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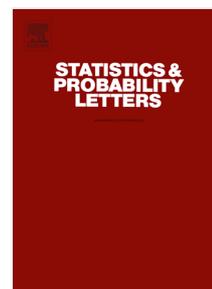
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# Moment inequalities for DVRL distributions, characterization and testing for exponentiality

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## Abstract

Our goal in this paper is to establish inequalities for the moments of decreasing variance residual life (DVRL) distributions. As a consequence we derive a new characterization of exponentiality. Then we use two of these inequalities to construct new tests for exponentiality versus DVRL. Pitman's asymptotic relative efficiency is employed to assess the performance of the tests. For some classes of life distributions our tests are better than, or well comparable with, other available tests. We carried out numerical simulations and produced a table for the critical values of one of the proposed test.

*Key words:* Life distributions, Decreasing variance residual life, Moment inequalities, Characterization of exponentiality, Testing for exponentiality, Asymptotic efficiency.

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## 1 Introduction

Suppose  $X$  is a nonnegative random variable which is interpreted as the lifetime of a device and has a distribution function  $F = \{F(x), x \geq 0\}$ , so its survival function is  $\bar{F} = 1 - F$ . We assume that all moments  $\mathbf{E}[X^k]$ ,  $k = 1, 2, \dots$  are finite and use the standard notations  $\mu = \mathbf{E}[X]$  for the mean value and  $\sigma^2 = \mathbf{Var}[X]$  for the variance of  $X$ . We need two characteristics, the conditional mean  $\mu(x) = \mathbf{E}[X - x | X \geq x]$  and the conditional variance

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$\sigma^2(x) = \mathbf{Var}[X - x|X \geq x]$ , usually called the mean residual life and the variance residual life.

We are interested in life distributions for which  $\sigma^2(x), x \geq 0$  is decreasing and use the notation DVRL. Our goal is to derive inequalities for combined moments of random variables similar to  $X$  and use them in constructing tests for exponentiality versus the class DVRL. Among previous works in this direction we mention the papers by Hollander and Proschan (1975), Dallas (1981), Launer (1984), Gupta (1987), Ahmad (2001), Abu-Youssef (2002), Ahmad and Mugdadi (2004) and Chin and Min (2006).

In Section 2 we establish inequalities for the moments of lifetimes whose distributions are in the class DVRL. As a corollary we derive a new characterization of the exponential distribution. In Section 3 we use some of the inequalities to construct new tests for exponentiality versus the class DVRL. In Section 4, Pitman's asymptotic relative efficiency is used to assess the performance of the tests. The conclusion is that our tests are well comparable and in some cases even better than other tests widely used in statistical practice. In Section 5 we describe how to apply the test. We use real data set as an illustration of our results.

## 2 Moment inequalities and characterization of exponentiality

The main assumption is that the random variable  $X \sim F$ , with  $F \in \text{DVRL}$ , has finite moments. We express this as follows:

$$m_k := \mathbf{E}[X^k] = \int_0^{\infty} x^k dF(x) < \infty \text{ for } k = 1, 2, \dots$$

We are going to show that there are relations/inequalities involving the moments of  $X$  and combined moments of the random variables  $X_1, X_2$  and  $Y$ , where  $X_1$  and  $X_2$  are independent copies of  $X$  and

$$Y = \min\{X_1, X_2\}.$$

We can easily interpret  $Y$  as being the lifetime of a parallel system formed by two elements each with lifetime  $X$ .

**Theorem 2.1** *Under the above assumptions and notations, the following inequalities hold:*

$$m_1 m_2 \leq 4 \mathbf{E} [X_2 Y^2] - \frac{8}{3} \mathbf{E} [Y^3], \quad (2.1)$$

$$m_2^2 \leq \frac{16}{3} \mathbf{E} [X_2 Y^3] - 4 \mathbf{E} [Y^4]. \quad (2.2)$$

