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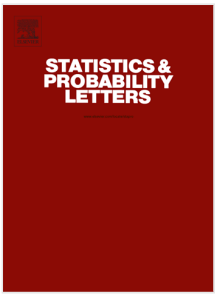

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SPECTRAL DENSITY ESTIMATION FOR LINEAR PROCESSES WITH DEPENDENT INNOVATIONS

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Abstract

This paper considers the problem of estimating the spectral density of a linear process whose innovations are uncorrelated and strong mixing. We prove that the Periodogram ordinates $I_n(\lambda_i)$ at any set of frequencies $\lambda_1, ..., \lambda_m,\ 0 < \lambda_1 < ... < \lambda_m < \pi$, are asymptotically independent exponential random variables with means $2\pi f(\lambda_i)$. Consequently the periodogram I_n is not consistent estimator of $2\pi f$. Consistent estimators can however be constructed by applying linear smoothing filters to the periodogram.

AMS Subject Classification: 62G05, 62G20, 62M15.

Keywords: spectral density estimation, periodogram, mixing.

1 Introduction

For a sequence $(X_t, t \in \mathbb{Z})$ of random variables we consider its periodogram, defined at the Fourrier frequencies $\omega_j = 2\pi j/n$, $\omega_j \in [-\pi, \pi]$, by

$$I_{n,X}(\omega_j) = \frac{1}{n} \Big| \sum_{t=1}^{n} X_t e^{-it\omega_j} \Big|^2, \tag{1}$$

then, if $(X_t, t \in \mathbb{Z})$ is a stationary time series with mean m and autocovariance $\gamma(.)$, $I_n(\omega_j)/2\pi$ appears to be the natural estimator of $f_X(\omega_j)$, the spectral density of $(X_t, t \in \mathbb{Z})$, defined by

$$f_X(\omega) = 1/(2\pi) \sum_{k=-\infty}^{+\infty} \gamma(k) \exp^{-ik\omega}, \quad \omega \in [-\pi, \pi]. \tag{2}$$

If $\gamma(.)$ is a absolutely summable autocovariance function of the process $(X_t, t \in \mathbb{Z})$, then f_X is continuous and the definition (1) is equivalent to

$$\begin{cases} I_{n,X}(0) = n|\overline{X}|^2 \\ I_{n,X}(\omega_j) = \sum_{|k|<n} \widehat{\gamma}(k) e^{-it\omega_j} \quad \text{if } \omega_j \neq 0, \end{cases}$$

where

$$\widehat{\gamma}(k) = 1/n \sum_{t=1}^{n-|k|} (X_t - \overline{X})(X_{t+|k|} - \overline{X}) \text{ and } \overline{X} = 1/n \sum_{t=1}^{n} X_t.$$

It is well known that the extensive studies of the periodogram and of closely related expressions produce several solutions to the problem of estimating the spectral density function of stationary random processes. It would be hard to compile a complete list. Here we only mention some representatives: Tukey (1949), Bartlett (1950), Parzen (1957), Grenander and Rosenblatt (1957), Hannan (1970), Anderson (1971), Brillinger (1981), Priestley (1981) and Brockwell and Davis (1991). The purpose of this paper is to investigate the asymptotic properties of spectral density estimates of linear process whose innovations are uncorrelated and strong mixing. Under more general conditions we obtain results quite analogous to the i.i.d case. More precisely, we are interested in the limit behaviour of the periodogram and the smooth-periodogram of $\{X_t, ..., X_n\}$, where $(X_t, t \in \mathbb{Z})$ is the linear process

$$X_t = \sum_{j=-\infty}^{+\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z} \tag{3}$$

and $(Z_t, t \in \mathbb{Z})$ is a strictly stationary sequence of uncorrelated random variables, with mean 0, variance $\sigma^2 > 0$, and spectral density f_Z, satisfying the strong mixing condition defined below.

Definition 1.1 *Denote the σ-fields generated by X_t, $t \leq 0$ and X_t, $t \geq k$, respectively, by $\mathcal{F}_{-\infty}^0$ and $\mathcal{F}_k^{+\infty}$. Then X_t is strong mixing or α-mixing if :*

$$\alpha(k) = \sup\{|P(A \cap B) - P(A)P(B)|, \ A \in \mathcal{F}_{-\infty}^0, \ B \in \mathcal{F}_k^{+\infty}\} \longrightarrow 0 \ \text{as} \ k \longrightarrow +\infty.$$

The case when the random variables Z_t are independently and identically distributed with mean 0 and variance σ^2 has been studied in Brockwell and Davis (1991). The paper is organized as follows. The essential results are contained in theorem 2.2 in section 2 where we prove that the periodogram ordinates at any set of different frequencies $\lambda_1, ..., \lambda_m \in [-\pi, \pi]$ are asymptotically independent exponential r.v.s with means $2\pi f(\lambda_i)$. Consequently, the periodogram is not consistent estimator of $2\pi f$. Consistent estimators can however be constructed, in section 3, by applying linear smoothing filters to the periodogram. The asymptotic behaviour of the resulting discrete spectral average estimators can be derived from the asymptotic behaviour of the periodogram.

