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INVERSE SPECTRAL POSITIVITY FOR SURFACES

PIERRE BÉRARD AND PHILIPPE CASTILLON

ABSTRACT. Let (M, g) be a complete non-compact Riemannian surface. We consider operators of the form $\Delta + aK - q$, where Δ is the non-negative Laplacian, K the Gaussian curvature, q a non-negative function, and a a positive real number. We address the question "What conclusions on (M, g) and q can one draw from the fact that the operator $\Delta + aK - q$ is non-negative" and we improve earlier results in particular in the cases $a = \frac{1}{4}$ and $a \in (0, \frac{1}{4})$.

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1. INTRODUCTION

Let (M, g) be a complete non-compact Riemannian surface. In the sequel, we will always implicitly assume that M is connected and orientable. We denote by Δ the non-negative Laplacian, by K the Gaussian curvature and by μ the measure associated with the metric g.

In this paper, we consider operators of the form $\Delta + aK - q$, where a is a positive parameter and q a non-negative function. Such operators appear naturally when one studies minimal (or constant mean curvature) immersions. Let us mention two examples. The Jacobi (stability) operator of an isometric minimal immersion $M \looparrowright \mathbb{R}^3$ into Euclidean 3-space is $\Delta + 2K$. More generally ([10], Section 3), the Jacobi operator of a minimal immersion $M \looparrowright \widehat{M}^3$ into a 3-manifold with scalar curvature $\widehat{S}$ can be written as $\Delta + K - (\frac{\widehat{S}}{2} + \frac{1}{2}|A|^2)$, where $|A|$ is the norm of the second fundamental form of the immersion.

More precisely, this paper is concerned with the following question: *What conclusions on the Riemannian surface (M, g), and on the function q, can one draw from the fact that the operator $\Delta + aK - q$ is non-negative on (M, g) ? i.e.* from the fact that the associated quadratic form is non-negative on Lipschitz functions with compact support in M (or equivalently on C^1-functions with compact support),

$$(\star) \qquad 0 \leq \int_M \left(|df|^2 + aKf^2 - qf^2\right) d\mu \qquad \forall f \in \mathrm{Lip}_0(M).$$

Our first result is the following.

Theorem 1.1. *Let (M, g) be a complete non-compact Riemannian surface, and let q be a non-negative locally integrable function on M. Assume that the operator $\Delta + aK - q$ is non-negative on M, and that either*

(i) $a \in (\frac{1}{4}, \infty)$, *or*
(ii) $a = \frac{1}{4}$, *and (M, g) has subexponential volume growth, or*
(iii) $a \in (0, \frac{1}{4})$, *and (M, g) has k_a-subpolynomial volume growth, with $k_a = 2 + \frac{4a}{1-4a}$*

Then,

(A) *The surface (M, g) has at most quadratic volume growth and is conformally equivalent to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ with the standard metrics.*
(B) *The function q is integrable on (M, g) and $\int_M q\, d\mu \leq 2\pi a\, \chi(M)$.*
(C) *If M is a cylinder, then (M, g) has at most linear volume growth and $q \equiv 0$.*

Definitions. Let $x \in M$, and let $V(r)$ denote the volume of the geodesic ball $B(x, r)$ for the metric g. We say that (M, g) has *subexponential volume growth* if

$$\limsup_{r\to\infty} \frac{\ln V(r)}{r} = 0.$$

We say that (M, g) has *polynomial volume growth of degree at most k* if

$$\limsup_{r\to\infty} \frac{V(r)}{r^k} < \infty.$$

We say that (M, g) has *k-subpolynomial volume growth* if

$$\limsup_{r\to\infty} \frac{V(r)}{r^k} = 0.$$

These definitions do not depend on the choice of the point $x \in M$. Note that if (M, g) has polynomial volume growth of degree at most k, then it has k'-subpolynomial volume growth for any $k' > k$.

Remarks

(1) Theorem 1.1 improves several known results (see the short historical account below). In particular, Case (i) was first treated in [4], with $q \equiv 0$; Cases (ii) and (iii) were first considered in [9], under stronger assumptions on (M, g).
(2) The assumptions and conclusions in the theorem are optimal, see Section 3.2 for more details.
(3) The main new idea in the proof of Theorem 1.1 is to introduce the function $\widehat{\chi}(t) := \sup\{\chi(B(s)) \mid s \in [t, \infty)\}$, the supremum of the Euler-Poincaré characteristics of open geodesic balls with radius at least t, whose jumps describe the large scale topology of M, see Section 2. We also introduce new functions to test the positivity of the quadratic form $(\star)$, see Lemmas 2.4 and 2.5.
(4) Theorem 1.1 can be extended to the case in which the operator $\Delta + aK - q$ is only assumed to have finite index. The conclusions, under the assumptions (i), (ii) or (iii), are that (M, g) is conformally equivalent to a closed Riemannian surface with a finite number of

points removed, and that q is integrable over (M, g). We refer to Section 4 for a precise statement and its proof.

(5) Another interesting situation, with applications to minimal and CMC surfaces, occurs when the potential q has a positive lower bound, *i.e.* $c = \inf q > 0$. Theorem 1.1 can be extended to this situation, improving the results of [7]. We refer to [1] for the details.

(6) Theorem 1.1 does not mention the case in which M is closed. We recall the arguments of [10]: when M is closed, one can plug the constant function $\mathbf{1}$ into $(\star)$, and conclude that $0 \leq \int_M q \leq 2\pi a\chi(M)$. If q does not vanish identically, then M is conformally equivalent to a round sphere. If $\chi(M) = 0$, then $q \equiv 0$. In this case, $\Delta + aK \geq 0$ and the constant function $\mathbf{1}$ minimizes the associated quadratic form because $\int_M K = 0$. It follows that $(\Delta + aK)\mathbf{1} = 0$, and hence $K \equiv 0$, *i.e.* (M, g) is a flat torus.

Short historical account on Theorem 1.1. The idea behind the proof of Theorem 1.1 goes back to [19] in which A. Pogorelov proves that orientable stable minimal surfaces in $\mathbb{R}^3$ are planes. For this purpose, he shows that a complete simply-connected surface, with non-positive curvature and non-negative operator $\Delta + 2K$, must be parabolic. Another proof consists in showing that such a simply-connected surface cannot be conformally equivalent to the unit disk. This method appears in [3, 10]. In the latter paper, D. Fischer-Colbrie and R. Schoen prove that there is no complete metric g on the unit disk $\mathbb{D}$, with $\Delta_g + aK_g \geq 0$ for some $a \geq 1$. They show that the set $I(\mathbb{D}, g) = \{a \geq 0 \mid \Delta_g + aK_g \geq 0\}$ is a closed interval, and they ask what is the value of the supremum of $I(\mathbb{D}, g)$. This question motivated [4] and the present paper. Pogorelov's result was extended to the case in which $\Delta + aK \geq 0$ for some $a > \frac{1}{4}$ by S. Kawai [14]. A more general setting (general topology and curvature, $a > \frac{1}{2}$) was considered by R. Gulliver and B. Lawson in [12]. A. Pogorelov's method was improved by T. Colding and W. Minicozzi [5] and, later on, by Ph. Castillon [4] who first proved Case (i) in Theorem 1.1 (with $q \equiv 0$). Cases (ii) and (iii) were first considered by J. Espinar and H. Rosenberg in [9], under more restrictive assumptions on (M, g). Case (i), with a potential q, was treated in [9] and [17] which also contain applications to constant mean curvature surfaces in 3-manifolds.