More generally, for any integer  $k \geq 2$ , we have:

$$\begin{aligned} \frac{k}{4} \mathbf{E} [X_1^2 X_2^2 Y^{k-1}] &\leq (k-1) \mathbf{E} [X_1 X_2^2 Y^k] - \frac{k(k-1)}{2(k+1)} \mathbf{E} [X_2^2 Y^{k+1}] \\ &- \frac{k(k-1)}{(k+1)} \mathbf{E} [X_1 X_2 Y^{k+1}] + \frac{4k(k-1)(k+1) + 8}{(k+1)(k+2)} \mathbf{E} [X_2 Y^{k+2}] \\ &- \frac{8 + k(k-1)(k+1)}{(k+1)(k+3)} \mathbf{E} [Y^{k+3}]. \end{aligned} \quad (2.3)$$

**Proof.** We prove the general inequality (2.3), assuming  $k \geq 2$ . Recall first that the functions  $\mu(x)$  and  $\sigma^2(x)$  can be expressed in terms of  $F$ , for all  $x > 0$  such that  $F(x) < 1$ , as follows:

$$\mu(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(u) \, du, \quad \sigma^2(x) = \frac{2}{\bar{F}(x)} \int_x^\infty \int_y^\infty \bar{F}(u) \, du \, dy - \mu^2(x).$$

Clearly,  $\mu(0) = \mu$  and  $\sigma^2(0) = \sigma^2$ .

Let us introduce two functions,  $v(x)$  and  $V(x)$ ,  $x \geq 0$ , where

$$v(x) = \int_x^\infty \bar{F}(z) \, dz \quad \text{and} \quad V(x) = \int_x^\infty v(u) \, du, \quad x \geq 0.$$

We need now the well-known fact that  $F \in \text{DVRL}$  if and only if the relation  $\sigma^2(x) \leq \mu^2(x)$  holds for  $x \geq 0$ . It can be shown that this is equivalent to the following:

$$F \in \text{DVRL} \quad \text{if and only if} \quad \bar{F}(x)V(x) \leq v^2(x), \quad \text{for } x \geq 0. \quad (2.4)$$

We multiply both sides of the inequality in (2.4) by  $x^k$ , integrate with respect to  $x$  over the interval  $(0, \infty)$  and introduce the notations  $I_1$  and  $I_2$  thus getting

$$I_1 := \int_0^\infty x^k \bar{F}(x)V(x) \, dx \leq \int_0^\infty x^k v^2(x) \, dx := I_2. \quad (2.5)$$

Integrating  $I_1$  by parts and using the fact that  $V'(x) = -v(x)$  gives this:

$$I_1 = \int_0^{\infty} x^k \bar{F}(x) V(x) dx = -x^k v(x) V(x) \Big|_0^{\infty} - \frac{1}{2} k x^{k-1} V^2(x) \Big|_0^{\infty} - I_2 + \frac{1}{2} k(k-1) \int_0^{\infty} x^{k-2} V^2(x) dx. \quad (2.6)$$

Now, the first two terms in the right-hand-side of (2.6) are equal to zero. What remains is combined with (2.5) thus getting

$$I_1 = -I_2 + \frac{1}{2} k(k-1) \int_0^{\infty} x^{k-2} V^2(x) dx \leq I_2.$$

This implies that

$$\int_0^{\infty} x^{k-2} V^2(x) dx \leq \frac{4I_2}{k(k-1)}. \quad (2.7)$$

The next is to work out the integral in the left-hand-side of (2.7). We find

$$\begin{aligned} \int_0^{\infty} x^{k-2} V^2(x) dx &= \frac{1}{4} \mathbf{E} \left[ \int_0^{\min\{X_1, X_2\}} x^{k-2} (X_1 - x)^2 (X_2 - x)^2 dx \right] \\ &= \frac{1}{4(k-1)} \mathbf{E} [X_1^2 X_2^2 Y^{k-1}] - \frac{1}{k} \mathbf{E} [X_1 X_2^2 Y^k] \\ &\quad + \frac{1}{2(k+1)} \mathbf{E} [X_2^2 Y^{k+1}] + \frac{1}{k+1} \mathbf{E} [X_1 X_2 Y^{k+1}] \\ &\quad - \frac{4}{k+2} \mathbf{E} [X_2 Y^{k+2}] + \frac{1}{k+3} \mathbf{E} [Y^{k+3}]. \end{aligned} \quad (2.8)$$