2 Convergences of the periodogram

In order to estimate $f_X(\omega)$ for arbitrary non-zero frequencies in the interval $[-\pi, \pi]$, we need to extend the domain of $I_{n,X}$ to the whole interval $[-\pi, \pi]$.
For any $\omega \in [-\pi, \pi]$ the periodogram is defined as follows :

$$I_{n,X}(\omega) = \begin{cases} I_{n,X}(\omega_k) & \text{if } \omega_k - \pi/n < \omega < \omega_k + \pi/n \quad \text{and} \quad 0 \leq \omega \leq \pi \\ I_{n,X}(-\omega) & \text{if } \omega \in [-\pi, 0). \end{cases}$$

Let $g(n, \omega)$ be the multiple of $2\pi/n$ closest to ω (the smaller one if there are two) and for $\omega \in [-\pi, 0]$ let $g(n, \omega) = g(n, -\omega)$. Then (1) can be rewritten as $I_{n,X}(\omega) = I_{n,X}(g(n, \omega))$. Let for $\omega_j \in [-\pi, \pi]$:

$$\begin{cases} A(\omega_j) = (2/n)^{\frac{1}{2}} \sum_{t=1}^{n} X_t \cos(\omega_j t) \\ B(\omega_j) = (2/n)^{\frac{1}{2}} \sum_{t=1}^{n} X_t \sin(\omega_j t) \end{cases} \tag{4}$$

so that :

$$I_{n,X}(\omega_j) = \frac{A^2(\omega_j) + B^2(\omega_j)}{2}.$$

Theorem 2.1 *Let $(Z_t, t \in \mathbb{Z})$ be a strictly stationary and strong mixing sequence of uncorrelated random variables, with mean 0 and variance $\sigma^2 > 0$. Suppose that for some $\gamma > 2$ and $m_\gamma \in \mathbb{R}^+$*

$$E|Z_t|^{2\gamma} = m_\gamma, \quad t \in \mathbb{Z},$$

(i) if the strong mixing coefficients are such that $\alpha(k) = O(k^{-\beta})$ for some $\beta > \gamma$, then for $0 < \lambda_1 < ... < \lambda_m < \pi$ the random vector $(I_{n,Z}(\lambda_1), ..., I_{n,Z}(\lambda_m))'$ converges in distribution as $n \to \infty$ to a vector of independent and exponentially distributed random variables, each with mean σ^2.
(ii) if the strong mixing coefficient verify $\sum_k \alpha(k)^{\frac{\gamma-2}{\gamma}} < \infty$, and $EZ_1^4 = \eta\sigma^4$, then

$$Cov(I_{n,Z}(\omega_j), I_{n,Z}(\omega_k)) = \begin{cases} n^{-1}(\eta-3)\sigma^4 + 2\sigma^4 + O(n^{-1}) & \text{if } \omega_j = \omega_k = 0 \text{ or } \pi, \\ n^{-1}(\eta-3)\sigma^4 + \sigma^4 + O(n^{-1}) & \text{if } 0 < \omega_j = \omega_k < \pi, \\ n^{-1}(\eta-3)\sigma^4 + O(n^{-1}) & \text{if } \omega_j \neq \omega_k. \end{cases}$$

Proof : For arbitrary frequency $\lambda \in (0, \pi)$ define $A(\lambda) := A(g(n, \lambda))$ *and* $B(\lambda) := B(g(n, \lambda))$, where $A(\omega_j)$ and $B(\omega_j)$ are given by (4) with Z_t replacing X_t. Since

$$I_{n,Z}(\lambda_j) = (A^2(\lambda_j) + B^2(\lambda_j))/2,$$

it suffices to show that

$$(A(\lambda_1), B(\lambda_1), ..., A(\lambda_m), B(\lambda_m))' \tag{5}$$

converges in distribution as $n \to \infty$ to a centered Gaussian random vector with covariance matrix $\sigma^2 I_{2m}$ where I_{2m} is the $2m$ x $2m$ identity matrix.
Let $\mu_1, \cdots, \mu_m, \mu_1', \cdots, \mu_m' \in \mathbb{R}$ be fixed and define for each $n \in \mathbb{N}$, $t = 1, \cdots, n$, the random variables

$$Y_{t,n} = \sum_{i=1}^{m} \sqrt{2}(\mu_i \cos \lambda_i t + \mu_i' \sin \lambda_i t) Z_t \text{ and } S_n = \sum_{t=1}^{n} Y_{t,n}.$$

The linear combination of the coordinates of (2) needed to use the Cramer-Wold theorem. The method of proof consists on defining large and small blocks, of sizes p_n and q_n, respectively, of variables and coupling them using the Bradley's lemma (1983). On what follows we will drop the subscript n on p_n and q_n to avoid heavy notation. Let $r \in \mathbb{N}$ be such that $r(p+q) \leq n < r(p+q+1)$ (again we should denote by r_n, what we do not for the same reason as before) and define

$$\begin{aligned} V_{1,n} &= Y_{1,n} + \cdots + Y_{p,n}, & V_{1,n}' &= Y_{p+1,n} + \cdots + Y_{p+q,n} \\ V_{2,n} &= Y_{p+q+1,n} + \cdots + Y_{2p+q,n}, & V_{2,n}' &= Y_{2p+q+1,n} + \cdots + Y_{2(p+q),n} \\ &\vdots & &\vdots \\ V_{r,n} &= Y_{(r-1)(p+q)+1,n} + \cdots + Y_{rp+(r-1)q,n}, & V_{r,n}' &= Y_{rp+(r-1)q+1,n} + \cdots + Y_{r(p+q),n}. \end{aligned}$$

Suppose that $r = O(n^c)$, $p = O(n^{1-c})$, $q = O(n^d)$, with $c, d \in (0, 1)$ suitably chosen, as it will be explained in course of proof.