To motivate our next theorem, we recall the following result due to D. Fischer-Colbrie and R. Schoen ([10], their Theorem 3 reformulated for our purpose),

Theorem. *Let N be a complete oriented 3-manifold of non-negative scalar curvature. Let M be an oriented complete non-compact stable minimal surface in N. Then M is conformally equivalent to the complex plane or to a cylinder. If M is a cylinder, and the absolute total curvature of M is finite, then M is flat and totally geodesic.*

They also point out (Remark 2, p. 207) that "the assumption of finite total curvature should not be essential". This is indeed the case: see [21], proof of Theorem 2, and [18] for a proof using L^2 harmonic 1-forms. The proof of the preceding theorem involves the operator $\Delta + K$. In the framework of the present paper, we can prove,

Theorem 1.2. *Let (M,g) be a complete non-compact 2-dimensional cylinder. Assume that the operator $\Delta + aK$ is non-negative on (M,g), and that either,*

(i) *$a > \frac{1}{4}$, or*
(ii) *$a = \frac{1}{4}$, and (M,g) has subexponential volume growth, or*
(iii) *$a \in (0, \frac{1}{4})$, and (M,g) has k_a-subpolynomial volume growth, with $k_a = 2 + \frac{4a}{1-4a}$.*

Then, (M,g) is flat, i.e. *$K \equiv 0$.*

Remark. Case (i) in Theorem 1.2 appears in [20], Section 3.3 under the assumption that $a \geq 1$; in [9], Theorem 6.3, under the restrictive assumption that $\int_M K^+ \, d\mu$ is finite; in [8] with a different proof. Note that Theorem 1.1 and Theorem 1.2, with $a = 1$, imply that the answer to the question raised by D. Fischer-Colbrie and R. Schoen in [10] is positive.

The next theorem provides an intrinsic version of the optimal length estimate of L. Mazet [15]. Note that this is a local result, we do not need M to be complete. It applies when M is a stable constant mean curvature surface, possibly with boundary, isometrically immersed into a simply connected space form $\widehat{M}$, see Corollary 6.2. Our proof follows the same ideas as in Mazet's paper. We clarify the argument by applying a transplantation method.

Theorem 1.3. *Let (M,g) be a Riemannian surface (possibly with boundary ∂M). Assume that the curvature satisfies $K \leq \alpha^2$ for some $\alpha > 0$. Let J be the operator $J = \Delta + aK - c$, with $a \in [\frac{1}{2}, \infty)$ and $c \geq (a+2)\alpha^2$.*

(i) *If J is non-negative in a geodesic ball $B(x,R)$ contained in $M \setminus \partial M$, then $R \leq \frac{\pi}{2\alpha}$.*
(ii) *Assume that the geodesic ball $B(x, \frac{\pi}{2\alpha})$ is contained in $M \setminus \partial M$. If J is non-negative in this ball, then $c = (a+2)\alpha^2$, $K \equiv \alpha^2$ and $B(x, \frac{\pi}{2\alpha})$ is covered by the hemisphere $S^2_+(\alpha^2)$ in the sphere with constant curvature α^2.*

In the proof of this theorem, we will use the following classical result. Let $\rho : (\widehat{M}, \hat{g}) \to (M,g)$ be a Riemannian covering. Let V be a locally integrable function on M, and let $\hat{V} = V \circ \rho$. According to [10], Theorem 1, $\Delta + V \geq 0$ on (M,g) implies that $\hat{\Delta} + \hat{V} \geq 0$ on $(\widehat{M}, \hat{g})$. It is a natural question to investigate under which conditions the converse statement holds. A partial answer is given by Proposition 2.5 in [17]. As a matter of fact, one can show that the converse statement holds provided that the group $\pi_1(\widehat{M})$ is co-amenable in the group $\pi_1(M)$. We defer the precise statement and its proof to [2] because they rely on techniques and ideas different from those used here.

The paper is organized as follows. In Section 2, we fix the notations and state some technical lemmas to be used later on. The proof of Theorem 1.1 is given in Section 3.1; the fact that the assumptions and conclusions in the theorem are optimal is explained in Section 3.2. The extension of Theorem 1.1 to finite index operators is given in Section 4. The proofs of Theorems 1.2 and 1.3 are given in the subsequent sections.

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2. Notations and preliminary results

In this section, we fix some notations which will be used throughout the paper, and we state some preliminary results.

2.1. **Notations.** In this paper, unless otherwise stated, (M, g) denotes a complete non-compact Riemannian surface. We also assume that M is connected and orientable.

2.1.1. The non-negative Laplacian for the metric g will be denoted by Δ, the Gauss curvature by K and the Riemannian measure by μ.

2.1.2. Let be given a reference point x_0 in M. We let $r(x)$ denote the Riemannian distance from the point x to the point x_0. We let $B(s)$ denote the *open* geodesic ball with center x_0 and radius s. For $t < s$, we let $C(t, s)$ denote the open set $C(t, s) = B(s) \setminus \overline{B}(t)$. The volume of the ball $B(s)$ is denoted by $V(s)$, the length of the boundary of $B(s)$ by $L(s)$. The length function is a priori only defined for $s \in \mathbb{R}_+ \setminus E$, where the set of exceptional values E is closed, of Lebesgue measure zero. On the set $R_+ \setminus E$, the function L is C^1 and satisfies the inequality (see [11]),

$$L'(t) \leq 2\pi\chi\big(B(t)\big) - \int_{B(t)} K \, d\mu, \tag{1}$$

where $\chi\big(B(t)\big)$ is the Euler-Poincaré characteristic of the ball $B(t)$. As a matter of fact, the function L can be extended to $\mathbb{R}_+$. This follows from the work of F. Fiala [11], P. Hartman [13] and K. Shiohama and M. Tanaka [22, 23]. More precisely, there exist two real functions H, J defined on $\mathbb{R}_+$, with H absolutely continuous on any compact subset, and J non-decreasing, such that $H(s) - J(s)$ coincides with $L(s)$ when s is not in E. The set E and the function J are defined in terms of the cut locus of the point x_0. The (extended) function L is not continuous in general ([13], Figure 1). However, it satisfies,

$$L(t^+) = L(t) \text{ and } L(t^-) \geq L(t), \quad \forall t > 0. \tag{2}$$

Furthermore, the function V is differentiable almost everywhere, and $V'(s) = L(s)$. From the formula $L = H - J$, one can deduce that

$$L(b) - L(a) \leq L(b^-) - L(a) \leq \int_a^b L'(t) \, dt, \quad \text{whenever } \ 0 \leq a < b. \tag{3}$$

Remark. In Fiala's paper, $M = \mathbb{R}^2$ and g is real analytic. In this case, the set E is discrete. Hartman's paper considers the case $(\mathbb{R}^2, g)$, with g smooth. The papers of Shiohama and Tanaka deal with the general case in which M may have finite or infinite topology. All these papers rely on a sharp analysis of the cut locus of a simple closed curve, and on the differential inequality (1) satisfied by the length function L away from the exceptional set E. This was initiated by Fiala, refined by Hartman and later by Shiohama-Tanaka

to take into account the transitions from a real analytic to a smooth metric, and from $\mathbb{R}^2$ to a general surface M.

2.1.3. We introduce the total curvature of the ball $B(s)$ to be

$$G(s) = \int_{B(s)} K(x)\, d\mu(x).$$

2.1.4. We denote by $\chi\big(B(s)\big)$ the Euler-Poincaré characteristic of the open ball $B(s)$. We introduce the function

$$\widehat{\chi}(s) = \sup\{\chi(B(t)) \mid t \in [s, \infty)\}. \tag{4}$$

Both functions are continuous to the left. The function $\widehat{\chi}$ is a non-increasing function from $[0, \infty)$ to $\mathbb{Z}$. It has at most countably many discontinuities. We write them as a sequence, finite possibly empty, or infinite tending to infinity,

$$\{t_j\}_{j=1}^{\overline{N}} = \{0 < t_1 < t_2 < \cdots < t_n < \cdots\}, \tag{5}$$

with $\overline{N} \in \mathbb{N} \cup \{\infty\}$, $\overline{N} = 0$ when the sequence is empty, and $\overline{N} = \infty$ when it is infinite. Note that this sequence depends on the reference point x_0.

