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Some Combinatorial Aspects of Lattice of Residual Mappings

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Abstract

We have developed and extended the study of some combinatorial aspects of lattice of isotone mappings to the case of lattice of residual mappings between two finite lattices of combinatorial structure. The algebraic characterization of residual mapping allows us to study the covering relation in the lattice of ultrametries and to give a way for describing the lattice of closure operators on a finite chain. In the final point, we give a necessary condition for a class of couples of closures and anti-closures to be a Galois connection on finite distributive lattice.

1 Introduction

The notions of residuated, residual mappings and Galois connections are equivalent under duality. They appear in many contexts and remain an important domain of research. Especially, they constitute a useful tool in several domains related with data analysis ([3],[14],[16],[23]). The study of combinatorial aspects of the ordered set of isotone mappings between two finite ordered sets studied by Duffus and Rival [12] has motivated the extension to the case of the lattice of Galois mappings in Yazı [29]. In the present paper we investigate the case of lattices of residual mappings between two finite lattices of combinatorial structure. This lattice will be denoted by $ResL(P, Q)$.

Section 3 is devoted to the case where (P, \leq) is a finite chain and (Q, \leq) is a lattice of partitions of finite set. We have extended the covering relation of the ordered set of isotone mappings between two finite ordered sets, studied by Duffus and Rival [12], to the lattice of ultrametries on a finite set taking values in finite chain. An example is given as illustration. In section 4, we consider the case where (P, \leq) and (Q, \leq) are both the same finite chain. We give a way for describing the set of closure operators on the finite chain $(k:1 < 2 < \dots < k)$ and we establish an isomorphism between the lattice of such operators ([22]) with that of the set of ordered s -tuples of intervals on the same chain induced by ordered refinement. We also give a method for computing the value of its Möbius function. Notice that in Dwinger [13]

it has been shown that the lattice of closer of (P, \leq) is a boolean lattice if and only if (P, \leq) is an ordinal with the greatest element. This result remains then true in our case. From this fact, our method can constitute an other method for computing the value of Möbius function of boolean lattice. In the final point, we use the inversion formula of Möbius for deriving the cardinality of each of its intervals. The last section concerns the case where $P = Q$ is a distributive lattice (not a chain). Among several characterisations of distributive lattice known in the litterature, we mainly used here the characterisation based on cleavage property (see Caspar et al. [8],[26]) which allowed us to get some results on closures of such lattice and to suggest a new simpler method for computing its value of Möbius function. These considered closures (resp, dual closures) being, under some condition, residuated (resp, residual), they play an important role for the characterisation of several types of lattices as mentionned in Blyth [5]. Before developing the study of these cases we give some basic definitions on ordered sets and lattices and some notations, which are needed in the sequel.

2 Definitions and Notations

Let be (P, \leq) an ordered set (or poset). Let be a and b two elements of P . a and b are said to be comparable if $a \leq b$ or $b \leq a$; if neither $a \leq b$ nor $b \leq a$ holds, a and b are incomparable, and we write $a \parallel b$. The subset of all elements $x \in P$ such that $a \leq x \leq b$ is called the interval $[a, b]$, with $a = \min [a, b]$ and $b = \max [a, b]$. We say that b covers a (denoted $a \prec b$) if $a \leq b$, $a \neq b$, and for any $x \in P$, $a \leq x \leq b$ implies $x = a$ or $x = b$. A subset C of (P, \leq) is called a chain, if its elements are pairwise comparable.

Example 1 For any non negative integer k ($k \geq 1$), we denote by $C(k)$ the chain $0 < 1 < \dots < k - 1$ and by \underline{k} the chain $1 < 2 < \dots < k$.

The principal ideal generated by $a \in P$ is $a \downarrow = \{x \in P : x \leq a\}$. The principal filter is dually defined and denoted by $a \uparrow$. Let be an other ordered set (Q, \leq) . The set of all mappings from P into Q is denoted by P^Q . It is again an ordered set with pointwise order on mappings: (for f and $g \in P^Q$, $f \leq g$ iff, for any $x \in P$, $f(x) \leq g(x)$). A mapping f from P^Q is said to be isotone, if $x \leq y$ implies $f(x) \leq f(y)$, antitone if $x \leq y$ implies $f(x) \geq f(y)$, for any $x, y \in P$. The set of all isotone mappings from P^Q (resp, antitone mappings from P^Q) is denoted by $Isot(P, Q)$ (resp, $Anti(P, Q)$). They are suborders of P^Q . A mapping f of P^Q is said to be resituated (resp, residual) if, for any $y \in Q$, $f^{-1}(y \downarrow)$ is a principal ideal of (P, \leq) (resp, $f^{-1}(y \uparrow)$ is a principal filter of (P, \leq)). The set of such mappings are denoted as $Res(P, Q)$ and $ResL(P, Q)$, respectively. They are suborders of $Isot(P, Q)$ for the pointwise order on mappings. A mapping f form P^Q is said to be a Galois mapping if it is residuated from P into Q^d (dual of Q). The set of such mappings denoted as $Gal(P, Q)$ is a suborder of $Anti(P, Q)$ for the pointwise order on mappings.

A closure operator φ of (P, \leq) is an operator satisfying the following conditions: for any $x, y \in P$

- $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ (Isotonicity)
- $x \leq \varphi(x)$ (Extensivity).

- $\varphi \circ \varphi(x) = \varphi(x)$ (Idempotence).

The set of such mappings denoted as $\Phi(P)$ is an ordered set for pointwise order on mappings.

A Galois connection between (P, \leq) and (Q, \leq) is a pair (f, g) antitone mappings $f : P \rightarrow Q$ and $g : Q \rightarrow P$ such that both $f \circ g$ and $g \circ f$ are extensive.

A poset (P, \leq) is a lattice, if for all $x, y \in P$ have join denoted $x \vee y$ and meet denoted $x \wedge y$.

Example 2 *The set of all partitions of the set X of cardinality n (denoted \mathcal{P}_n) endowed with the order refinement is a lattice.*

(P, \leq) is a complete lattice if, any subset X of P has a join and a meet, denoted by $\vee X$ and $\wedge X$, respectively.

The least element of (P, \leq) is denoted by 0_P (if it exists) and its greatest element denoted by 1_P (if it exists).

An element j of (P, \leq) is a join-irreducible if, for any subset X of P , $j = \vee X$ implies $j \in X$. Dually, $i \in P$ is a meet-irreducible element, if $i = \wedge X$ implies $i \in X$. Let $\mathcal{J}(P)$ and $\mathcal{I}(P)$ be respectively the sets of all the join-irreducibles and all meet-irreducibles of (P, \leq) . If (P, \leq) is finite, $\mathcal{J}(P)$ is the set of elements j of P that covers a unique element denoted by j^- and $\mathcal{I}(P)$ is the set of elements $i \in P$ that are covered by a unique element denoted by i^+ . If for $j \in \mathcal{J}(P)$, $j^- = 0_P$, j is called atom of (P, \leq) ; and the set of such elements is denoted by $\mathcal{A}(P)$. Dually, if for $i \in \mathcal{I}(P)$, $i^+ = 1_P$, i is called coatom (dual-atom); and the set of such elements is denoted by $\mathcal{C}(P)$. For other definitions not recalled here see [3] and [11].