Using recursively the Bradley's Lemma (1983) with $p = 2\gamma,\ \gamma > 2$, we may conclude that there exist independent variables $W_{1,n}, \cdots, W_{r,n}$, with distributions $P_{W_{j,n}} = P_{V_{j,n}}$ and such that

$$P(|V_{j,n} - W_{j,n}| > \xi_n) \leq 11\left(\frac{\|V_{j,n} + C_n\|_{2\gamma}}{\xi_n}\right)^{\frac{2\gamma}{4\gamma+1}} \alpha(q)^{\frac{4\gamma}{4\gamma+1}}, \tag{6}$$

where $\xi_n \in (0, \|V_{j,n} + C_n\|_{2\gamma}]$ and $\|V_{j,n} + C_n\|_{2\gamma} > 0$. Now

$$\begin{aligned} \|Y_{t,n}\|_{2\gamma}^{2\gamma} &= E\Big(\sum_{t=1}^{m} \sqrt{2}(\mu_i \cos \lambda_i t + \mu_i{}' \sin \lambda_i t) Z_t\Big)^{2\gamma} \\ &\leq 2^{\gamma} E(Z_t^{2\gamma})\Big(\sum_{i=1}^{m} |\mu_i| + |\mu_i{}'|\Big)^{2\gamma} \\ &:= A. \end{aligned} \tag{7}$$

Choosing in (6) $C_n = 2pA^{\frac{1}{2\gamma}}$ it follows that, for n large enough,

$$pA^{\frac{1}{2\gamma}} \leq \|V_{j,n} + C_n\|_{2\gamma} \leq 3pA^{\frac{1}{2\gamma}}.$$

So that, for n large enough,

$$P(|V_{j,n} - W_{j,n}| > \xi_n) = O\left(\frac{p^{\frac{2\gamma}{4\gamma+1}} \alpha(q)^{\frac{4\gamma}{4\gamma+1}}}{\xi_n^{\frac{2\gamma}{4\gamma+1}}}\right). \tag{8}$$

If we consider now

$$\Delta_n = \frac{\sum_{j=1}^{r} W_{j,n}}{\sqrt{rp}} - \frac{\sum_{j=1}^{r} V_{j,n}}{\sqrt{rp}},$$

it follows from (8)

$$P(|\Delta_n| > \epsilon) \leq \sum_{j=1}^{r} P\left(\big|(V_{j,n} - W_{j,n})\big| > \epsilon\sqrt{\frac{p}{r}}\right) = O\big(p^{\frac{\gamma}{4\gamma+1}} r^{\frac{5\gamma+1}{4\gamma+1}} \alpha(q)^{\frac{4\gamma}{4\gamma+1}}\big).$$

Taking account of the convergence rate supposed for the strong mixing coefficients, it follows that

$$P(|\Delta_n| > \epsilon) = O\Big(n^{\frac{\gamma+4c\gamma+c-4d\beta\gamma}{4\gamma+1}}\Big). \tag{9}$$

As $c < 1$ and $\beta > \gamma$, (9) converges to zero provided that $d > \frac{5\gamma+1}{4\gamma^2}$, which is verified if

$$d > \frac{11}{16}. \tag{10}$$

We prove now the asymptotic normality of $(rp)^{-1/2} \sum_{j=1}^{r} W_{j,n}$ verifying the Lyapounov condition. For this we will verify that, for some $\rho > 2$,

$$Z_n = \frac{\sum_{j=1}^{r} E(|W_{j,n}|^{\rho})}{(rVar(W_{j,n}))^{\frac{\rho}{2}}} \longrightarrow 0. \tag{11}$$

First, we have

$$Var(W_{1,n}) = Var\Big(\sum_{t=1}^{p}\sum_{i=1}^{m}\sqrt{2}(\mu_i \cos\lambda_i t + \mu_i{}' \sin\lambda_i t)Z_t\Big) = 2\sigma^2\sum_{t=1}^{p}\Big(\sum_{i=1}^{m}(\mu_i \cos\lambda_i t + \mu_i{}' \sin\lambda_i t)\Big)^2$$

so, it's easy to check that $(rVar(W_{1,n}))^{\frac{\rho}{2}} \sim (rp)^{\frac{\rho}{2}}$.

Now, we shall control the numerator in (11) by using Yokoyama's inequality (1980). We have show in (7) that $E(|Y_{1,n}|^{2\gamma}) < \infty$, then it suffices to verify that there exists $2 < \rho < 2\gamma$ such that

$$\sum_n (n+1)^{\frac{\rho}{2}-1}\alpha(n)^{\frac{2\gamma-\rho}{2\gamma}} < \infty.$$

Given the assumptions made on the mixing coefficients, the convergence of the series follows from $\beta > \frac{\rho\gamma}{2\gamma-\rho}$.