At the discontinuity $t_n, n \geq 1$, the function $\widehat{\chi}$ has a jump

$$\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+), \text{ with } \omega_n \in \mathbb{N}, \omega_n \geq 1. \tag{6}$$

Therefore,

$$\begin{cases} \widehat{\chi}(s) = 1 \text{ for } s \in [0, t_1], \text{ and} \\ \widehat{\chi}(s) = 1 - (\omega_1 + \cdots + \omega_n) \leq -(n-1) \text{ for } s \in (t_n, t_{n+1}]. \end{cases} \tag{7}$$

The function $\widehat{\chi}$ somehow controls the large scale topology of M as the following lemma shows.

Lemma 2.1. *Let (M, g) be a complete Riemannian surface. Let $\{t_j\}_{j=1}^{\overline{N}}$ be the discontinuities of the function $\widehat{\chi}$, with jumps $\{\omega_j\}$. Let $\chi(M)$ be the Euler-Poincaré characteristic of M, with $\chi(M) = -\infty$ if M does not have finite topology. Then*

$$1 - \sum_{n=1}^{\overline{N}} \omega_n \leq \chi(M).$$

Proof. We apply Lemma 1.4 in [4].
$\diamond$ If M has finite topology, then there exists a value s_0 such that $\chi(B(s)) \leq \chi(M)$ for all $s \geq s_0$. By (7), this implies that $1 - \sum_{n=1}^{\overline{N}} \omega_n \leq \widehat{\chi}(s_0) \leq \chi(M)$.
$\diamond$ Otherwise, $\chi(B(s))$ tends to minus infinity when s tends to infinity, so does $\widehat{\chi}(s)$, and formula (7) implies that $1 - \sum_{n=1}^{\overline{N}} \omega_n = -\infty$. □

2.1.5. As mentioned earlier, the Euler-Poincaré characteristic of balls is related to the length function and to the total curvature of balls. More precisely, we have the inequalities,

$$\text{(8)}\quad \begin{cases} \text{For all } 0 \le a < b, \\ L(b^-) - L(a) \le 2\pi(b-a)\sup\{\chi(B(s)) \mid s \in [a,b]\} - \int_a^b G(s)\,ds, \\ L(b^-) - L(a) \le 2\pi(b-a)\widehat{\chi}(a) - \int_a^b G(s)\,ds, \end{cases}$$

which follow by integrating the inequality (1) satisfied by $L'(t)$ for $t \in]0,\infty[\backslash E$ ([11], p. 326-328; [13], Proposition 6.1; [22], Proposition 3.7). Note that we can substitute $L(b^-)$ by $L(b)$ in (8), because of inequality (2)

2.2. **Technical lemmas. Definition**. Let $0 \le R < S$. We say that a function $\xi : [R,S] \to \mathbb{R}$ is *admissible* in the interval $[R,S]$ if

$$\text{(9)}\quad \begin{cases} \xi \text{ is } C^1 \text{ and piecewise } C^2 \text{ in } [R,S], \\ \xi \ge 0,\ \xi' \le 0 \text{ and } \xi'' \ge 0. \end{cases}$$

The next two lemmas extend Lemma 1.8 in [4], whose proof uses the method of [5].

Lemma 2.2. *For all $0 \le a < b$, and for all admissible functions ξ on $[a,b]$,*

$$\text{(10)}\quad \begin{cases} \int_{C(a,b)} K(x)\xi^2(r(x))\,d\mu(x) \le & \xi^2 G\Big|_a^b - 2\pi\widehat{\chi}(a)\xi^2\Big|_a^b + (\xi^2)'L\Big|_a^{b^-} \\ & - \int_{C(a,b)} (\xi^2)''(r(x))\,d\mu(x). \end{cases}$$

Note that in the right-hand side of (10) one can substitute $(\xi^2)'L\big|_a^{b^-}$ by $(\xi^2)'L\big|_a^b$, using (2) and the fact that ξ' is non-positive.

Proof. We sketch the proof for completeness. First assume that ξ is C^2. By the co-area formula,

$$\int_{C(a,b)} K\xi^2(r)\,d\mu = \int_a^b \xi^2(t)G'(t)\,dt,$$

where $G(t)$ is the total curvature of the ball $B(t)$. Introduce the function $H(t) := \int_a^t G(s)\,ds$, and integrate the preceding equality by parts twice to get,

$$\int_{C(a,b)} K\xi^2(r)\,d\mu = \xi^2 G\Big|_a^b - (\xi^2)'(b)H(b) + \int_a^b H(t)(\xi^2)''(t)\,dt.$$

One can estimate H in the right-hand side using (8) and the signs of ξ and its derivatives. After some computations and applying the co-area formula once more, one obtains,

$$\int_{C(a,b)} K\xi^2(r)\,d\mu \le \{\xi^2 G - 2\pi\widehat{\chi}(a)\xi^2 + (\xi^2)'L\}\Big|_a^b - \int_{C(a,b)} (\xi^2)''(r)\,d\mu.$$

This proves the lemma when ξ is C^2. The fact that the lemma holds for C^1 and piecewise C^2 functions ξ follows by cutting the interval into subintervals in which ξ is C^2. Apply the preceding method in each sub-interval $(c,d) \subset (a,b)$ using an inequality similar to (8) with $\widehat{\chi}(a)$ in place of $\widehat{\chi}(c)$; use the fact

that ξ and $\widehat{\chi}$ are non-increasing, and the inequality $L(t^-) \geq L(t^+) = L(t)$ to conclude. □

Taking into account the discontinuities $\{t_n\}_{n\geq 1}$ of the function $\widehat{\chi}$, see Section 2.1, formula (7), we have the following lemma.

Lemma 2.3. *Let $\{t_j\}_{j=1}^{\overline{N}}$ be the discontinuities of the function $\widehat{\chi}$. Define the index $N(R)$ to be the largest integer n such that $t_n \leq R$. Let $t_0 = 0$. Let ξ be an admissible function in the interval $[R, Q]$. Then,*

$$\left\{\begin{array}{ll} \int_{C(R,Q)} K\xi^2(r)\,d\mu \leq & 2\pi\Big[\xi^2(R)\widehat{\chi}(t_{N(R)}) - \xi^2(Q)\widehat{\chi}(t_{N(Q)}) \\ & -\sum_{n=N(R)+1}^{N(Q)} \omega_n \xi^2(t_n)\Big] \\ & +\xi^2 G\Big|_R^Q + (\xi^2)' L\Big|_R^Q - \int_{C(R,Q)} (\xi^2)''(r)\,d\mu. \end{array}\right. \tag{11}$$

Taking $R = 0$ and assuming that $\xi(Q) = 0$, we have the inequality

$$\left\{\begin{array}{ll} \int_{B(Q)} K\xi^2(r)\,d\mu \leq & 2\pi\left\{\xi^2(0) - \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n)\right\} \\ & -\int_{B(Q)} (\xi^2)''(r)\,d\mu. \end{array}\right. \tag{12}$$

In particular, assuming that $\xi(Q) = 0$, we have the inequality

$$\int_{B(Q)} K(x)\xi^2(r)\,d\mu \leq 2\pi\xi^2(0) - \int_{B(Q)} (\xi^2)''(r)\,d\mu. \tag{13}$$

Proof. To prove (11), split the integral $\int_{C(R,Q)} K\xi^2(r)\,d\mu$ into a sum,

$$\int_{C(R,Q)} = \int_{C(R,t_{N(R)+1})} + \sum_{n=N(R)+1}^{N(Q)-1} \int_{C(t_n,t_{n+1})} + \int_{C(t_{N(Q)},Q)},$$

apply Lemma 2.2 and use (7). To establish the last two inequalities, use the fact that $\xi(Q) = 0$ and $G(0) = L(0) = 0$. □

The next two lemmas provide admissible functions which we will plug into $(\star)$ later on.