3 Covering relation in lattice of ultrametrics

We study the covering relation in the lattice of ultrametrics on X , set of cardinality n ($|X| = n$), with values in the chain $C(k)$, $k \leq n$. This lattice will be denoted $\mathcal{U}_{X, C(k)}$. We first recall some basic results on ordered set $ResL(P, Q)$ when (P, \leq) and (Q, \leq) are both lattices and some other related fields, precisely those that are well known in data analysis.

Theorem 1 (*Shmuley [25]*)

Let be two ordered sets (P, \leq) and (Q, \leq) . If, (P, \leq) and (Q, \leq) are complete lattices, then the ordered sets $Res(P, Q)$, $ResL(P, Q)$ and $GaL(P, Q)$ are also complete lattices.

Furthermore, the \wedge -complete morphisms between (P, \leq) and (Q, \leq) (resp, the \vee -complete morphisms) constitute exactly $ResL(P, Q)$ (resp, $Res(P, Q)$).

This result is due to Shmuley [25].

Some results of Blyth and Janowitz (Exercise 2.21 page 37 [7], Exercise 4.1 page 32 [4] and Exercise 1.10 page [7]) are summarized in the following statement.

Theorem 2 (*Blyth and Janowitz [4], Blyth [7]*)

- Any morphism lattice between (P, \leq) and (Q, \leq) is isotone if and only if, (P, \leq) is a chain.
- If (P, \leq) and (Q, \leq) are both chains, then $ResL(P, Q)$ is a lattice, sublattice of $Isot(P, Q)$.
- Any isotone mapping from \underline{m} into \underline{n} is residual if and only if, $f(m) = n$.

Some other results of Barthelemy, Leclerc and Janowitz ([2], [16], [17], [19]) are summarized in the following statement.

Theorem 3 - *The set of all $C(k)$ -dendogram on X (denoted $\mathcal{D}_{C(k)}$) is exactly the set of residual mappings from $C(k)$ into \mathcal{P}_n .*

- *The sets $\mathcal{D}_{C(k)}$ and $\mathcal{U}_{X,C(k)}$ are in one-to-one correspondence.*

- *$\mathcal{D}_{C(k)}$ and $\mathcal{U}_{X,C(k)}$ are lattices for pointwise order on mappings. Furthermore, they are dually isomorphic.*

Proposition 4 *$ResL(C(k), \mathcal{P}_n)$ is an sublattice of $Isot(C(k), \mathcal{P}_n)$ preserving covers. i.e.,*

$$\text{for two elements } f \text{ and } g \in ResL(C(k), \mathcal{P}_n), \quad \underset{ResL(C(k), \mathcal{P}_n)}{f \prec g} \iff \underset{Isot(C(k), \mathcal{P}_n)}{f \prec g} .$$

Proof. Let be two elements f and g of $ResL(C(k), \mathcal{P}_n)$. Then $f, g, f \wedge g$ and $f \vee g$ are elements of $Isot(C(k), \mathcal{P}_n)$. It follows from Theorem 2 that $f \wedge g$ and $f \vee g$ are morphisms lattices. Furthermore, one has $f \wedge g(k-1) = f(k-1) \wedge g(k-1) = 1_{\mathcal{P}_n} \wedge 1_{\mathcal{P}_n} = 1_{\mathcal{P}_n}$ and $f \vee g(k-1) = f(k-1) \vee g(k-1) = 1_{\mathcal{P}_n} \vee 1_{\mathcal{P}_n} = 1_{\mathcal{P}_n}$. From Theorem 1, $f \wedge g$ and $f \vee g$ are then the elements of $ResL(C(k), \mathcal{P}_n)$. For the sequel, assume that $f \prec g$ in $ResL(C(k), \mathcal{P}_n)$ such that there exists $h \in Isot(C(k), \mathcal{P}_n)$ satisfying $f < h < g$. So, one has $1_{\mathcal{P}_n} = f(k-1) \leq h(k-1) \leq g(k-1) = 1_{\mathcal{P}_n}$, therefore $h(k-1) = 1_{\mathcal{P}_n}$. It result from Theorem 2 that the element h belongs to $ResL(C(k), \mathcal{P}_n)$, a contradiction. In other words, the covers in $ResL(C(k), \mathcal{P}_n)$ is the restriction of the covers in $Isot(C(k), \mathcal{P}_n)$ to $ResL(C(k), \mathcal{P}_n)$. ■

Theorem 5 *Let be u and v two elements of $\mathcal{U}_{X,C(k)}$ such that $u \prec v$. Then ,for all elements $x, y \in X$, we have $u(x, y) \in \{v(x, y), v(x, y) + 1\}$.*

Proof. We first observe that for any chain of elements of \mathcal{P}_n including the least and the greatest refined partition $0_{\mathcal{P}_n}$ and $1_{\mathcal{P}_n}$,i.e., $0_{\mathcal{P}_n} \leq \pi_1 \leq \pi_2 \leq \dots \leq \pi_i \leq \pi_{i+1} \leq \dots \leq 1_{\mathcal{P}_n}$, and for all elements x and y of X , there exists a smallest integer i such that x and y are both in the same block in partition π_i , and a greatest integer j such that they are separated in partition π_j with $j = i + 1$. So, if we consider the residual mappings from $C(k)$ into \mathcal{P}_n associated with the ultrametrics u and v respectively, one has from Theorem 3 and Theorem 4 $\underset{\mathcal{U}_{X,C(k)}}{v \prec u} \iff \underset{ResL(C(k), \mathcal{P}_n)}{f \prec g} \iff \underset{Isot(C(k), \mathcal{P}_n)}{f \prec g}$, and this, in view of Proposition