Choosing $\rho = \gamma$, the convergence of the serie follows from the fact that $\beta > \gamma$.

Now, we may apply Yokoyama's inequality to derive $E(|W_{j,n}|^{\rho}) = O(\rho^{\frac{\rho}{2}})$, so that $Z_n = O\Big(\frac{rp^{\frac{\rho}{2}}}{(rp)^{\frac{\rho}{2}}}\Big) = O(n^{c(1-\frac{\rho}{2})})$. This converge to zero, since $\rho > 2$. This proves the asymptotic normality of $(rp)^{-1/2}\sum_{j=1}^{r} W_{j,n}$ from which follows the asymptotic normality of $(rp)^{-1/2}\sum_{j=1}^{r} V_{j,n}$.

To finish our proof write

$S_n = \sum_{j=1}^{r} V_{j,n} + \sum_{j=1}^{r} V_{j,n}^{'} + R_n$.

Using the same coupling technique it is easy to check that

$$Var\left(\frac{\sum_{j=1}^{r} V_{j,n}^{'}}{\sqrt{rp}}\right) = O\Big(\frac{rq}{rp}\Big) = O(n^{d-1+c}),$$

as the $V_{j,n}^{'}$ are sums of q variables, and $Var\left(\frac{R_n}{\sqrt{rp}}\right) = O(n^{c-1})$, for analogous reasons. Thus $(rp)^{-1/2}R_n$ converges in probability to zero. The term $(rp)^{-1/2}\sum_{j=1}^{r} V_{j,n}$ also converges in probability to zero if

$$c < 1 - d. \tag{12}$$

Taking account of (10), we derive $c < \frac{5}{16}$, so (i) is proved.

(ii) By definition of $I_{n,Z}(\omega_j)$, we have

$$I_{n,Z}(\omega_j) = n^{-1}\sum_{s=1}^{n}\sum_{t=1}^{n} Z_s Z_t e^{i\omega_j(t-s)},$$

hence

$$E(I_{n,Z}(\omega_j)I_{n,Z}(\omega_k)) = n^{-2}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n} E(Z_s Z_t Z_u Z_v)e^{i\omega_j(t-s)}e^{i\omega_k(v-u)}. \tag{13}$$

We shall examine many cases

1) If $s = t = u = v$, then

$$T_1 = n^{-2}\sum_{s=t=u=v} E(Z_s^4) = n^{-1}\eta\sigma^4. \tag{14}$$

2) If $s = t \neq u = v$ then

$$T_2 = n^{-2}\sum_{s\neq u} E(Z_s^2 Z_u^2).$$

Applying the Davydov's inequality (1968) we obtain,

$$|Cov(Z_s^2, Z_u^2)| \leq \frac{2\gamma}{\gamma-2}\|Z_s^2\|_\gamma \|Z_u^2\|_\gamma \big(2\alpha(s-u)\big)^{\frac{\gamma-2}{\gamma}}.$$

Hence,

$$E(Z_s^2 Z_u^2)) + \sigma^4 \leq \frac{2\gamma}{\gamma-2}\|Z_s^2\|_\gamma \|Z_u^2\|_\gamma \big(2\alpha(s-u)\big)^{\frac{\gamma-2}{\gamma}}, \tag{15}$$

and

$$n^{-2}\sum_{s\neq u} E(Z_s^2 Z_u^2) - (1-\frac{1}{n})\sigma^4 \leq n^{-2}\frac{2\gamma}{\gamma-2} m_\gamma^{2/\gamma} \sum_{s\neq u} \big(2\alpha(s-u)\big)^{\frac{\gamma-2}{\gamma}},$$

but,

$$\sum_{s=2}^{n}\sum_{s\neq u} \big(2\alpha(s-u)\big)^{\frac{\gamma-2}{\gamma}} = 2\sum_{s=2}^{n}\sum_{u=1}^{s-1} \big(2\alpha(s-u)\big)^{\frac{\gamma-2}{\gamma}} = 2\sum_{s=2}^{n}\sum_{v=1}^{s-1} \big(2\alpha(v)\big)^{\frac{\gamma-2}{\gamma}}.$$

Then

$$T_2 - (1-\frac{1}{n})\sigma^4 \leq \frac{2}{n^2}\frac{2\gamma}{\gamma-2} m_\gamma^{2/\gamma} \sum_{s=2}^{n}\sum_{v=1}^{s-1} \big(2\alpha(v)\big)^{\frac{\gamma-2}{\gamma}}.$$