Lemma 2.4. *Fix $0 < R < 5R < Q$, and define the function $\xi_{\alpha,\beta,R,Q}$ by*

$$\xi_{\alpha,\beta,R,Q}(t) = \left\{\begin{array}{l} e^{(1-\frac{t}{2R})^2} \text{ for } 0 \leq t \leq R, \\ \beta(e^{-\alpha t} - e^{-\alpha Q}) \text{ for } R \leq t \leq Q. \end{array}\right.$$

Then, there exists a unique choice $\alpha(R, Q)$, $\beta(R, Q)$ of the parameters α, β such that the corresponding function $\xi_{R,Q}$ is admissible in the interval $[0, Q]$. Furthermore,

$$1 \leq 4R\,\alpha(R, Q) \leq 2 \text{ and } 1 \leq \beta(R, Q) \leq 10.$$

Lemma 2.5. *For $a \in (0, \frac{1}{4})$, let $\alpha = \frac{2a}{1-4a}$ and $\beta = \frac{a}{1-4a}$. For $0 < R < Q$ and $0 < \delta, \epsilon$, let $\xi_{\delta,\epsilon,R,Q}$ be the function,*

$$\xi(t) = \left\{\begin{array}{l} (1 + \frac{t}{R})^{-\beta}, \text{ for } t \in [0, R], \\ \delta\Big((1 + \epsilon t)^{-\alpha} - (1 + \epsilon Q)^{-\alpha}\Big) \text{ for } t \in [R, Q]. \end{array}\right. \tag{14}$$

There exists a positive constant $C(\alpha,\beta) > 1$, such that for For $0 < R \leq C(\alpha,\beta)R < Q$, there is a unique choice $\delta(R,Q)$ and $\epsilon(R,Q)$ of the parameters δ, ϵ, such that the function $\xi_{R,Q}$ defined by equation (14) is admissible in the interval $[0,Q]$. Furthermore, there exist positive constants c_1, c_2 such that

$$1 \leq 6R\,\epsilon(R,Q) \leq 2 \quad \textit{and} \quad c_1 \leq \delta(R,Q) \leq c_2.$$

We leave the proofs of these lemmas to the reader.

3. Proof and optimality of Theorem 1.1

3.1. **Proof of Theorem 1.1.** In Step 1, we make some preparation. In Step 2, we prove that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$, and that q is integrable and satisfies $\int_M q\,d\mu \leq 2\pi a\chi(M)$. We can actually finish the proof of the theorem in the Case (i). In Step 3, we prove that (M,g) has at most quadratic volume growth and, under the assumption that M is a cylinder, that it has at most linear volume growth. Steps 2 and 3 both follow from adequate choices of test functions (using Lemma 2.4 and 2.5) in the stability condition $(\star)$, depending on the case at hand (i), (ii) or (iii).

3.1.1. *Step 1.* We choose an admissible function ξ on $[0,Q]$, with $\xi(Q) = 0$, and we apply the stability condition $(\star)$ to the Lipschitz function $\xi(r)$, where r is the Riemannian distance to some given point $x_0 \in M$. We obtain,

$$(a) \qquad 0 \leq \int_{B(Q)} q\xi^2(r)\,d\mu \leq \int_{B(Q)} \left\{(\xi')^2(r) + aK\xi^2(r)\right\} d\mu.$$

Because ξ is admissible in $[0,Q]$ and $\xi(Q) = 0$, we can apply Lemma 2.3, inequality (12), and we obtain.

$$(b) \qquad \begin{cases} \int_{B(Q)} q\xi^2(r)\,d\mu + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n) \leq 2\pi a\xi^2(0) \\ \qquad\qquad + \int_{B(Q)} \left\{(1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r)\right\} d\mu, \end{cases}$$

where we have used the notations of Lemma 2.3. Inequality (b) holds for all admissible functions ξ in $[0,Q]$ which vanish at Q.

Recall (Section 2.1.4) that the points of discontinuity of the function $\widehat{\chi}$ form a sequence $\{t_n\}_{n=1}^{\overline{N}}$ which is either finite possibly empty, or infinite tending to infinity, with stopping index $\overline{N} \in \mathbb{N} \cup \{\infty\}$.

We fix N to be either the stopping index $\overline{N}$, if $\overline{N} \in \mathbb{N}$, or any fixed integer otherwise. We also fix some R, with $0 < R < Q$. For Q is large enough, $Q > t_N$ and $Q \geq C(\xi)R$, inequality (b) implies that

$$(c) \qquad \begin{cases} \int_{B(R)} q\xi^2(r)\,d\mu + 2\pi a \sum_{n=1}^{N} \omega_n \xi^2(t_n) \leq 2\pi a\xi^2(0) \\ \qquad\qquad + \int_{B(Q)} \left\{(1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r)\right\} d\mu, \end{cases}$$

where this inequality holds for any admissible function ξ in $[0,Q]$ vanishing at Q, and for any fixed N and R as above.

The idea is now to apply (c) to a function ξ which is well adapted to the case at hand, (i), (ii) or (iii), and to the assertion we want to prove.

3.1.2. *Step 2.* We will now show that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ and that q in integrable over (M,g). We consider the cases (i), (ii) and (iii) separately.

Case (i) Here, $a \in (\frac{1}{4}, \infty)$. Choose $\xi(t) = (1-\frac{t}{Q})^\alpha$ for $t \in [0,Q]$, with $\alpha \geq 1$. Then,

$$(d1) \qquad (1-2a)(\xi')^2 - 2a\xi\xi'' = -\frac{\alpha[(4a-1)\alpha - 2a]}{Q^2}(1-\frac{t}{Q})^{2\alpha-2}.$$

We now fix some $\alpha > \frac{2a}{4a-1}$ and plug the above equality into inequality (c) and obtain, for all R and N fixed,

$$(e1) \qquad \begin{cases} \int_{B(R)} q\,\xi^2(r)\,d\mu + 2\pi a \sum_{n=1}^N \omega_n \xi^2(t_n) \\ \qquad + \frac{\alpha[(4a-1)\alpha-2a]}{Q^2} \int_{B(Q)} (1-\frac{r}{Q})^{2\alpha-2}\,d\mu \leq 2\pi a. \end{cases}$$

Note that the three terms in the left-hand side of (e1) are non-negative. In particular, we obtain that

$$\sum_{n=1}^N \omega_n (1-\frac{t_n}{Q})^{2\alpha} \leq 1.$$

Letting Q tend to infinity, we obtain that

$$\sum_{n=1}^N \omega_n \leq 1$$

for any fixed N as above, hence $\overline{N} \leq 1$ and $\omega_1 = 1$ if $\overline{N} = 1$. This proves that $0 \leq 1 - \sum_{n=1}^{\overline{N}} \omega_n$ and hence, by Lemma 2.1, that M is homeomorphic to either $\mathbb{C}$ or $\mathbb{C}^\bullet$.

From (e1) and the previous conclusions, we can choose $N = \overline{N}$ and we obtain,

$$(1-\frac{R}{Q})^\alpha \int_{B(R)} q\,d\mu \leq 2\pi a (1 - \sum_{n=1}^N \omega_n \xi^2(t_n)).$$

Letting Q tend to infinity and using Lemma 2.1, this proves Assertion (B) in the Case (i).

From (e1), we finally infer that there exists a positive constant C_α such that

$$Q^{-2} V(\frac{Q}{2}) \leq \begin{cases} C_\alpha, & \text{if } M \sim \mathbb{C}, \\ C_\alpha\{1-(1-\frac{t_1}{Q})^{2\alpha}\}, & \text{if } M \sim \mathbb{C}^\bullet, \end{cases}$$

where $\sim$ stands for "homeomorphic". Indeed, if $M \sim \mathbb{C}^\bullet$, we have $\omega_1 = 1$. This proves that (M,g) has at most quadratic volume growth in general, and at most linear volume growth when $\chi(M) = 0$.

Note that this completes the proof of Theorem 1.1 in the Case (i). In the next steps, we will concentrate on the cases (ii) and (iii).