Similarly, integrating  $I_2$  by parts, one gets

$$\begin{aligned} I_2 &= \int_0^{\infty} x^k v^2(x) dx = \frac{2}{k+1} \int_0^{\infty} x^{k+1} v(x) \bar{F}(x) dx \\ &= \frac{2}{k+1} \mathbf{E} \int_0^{\min\{X_1, X_2\}} x^{k+1} (X_2 - x) dx \\ &= \frac{2}{(k+1)(k+2)} \mathbf{E} [X_2 Y^{k+2}] \end{aligned}$$

$$-\frac{2}{(k+1)(k+3)} \mathbf{E} [Y^{k+3}]. \quad (2.9)$$

hence the required inequality (2.3) follows from relations (2.7) – (2.9).

Let us work a little more with the integrals  $I_1$  and  $I_2$ , see (2.5). Since  $v(\infty) = 0$ ,  $V(\infty) = 0$  and since all moments of  $X$  and of  $F$  are finite, we have that  $\lim_{x \rightarrow \infty} x^k v(x) V(x) = 0$  for  $k = 0, 1, 2, \dots$ . We also have that  $v(0) = \mu = m_1$ ,  $V(0) = \frac{1}{2} m_2$ . Notice that inequality (2.5) is true for any nonnegative  $k$ . We take  $k = 0$  and then  $k = 1$  and work either directly with (2.5), or use (2.6) whose right-hand-side becomes simple. This together with (2.9) can be summarized as follows:

If  $k = 0$ , then  $I_1 = \frac{1}{2} m_1 m_2 - I_2$ ,  $I_2 = \mathbf{E}[X_2 Y^2] - \frac{2}{3} \mathbf{E}[Y^3]$ , hence we arrive at (2.1). If  $k = 1$ , then  $I_1 = \frac{1}{2} V^2(0) - I_2 = \frac{1}{2} \cdot \frac{1}{4} m_2^2 - I_2$ ,  $I_2 = \frac{1}{3} \mathbf{E}[X_2 Y^3] - \frac{1}{4} \mathbf{E}[Y^4]$  and we arrive at (2.2).

This completes the proof of Theorem 2.1.

As a consequence, we derive a new characterization property of the exponential distribution.

**Theorem 2.2** *Suppose  $F$  is the distribution function of a nonnegative random variable  $X$  with finite moments  $m_k$ ,  $k = 1, 2, \dots$  and such that the conditional variance  $\sigma^2(x) = \mathbf{Var}[X - x | X \geq x]$ ,  $x \geq 0$  is a decreasing function. Let  $X_1$  and  $X_2$  be independent copies of  $X$  and  $Y = \min\{X_1, X_2\}$ . Then:*

(a)  *$F$  is exponential if and only if  $m_1 m_2 = 4 \mathbf{E}[X_2 Y^2] - \frac{8}{3} \mathbf{E}[Y^3]$ .*

(b)  *$F$  is exponential if and only if  $m_2^2 = \frac{16}{3} \mathbf{E}[X_2 Y^3] - 4 \mathbf{E}[Y^4]$ .*

**Hint.** For the ‘if’ part, let us assume that  $X \sim \text{Exp}(\lambda)$ . Then  $Y \sim \text{Exp}(2\lambda)$ , however we keep in mind that  $Y$  is not independent of  $X_1$  and  $X_2$ . We easily find  $m_1$ ,  $m_2$ ,  $\mathbf{E}[Y^3]$ ,  $\mathbf{E}[Y^4]$  and use conditioning arguments, or another way, to calculate  $\mathbf{E}[X_2 Y^2]$  and  $\mathbf{E}[X_2 Y^3]$ . A substitution shows that the equalities in (a) and (b) hold true. For the ‘only if’ part, we start with the two equalities in (a) and (b) and follow the proof of the inequalities (2.1) and (2.2) thus arriving at relation (2.4). It remains only to mention that equality in (2.4) holds only if  $F$  is exponential, see e.g. Dallas (1981).

### 3 Application to hypothesis testing

#### 3.1 Construction of tests versus DVRL

Suppose the lifetime  $X$  of a device has a distribution function  $F$  which is unknown. We have in our disposal a random sample  $X_1, X_2, \dots, X_n$  of independent observations from  $F$ . We want to test the null hypothesis  $H_0$  against its alternative  $H_1$ , where

$H_0$  :  $F$  is exponential, versus

$H_1$  :  $F$  belongs to the class DVRL and  $F$  is not exponential.