Taking account of $\sum_k \alpha(k)^{\frac{\gamma-2}{\gamma}} < \infty$, we have

$$T_2 = (1-\frac{1}{n})\sigma^4 + O(n^{-1}), \tag{16}$$

3) If $s = u \neq t = v$ then

$$T_3 = n^{-2}\sum_{s\neq t} E(Z_s^2 Z_t^2) e^{i(\omega_j+\omega_k)(t-s)},$$

so it follows from (15) that

$$T_3 \leq \frac{2}{n^2}\frac{2\gamma}{\gamma-2} m_\gamma^{2/\gamma} \sum_{s=2}^{n}\sum_{v=1}^{s-1} \big(2\alpha(v)\big)^{\frac{\gamma-2}{\gamma}} + \frac{1}{n^2}\sigma^4 |\sum_{s=1}^{n} e^{i(\omega_j+\omega_k)s}|^2 - \frac{1}{n}\sigma^4,$$

hence for the same reasons, we have

$$T_3 = -\frac{1}{n}\sigma^4 + O(n^{-1}). \tag{17}$$

4) If $s = v \neq t = u$ then

$$\begin{aligned} T_4 &= n^{-2}\sum_{s\neq t} E(Z_s^2 Z_t^2) e^{i(\omega_j-\omega_k)(t-s)} \\ &\leq \frac{2}{n^2}\frac{2\gamma}{\gamma-2} m_\gamma^{2/\gamma} \sum_{s=2}^{n}\sum_{v=1}^{s-1} \big(2\alpha(v)\big)^{\frac{\gamma-2}{\gamma}} + \frac{1}{n^2}\sigma^4 |\sum_{s=1}^{n} e^{i(\omega_j-\omega_k)s}|^2 - \frac{1}{n}\sigma^4, \end{aligned}$$

thus

$$T_4 = -\frac{1}{n}\sigma^4 + O(n^{-1}). \tag{18}$$

5) If $s = t$, $t \neq u$ and $u \neq v$ then

$$\begin{aligned} |T_5| &= \Big| \frac{3!}{n^2} \sum_{s=1}^{n} \sum_{u=s+1}^{n} \sum_{v=u+1}^{n} E(Z_s^2 Z_u Z_v) e^{i\omega_k (v-u)} \Big| \\ &\leq \frac{3!}{n^2} \sum_{s=1}^{n} \sum_{u=s+1}^{n} \Big| Cov(Z_s^2, Z_u \sum_{v=u+1}^{n} Z_v) \Big| \\ &\leq \frac{3!}{n^2} \sum_{s=1}^{n} \sum_{k=1}^{n-s} \alpha(k). \end{aligned}$$

It follows that

$$T_5 = O(n^{-1}). \tag{19}$$

Using the same technique we can check the same result in the cases where $s = u \neq t \neq v$, $s = v \neq u \neq t$, $t = u \neq v \neq s$, $t = v \neq u \neq t$ and $u = v \neq s \neq t$, thus

$$T_i = O(n^{-1}) \text{ for } i = 6, .., 10. \tag{20}$$

6) If $s \neq t = u = v$ then

$$\begin{aligned} |T_{11}| &= \Big| \frac{2}{n^2} \sum_{s<t} E(Z_s Z_t^3) e^{i\omega_j (t-s)} \Big| \\ &\leq \frac{2}{n^2} \sum_{s=1}^{n} \sum_{t=s+1}^{n} |Cov(Z_s, Z_t^3)| \\ &\leq \frac{2}{n^2} \sum_{s=1}^{n} \sum_{k=1}^{n-s} \alpha(k). \end{aligned}$$

It follows that

$$T_{11} = O(n^{-1}). \tag{21}$$

We obtain the same result in the cases where $t \neq s = u = v$, $u \neq s = t = v$ and $v \neq s = t = u$, thus

$$T_i = O(n^{-1}) \text{ for } i = 12, 13, 14. \tag{22}$$

7) If $s \neq t$, $s \neq u$ and $s \neq v$ then ,

$$\begin{aligned} |T_{15}| &= \Big| n^{-2} \sum_{s=1}^{n} \sum_{t=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} E(Z_s Z_t Z_u Z_v) e^{i\omega_j (t-s)} e^{i\omega_k (v-u)} \Big| \\ &= \frac{4!}{n^2} \Big| \sum_{s=1}^{n} \sum_{t=s+1}^{n} \sum_{u=t+1}^{n} \sum_{v=u+1}^{n} E(Z_s Z_t Z_u Z_v) e^{i\omega_j (t-s)} e^{i\omega_k (v-u)} \Big| \\ &\leq \frac{4!}{n^2} \sum_{s=1}^{n} \sum_{t=s+1}^{n} \Big| Cov(Z_s, Z_t \sum_{u=t+1}^{n} \sum_{v=u+1}^{n} Z_u Z_v) \Big| \\ &\leq \frac{4!}{n^2} \sum_{s=1}^{n} \sum_{k=1}^{n-s} \alpha(k), \end{aligned}$$

and thus

$$T_{15} = O(n^{-1}). \tag{23}$$

Taking account of (14), (16), (17), (18), (19), (20), (21), (22), and (23) it follows that

$$cov(I_{n,Z}(\omega_j)I_{n,Z}(\omega_k)) = \sum_{i=1}^{15} T_i - \sigma^2 = \begin{cases} n^{-1}(\eta-3)\sigma^4 + \sigma^4 + O(n^{-1}) & \text{if } \omega_i = \omega_k \text{ and } 0 < \omega_i < \pi, \\ n^{-1}(\eta-3)\sigma^4 + 2\sigma^4 + O(n^{-1}) & \text{if } \omega_i = \omega_k \text{ and } \omega_i = 0 \text{ or } \pi, \\ n^{-1}(\eta-3)\sigma^4 + O(n^{-1}) & \text{if } \omega_i \neq \omega_k, \end{cases}$$

the proof of theorem is therefore complete.