Case (ii). Here $a = \frac{1}{4}$ and (M,g) has subexponential volume growth. We choose $\xi(t) = e^{-\alpha t} - e^{-\alpha Q}$ in $[0,Q]$ for some $\alpha > 0$. Then,

$$(d2) \qquad (1-2a)(\xi')^2 - 2a\xi\xi'' = \frac{1}{2}\alpha^2 e^{-\alpha Q} e^{-\alpha t}.$$

Plugging (d2) into (c), we obtain, for all R and N fixed,

$$(e2)\quad \int_{B(R)} q\xi^2(r)\,d\mu + \frac{\pi}{2}\sum_{n=1}^{N}\omega_n\xi^2(t_n) \leq \frac{\pi}{2}\xi^2(0) + \frac{1}{2}\alpha^2 e^{-\alpha Q}\int_{B(Q)} e^{-\alpha r}\,d\mu.$$

We have the following lemma,

Lemma 3.1. *If (M,g) has subexponential volume growth, then for any positive α,*

$$\lim_{Q\to\infty} e^{-\alpha Q}\int_{B(Q)} e^{-\alpha r}\,d\mu = 0.$$

We leave the proof of the lemma to the reader.

Let Q tend to infinity in (e2), and use Lemma 3.1 to obtain,

$$(f2)\qquad \int_{B(R)} qe^{-\alpha r}\,d\mu + \frac{\pi}{2}\sum_{n=1}^{N}\omega_n e^{-2\alpha t_n} \leq \frac{\pi}{2},$$

and inequality (f2) holds for all $\alpha > 0$ and N, R as above. Letting α tend to zero, we can conclude as in Case (i) that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$, and that q is integrable, with $\int_M q\,d\mu \leq \frac{\pi}{2}\chi(M)$.

Note that, unlike in Case (i), we cannot, at this stage, obtain quadratic volume growth (see Step 3).

Case (iii). Here $a \in (0, \frac{1}{4})$ and (M,g) has k_a-subpolynomial volume growth, with $k_a = 2 + \frac{4a}{1-4a}$. We choose $\xi(t) = (1+\epsilon t)^{-\alpha} - (1+\epsilon Q)^{-\alpha}$ in $[0,Q]$, with $\alpha = \frac{2a}{1-4a}$ and some $\epsilon > 0$. Then,

$$(d3)\qquad (1-2a)(\xi')^2 - 2a\xi\xi'' = 2a\alpha(\alpha+1)\epsilon^2(1+\epsilon Q)^{-\alpha}(1+\epsilon t)^{-\alpha-2}.$$

Plugging (d3) into (c), we obtain,

$$(e3)\qquad \begin{aligned}\textstyle\int_{B(R)} q\xi^2(r)\,d\mu + 2\pi a\sum_{n=1}^{N}\omega_n\xi^2(t_n) &\leq 2\pi a\xi^2(0)\\ &+2a\alpha(\alpha+1)\epsilon^2(1+\epsilon Q)^{-\alpha}\textstyle\int_{B(Q)}(1+\epsilon r)^{-\alpha-2}\,d\mu.\end{aligned}$$

We have the following lemma,

Lemma 3.2. *Let (M,g) be a Riemannian surface with k_a-subpolynomial volume growth, with $k_a = 2 + \frac{4a}{1-4a}$. Then, for $\alpha = \frac{2a}{1-4a}$ and any $\epsilon > 0$,*

$$\lim_{Q\to\infty}(1+\epsilon Q)^{-\alpha}\int_{B(Q)}(1+\epsilon r)^{-\alpha-2}\,d\mu = 0.$$

We leave the proof of this lemma to the reader.

Since both terms in the left-hand side of (e3) are non-negative, letting Q tend to infinity and using Lemma 3.2, we obtain,

$$(f3)\qquad \int_{B(R)} q(1+\epsilon r)^{-2\alpha}\,d\mu + 2\pi a\sum_{n=1}^{N}\omega_n(1+\epsilon t_n)^{-2\alpha} \leq 2\pi a,$$

and inequality (f3) holds for any $\epsilon > 0$. Letting ϵ tend to zero, we obtain,

$$\int_{B(R)} q\,d\mu + 2\pi a\sum_{n=1}^{N}\omega_n \leq 2\pi a,$$

and we can conclude as in the previous cases that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ and that Assertion (B) holds.

3.1.3. *Step 3.* We now show that (M,g) has at most quadratic volume growth, and at most linear volume growth when $\chi(M)=0$. We have already dealt with Case (i) in Step 2. We now consider Cases (ii) and (iii).

Case (ii). Here, $a=\frac{1}{4}$ and (M,g) has subexponential volume growth. We choose the function ξ to be $\xi_{R,Q}$ as given by Lemma 2.4,

$$\xi(t)=\begin{cases} e^{(1-\frac{t}{2R})^2}, & \text{for } t\in[0,R] \\ \beta(e^{-\alpha t}-e^{-\alpha Q}), & \text{for } t\in[R,Q], \end{cases}$$

with $0<R<5Q$ and α,β given by the lemma, so that ξ is admissible in $[0,Q]$ and vanishes at Q. We apply (c) again (making $q\equiv 0$ which is sufficient for our estimates). For this purpose, we compute,

$$(p2)\qquad (\xi')^2-\xi\xi''=\begin{cases} -\frac{1}{2R^2}e^{2(1-\frac{t}{2R})^2}, & \text{for } t\in[0,R] \\ \alpha^2\beta^2 e^{-\alpha Q}e^{-\alpha t}, & \text{for } t\in[R,Q], \end{cases}$$

and we obtain,

$$(q2)\qquad \begin{cases} \frac{1}{4R^2}\int_{B(R)} e^{2(1-\frac{r}{2R})^2}\,d\mu \le \frac{\pi}{2}\{e^2-\omega e^{2(1-\frac{t_1}{2R})^2}\} \\ \qquad\qquad +\frac{1}{2}\alpha^2\beta^2 e^{-\alpha Q}\int_{C(R,Q)} e^{-\alpha r}\,d\mu, \end{cases}$$

where $\omega=0$ if $M\sim\mathbb{C}$ and $\omega=1$ if $M\sim\mathbb{C}^\bullet$, and we have chosen $R>t_1$. We fix $R>t_1$ and we let Q tend to infinity, using the fact that α and β remain controlled and that the second term in the right-hand side of (q2) goes to zero when Q tends to infinity because (M,g) has subexponential volume growth (Lemmas 2.4 and 3.1). Finally, we obtain,

$$R^{-2}V(R)\le C\{1-\omega e^{-\frac{t_1}{R}(2-\frac{t_1}{2R})}\}\le C,$$

for some constant C independent of R. This gives at most quadratic volume growth in general, and at most linear volume growth when $\chi(M)=0$.