We suggest to use the first two inequalities, (2.1) and (2.2), established in Theorem 2.1. For this purpose we introduce the following two quantities:

$$M^{(1)} = 4\mathbf{E} [X_2 Y^2] - \frac{8}{3}\mathbf{E} [Y^3] - m_1 m_2, \quad (3.1)$$

$$M^{(2)} = \frac{16}{3}\mathbf{E} [X_2 Y^3] - 4\mathbf{E} [Y^4] - m_2^2. \quad (3.2)$$

According to Theorem 2.2, if  $F$  is exponential, this is hypothesis  $H_0$ , we have  $M^{(1)} = 0$  and  $M^{(2)} = 0$ . Hence, under hypothesis  $H_1$ , in view of (2.1) and (2.2), we have  $M^{(1)} > 0$  and  $M^{(2)} > 0$ . This motivates us to use the above quantities  $M^{(1)}$  and  $M^{(2)}$  as measures of departure of  $F$  from the exponential distribution.

We use the sample  $X_1, \dots, X_n$  and define  $Y_{ij} = \min\{X_i, X_j\}$ ,  $i, j = 1, 2, \dots, n$ . Since we do not know  $F$ , and hence  $M^{(1)}$  and  $M^{(2)}$ , we have to replace them by appropriate estimators, say  $\hat{M}_n^{(1)}$  and  $\hat{M}_n^{(2)}$ , based on the sample  $X_1, \dots, X_n$ . We take

$$\hat{M}_n^{(1)} = \frac{1}{n(n-1)} \sum_{i \neq j} \left\{ 4X_j Y_{ij}^2 - \frac{8}{3} Y_{ij}^3 - X_i X_j^2 \right\}, \quad (3.3)$$

$$\hat{M}_n^{(2)} = \frac{1}{n(n-1)} \sum_{i \neq j} \left\{ \frac{16}{3} X_j Y_{ij}^3 - 4Y_{ij}^4 - X_i^2 X_j^2 \right\}. \quad (3.4)$$

It is interesting to note the difference between the pair  $M^{(1)}, \hat{M}_n^{(1)}$  and the pair  $M^{(2)}, \hat{M}_n^{(2)}$ . Clearly,  $M^{(1)}$  involves combined moments of total order equal to 3, while the total order of combined moments in  $M^{(2)}$  is equal to 4. The estimators  $\hat{M}_n^{(1)}$  and  $\hat{M}_n^{(2)}$  involve empirical moments of total order equal to 3

and 4, respectively. It can be shown that  $\hat{M}_n^{(1)} \xrightarrow{P} M^{(1)}$  and  $\hat{M}_n^{(2)} \xrightarrow{P} M^{(2)}$  as  $n \rightarrow \infty$ , where  $\xrightarrow{P}$  stands for convergence in probability.

Let us propose now the test statistics. If the mean value  $\mu$  of  $F$  is known to us, we use  $\hat{M}_n^{(1)}/\mu^3$  and  $\hat{M}_n^{(2)}/\mu^4$  as scale-invariant test statistics. If  $\mu$  is unknown, we replace it by the sample mean  $\bar{X}$  and in this case the scale-invariant test statistics are denoted by  $\hat{T}_n^{(1)}$  and  $\hat{T}_n^{(2)}$  and defined by

$$T_n^{(1)} = \hat{M}_n^{(1)}/\bar{X}^3 \quad \text{and} \quad T_n^{(2)} = \hat{M}_n^{(2)}/\bar{X}^4. \quad (3.5)$$