Theorem 2.2 *Let $(X_t, t \in \mathbb{Z})$ be the linear process,*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, and $(Z_t, t \in \mathbb{Z})$ is a strictly stationary and strong mixing sequence of uncorrelated random variables, with mean 0 and variance $\sigma^2 > 0$.

Suppose that for some $\gamma > 2$ and $m_\gamma \in \mathbb{R}^+$, $E|Z_t|^{2\gamma} = m_\gamma$, $t \in \mathbb{Z}$.
(i) If the strong mixing coefficients are such that $\alpha(k) = O(k^{-\beta})$ for some $\beta > \gamma$, if $f_X(\lambda) > 0$ for all $\lambda \in [-\pi, \pi]$, then for $0 < \lambda_1 < ... < \lambda_m < \pi$ the random vector $(I_{n,X}(\lambda_1), ..., I_{n,X}(\lambda_m))'$ converges in distribution to a vector of independent and exponentially distributed random variables, the i^{th} component of which has mean $2\pi f_X(\lambda_i), i = 1, ..., m$.
(ii) If the strong mixing coefficient verify $\sum_k \alpha(k)^{\frac{\gamma-2}{\gamma}} < \infty$, if $EZ_1^4 = \eta\sigma^4 < \infty$, $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{1/2} < \infty$, and $\omega_j = 2\pi j/n \geq 0$ $\omega_k = 2\pi k/n \geq 0$, then

$$Cov(I_{n,X}(\omega_j), I_{n,X}(\omega_k)) = \begin{cases} 2(2\pi)^2 f_X^2(\omega_j) + O(n^{-1/2}) & \text{if } \omega_j = \omega_k = 0 \text{ or } \pi, \\ (2\pi)^2 f_X^2(\omega_j) + O(n^{-1/2}) & \text{if } 0 < \omega_j = \omega_k < \pi, \\ O(n^{-1/2}) & \text{if } 0 < \omega_j \neq \omega_k. \end{cases}$$

Proof : From Theorem 10.3.1 page 346 of Brockwell and Davis (1991), we have

$$I_{n,X}(\lambda) = |\psi(e^{-ig(n,\lambda)})|^2 I_{n,Z}(\lambda) + R_n(g(n,\lambda)), \tag{24}$$

where $\psi(e^{-i\lambda}) = \sum_{j=-\infty}^{+\infty} \psi_j e^{-ij\lambda}$, $-\pi \leq \lambda \leq \pi$, $sup_{\lambda\in[-\pi,\pi]} E|R_n(g(n,\lambda))| \to 0$.
It's well known that

$$f_X(\lambda) = |\psi(e^{-i\lambda})|^2 f_Z(\lambda), \ -\pi \leq \lambda \leq \pi,$$

so

$$\begin{aligned} (24) &= \frac{f_X(g(n,\lambda))}{f_Z(g(n,\lambda))} I_{n,Z}(\lambda) + R_n(g(n,\lambda)) \\ &= \frac{2\pi}{\sigma^2} f_X(g(n,\omega)) I_{n,Z}(\omega) + R_n(g(n,\lambda)). \end{aligned}$$

Since $f_X(g(n,\lambda)) \to f_X(\lambda)$ and $R_n(g(n,\lambda)) \to^P 0$, the result (i) follows immediately from Theorem 2.1. Now if $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{1/2} < \infty$ and $EZ_1^4 < \infty$ then from (24) we have

$$\begin{aligned} Var(I_{n,X}(\omega_k)) = \ & (2\pi f_X(\omega_k)/\sigma^2)^2 Var(I_{n,Z}(\omega_k)) + Var(R_n(\omega_k)) \\ & + 2(2\pi f_X(\omega_k)/\sigma^2) Cov(I_{n,Z}(\omega_k), R_n(\omega_k)). \end{aligned} \tag{25}$$

Since $Var(R_n(\omega_k)) \leq E|R_n(\omega_k)|^2 = O(n^{-1})$ and since $Var(I_{n,Z}(\omega_k))$ is bouded uniformly in ω_k, the Cauchy-Schwarz inegality implies that $Cov(I_{n,Z}(\omega_k), R_n(\omega_k)) = O(n^{-1/2})$. It therefore follows from (25) and Theorem 2.1 that

$$Var(I_{n,X}(\omega_k)) = \begin{cases} 2(2\pi)^2 f_X^2(\omega_k) + O(n^{-1/2}) & \text{if } \omega_k = 0 \, or \, \pi, \\ (2\pi)^2 f_X^2(\omega_k) + O(n^{-1/2}) & \text{if } 0 < \omega_k < \pi, \end{cases}$$

a similar argument also gives

$$Cov(I_{n,X}(\omega_j), I_{n,X}(\omega_k)) = O(n^{-1/2}) \quad if \quad \omega_j \neq \omega_k.$$