2.1 in Duffus and Rival [12], is equivalent to: $(\exists \lambda_0 \in C(k)$ such that $f(\lambda_0) \prec g(\lambda_0)$ and $f(\lambda) = g(\lambda)$ for all $\lambda \neq \lambda_0$). Setting C_{λ_0} the chain of \mathcal{P}_n consisting of all elements $f(\lambda)$ and $g(\lambda)$ where $\lambda \in \{0, 1, 2, \dots, k-1\}$ including $0_{\mathcal{P}_n}$ and $1_{\mathcal{P}_n}$, i.e. , $C_{\lambda_0} : 0_{\mathcal{P}_n} \leq f(0) \leq f(1) \leq \dots \leq f(\lambda_0 - 1) \leq f(\lambda_0) \prec g(\lambda_0) \leq g(\lambda_0 + 1) \leq \dots \leq g(k-1) = f(k-1) = 1_{\mathcal{P}_n}$. For all x and y of X , we can distinguish two cases. Firstly $g(\lambda_0)$ is the smallest partition in C_{λ_0} such that x and y belongs to the same block; from above, $f(\lambda_0)$ is then the greatest partition in C_{λ_0} such that x and y are separated. In this case ,we show then that $\lambda_0 = \min \{\lambda : \lambda \in \{0, \dots, k-1\}\}$ such that $x, y \in$ to the same block of $g(\lambda)$. Indeed, if there exist $\lambda_1 \leq \lambda_0 - 1$ such that x and y belongs to the same block of $g(\lambda_1)$, this gives $g(\lambda_1) \leq g(\lambda_0 - 1) = f(\lambda_0 - 1) \leq f(\lambda_0)$, and therefore x and y belongs to the same block in the partition $f(\lambda_0)$, a contradiction. Hence, $v(x, y)$ is equal λ_0 , and the smallest value $\lambda \in \{0, \dots, k-1\}$ such that x and y belongs to the same block in partition $f(\lambda)$ is equal $\lambda_0 + 1$, i.e., $u(x, y) = v(x, y) + 1$. In the second case where $g(\lambda_0)$ is not the smallest partition

in C_{λ_0} such that x and y are in the same block. Clearly, one has $u(x, y) = v(x, y)$. Finally, $u(x, y) \in \{v(x, y), v(x, y) + 1\}$. ■

The converse is not true. As illustration, we consider three ultrametrics u, v and w defined over $X = \{a, b, c, d\}$ with values in the chain $C(4) = 0 < 1 < 2 < 3$, and represented by the following tables, respectively.

		b	c	d
$u :$	a	0	2	2
	b		2	2
	c			0

		b	c	d
$v :$	a	1	2	2
	b		2	2
	c			0

		b	c	d
$w :$	a	1	3	3
	b		3	3
	c			0

Clearly, one has for all $x, y \in X$, $w(x, y) = u(x, y)$ or $u(x, y) + 1$, but $u < v < w$, i.e; w does not cover u .

Proposition 6 *Let be an element of the lattice $ResL(C(k), \mathcal{P}_n)$ and u its associated ultrametric. Let be i belongs to $\{0, 1, \dots, k - 2\}$ such that the length of the interval $[f(i), f(i + 1)]$ is greatest than two. Then, for every element $\sigma \in \mathcal{P}_n$ satisfying $f(i) \prec \sigma < f(i + 1)$, the ultrametric associated to the residual mapping g from $C(k)$ into \mathcal{P}_n described by*

$$g(j) = \begin{cases} f(j) & j \in \{0, 1, \dots, k - 2\} \text{ and } j \neq i \\ \sigma & j = i \end{cases}$$

is covered by u , and the number of such ultrametrics is equal to the number of elements that covers $f(i)$ in the sublattice $[f(i), f(i + 1)]$.

Proof. Before starting the proof, we recall, on the one hand a classical result on combinatorial lattice partitions \mathcal{P}_n . Let be $\pi = \pi_1/\pi_2/\dots/\pi_r$ and $\sigma = \sigma_1/\sigma_2/\dots/\sigma_s$ two elements of \mathcal{P}_n with r and s their numbers of blocks respectively. Assume that $\sigma < \pi$, then the interval $[\sigma, \pi]$ is isomorphic to the direct product of lattice partitions $\mathcal{P}_{r_1} \times \mathcal{P}_{r_2} \times \dots \times \mathcal{P}_{r_s}$ with $\sum_{i=1}^s r_i = r$. On the other hand, g before defined is clearly an element of $ResL(C(k), \mathcal{P}_n)$; if, furthermore r and s are the number of blocks of $f(i)$ and $f(i + 1)$ respectively, it follows, from this, that the interval $[f(i), f(i + 1)]$ is isomorphic to the direct product of lattices $\mathcal{P}_{r_1} \times \mathcal{P}_{r_2} \times \dots \times \mathcal{P}_{r_s}$ with $\sum_{i=1}^s r_i = r$. As well known, $r_i(r_i - 1)/2$ is the number of atoms of \mathcal{P}_{r_i} . Then, the number of elements that covers $f(i)$ in the interval $[f(i), f(i + 1)]$ is equal $\sum_{i=1}^s r_i(r_i - 1)/2$. ■

Notice that the dual result can be stated by the same argument above, i.e; if, u is the ultrametric associated with f , then, for $\sigma \in \mathcal{P}_n$ such that $f(i) < \sigma \prec f(i + 1)$, the ultrametric v associated to the residual mapping g from $C(k)$ into \mathcal{P}_n described by

$$g(j) = \begin{cases} f(j) & j \in \{0, 1, \dots, k - 1\} \text{ and } j \neq i + 1 \\ \sigma & j = i + 1 \end{cases}$$

covers u , and the number of such ultrametrics is equal to the number of elements covered by $f(i + 1)$ in the sublattice $[f(i), f(i + 1)]$, i.e, $\sum_{i=1}^s (2^{(r_i-1)} - 1)$ ([1],[24],[27]).

4 Closures on finite chain and Möbius function

Let be given k a non-negative integer and $r \leq k$. In this section, we introduce two ordered sets denoted $P_{r,k}$ ($P_{r,k} \subset \mathcal{P}_k$) and $\Pi_{r,k}$ consisting of ordered r -tuples of intervals

of the chain \underline{k} and the ordered r -partitions of the integer k , respectively. We first show that the disjoint unions $\bigcup_{1 \leq r \leq k} P_{r,k}$, $\bigcup_{1 \leq r \leq k} \Pi_{r,k}$ and the ordered set as denoted $\underline{Isot}(\underline{r}, \underline{k}) = \{f : \underline{r} \rightarrow \underline{k} \text{ strictly isotone with } f(r) = k\}$ are pairwise in one-to-one correspondence [1]. After, we give a way for described a subset of $ResL(\underline{k}, \underline{k})$ denoted $Res^{(r)}\underline{k}$; and we show that the set of closure operators of \underline{k} denoted $\Phi(\underline{k})$ is exactly the disjoint union of $Res^{(r)}\underline{k}$. Thus, we establish an order isomorphism between $\Phi(\underline{k})$ endowed with pointwise order on mappings and $\bigcup_{1 \leq r \leq k} P_{r,k}$ induced by ordered of refinement. At last, we give a method for computing directly the Möbius function of the ordered set $\Phi(\underline{k})$; and by inversion formula Möbius, we deduce the cardinality of any interval of $\Phi(\underline{k})$. Before developing the study of $ResL(\underline{k}, \underline{k})$, let us recall that for any isotone mapping f from \underline{k} into the ordered set (Q, \leq) ; its image denoted Imf is a chain of (Q, \leq) .