### 3.2 Asymptotic properties

We follow the general approach of using  $U$ -statistics. For any two variables,  $X_i$  and  $X_j$ , from the sample  $X_1, X_2, \dots, X_n$  of independent observations, we define the functions

$$\phi^{(1)}(X_1, X_2) = 4X_2Y^2 - \frac{8}{3}Y^3 - X_1X_2^2, \quad (3.6)$$

$$\phi^{(2)}(X_1, X_2) = \frac{16}{3}X_2Y^3 - 4Y^4 - X_1^2X_2^2. \quad (3.7)$$

The functions  $\phi^{(1)}(X_1, X_2)$  and  $\phi^{(2)}(X_1, X_2)$  are not symmetric and we need their symmetrization. We take

$$\bar{\phi}^{(1)}(X_1, X_2) = \frac{1}{2} \{ \phi^{(1)}(X_1, X_2) + \phi^{(1)}(X_2, X_1) \}$$

and similarly we define  $\bar{\phi}^{(2)}(X_1, X_2)$  in terms of  $\phi^{(2)}(X_1, X_2)$ . Then the test statistics  $\hat{M}_n^{(1)}$  and  $\hat{M}_n^{(2)}$  are equivalent to  $U$ -statistics  $U_n^{(1)}$  and  $U_n^{(2)}$  of order 2, where

$$U_n^{(1)} = \frac{1}{\binom{n}{2}} \sum_{i < j}^n \bar{\phi}^{(1)}(X_i, X_j), \quad U_n^{(2)} = \frac{1}{\binom{n}{2}} \sum_{i < j}^n \bar{\phi}^{(2)}(X_i, X_j). \quad (3.8)$$

Based on classical results of Hoeffding's type, see Serfling (1980), Randles (1982) or Severini (2005), the following two theorems summarize the asymptotic properties of  $T_n^{(1)}$  and  $T_n^{(2)}$  as defined by (3.5). We use also the notations  $T^{(1)} = M^{(1)}/\mu^3$  and  $T^{(2)} = M^{(2)}/\mu^4$ .

**Theorem 3.1** *If  $n \rightarrow \infty$ , then  $\sqrt{n}(T_n^{(1)} - T^{(1)})$  is asymptotically normal with mean 0 and variance  $B_1^2$ , where*

$$B_1^2 = \frac{1}{\mu^6} \mathbf{Var} \left\{ 4 \int_0^{X_1} \int_y^\infty yu \, dF(u) \, dy - \frac{2}{3} X_1^3 \bar{F}(X_1) + 4 X_1 \int_0^{X_1} u^2 \, dF(u) - \frac{16}{3} \int_0^{X_1} u^3 \, dF(u) - X_1^2 m_1 - X_1 m_2 \right\}. \quad (3.9)$$

**Theorem 3.2** *If  $n \rightarrow \infty$ , then  $\sqrt{n}(T_n^{(2)} - T^{(2)})$  is asymptotically normal with mean 0 and variance  $B_2^2$ , where*

$$B_2^2 = \frac{1}{\mu^8} \mathbf{Var} \left\{ 8 \int_0^{X_1} \int_y^\infty y^2 u \, dF(u) \, dy - \frac{4}{3} X_1^4 \bar{F}(X_1) + \frac{8}{3} X_1 \int_0^{X_1} u^3 \, dF(u) - 4 \int_0^{X_1} u^4 \, dF(u) - X_1^2 m_2 \right\}. \quad (3.10)$$

**Proof.** The proof of Theorem 3.1 and Theorem 3.2 is based on the same idea. First, as  $n \rightarrow \infty$ , we have  $\bar{X} \xrightarrow{P} \mu$ ,  $\hat{M}_n^{(1)} \xrightarrow{P} M^{(1)}$ ,  $\hat{M}_n^{(2)} \xrightarrow{P} M^{(2)}$  and we derive that also  $T_n^{(1)} \xrightarrow{P} T^{(1)}$ ,  $T_n^{(2)} \xrightarrow{P} T^{(2)}$ . We use now results from Serfling (1980) and Randles (1982) to conclude that, as  $n \rightarrow \infty$ ,  $T_n^{(1)}$  and  $\hat{M}_n^{(1)}/\mu^3$ , have asymptotically the same normal distribution  $N(0, B_1^2)$ , and similarly  $T_n^{(2)}$  and  $\hat{M}_n^{(2)}/\mu^4$ , have asymptotically the same normal distribution  $N(0, B_2^2)$ . The variance  $B_i^2$ ,  $i = 1, 2$ , has the following explicit form:  $B_i^2 = m^2 \mathbf{Var}[\psi^{(i)}(X_1)]$ , where  $m = 2$ , is the order of the  $U$ -statistics, see (3.8), and

$$\psi^{(i)}(X_1) = \frac{1}{2} \left\{ \mathbf{E}[\phi^{(i)}(X_1, X_2)|X_1] + \mathbf{E}[\phi^{(i)}(X_2, X_1)|X_1] \right\}, \quad i = 1, 2. \quad (3.11)$$