Case (iii). Here, $a\in(0,\frac{1}{4})$ and (M,g) has k_a-subpolynomial volume growth, with $k_a=2+\frac{4a}{1-4a}$. We choose the function ξ to be $\xi_{R,Q}$ as given by Lemma 2.5,

$$\xi(t)=\begin{cases} (1+\frac{t}{R})^{-\beta}, & \text{for } t\in[0,R], \beta=\frac{a}{1-4a}, \\ \delta\{(1+\epsilon t)^{-\alpha}-(1+\epsilon Q)^{-\alpha}\}, & \text{for } t\in[R,Q], \alpha=\frac{2a}{1-4a}, \end{cases}$$

with $0<R\ll Q$ and δ,ϵ given by the lemma, so that ξ is admissible in $[0,Q]$ and vanishes at Q. We apply (c) again (making $q\equiv 0$ which is sufficient for our estimates). For this purpose, we compute,

$$(p3)\quad (1-2a)(\xi')^2-2a\xi\xi''=\begin{cases} -\frac{a\beta}{R^2}(1+\frac{t}{R})^{-2\beta-2}, & \text{for } t\in[0,R], \\ 2a\alpha(\alpha+1)\delta^2\epsilon^2(1+\epsilon Q)^{-\alpha}(1+\epsilon t)^{-\alpha-2}, \\ \quad \text{for } t\in[R,Q], \end{cases}$$

and we obtain,

$$(q3)\qquad \begin{cases} \frac{a\beta}{R^2}\int_{B(R)}(1+\frac{r}{R})^{-2\beta-2}\,d\mu \le 2\pi a\{1-\omega(1+\frac{t_1}{R})^{-2\beta}\} \\ \qquad +2a\alpha(\alpha+1)\delta^2\epsilon^2(1+\epsilon Q)^{-\alpha}\int_{C(R,Q)}(1+\epsilon r)^{-\alpha-2}\,d\mu, \end{cases}$$

where $\omega = 0$ if $M \sim \mathbb{C}$ and $\omega = 1$ if $M \sim \mathbb{C}^\bullet$, and where we have chosen $R > t_1$. We fix $R > t_1$ and we let Q tend to infinity, using the fact that δ and ϵ remain controlled and that the second term in the right-hand side of (q3) goes to zero when Q tends to infinity because (M, g) has k_a-subpolynomial volume growth (Lemmas 2.5 and 3.2). Finally, we obtain,

$$R^{-2}V(R) \le C\{1-\omega(1+\frac{t_1}{R})^{-2\beta}\} \le C,$$

for some constant C independent of R. This gives at most quadratic volume growth in general and at most linear volume growth when $\chi(M) = 0$.

3.1.4. *Conclusion.* In the three cases (i), (ii) and (iii), we have proved,

- $\diamond$ M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ (Step 2),
- $\diamond$ (M, g) has at most quadratic volume growth, and hence ([4], Proposition 2.3) (M, g) is conformally equivalent to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ (Step 3),
- $\diamond$ q is integrable and $\int_M q\,d\mu \le 2\pi a\chi(M)$; in particular, $q \equiv 0$ when $\chi(M) = 0$ (Step 2),
- $\diamond$ (M, g) has at most linear volume growth when $\chi(M) = 0$.

The proof of Theorem 1.1 is therefore complete. □

3.2. **The assumptions and conclusions in Theorem 1.1 are optimal.** $\diamond$ The example of the hyperbolic plane shows that Assertion (A) in Theorem 1.1 cannot hold when $a = \frac{1}{4}$ without an extra assumption on (M, g).

$\diamond$ The examples of the hyperbolic planes of curvature $-c^2, c > 0$, show that the assumptions in Case (ii) are optimal.

$\diamond$ Consider the unit disk $\mathbb{D}$ with the conformal metric $h_\alpha = \big(\frac{2}{1-|z|^2}\big)^{2\alpha}|dz|^2$ for $\alpha \ge 1$. The metric h_1 is the hyperbolic metric with constant curvature -1. When $\alpha > 1$, the metric h_α is a complete conformal metric on $\mathbb{D}$, with negative curvature. A simple computation shows that it has polynomial volume growth of degree $2 + \frac{1}{\alpha-1}$. Given a complete Riemannian surface (M, g), let $a_+(M, g)$ denote the supremum of the numbers a such that $\Delta + aK \ge 0$. It is proved in [4] that this supremum is achieved (Proposition 1.1) and that $a(\alpha) := a_+(\mathbb{D}, h_\alpha)$ is equal to $\frac{1}{4\alpha}$ (Proposition 4.3). It follows that the volume growth of $(\mathbb{D}, h_\alpha)$ is polynomial with degree equal to $2 + \frac{4a(\alpha)}{1-4a(\alpha)}$. This shows that the assumptions in Case (iii) are optimal.

4. Generalization to finite index

Theorem 4.1. *Let (M, g) be a complete non-compact Riemannian surface. Let q be a locally integrable, non-negative function on M. Assume that the operator $\Delta + aK - q$ has finite index on M. Assume furthermore that either,*

(i) *$a > \frac{1}{4}$, or*
(ii) *$a = \frac{1}{4}$, and (M, g) has subexponential volume growth, or*

(iii) $a \in (0, \frac{1}{4})$, *and* (M, g) *has* k_a*-subpolynomial volume growth, with* $k_a = 2 + \frac{4a}{1-4a}$.

Then, (M, g) *has finite topology and at most quadratic area growth. In particular,* (M, g) *is conformally equivalent to a closed Riemannian surface with finitely many points removed. Furthermore, the function* q *is integrable on* (M, g).

Proof. We first recall what "finite index" means. Let Ω be a smooth, relatively open compact domain in M. By elliptic theory, the operator $\Delta + aK - q$ has finitely many negative eigenvalues in Ω (with Dirichlet boundary conditions on $\partial\Omega$). Call $\mathrm{Ind}(\Omega)$ the number of these negative eigenvalues. This is a non-decreasing function of Ω with respect to inclusion. If $\sup\{\mathrm{Ind}(\Omega) \mid \Omega \Subset M\}$ is finite, then we say that the operator $\Delta + aK - q$ has finite index in M.

It is a well-known fact that if $\Delta + aK - q$ has finite index on M, then it is stable outside a compact set, *i.e.* that

$$0 \leq \int_M \left(|df|^2 + aKf^2 - qf^2\right) d\mu, \tag{a}$$

for any Lipschitz function f with compact support in $M \setminus \overline{B}(R_0)$, for some positive radius R_0 (see [6] for a proof and for a converse statement). Choose

$$0 < R_0 < R_1 := R_0 + 1 < R_2 := R_0 + 2 < R < Q, \tag{b}$$

and a function ξ which is admissible in the interval $[R_2, Q]$ and vanishes at Q. Define the function η as follows,

$$\eta(t) = \begin{cases} 0, & \text{for } \ t \in [0, R_1], \\ \xi(R_2)(t - R_1), & \text{for } \ t \in [R_1, R_2], \\ \xi(t), & \text{for } \ t \in [R_2, Q]. \end{cases} \tag{c}$$

We shall choose ξ later on, depending on the assumptions we are working with –Cases (i), (ii) or (iii)– as in the proof of Theorem 1.1. Consider the function $\eta(r)$. This is a Lipschitz function with compact support in $M \setminus B(R_0)$ and we can therefore plug it into (a). Using (c), we find that,

$$\begin{aligned} 0 \leq \textstyle\int_{C(R_2,Q)} q\xi^2(r)\, d\mu \leq \int_{C(R_2,Q)} \left\{(\xi')^2(r) + aK\xi^2(r)\right\} d\mu \\ + C_1\xi^2(R_2), \end{aligned} \tag{d}$$

for some constant C_1 which depends only on a, the geometry of (M, g) and the values of K in the ball $\bar{B}(R_2)$.

We now apply Lemma 2.3 in $[R_2, Q]$ and we define the indices N_2 and $N(Q)$ by the inequalities

$$t_{N_2} \leq R_2 < t_{N_2+1} \ \text{ and } \ t_{N(Q)} \leq Q < t_{N(Q)+1},$$

so that we can plug (11) into (d) and we obtain the inequality,

$$\begin{cases} \int_{C(R_2,Q)} q\xi^2(r)\, d\mu + 2\pi a \sum_{n=N_2+1}^{N(Q)} \omega_n \xi^2(t_n) \leq \\ \qquad \int_{C(R_2,Q)} \left\{(1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r)\, d\mu\right\} \\ \qquad\qquad + C_2\Big(\xi^2(R_2) + (\xi')^2(R_2)\Big), \end{cases} \tag{e}$$

for some positive constant C_2 which only depends on a, on the geometry of (M, g) and the values of K in $\bar{B}(R_2)$.

We can now proceed from inequality (e) as we proceeded from inequality (b) in the proof of Theorem 1.1. We leave the details to the reader. □

5. Proof of Theorem 1.2

By Theorem 1.1, we already know that (M, g) has at most linear volume growth, so that Theorem 1.2 follows from the following proposition.