If $a_1 < a_2 < \dots < a_r$ is the image of f and $I = I_1/I_2/\dots/I_r$ the canonical partition associated with it, we ordered the set of these blocks $\{I_1, I_2, \dots, I_r\}$ by setting:

$$I_i \leq I_j \Leftrightarrow a_i < a_j$$

A such partition denoted $I = I_1/I_2/\dots/I_r$ is an ordered r -tuples of intervals of the chain \underline{k} ; that is, $I \in P_{r,k}$. As well known, an isomorphism between the ordered set $\{I_1, I_2, \dots, I_r\}$ and Imf is established (Stanley [28]).

Proposition 1 *The three following sets $P_{r,k}$, $\Pi_{r,k}$ and $\underline{Isot}(\underline{r}, \underline{k})$ are pairwise in one-to-one correspondence.*

Proof. The bijection between $\Pi_{r,k}$ and $P_{r,k}$ is obtained by the mapping ψ defined:

$$k = k_1 + k_2 + \dots + k_r \longrightarrow \psi(k) = [1, k_1] / [k_1 + 1, k_1 + k_2] / \dots / [k_1 + k_2 + \dots + k_{r-1} + 1, k]$$

whose converse denoted ψ^{-1} is given by

$$I = I_1/I_2/\dots/I_r \longrightarrow \psi^{-1}(I) = |I_1| + |I_2| + \dots + |I_r|$$

where $|I_i|$ is the cardinality of I_i . The bijection between $\underline{Isot}(\underline{r}, \underline{k})$ and $P_{r,k}$ is obtained by means of the mapping ϕ defined:

$$\sigma \longrightarrow \phi(\sigma) = [1, \sigma(1)] / [\sigma(1) + 1, \sigma(2)] / \dots / [\sigma(r-1) + 1, \sigma(r) = k]$$

Indeed, let be $I = I_1/I_2/\dots/I_r$ an element of $P_{r,k}$. The mapping σ from \underline{r} into \underline{k} described by: $\sigma(i) = \max(I_i)$, for every $i \in \{1, 2, \dots, k\}$ is well the unique mapping strictly isotone from \underline{r} into \underline{k} such that $\phi(\sigma) = [1, \sigma(1)] / [\sigma(1) + 1, \sigma(2)] / \dots / [\sigma(r-1) + 1, \sigma(r) = k]$. Hence, the one-to-one correspondence between $P_{r,k}$ and $\underline{Isot}(\underline{r}, \underline{k})$ follows, implying then the one-to-one correspondence between $\underline{Isot}(\underline{r}, \underline{k})$ and $\Pi_{r,k}$ whose cardinality (see [1]) equal C_{k-1}^{r-1} . ■

This binomial coefficient allows us to give in the last point in this section the value of Möbius function of lattice $\Phi(\underline{k})$. Now, we are able to describe the subset denoted $Res^{(r)}\underline{k}$ with the following way: an element $g \in Res^{(r)}\underline{k}$ is a mapping from \underline{k} into itself whose cardinality of its image equal r ($|Im g| = r$), and such that there exists $\sigma \in \underline{Isot}(\underline{r}, \underline{k})$

$$g(j) = \begin{cases} \sigma(1) & j \in [1, \sigma(1)] \\ \sigma(2) & j \in [\sigma(1) + 1, \sigma(2)] \\ \vdots & \vdots \\ \sigma(r-1) & j \in [\sigma(r-2) + 1, \sigma(r-1)] \\ \sigma(r) = k & j \in [\sigma(r-1) + 1, \sigma(r) = k] \end{cases}$$

From proposition 3.2, g is clearly an element of $ResL(\underline{k}, \underline{k})$ which canonical partition associated denoted : $I_g = [1, \sigma(1)] / [\sigma(1) + 1, \sigma(2)] / \dots / [\sigma(r-1) + 1, \sigma(r) = k]$. Conversely, let be $I = I_1/I_2/\dots/I_r$ an element of $P_{r,k}$ the unique element denoted g_I belongs to $Res^{(r)}\underline{k}$ associated with I is given by:

$$g_I(j) = \begin{cases} \max I_1 & j \in I_1 \\ \max I_2 & j \in I_2 \\ \vdots & \vdots \\ \max I_r & j \in I_r \end{cases}$$

It follows from proposition 4.1, that $I = [1, \sigma(1)] / [\sigma(1) + 1, \sigma(2)] / \dots / [\sigma(r-1) + 1, \sigma(r) = k]$. So, for every $r \in \{1, 2, \dots, k\}$, $Res^{(r)}\underline{k}$ and $P_{r,k}$ are in one-to-one correspondence, and then their disjoint union; i.e; $\bigcup_{1 \leq r \leq k} Res^{(r)}\underline{k}$ and $\bigcup_{1 \leq r \leq k} P_{r,k}$ are also in one-to-one correspondence.

Proposition 2 *Let be an element f of $Res^{(r)}\underline{k}$. Then f satisfies the following properties*

1. For all $j \in \underline{k}$, $j \leq f(j)$ (extensivity)
2. For all $j \in \underline{k}$, $f \circ f(j) = f(j)$ (idempotence)

Proof. By definition of $Res^{(r)}\underline{k}$, f belonging to $Res^{(r)}\underline{k}$ is given by

$$f(j) = \begin{cases} \sigma(1) & j \in [1, \sigma(1)] \\ \sigma(2) & j \in [\sigma(1) + 1, \sigma(2)] \\ \vdots & \vdots \\ \sigma(r-1) & j \in [\sigma(r-2) + 1, \sigma(r-1)] \\ \sigma(r) = k & j \in [\sigma(r-1) + 1, \sigma(r) = k] \end{cases}$$

where $\sigma \in \underline{Isot}(r, \underline{k})$. f is clearly extensive and isotone (by construction); and on the other hand, one has for all $j \in [\sigma(i-1), \sigma(i)]$, $f(j) = \sigma(i)$, since $\sigma(i) \in [\sigma(i-1), \sigma(i)]$ implies $f \circ \sigma(i) = \sigma(i)$; that is $f \circ f(j) = f(j)$. Hence, $f \in \Phi(\underline{k})$. Conversely, we show that any closure operator f over \underline{k} which taking r values ($r \leq k$) is an element of $Res^{(r)}\underline{k}$. Indeed, if $a_1 < a_2 < \dots < a_r$ is the image of f , by isotonicity, there exists an ordered r -tuples of intervals of the chain \underline{k} ; i.e, $I = I_1/I_2/\dots/I_r$ such that

$$f(j) = \begin{cases} a_1 & j \in I_1 \\ a_2 & j \in I_2 \\ \vdots & \vdots \\ a_r & j \in I_r \end{cases}$$

Thus, for all $i \in \{1, 2, \dots, r\}$, and all $j \in I_i$, one has, by extensivity $j \leq f(j) = a_i$, and then a_i is an upper bound of I_i . Moreover, the idempotence of f implies that $f(a_i) = a_i$; i.e, $a_i \in I_i$. Hence, $a_i = \max I_i$. Therefore, $f \in Res^{(r)}\underline{k}$. ■

As a consequence $\Phi(\underline{k}) = \bigcup_{1 \leq r \leq k} Res^{(r)}\underline{k}$.