The next step is to use (3.11) and the functions  $\phi^{(i)}(X_1, X_2)$  defined by (3.6) and (3.7). After some transformations we find that

$$\psi^{(1)}(X_1) = \frac{1}{2} \left\{ 8 \int_0^{X_1} \int_y^\infty yu \, dF(u) \, dy - \frac{4}{3} X_1^3 \bar{F}(X_1) + 4 X_1 \int_0^{X_1} u^2 \, dF(u) - \frac{16}{3} \int_0^{X_1} u^3 \, dF(u) - X_2 m_1 - X_1 m_2 \right\}, \quad (3.12)$$

$$\psi^{(2)}(X_1) = \frac{1}{2} \left\{ 16 \int_0^{X_1} \int_y^\infty y^2 u \, dF(u) \, dy - \frac{8}{3} X_1^4 \bar{F}(X_1) + \frac{16}{3} X_1 \int_0^{X_1} u^3 \, dF(u) - 8 \int_0^{X_1} u^4 \, dF(u) - 2 X_1^2 m_2 \right\}. \quad (3.13)$$

Finally, taking the variance of (3.12), we arrive at the value  $B_1^2$  as given in (3.9). Similarly,  $B_2^2$ , given in (3.10) is obtained by taking the variance of (3.13).

**Corollary** *If the null hypothesis  $H_0$  is true, e.g., the life distribution  $F$  is exponential with parameter 1, then, as  $n \rightarrow \infty$ , the limiting distribution of  $\sqrt{n}(T_n^{(i)} - T^{(i)})$ ,  $i = 1, 2$ , is normal,  $N(0, \sigma_i^2)$ , with null-variances  $\sigma_1^2 = 4/3$  and  $\sigma_2^2 = 56/27$ .*

**Hint.** Using the explicit expressions for the exponential distribution function and its density, and after a series of calculations (we do not include here the technical details), we find that  $\sigma_1^2 = 4/3$  and  $\sigma_2^2 = 56/27$ .

#### 4 Pitman's asymptotic efficiency

To compare the goodness of the test statistics  $T_n^{(i)}$ ,  $i = 1, 2$ , we use the concept of Pitman's asymptotic efficiency (PAE); see e.g. Nikitin (1995). Here are some details. Let  $F(x; \theta_n)$  be a sequence of alternative distribution functions, where  $x > 0$ ,  $\theta_n = \theta_0 + c/\sqrt{n}$  and  $c$  is a fixed nonnegative number. If  $c = 0$ , then  $\theta_0$  will correspond to the exponential distribution. We assume that  $F$  has a density function,  $f$ , and further, that  $f$  is smooth.

If  $T$  is a test statistic, its PAE is given by

$$PAE(T, F(\theta_0)) = \frac{1}{\sigma_0^2} \lim_{n \rightarrow \infty} \left\{ \frac{d}{d\theta} \mathbf{E}[T_n] \right\}, \quad (4.1)$$

where  $\sigma_0^2$  is the asymptotic variance corresponding to the null hypothesis.

Hence for our tests  $\hat{M}_n^{(1)}$  and  $\hat{M}_n^{(2)}$ , see (3.3) and (3.4), we take  $n \rightarrow \infty$ , in which case  $\theta \rightarrow \theta_0$ , thus finding the following expressions:

$$\begin{aligned} \lim_{\theta \rightarrow \theta_0} \frac{d\{\hat{M}_n^{(1)}(\theta)\}}{d\theta} &= 8 \int_0^\infty \int_x^\infty xu \left\{ \bar{F}'(x, \theta_0) f(u, \theta_0) + \bar{F}(x, \theta_0) f'(u, \theta_0) \right\} du dx \\ &\quad - 16 \int_0^\infty x^2 \bar{F}(x, \theta_0) \bar{F}'(x, \theta_0) dx - 2 \int_0^\infty \bar{F}'(x, \theta_0) dx \\ &\quad - 2 \int_0^\infty x \bar{F}'(x, \theta_0) dx, \end{aligned} \quad (4.2)$$