3 Convergences of the periodogram smoothing

Let $(X_t, t \in \mathbb{Z})$ be the linear process defined by (3). We consider the class of estimators of the form

$$\widehat{f}(w_j) = (2\pi)^{-1} \sum_{|k| \leq m_n} W_n(k) I_{n,X}(\omega_{j+k}), \tag{26}$$

where $(m_n, n \in \mathbb{N})$ is a sequence of positive integers such that

$$m_n \longrightarrow \infty, \ \frac{m_n}{\sqrt{n}} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and $(W_n, n \in \mathbb{N})$ is a sequence of weight functions satisfying the following conditions :

$$W_n(k) = W_n(-k), \ W_n(k) \geq 0, \quad \text{for all} \quad k, \ \sum_{|k| \leq m_n} W_n(k) = 1, \text{ and } \sum_{|k| \leq m_n} W_n^2(k) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

If $\omega_{j+k} \notin [-\pi, \pi]$, the term $I_{n,X}(\omega_{j+k})$ in (26) will be evaluated by defining $I_{n,X}$ to have period 2π. The same convention will be used to define $f_X(\omega)$, $\omega \notin [-\pi, \pi]$.
Now, for any $\omega \in [-\pi, \pi]$, we define the estimator of the spectral density of $(X_t, t \in \mathbb{Z})$ as follows

$$\widehat{f}(\omega) = \widehat{f}(g(n, \omega))$$

with $\widehat{f}(\omega_j)$ defined by (26).

Theorem 3.1 *Let $(X_t, t \in \mathbb{Z})$ be the linear process,*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

where $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{1/2} < \infty$ and $(Z_t, t \in \mathbb{Z})$ is a strictly stationary and strong mixing sequence of uncorrelated random variables, with mean 0 and variance $\sigma^2 > 0$. Then for λ, $\omega \in [0, \pi]$,
(a) $lim_{n \to \infty} E\widehat{f}(\omega) = f_X(\omega)$

and

$$(b) \ lim_{n \to \infty} \Big(\sum_{|j| \leq m_n} W_n^2(j) \Big)^{-1} Cov(\widehat{f}(\omega), \widehat{f}(\lambda)) = \begin{cases} 2f_X^2(\omega) & if \, \omega = \lambda = 0 \, or \, \pi, \\ f_X^2(\omega) & if \, 0 < \omega = \lambda < \pi, \\ 0 & if \, \omega \neq \lambda. \end{cases}$$

Proof : The proof is similar to that of the theorem 10.4.1 page 351 of Brockwell and Davis (1991).

Remark 3.2 *By theorem 2.2, the random variables* $I_{n,X}(\omega_j+\omega_k)/(\pi f_X(\omega_j+\omega_k))$, $-j<k<n/2-j$, *are approximately independent and distributed as chi-squared with 2 degrees of freedom. As mentioned in Brockwell and Davis (1991), this suggests approximating the distribution of* $\nu\widehat{f}(\omega_j)/f_X(\omega_j)$ *by the chi-squared with* $\nu=2/(\sum_{|k|\leq m_n}W_n^2(k))$ *degrees of freedom, then the interval*

$$\left(\frac{\nu\widehat{f}(\omega_j)}{\chi^2_{0.975}(\nu)},\ \frac{\nu\widehat{f}(\omega_j)}{\chi^2_{0.025}(\nu)}\right),\ 0<\omega_j<\pi, \tag{27}$$

is an approximate 95% *confidence interval for* $f_X(\omega_j)$. *By taking logaritms in (27) we obtain the* 95% *confidence interval for* $lnf_X(\omega_j)$*:*

$$\left(ln\widehat{f}(\omega_j)+ln\nu-ln\chi^2_{0.975}(\nu),\ ln\widehat{f}(\omega_j)+ln\nu-ln\chi^2_{0.025}(\nu)\right),\ 0<\omega_j<\pi, \tag{28}$$

4 Example

Let the MA(1) process

$$X_t=Z_t-0.6Z_{t-1}, \tag{29}$$

where $Z_t=U_tU_{t-1}$, with $(U_t,t\in\mathbb{Z})$ a sequence of i.i.d random variables with distribution $\mathcal{N}(0,1)$. The periodogram of 200 observations generated from $(X_t,t\in\mathbb{Z})$ is displayed in figure 1.

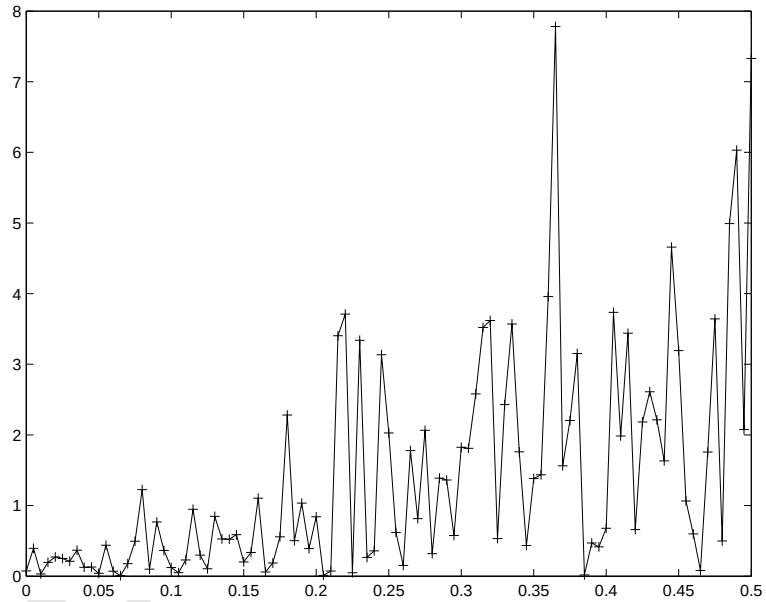

Figure 1: The periodogram $I_{200}(2\pi c)$, $0<c\leq 0.5$, of the simulated MA(1) series.