It remains to show that $\Phi(\underline{k})$ for pointwise order between mappings and $\bigcup_{1 \leq r \leq k} P_{r,k}$ induced by the order of refinement are isomorphic .

Proposition 3 Let be f and g two elements of $\Phi(\underline{k})$ such that $\text{Im } g \subseteq \text{Im } f$. Then $f \leq g$.

Proof. Let be $j \in \underline{k}$. By extensivity of g , one has $j \leq g(j)$, and by isotonicity of f , it follows that $f(j) \leq f \circ g(j)$. Since $g(j) \in \text{Im } f$, so the idempotence of f implies $f(j) \leq g(j)$; i.e., $f \leq g$ ([21]). ■

Proposition 4 Let be I and J two elements of $\bigcup_{1 \leq r \leq k} P_{r,k}$. Let be f_I and g_J the elements of $\Phi(\underline{k})$ associated to I and J , respectively. Then, $I \leq J$ implies $f_I \leq g_J$.

Proof. Consider $I = I_1/I_2/\dots/I_s$ and $J = J_1/J_2/\dots/J_t$. The hypothesis that $I \leq J$ implies $t \leq s$ and there exists $\sigma \in \underline{Isot}(t, s)$ such that

$$\begin{aligned} J_1 &= I_1/I_2/\dots/I_{\sigma(1)} \\ J_2 &= I_{\sigma(1)+1}/I_{\sigma(1)+2}/\dots/I_{\sigma(2)} \\ &\vdots \\ J_t &= I_{\sigma(t-1)+1}/I_{\sigma(t-1)+2}/\dots/I_{\sigma(t)=s} \end{aligned}$$

Thus, the mappings $f_I \in \text{Res}^{(t)}\underline{k}$ and $g_J \in \text{Res}^{(s)}\underline{k}$ associated to I and J respectively are given by

$$f_I(j) = \begin{cases} \max I_1 & j \in I_1 \\ \max I_2 & j \in I_2 \\ \vdots & \vdots \\ \max I_s & j \in I_s \end{cases}$$

and

$$g_J(j) = \begin{cases} \max J_1 & j \in J_1 \\ \max J_2 & j \in J_2 \\ \vdots & \vdots \\ \max J_t & j \in J_t \end{cases}$$

By construction, $\text{Im } g$ is obviously included in $\text{Im } f$ and therefore, from proposition 4.3, $f_I \leq g_J$. ■

Proposition 5 Let be two elements f and g of $\Phi(\underline{k})$ such that $\text{Im } g \subseteq \text{Im } f$. Let be I_f and I_g the canonical partitions of the chain \underline{k} associated with f and g respectively. Then $I_f \leq I_g$.

Proof. By definition of f and g , there exists $\sigma \in \underline{Isot}(s, k)$ and $\gamma \in \underline{Isot}(t, k)$ such that

$$f(j) = \begin{cases} \sigma(1) & j \in [1, \sigma(1)] \\ \sigma(2) & j \in [\sigma(1) + 1, \sigma(2)] \\ \vdots & \vdots \\ \sigma(s) = k & j \in [\sigma(s-1) + 1, \sigma(s) = k] \end{cases}$$

and

$$g(j) = \begin{cases} \gamma(1) & j \in [1, \gamma(1)] \\ \gamma(2) & j \in [\gamma(1) + 1, \gamma(2)] \\ \vdots & \vdots \\ \gamma(t) = k & j \in [\gamma(t-1) + 1, \gamma(t) = k] \end{cases}$$

where $I_f = [1, \sigma(1)] / [\sigma(1) + 1, \sigma(2)] / \dots / [\sigma(s-1) + 1, \sigma(s) = k]$ and $I_g = [1, \gamma(1)] / [\gamma(1) + 1, \gamma(2)] / \dots / [\gamma(t-1) + 1, \gamma(t) = k]$. Since $\text{Im } g \subseteq \text{Im } f$, then the values $\gamma(l-1)$ and $\gamma(l)$, ($l \in \{2, 3, \dots, t\}$) belongs to $\text{Im } f$, and by idempotence of f , $f(\gamma(l-1) + 1) = \gamma(l-1) + 1$ and $f(\gamma(l)) = \gamma(l)$. Because of isotonicity of f , every $i \in [\gamma(l-1) + 1, \gamma(l)]$ is invariant under f ; i.e., $[\gamma(l-1) + 1, \gamma(l)] = [\sigma(i-1) + 1, \sigma(i)] / [\sigma(i) + 1, \sigma(i+1)] / \dots / [\sigma(m-1) + 1, \sigma(m)]$, where $\gamma(l-1) = \sigma(i-1)$, $\sigma(m) = \gamma(l)$ and $1 \leq i \leq m \leq s$. ■

Then, the order isomorphism between $\Phi(\underline{k})$ and $\bigcup_{1 \leq r \leq k} P_{r,k}$ is established.

Note that in Morgan [22] it has been given a necessary and sufficient condition on (P, \leq) in order that $\Phi(P)$ be a complete lattice: if (P, \leq) is a complete lattice then $\Phi(P)$ is a complete lattice. In particular if $P = \underline{k}$. Consequently the ordered set $\bigcup_{1 \leq r \leq k} P_{r,k}$ is a lattice.

Now, we propose to give a method for computing the value of Möbius function of $\Phi(\underline{k})$. For every $s \in \{1, 2, \dots, k\}$, let be $I^{(s)}$ denote an ordered $(k-s+1)$ -tuples of intervals of the chain \underline{k} of the following form: $I^{(s)} = /1234\dots s//s+1//s+2//\dots//k/$; with this notation, the bounds of the lattice $\bigcup_{1 \leq r \leq k} P_{r,k}$ are obtained by taking $s = 1$ and $s = k$; that is, $I^{(1)} = /1//2//\dots//k/ = \min_{1 \leq r \leq k} \bigcup_{r,k} P_{r,k}$ (denoted simply 0_k) and $I^{(k)} = /123\dots k/ = \max_{1 \leq r \leq k} \bigcup_{r,k} P_{r,k}$ (denoted simply 1_k).