Proposition 5.1. *Let (M, g) be a complete cylinder with 2-subpolynomial volume growth. If, for some $a > 0$, the operator $\Delta + aK$ is non-negative on M, then the Gaussian curvature K is non-negative.*

Indeed, assuming the proposition, we can apply Cohn-Vossen's theorem and conclude that $0 \leq \int_M K\, d\mu \leq 2\pi\chi(M)$, so that indeed, $K \equiv 0$.

Proof. For some $a > 0$, we have

$$(a) \qquad 0 \leq \int_M \Big(|df|^2 + aKf^2\Big)\, d\mu, \qquad \forall f \in \mathrm{Lip}_0(M).$$

Choose some $x \in M$. We want to prove that $\kappa := K(x) \geq 0$. We fix this point x and we take the distance function and the geodesic balls with respect to this point.

$\diamond$ According to the assumption that M is a cylinder, the function $\widehat{\chi}(t)$ (considering balls centered in x) has at least one discontinuity at some $t_1 > 0$ (which depends on the choice of x), with $\omega_1 \geq 1$.

$\diamond$ Let $0 < \alpha < 1$ and $0 < R < t_1 < Q$. Define the function ξ (with parameters α, R, Q) to be

$$(b) \qquad \xi(t) = \begin{cases} 1 - \alpha\frac{t}{R} & \text{for } \ t \in [0, R], \\ (1-\alpha)\frac{Q-t}{Q-R} & \text{for } \ t \in [R, Q]. \end{cases}$$

$\diamond$ Use the function $\xi(r)$ to test the positivity condition (a). Straightforward computations give

$$(c) \qquad \int_{B(Q)} (\xi')^2(r)\, d\mu = \frac{\alpha^2}{R^2} V(R) + \Big(\frac{1-\alpha}{Q-R}\Big)^2 \Big(V(Q) - V(R)\Big).$$

Applying Lemma 2.3 to the ball $B(R)$ and to the set $C(R, Q)$, another computation yields

$$\begin{cases} \int_{B(Q)} K\xi^2(r)\, d\mu \leq & -\frac{2\alpha^2}{R^2} V(R) + 2(1-\alpha)\frac{R-\alpha Q}{R(Q-R)} L(R) \\ & +2\pi\Big(1 - \omega_1(1-\alpha)^2(\frac{Q-t_1}{Q-R})^2\Big) \\ & -2(\frac{1-\alpha}{Q-R})^2\Big(V(Q) - V(R)\Big), \end{cases}$$

and hence, since $\omega_1 \geq 1$,

$$(d) \qquad \begin{cases} \int_{B(Q)} K\xi^2(r)\, d\mu \leq & -\frac{2\alpha^2}{R^2} V(R) + 2(1-\alpha)\frac{R-\alpha Q}{R(Q-R)} L(R) \\ & +2\pi\Big(1-(1-\alpha)^2(\frac{Q-t_1}{Q-R})^2\Big) \\ & -2(\frac{1-\alpha}{Q-R})^2\Big(V(Q)-V(R)\Big). \end{cases}$$

Finally, we obtain that for $a > 0$ and the above choice (b) of ξ,

$$(e) \qquad \begin{cases} 0 \leq \int_{B(Q)} \Big((\xi')^2(r) + aK\xi^2(r)\Big)\, d\mu \leq (1-2a)\alpha^2 \frac{V(R)}{R^2} \\ \qquad +(1-2a)(\frac{1-\alpha}{Q-R})^2\Big(V(Q)-V(R)\Big) + 2a(1-\alpha)\frac{R-\alpha Q}{Q-R}\frac{L(R)}{R} \\ \qquad +2\pi a\Big(1-(1-\alpha)^2(\frac{Q-t_1}{Q-R})^2\Big). \end{cases}$$

$\diamond$ Inequality (e) holds for all choices of $\alpha \in (0,1)$ and $0 < R < t_1 < Q$. Recall that the cylinder is assumed to have 2-subpolynomial volume growth. Letting Q tend to infinity, we find that

$$(f) \qquad \begin{cases} 0 \leq \int_{B(Q)} \Big((\xi')^2(r) + aK\xi^2(r)\Big)\, d\mu \leq \\ \qquad (1-2a)\alpha^2 \frac{V(R)}{R^2} - 2a\alpha(1-\alpha)\frac{L(R)}{R} \\ \qquad +2\pi a\alpha(2-\alpha), \end{cases}$$

for all $\alpha \in (0,1)$ and $R \in (0, t_1)$.

$\diamond$ We now use the classical expansions for the length and area of small geodesic balls $B(x,R)$,

$$(g) \qquad \begin{cases} L(R) = 2\pi R\Big(1 - \frac{\kappa}{6}R^2 + R^2\epsilon_1(R)\Big), \\ V(R) = \pi R^2\Big(1 - \frac{\kappa}{12}R^2 + R^2\epsilon_2(R)\Big), \end{cases}$$

where κ is the value of the curvature at x.

$\diamond$ Plugging (g) into (f), we find that

$$(h) \qquad \begin{cases} 0 \leq \alpha^2 + \frac{\kappa R^2}{12}\alpha\Big(8a - (1+6a)\alpha\Big) \\ \qquad +\alpha R^2\Big((1-2a)\alpha\epsilon_2(R) - 4a(1-\alpha)\epsilon_1(R)\Big). \end{cases}$$

Dividing by α and letting α tend to zero gives

$$0 \leq \frac{2\kappa}{3}R^2(1+\epsilon(R)),$$

which implies that $\kappa := K(x) \geq 0$. □

6. Proof of Theorem 1.3

Let us first consider the case of the sphere with constant curvature α^2, $M_0 = S^2(\alpha^2)$. In the sequel, the subscript 0 refers to M_0. Let $J_0 = \Delta_0 + a\alpha^2 - c$. The operator J_0 is non-negative in the ball $B_0(R)$ if and only if the first Dirichlet eigenvalue of the Laplacian Δ_0 in this ball satisfies $\lambda_1(B_0(R)) \geq c - a\alpha^2$. Since $c \geq (a+2)\alpha^2$, it follows that J_0 non-negative in the ball $B_0(R)$ implies that $\lambda_1(B_0(R)) \geq \lambda_1(B_0(\frac{\pi}{2\alpha}))$, and hence that

$R \leq \frac{\pi}{2\alpha}$, because $\lambda_1(B_0(R))$ is a decreasing function of R. If $J_0 \geq 0$ in $B_0(\frac{\pi}{2\alpha})$, then $c = (a+2)\alpha^2$, since all previous inequalities become equalities. Recall that the first Dirichlet eigenfunction for the Laplacian Δ_0 in the hemisphere $B_0(\frac{\pi}{2\alpha})$ is $\cos(\alpha r_0)$, up to a scaling factor, where r_0 is the distance function to a point on the sphere.

Proof of Theorem 1.3. Recall that this theorem is of a local nature. We first state a lemma.

Lemma 6.1. *Let (M, g) be a Riemannian surface. Assume that the curvature satisfies $K \leq \alpha^2$ for some $\alpha > 0$. Let J be the operator $J = \Delta + aK - c$, with $a \in [\frac{1}{2}, \infty)$ and $c \geq (a+2)\alpha^2$. Assume furthermore that the ball $B(x, \frac{\pi}{2\alpha})$ is contained in M, for some $x \in M$. Then the least eigenvalue of the operator J with Dirichlet boundary conditions in this ball is non-positive. If J has least Dirichlet eigenvalue 0 in the ball $B(x, \frac{\pi}{2\alpha})$, then $c = (a+2)\alpha^2$, $K \equiv \alpha^2$ and $B(x, \frac{\pi}{2\alpha})$ is covered by the hemisphere $S^2_+(\alpha^2)$ in the sphere with constant curvature α^2.*

Clearly, the lemma implies the theorem. Indeed, Assertion (i) follows from the lemma and from the monotonicity of eigenvalues with respect to domain inclusion. Assertion (ii) follows immediately.