$$\lim_{\theta \rightarrow \theta_0} \frac{d\{\hat{M}_n^{(2)}(\theta)\}}{d\theta} = 16 \int_0^\infty \int_x^\infty x^2 u \left\{ \bar{F}'(x, \theta_0) f(u, \theta_0) + \bar{F}(x, \theta_0) f'(u, \theta_0) \right\} du dx - 32 \int_0^\infty x^3 \bar{F}(x, \theta_0) \bar{F}'(x, \theta_0) dx - 8 \int_0^\infty x \bar{F}'(x, \theta_0) dx. \quad (4.3)$$

Pitman's asymptotic relative efficiency is usually calculated for certain specific tests and specific families of life distributions. As an illustration we have chosen three distributions widely used in reliability analysis and calculated their PAE's. The distributions of interest are:

- (a) Linear failure rate distribution  $F_1$ :  $\bar{F}_1(x; \theta) = e^{-x - (\theta/2)x^2}$ ,  $x > 0$ ,  $\theta > 0$ .
- (b) Weibull distribution  $F_2$ :  $\bar{F}_2(x; \theta) = e^{-x^\theta}$ ,  $x > 0$ ,  $\theta > 1$ .
- (c) Makeham distribution  $F_3$ :  $\bar{F}_3(x; \theta) = e^{-x - \theta(x + e^{-x})}$ ,  $x > 0$ ,  $\theta > 0$ .

It is worth mentioning that the Weibull distribution with parameter  $\theta > 1$  has a decreasing VRL, and this property holds also for the two others.

The quality of the tests  $T_n^{(1)}$ ,  $T_n^{(2)}$  is best seen when comparing them with other available tests. A well-known and widely used test has been described by Hollander and Proschan (1975). These authors proposed a test,  $V^*$ , for exponentiality versus DMRL (decreasing **mean** residual life). Recently, Abu-Youssef (2002) proposed a test  $\hat{\Delta}_n$  for DMRL based on a moment inequality. The results of our calculations of the Pitman asymptotic efficiency, together with the calculations of Hollander and Proschan (1975) and Abu-Youssef (2002) are summarized in Table 1.

These calculations clearly indicate that the tests proposed in this paper are well comparable with other tests widely used in statistical practice. In some cases, e.g. for life distributions with linear failure rate, the test  $T_n^{(2)}$  is better than other tests.

Table 1  
Pitman's asymptotic efficiency (PAE) of  $V^*$ ,  $\hat{\Delta}_n$ ,  $T_n^{(1)}$  and  $T_n^{(2)}$

Distribution	$V^*$	$\hat{\Delta}_n$	$T_n^{(1)}$	$T_n^{(2)}$
Linear failure rate, $F_1$	0.906	0.919	0.866	1.389
Weibull, $F_2$	0.846	0.710	0.532	0.638
Makeham, $F_3$	0.242	0.201	0.143	0.153

**Remark.** Comparing the two tests  $T_n^{(1)}$  and  $T_n^{(2)}$  we see that  $T_n^{(2)}$  is better. This can be explained by the fact that they both involve combined moments of  $X_1, X_2$  and  $Y$ , however for  $T_n^{(2)}$  the total order of moments is 4, while for  $T_n^{(1)}$  the order is 3. This observations suggests to look for tests based on combined

moments of a higher total order with the expectation to get a better efficiency. We conjecture that this can be achieved by using the inequality (2.3) for  $k = 2, 3$  or more. The test statistics  $T_n^{(k)}$  will be of the form  $T_n^{(k)} = \hat{M}_n^{(k)} / \bar{X}^{k+2}$ , where

$$\begin{aligned} \hat{M}_n^{(k)} = & \frac{1}{n(n-1)} \sum_{i \neq j} \sum \left[ X_i X_j^2 Y_{ij}^k - \frac{k}{2(k+1)} X_j^2 Y_{ij}^{k+1} - \frac{k}{k+1} X_i X_j Y_{ij}^{k+1} \right. \\ & \left. + \frac{4k(k+1)+8}{(k+1)(k+2)} X_j Y_{ij}^{k+2} - \frac{8+k(k+1)}{(k+1)(k+3)} Y_{ij}^{k+3} - \frac{k X_1^2 X_2^2 Y_{ij}^{k-1}}{4(k-1)} \right]. \end{aligned}$$