Proposition 6 *Let be s an element of the set $\{1, 2, \dots, k\}$. Then the interval $[0_k, I^{(s)}]$ of the lattice $\bigcup_{1 \leq r \leq k} P_{r,k}$ and the lattice $\bigcup_{1 \leq r \leq s} P_{r,s}$ are isomorphic.*

Proof. We first remark that the interval $[0_k, I^{(s)}]$ consists of elements I of $\bigcup_{1 \leq r \leq k} P_{r,k}$ of the form $I = I_1/I_2/\dots/I_m//s+1//s+2//\dots//k/$, where $I_1/I_2/\dots/I_m$ is an ordered m -tuples of intervals of the chain \underline{s} , $m \leq s$. Let be then the mapping ψ defined from $[0_k, I^{(s)}]$ into $\bigcup_{1 \leq r \leq s} P_{r,s}$ by

$$I = I_1/I_2/\dots/I_m//s+1//s+2//\dots//k/ \longrightarrow \psi(I) = I_1/I_2/\dots/I_m$$

Clearly, ψ as defined is bijectif. On the other hand, the restriction to $\bigcup_{1 \leq r \leq s} P_{r,s}$ of the order refinement implies that for J and K two elements of $[0_k, I^{(s)}]$; i.e., $J = J_1//s+1//s+2//\dots//k/$ and $K = K_1//s+1//s+2//\dots//k/$, where J_1 and K_1 are two elements of $\bigcup_{1 \leq r \leq s} P_{r,s}$, $J_1 \leq K_1 \Leftrightarrow J \leq k$. Then, the isomorphism is established. ■

Remark 7 *It has been shown above that these lattices $\Phi(\underline{k})$ and $\bigcup_{1 \leq r \leq k} P_{r,k}$ are isomorphics; then, the Möbius function of these two lattice are the same. If we denote by μ this function, one has $\mu(\Phi(\underline{k})) = \mu(\bigcup_{1 \leq r \leq k} P_{r,k}) = \mu(0_k, 1_k)$*

Now, we are able to state the following result (see [9],[10],[27]):

Theorem 8 $\mu(0_k, 1_k) = (-1)^{k-1}$.

Proof. For any element s belongs $\{0, 1, 2 \dots k-1\}$, let be $\sigma(s)$ denote an element of $\bigcup_{1 \leq r \leq k} P_{r,k}$ the rank of which is s , i.e., an ordered $(k-s)$ -tuples of intervals of the chain \underline{k} . (Notice that

$\sigma(0) = 0_k$ and $\sigma(k-1) = 1_k$). We proceed by induction on the rank of $\bigcup_{1 \leq r \leq k} P_{r,k}$. It is easily to verify that for $k = 2$ and $k = 3$, $\mu(\sigma(0), \sigma(2)) = (-1)^2$ and $\mu(\sigma(0), \sigma(3)) = (-1)^3$. Assume that the assertion is true until $k-2$; i.e, for all $s \leq k-2$, $\mu(\sigma(0), \sigma(s)) = (-1)^s$. From the relation

$$\sum_{\sigma(0) \leq \sigma(i) \leq \sigma(k-1)} \mu(\sigma(0), \sigma(i)) = 0,$$

it follows that

$$\begin{aligned} \mu(\sigma(0), \sigma(k-1)) &= - \sum_{\sigma(0) \leq \sigma(i) < \sigma(k-1)} \mu(\sigma(0), \sigma(i)) \\ \mu(\sigma(0), \sigma(k-1)) &= - \sum_{\sigma(0) \leq \sigma(i) \leq \sigma(k-2)} \mu(\sigma(0), \sigma(i)) - \sum_{\{\sigma(i) \parallel \sigma(k-2)\}} \mu(\sigma(0), \sigma(i)) \end{aligned}$$

for some σ of rank equal $k-2$, and $\sum_{\sigma(0) \leq \sigma(i) \leq \sigma(k-2)} \mu(\sigma(0), \sigma(i)) = 0$. This relation above may be written by choosing among all elements σ having the same rank $k-2$, the element $I^{(k-1)}$ defined before; i.e, $I^{(k-1)} = /123\dots k-1//k/$, and we get

$$\mu(\sigma(0), \sigma(k-1)) = - \sum_{\{\sigma(i) \parallel I^{(k-1)}\}} \mu(\sigma(0), \sigma(i))$$

If we let W_i denote the set consisting of elements of the lattice $\bigcup_{1 \leq r \leq k} P_{r,k}$ the rank of which is i and which are incomparable with $I^{(k-1)}$, i.e., $W_i = \{\sigma(i), \sigma(i) \parallel I^{(k-1)}\}$. One has, from propositions 4.1 and 4.6, the cardinality of each W_i , $i = \{0, 1, \dots, k-2\}$ is equal to $C_{k-1}^{k-(i+1)} - C_{k-2}^{k-(i+2)}$; that is

$$\begin{aligned} |W_0| &= C_{k-1}^{k-1} - C_{k-2}^{k-2} \\ |W_1| &= C_{k-1}^{k-2} - C_{k-2}^{k-3} \\ &\vdots \\ |W_{k-2}| &= C_{k-1}^1 - C_{k-2}^0. \end{aligned}$$

Thus, by hypothesis induction we get

$$\mu(\sigma(0), \sigma(k-1)) = - (|W_0| \cdot 1 + |W_1| \cdot (-1) + \dots + |W_{k-3}| \cdot (-1)^{k-3} + |W_{k-2}| \cdot (-1)^{k-2})$$

or equivalently

$$\begin{aligned} \mu(\sigma(0), \sigma(k-1)) &= - \left[(C_{k-1}^{k-1} - C_{k-2}^{k-2}) \cdot 1 + (C_{k-1}^{k-2} - C_{k-2}^{k-3}) \cdot (-1) + \dots + (C_{k-1}^1 - C_{k-2}^0) \cdot (-1)^{k-2} \right] \\ &= - \left(\sum_{i=1}^{k-1} C_{k-1}^i (-1)^{k-1-i} - \sum_{i=0}^{k-2} C_{k-2}^i (-1)^{k-2-i} \right), \text{ with } \sum_{i=0}^{k-2} C_{k-2}^i (-1)^{k-2-i} = 0 \\ &= - \left(\sum_{i=0}^{k-1} C_{k-1}^i (-1)^{k-1-i} - C_{k-1}^0 (-1)^{k-1} \right), \text{ with } \sum_{i=0}^{k-1} C_{k-1}^i (-1)^{k-1-i} = 0. \end{aligned}$$

Hence,

$$\mu(\sigma(0), \sigma(k-1)) = C_{k-1}^0 (-1)^{k-1} = (-1)^{k-1}.$$

■

We end this section with the use principle of Möbius inversion on $\bigcup_{1 \leq r \leq k} P_{r,k}$ for computing the cardinality of all interval of form $[\sigma(s), 1_k]$, $s \in \{0, 1, \dots, k-1\}$ ([9],[10]).

