \int \frac{\mathbb{E}^x \{(V-t)^2 I[V > t > 0 \text{ or } V < t < 0]\}}{2\lambda} G^{(3)}(x+t) dt
\end{aligned}$$

where \mathbb{E}^x denotes $\mathbb{E}\{\cdot | W = x\}$ and $V = W' - W$. Thus, we in fact compare the infinitesimal operator of this jump process with the generator of the Ornstein-Uhlenbeck diffusion.

The antisymmetric function $F(w, w') = (w' - w)(f(w') + f(w))$ in fact also (almost) calculates this infinitesimal operator, but with the first step of the Taylor's expansion already carried out.

In the case of Stein (1995) and Meckes (2006), we have a family of Markov jump processes $\{Z_m; m \in \mathbb{N}\}$ with infinitesimal operators

$$(\mathcal{B}_m G)(x) = \frac{\mathbb{E}^x G(W_m) - G(x)}{\varepsilon_m \vartheta}$$

where $\varepsilon_m^{-1} \lambda^{-1}$ are the jump rates and by letting $\varepsilon \rightarrow 0$ we show that the processes Z_m converge to a diffusion process with infinitesimal operator

$$(\mathcal{B}G)(x) = (1 + \frac{1}{2}\lambda^{-1}\mathbb{E}^x E)G''(x) - xG'(x)$$

for some specific random variable E . This diffusion has the same linear drift as the Ornstein-Uhlenbeck diffusion, but a non-constant diffusion rate. Stein's method now allows us to state through the approaches in Stein (1995) and Meckes (2006) that the less $\mathbb{E}^x E$ fluctuates around zero, the nearer the stationary distribution of the process is to the stationary distribution of Ornstein-Uhlenbeck diffusion, that is, to the standard normal distribution.

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REFERENCES

- A. D. Barbour (1988). Stein's method and Poisson process convergence. *J. Appl. Probab.* **25A**, 175–184.
- A. D. Barbour, L. Holst, and S. Janson (1992). *Poisson approximation*, volume 2 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York. ISBN 0-19-852235-5. Oxford Science Publications.
- S. Chatterjee and J. Fulman (2006). Exponential approximation by exchangeable pairs and spectral graph theory. *Preprint*. Available at <http://arxiv.org/abs/math/0605552>.
- S. Chatterjee, P. Diaconis, and E. Meckes (2005). Exchangeable pairs and Poisson approximation. *Probab. Surv.* **2**, 64–106. Available at www.arxiv.org/abs/math.PR/0411525.
- P. Diaconis (1977). The distribution of leading digits and uniform distribution mod 1. *Ann. Probability* **5**, 72–81.
- J. Fulman (2004a). Stein's method and non-reversible Markov chains. In *Stein's method: expository lectures and applications*, volume 46 of *IMS Lecture Notes Monogr. Ser.*, pages 69–77. Inst. Math. Statist.
- J. Fulman (2004b). Stein's method, Jack measure, and the Metropolis algorithm. *J. Combin. Theory Ser. A* **108**, 275–296.
- E. Meckes (2006). Linear functions on the classical matrix groups. *Preprint*. Available at www.arxiv.org/math.PR/0509441.
- Y. Rinott and V. Rotar (1997). On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U -statistics. *Ann. Appl. Probab.* **7**, 1080–1105.
- A. Röllin (2007). Translated Poisson approximation using exchangeable pair couplings. *Ann. Appl. Probab.* **17**, 1596–1614.
- C. Stein (1986). *Approximate computation of expectations*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 7. Institute of Mathematical Statistics, Hayward, CA.

C. Stein (1995). The accuracy of the normal approximation to the distribution of the traces of powers of random orthogonal matrices. Technical Report No. 470, Stanford University Department of Statistics.