Proof of the lemma.
$\diamond$ First observe that we can reduce to the case $a \in [\frac{1}{2}, 2]$. Indeed, if $a > 2$, then for any $a' \in [\frac{1}{2}, 2]$, we can write

$$\Delta + aK - c = \Delta + a'K + (a - a')K - c \leq \Delta + a'K - c',$$

where $c' = c + (a' - a)\alpha^2 \geq (a' + 2)\alpha^2$. Moreover, if $c' = (a' + 2)\alpha^2$, then $c = (a + 2)\alpha^2$.

$\diamond$ Assume that $a \in [\frac{1}{2}, 2]$. Let $A := \frac{\pi}{2\alpha}$. Because $K \leq \alpha^2$, the map $\exp_x : T_xM \to M$ is a local diffeomorphism on the ball $D(0, A)$. Let $\tilde{g} = \exp_x^* g$ be the pulled-back metric to T_xM. Let μ_1 be the least Dirichlet eigenvalue of $\Delta + aK - c$ in $B(x, A)$. Then, $\Delta + aK - c - \mu_1 \geq 0$ in $B(x, A)$ and hence, there exists a positive function $u : B(x, A) \to \mathbb{R}$ such that $(\Delta + aK - c - \lambda_1)u = 0$ (see [10]). Let $\tilde{u} = u \circ \exp_x$. Because $\exp_x$ is a local isometry, we have $(\tilde{\Delta} + a\tilde{K} - c - \mu_1)\tilde{u} = 0$ and hence the least Dirichlet eigenvalue $\tilde{\mu}_1$ of $\tilde{\Delta} + a\tilde{K} - c$ satisfies $\tilde{\mu}_1 \geq \mu_1$. To show that μ_1 is non-positive, it suffice to show that $\tilde{\mu}_1$ is non-positive. We have reduced to the simply-connected case.

$\diamond$ We now work in the simply-connected disk $D(0, A)$, with a metric (also denoted) g such that $K \leq \alpha^2$. We denote by $L(r)$ the length of $\partial D(0, r)$ for this metric and we let $L_0(r)$ be the corresponding length on the sphere, $L_0(r) = 2\pi\frac{\sin(\alpha r)}{\alpha}$. By Bishop's comparison theorem, we have that

$$L(r) \geq L_0(r). \tag{a}$$

We now use Pogorelov's trick. Let $\xi : [0, A] \to \mathbb{R}$ be a C^2 function such that $\xi(0) = 1$ and $\xi(A) = 0$. We compute the quadratic form Q associated with $J = \Delta + aK - c$ on the function $\xi(r)$, where r is the geodesic distance to 0

in $D(0,A)$. We also introduce the total curvature of $D(0,r)$,

$$(b) \qquad G(r) = \int_{D(r)} K\, d\mu,$$

with respect to the Riemannian measure in $D(0,A)$. Applying the co-area formula,

$$\begin{aligned} Q(\xi(r)) &= \textstyle\int_{D(0,A)} \left(|d\xi(r)|^2 + (aK-c)\xi^2(r)\right) d\mu \\ &= \textstyle\int_0^A \left((\xi')^2 - c\xi^2\right) L\, dt + a\int_0^A G'\xi^2\, dt, \end{aligned}$$

and we compute the second integral in the right-hand side by integration by parts,

$$(c) \qquad Q(\xi(r)) = \int_0^A \Big((\xi')^2 - c\xi^2\Big) L\, dt - a\int_0^A G(\xi^2)'\, dt.$$

By the Gauss-Bonnet formula, we have $G(t) = 2\pi - L'(t)$ so that (c) becomes, after another integration by parts,

$$Q(\xi(r)) = \int_0^A \Big((\xi')^2 - c\xi^2 - a(\xi^2)''\Big) L\, dt + 2\pi a.$$

Finally, we obtain

$$(d) \qquad Q(\xi(r)) = \int_0^A \Big((1-2a)(\xi')^2 - 2a\xi\xi'' - c\xi^2\Big) L\, dt + 2\pi a,$$

for any function $\xi : [0,A] \to \mathbb{R}$ which is C^2 and such that $\xi(0) = 1$ and $\xi(A) = 0$.

We now use the test function $\eta(r) = \cos(\alpha r)$ in formula (d), where r is the Riemannian distance to the center of the ball, *i.e.* we transplant the first eigenfunction of the hemisphere to a function on the ball $D(0,A)$.

$$Q(\eta(r)) = \int_0^A \Big((1-2a)\alpha^2 \sin^2(\alpha t) + (2a\alpha^2 - c)\cos^2(\alpha t)\Big) L(t)\, dt + 2\pi a,$$

and hence

$$(e) \quad Q(\eta(r)) \le \int_0^A \Big((1-2a)\alpha^2 \sin^2(\alpha t) + (a-2)\alpha^2 \cos^2(\alpha t)\Big) L(t)\, dt + 2\pi a,$$

where we have used the fact that $c \ge (2+a)\alpha^2$. Recall that $a \in [\frac{1}{2}, 2]$. Using the inequality (a), we find that

$$(f) \qquad \begin{cases} Q(\eta(r)) \le 2\pi a \\ \qquad + \int_0^A \left((1-2a)\alpha^2 \sin^2(\alpha t) + (a-2)\alpha^2 \cos^2(\alpha t)\right) L_0(t)\, dt. \end{cases}$$

The right-hand side of (f) is zero because this is the value of the quadratic form of the operator $J_0 = \Delta - 2\alpha^2$ on the hemisphere $S^2_+(\alpha^2) = B_0(\frac{\pi}{2\alpha})$. We conclude that $Q(\eta(r)) \le 0$ and hence that the least Dirichlet eigenvalue of $\Delta + aK - c$ in $D(0,A)$ is non-positive, as stated in the lemma. If this eigenvalue is zero, then $Q(\eta(r)) = 0$, and we must have equality in both (e) and (f), *i.e.* $c = (2+a)\alpha^2$ and $L(t) \equiv 2\pi \frac{\sin(\alpha t)}{\alpha}$. We then deduce that $G(t) \equiv 2\pi(1 - \cos(\alpha t))$. Since $K \le \alpha^2$, integrating K we find that $K \equiv \alpha^2$ and hence we conclude that $D(0,A) = S^2_+(\alpha^2)$. This proves the lemma. □

As a corollary of Theorem 1.3, we obtain Mazet's estimates.

Corollary 6.2. *Let $(M,g) \looparrowright (\widehat{M},\hat{g})$ be an isometric immersion with constant mean curvature H in a simply connected space form with constant sectional curvature κ. Assume furthermore that $H^2+\kappa > 0$ and that the immersion is (strongly) stable. Then,*

$$d_g(x,\partial M) \leq \frac{\pi}{2\sqrt{H^2+\kappa}},$$

where $d_g(x,\partial M)$ is the distance from $x \in M$ with respect to the metric g to the boundary of M, with equality if and only if M is the hemisphere of a sphere of mean curvature H in $\widehat{M}$.

Proof. The Jacobi operator of the immersion is $J = \Delta - |A|^2 - \widehat{\mathrm{Ric}}(n)$, where A is the second fundamental form of the immersion and n the unit normal along the immersion. By the Gauss equation, we find that $J = \Delta + 2K - 4(H^2+\kappa)$ and that $K = H^2+\kappa-\frac{1}{4}(k_1-k_2)^2$, where k_i are the principal curvatures. We can apply Theorem 1.3 with $a=2$ and $\alpha^2 = H^2+\kappa$. For the equality case, note that equality implies that M is totally umbilic. □

Remarks.

(1) This corollary provides a unified proof of Theorem 3.1 and Corollary 3.2 in Mazet's paper [15], without using Lawson's correspondence.
(2) The proof of Theorem 1.3 is simpler than that of Theorem 3.1 in [15], but it uses the same idea which goes back to A. Pogorelov [19].

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