Proposition 9 *Let be $\sigma(s)$ and $\gamma(s)$ denote two elements of $\bigcup_{1 \leq r \leq k} P_{r,k}$ of rank s . Then*

1. *The intervals $[\sigma(s), 1_k]$ and $[\gamma(s), 1_k]$ are isomorphic.*
2. *The number of such intervals is equal to C_{k-1}^{k-1-s} .*

Proof. Because $\sigma(s)$ and $\gamma(s)$ have the same number of blocks, the intervals $[\sigma(s), 1_k]$ and $[\gamma(s), 1_k]$ are then isomorphics. For the second assertion, we have from proposition 4.1, that there exists exactly C_{k-1}^{k-1-s} of elements of rank s , and therefore, C_{k-1}^{k-1-s} intervals of form $[\sigma(s), 1_k]$. ■

Now, we apply the principle of Möbius inversion to the lattice $\bigcup_{1 \leq r \leq k} P_{r,k}$. To do this, define two mappings F and G from $\bigcup_{1 \leq r \leq k} P_{r,k}$ into \mathbb{N}

$$F : \sigma(s) \longrightarrow F(\sigma(s)) = \left| \left\{ \sigma \in \bigcup_{1 \leq r \leq k} P_{r,k} : \text{rank}(\sigma) = s \text{ and } \sigma \leq \sigma(s) \right\} \right|$$

clearly, $F(\sigma(s)) = 1$. And

$$G : \gamma(t) \longrightarrow G(\gamma(t)) = \sum_{\sigma(s) \leq \gamma(t)} F(\sigma(s))$$

Thus, the principle of Möbius inversion on $\bigcup_{1 \leq r \leq k} P_{r,k}$ gives:

$$1 = F(\gamma(t)) = \sum_{\sigma(s) \leq \gamma(t)} \mu(\sigma(s), \gamma(t)) G(\sigma(s)). \quad (4.1)$$

In the case $\gamma(t) = 1_k$, the above relation can be written

$$1 = \sum_{\{\sigma(s): s \in \{0, 1, \dots, k-1\}\}} \mu(\sigma(s), 1_k) G(\sigma(s)). \quad (4.2)$$

From proposition 4.1, 4.9 and theoreme 4.8 the equality above becomes

$$1 = \sum_{s \in \{0, 1, \dots, k-1\}} C_{k-1}^{k-1-s} (-1)^{k-1-s} G(\sigma(s)). \quad (4.3)$$

We can remark in the definition of $G(\sigma(s))$ that it is exactly the number of elements of $\bigcup_{1 \leq r \leq k} P_{r,k}$ smallest than $\sigma(s)$; i.e, $G(\sigma(s)) = |[0_k, \sigma(s)]|$, we finally get

$$1 = \sum_{s \in \{0, 1, \dots, k-1\}} C_{k-1}^{k-1-s} (-1)^{k-1-s} |[0_k, \sigma(s)]|. \quad (4.4)$$

Proposition 10 For every $s \in \{0, 1, \dots, k-1\}$, the cardinality of $[0_k, \sigma(s)]$ is equal to 2^s .

Proof. We use induction on k . If we let $k = 1$ and $k = 2$ in the equality (4.3) of preceding proof, we obtain for $k = 1$, $1 = C_0^0 G(\sigma(0))$; i.e., $G(\sigma(0)) = 2^0$. For $k = 2$, $1 = \sum_{s=0}^1 C_1^{1-s} (-1)^{1-s} = -1 + G(\sigma(1))$; i.e., $G(\sigma(1)) = 2^1$. Assume that $G(\sigma(s)) = 2^s$ is true for every $s \in \{0, 1, \dots, k-2\}$, for $s = k-1$ one has $1 = \sum_{s=0}^{k-1} C_{k-1}^{k-1-s} (-1)^{k-1-s} G(\sigma(s)) = \sum_{s=0}^{k-2} C_{k-1}^{k-1-s} (-1)^{k-1-s} G(\sigma(s)) + C_{k-1}^0 G(\sigma(k-1))$. Thus, by hypothesis induction, we get $1 = \sum_{s=0}^{k-2} C_{k-1}^{k-1-s} (-1)^{k-1-s} 2^s + C_{k-1}^0 G(\sigma(k-1)) = 1 - \sum_{s=0}^{k-1} C_{k-1}^{k-1-s} (-1)^{k-1-s} 2^s + C_{k-1}^0 G(\sigma(k-1)) = 1 - 2^{k-1} + C_{k-1}^0 G(\sigma(k-1))$; i.e., $G(\sigma(k-1)) = |[0_k, \sigma(k-1)]| = 2^{k-1}$. ■

5 Closures on finite distributive lattice

In section 3, we give some results on closure of distributive lattice. Thus, let be given φ a closure operator of a lattice (P, \leq) such that there exist an element $c \in \mathcal{C}(P)$ which closed under φ ($\varphi(c) = c$). The mapping denoted φ defined by $\varphi(x) = \varphi(cx)$, $x \in P$, is again a closure operator of (P, \leq) . If, in addition (P, \leq) is distributive, we get an upper bound of the cardinality of the subset $\{x \in P : \varphi(x) = 1\}$. With similar argument above, we get also an upper bound of the subset $\{x \in P : (x) = 0\}$, where is a dual φ -closure of (P, \leq) such that there exist a $A \in \mathcal{A}(P)$ which closed under φ ($\varphi(A) = A$).

After, we give a sufficient condition in order that the following residual-closure constitutes an Galois connection between (P, \leq) and its dual $(P, \varphi(x) = \varphi(cx), (x) = (ax))$. At last, we give a new simpler method for computing the Mobius function of distributive lattice. In what follow, a useful characterisation based on the cleavage property is mainly applied 9. 32

Thus, one has the following result.

Theorem 1 ([8],[20],[26])

A lattice (P, \leq) is distributive, if and only if, for every element $j \in \mathcal{J}(P)$ (j join-irreducible), there exists a unique element $i \in \mathcal{I}(P)$ such that P is the disjoint sum of $j \uparrow$ and $i \downarrow$; that is, $P = j \uparrow + i \downarrow$. As consequence in this case, one has $j \wedge i = j^-$ and $j \vee i = i^+$.

Notice that this theorem can be stated by taking the cleavage property dualised.

Proposition 2 Let be (P, \leq) a lattice and φ an closure operator on (P, \leq) such that there exist an element $c \in \mathcal{C}(P)$ which is closed under φ . Then, the mapping denoted φ_c defined by: $\varphi_c(x) = \varphi(c \vee x)$ for all $x \in P$, is again a closure on the same lattice. If, in addition (P, \leq) is distributive, one has: $|\{x \in P : \varphi_c(x) = 1_P\}| \leq |[j_c, 1_P]|$ where j_c is the unique element belongs to $\mathcal{J}(P)$ which corresponds to c in the decomposition $P = j_c \uparrow + c \downarrow$. If we let $r_m = |\{x \in P : \varphi_c(x) = 1_P \text{ and } \text{rank}(x) = m\}|$, then $\sum_{m \geq 0} r_m \leq \min_{\{c \in \mathcal{C}(P) : \varphi(c) = c\}} |[j_c, 1_P]|$.