## 5 Applying the test

To carry out the test, we use the available sample  $X_1, \dots, X_n$  and calculate, for  $i = 1$  and  $i = 2$ , the value  $t_n = \sqrt{n} T_n^{(i)} / \sigma_i$ , where  $\sigma_i^2$  is the variance of the distribution corresponding to the null hypothesis. Now given  $\alpha$ , a significance level, we compare  $t_n$  and the normal variate value  $z_{1-\alpha}$ . If  $t_n$  exceeds  $z_{1-\alpha}$ , then we reject the hypothesis  $H_0$ . Otherwise, we accept the null hypothesis  $H_0$ .

Comparing  $T_n^{(1)}$  and  $T_n^{(2)}$ , we may suggest that the test  $T_n^{(2)}$  is better. To numerically illustrate the test  $T_n^{(2)}$ , we have simulated the lower and the upper percentile points for the significance level  $\alpha = 0.01, 0.05$  and  $0.10$ . The calculation of the test  $T_n^{(2)}$  is based on 5,000 simulated samples from the standard exponential distribution. Table 2 shows the critical values for the test statistic  $T_n^{(2)}$ .

As an illustration, we have estimated the power of  $T_n^{(2)}$  when alternatives are the Linear failure rate and the Weibull distributions. Our findings are summarized in Table 3.

Finally, as an illustration, we consider a real data set representing 40 patients suffering from blood cancer. We use the data as given in Hindi and Abouammoh (2001). The ordered life times (in days) are provided in Table 4. Based on this data set, the value of the test statistic  $T_n^{(2)}$  is equal to 0.3168. This value is greater than the critical value in Table (2) at 90% upper percentile, hence, we reject the null hypothesis  $H_0$  in favor of the alternative  $H_1$ . This means that the data set comes from DVRL distribution. This is agreeing with the conclusion of Hindi and Abouammoh (2001).

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Table 2  
 Critical values for the test statistic  $T_n^{(2)}$

$n$	0.01	0.05	.0.10	0.90	0.95	0.99
10	-1.4908	-0.8347	-0.5621	0.3653	0.4471	0.6366
15	-1.4868	-0.8082	-0.5085	0.3578	0.4236	0.5758
25	-1.4791	-0.7950	-0.4761	0.3382	0.3957	0.5188
30	-1.4241	-0.6750	-0.4190	0.3331	0.3878	0.5106
35	-1.3629	-0.6738	-0.4271	0.3224	0.3701	0.4739
40	-1.3617	-0.6627	-0.4102	0.3122	0.3670	0.4594
45	-1.3020	-0.6392	-0.3929	0.3032	0.3523	0.4467
50	-1.2127	-0.6026	-0.3692	0.2924	0.3412	0.4223
60	-1.1737	-0.5789	-0.3737	0.2854	0.3266	0.4088
70	-1.1349	-0.5645	-0.3481	0.2697	0.3206	0.4040
80	-1.1220	-0.5271	-0.3275	0.2596	0.3018	0.3786
90	-0.9280	-0.5000	-0.3130	0.2511	0.2934	0.3709
100	-0.8508	-0.4552	-0.2989	0.2389	0.2704	0.3527

Table 3  
 Power estimates for the test statistic  $T_n^{(2)}$

Distribution	Parameter	Sample size $n$			
		$\theta$	20	30	40
$F_1$	1		0.973	0.977	0.975
Linear failure rate	2		0.994	0.998	1.000
	3		1.000	1.000	1.000
Weibull	$F_2$	1	0.944	0.945	0.950
	2		0.999	1.000	1.000
	3		1.000	1.000	1.000

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Table 4

*Life times for 40 patients suffering blood cancer*

115	181	255	418	441	461	516	739	743	789	807	865
924	983	1024	1062	1063	1165	1191	1222	1222	1251	1277	1290
1357	1369	1408	1455	1478	1549	1578	1578	1599	1603	1605	1696
1735	1799	1815	1852								

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Figure