Next we use a set of weights $(\frac{1}{231},\frac{3}{231},\frac{6}{231},\frac{9}{231},\frac{12}{231},\frac{15}{231},\frac{18}{231},\frac{20}{231},\frac{21}{231},\frac{21}{231},\frac{21}{231},\frac{20}{231},\frac{18}{231},\frac{15}{231},\frac{12}{231},\frac{9}{231},\frac{6}{231},\frac{3}{231},\frac{1}{231})$ producting the smoother spectral estimate. This particular weight function is obtained by successive application of the filters $(W_n(k)=(2m+1)^{-1},\ |k|\leq m=1,3,5)$. This smoother spectral estimate and the true spectral density are shown in figure 2.

In figure 3, we have plotted the confidence interval (28) for the data of (29) using the spectral estimate displayed in figure 2.

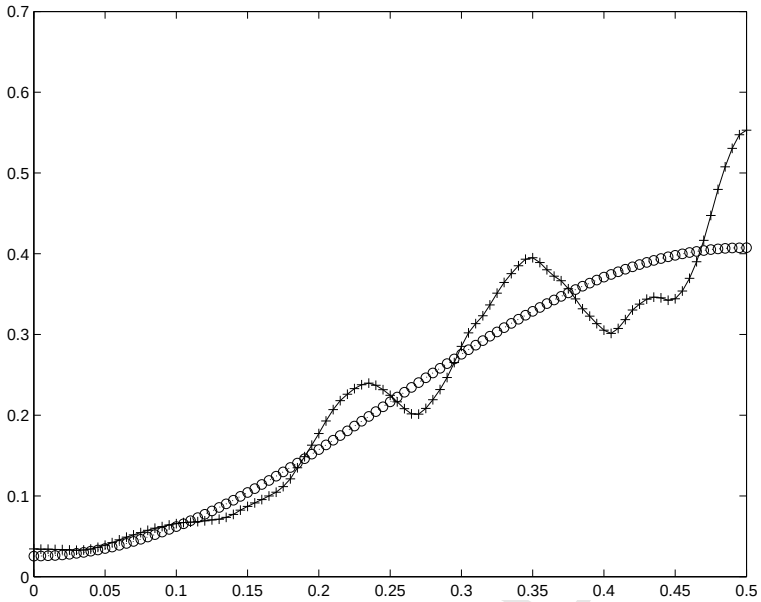

Figure 2: The spectral estimate $\widehat{f}(2\pi c)$, $0 < c \leq 0.5$, obtained with the inset weight function. The true function is also shown.

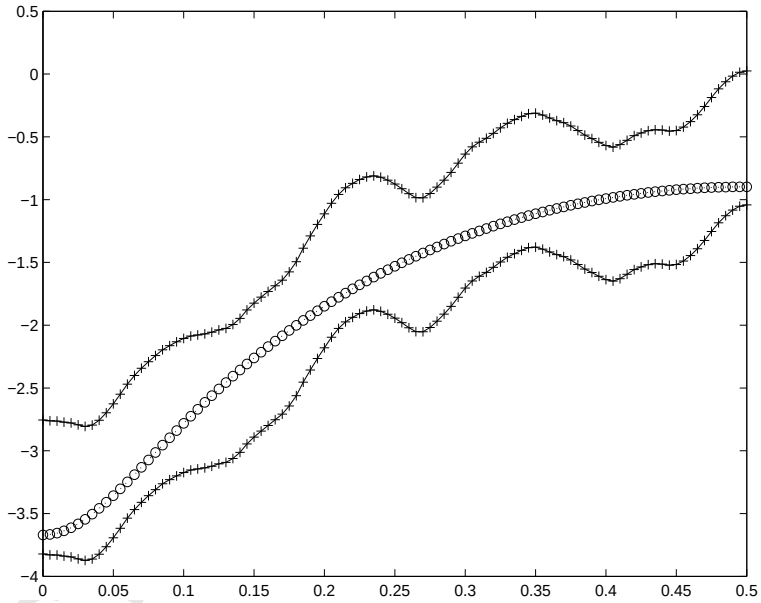

Figure 3: 95% confidence interval for $lnf_X(2\pi c)$ based on the spectral estimates of figure 2 and a χ^2 approximation. The true function is also shown.

Using the weights specified above, we obtain

$$\sum_{|k|\leq m} W_n^2(k) = 0.07052 \text{ and } \nu = 28.36$$

so that (28) reduces to that interval

$$\Big(ln\widehat{f}(\omega_j) - 0.450,\, ln\widehat{f}(\omega_j) + 0.617\Big).$$

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