Proof. For $c \in \mathcal{C}(P)$ and $x \in P$, one has

$$c \vee x = \begin{cases} c & x \leq c \\ 1_P & x \not\leq c \end{cases}$$

thus

$$\varphi_c(x) = \begin{cases} \varphi(c) = c & x \leq c \\ \varphi(1_P) = 1_P & x \not\leq c \end{cases}$$

Clearly, φ_c as defined above is again a closure operator of (P, \leq) . If (P, \leq) is a distributive lattice; from theorem 5.1, there corresponds to c one and only one element denoted $j_c \in \mathcal{J}(P)$ such that $P = j_c \uparrow + c \downarrow$; that is, for all $x \in P$, $x \leq c$ is equivalent to $x \not\leq j_c$. Hence,

$$\varphi_c(x) = \begin{cases} \varphi(c) = c & x \not\leq j_c \\ \varphi(1_P) = 1_P & x \geq j_c \end{cases}$$

Therefore, $\{x \in P : \varphi(x) = 1_P\} = \{x \in P : x \geq j_c\}$; since $\{x \in P : \varphi(x) = 1_P\} \subseteq \{x \in P : \varphi_c(x) = 1_P\}$, then $|\{x \in P : \varphi(x) = 1_P\}| \leq |[j_c, 1_P]|$. If we let $r_n = |\{x \in P : \varphi(x) = 1_P \text{ and } \text{rank}(x) = n\}|$, we get finally $\sum_{m \geq 0} r_m \leq \min_{\{c \in \mathcal{C}(P) : \varphi(c) = c\}} |[j_c, 1_P]|$. ■

By the same argument above, we get the dual result, i.e, if ψ is a dual-closure of (P, \leq) such that $\psi(a) = a$ for some $a \in \mathcal{A}(P)$, one has the following inequality $|\{x \in P : \psi(x) = 0_P\}| \leq |[0_P, i_a]|$ where i_a is the unique element belongs to $\mathcal{I}(P)$ such that $P = a \uparrow + i_a \downarrow$. If we let $S_m = |\{x \in P : \psi(x) = 0_P \text{ and } \text{rank}(x) = m\}|$, then $\sum_{m \geq 0} S_m \leq \min_{\{a \in \mathcal{A}(P) : \psi(a) = a\}} |[0_P, i_a]|$.

Proposition 3 *Let be (P, \leq) a distributive lattice. Let be $a \in \mathcal{A}(P)$ and $c \in \mathcal{C}(P)$. Then, the following conditions are equivalent*

1. *Each element of the pair (a, c) is the complement of the other. (this is equivalent to: a/c).*
2. *P is the disjoint sum of $a \uparrow$ and $c \downarrow$; that is, $P = a \uparrow + c \downarrow$.*

Proof. 1 \Rightarrow 2. Let be $a \in \mathcal{A}(P)$. By theorem 5.1, there exists an unique element $i \in \mathcal{I}(P)$ such that $P = a \uparrow + i \downarrow$; since a and c are incomparable, it follows that $c \in i \downarrow$. This implies $c = i$; i.e, $P = a \uparrow + c \downarrow$.

2 \Rightarrow 1. These consequences of theorem 5.1 allows us to write a $a \vee c = c^+ = 1_P$ and $a \wedge c = a^- = 0_P$, where c^+ (resp. a^-) is the unique element which covers c (resp. is the unique element covered by a). Therefore, each element of of the pair (a, c) is the complement of the other. ■

Proposition 4 *Let be (P, \leq) a distributive lattice. Let $a \in \mathcal{A}(P)$ and $c \in \mathcal{C}(P)$ such that each element of the pair (a, c) is the complement of the other. Then the mappings φ_c and ψ_a defined above constitute a resituated-residual pair between (P, \leq) and itself, or equivalently, a Galois connection between (P, \leq) and its dual P^d .*

Proof. By proposition 5.3, one has $P = a \uparrow + c \downarrow$; thus, the mappings

$$\varphi_c(x) = \begin{cases} 1_P & x \geq a \\ c & x \leq c \end{cases}$$

and

$$\psi_a(x) = \begin{cases} a & x \geq a \\ 0_P & x \leq c \end{cases}$$

verify the Pickert's relation dualized; that is, for all element x and y belongs to P , $x \leq \varphi_c(y) \iff \psi_a(x) \leq y$. Indeed, it suffices to remark that for all $y \in P$,

$$\psi_a \circ \varphi_c(y) = \begin{cases} a & y \geq a \\ 0_P & y \leq c \end{cases}$$

That is, $\psi_a \circ \varphi_c(y) \leq y$ and for all $x \in P$

$$\varphi_c \circ \psi_a(x) = \begin{cases} 1_P & x \geq a \\ 0_P & x \leq c \end{cases}$$

That is, $x \leq \varphi_c \circ \psi_a(x)$. Therefore, from $x \leq \varphi_c(y)$, one has by isotonicity of ψ_a , $\psi_a(x) \leq \psi_a \circ \varphi_c(y) \leq y$; thus, $\psi_a(x) \leq y$. Also, from $\psi_a(x) \leq y$, one has; by isotonicity of φ_c , $\varphi_c \circ \psi_a(x) \leq \varphi_c(y)$; thus, $x \leq \varphi_c(y)$. Then, the pair (φ_c, ψ_a) constitutes a resituated-residual pair between (P, \leq) and itself. ■

We end this section with the following result:

Proposition 5 *Let be (P, \leq) a distributive lattice non complemented with $|P| \geq 3$ and μ denote its Möbius function. Then $\mu(0_P, 1_P) = 0$.*

Proof. Since (P, \leq) is distributive lattice non complemented with $|P| \geq 3$, it follows that $\bigvee_{a \in \mathcal{A}(P)} a < 1$ (see Exercise 7 page 18 [3]). From this inequality above, there exists an element $c \in \mathcal{C}(P)$ such that $\bigvee_{a \in \mathcal{A}(P)} a \leq c$, or equivalently $a < c$ for all $a \in \mathcal{A}(P)$. Thus, by theorem 5.1 there corresponds to c a unique element $j \in \mathcal{J}(P) \setminus \mathcal{A}(P)$ such that $P = j \uparrow + c \downarrow$ with $c \wedge j = j^- \neq 0_P$. Hence, the subset denoted $\mathcal{E}_{c,j}$ described by $\{x \in P, \text{ such that } x \wedge c = 0_P\}$ is reduced to the element 0_P . Then, the dual Wesneir's formula ([9],[24],[28]) allows us to state:

$$0 = \sum_{x: x \wedge c = 0_P} \mu(x, 1_P) = \mu(0_P, 1_P).$$

Finally, we find again the well known result on non-complemented distributive lattice: $\mu(0_P, 1_P) = 0$. ■

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