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.

INVERSE SPECTRAL POSITIVITY FOR SURFACES

PIERRE BÉRARD AND PHILIPPE CASTILLON

ABSTRACT. Let (M, g) be a complete non-compact Riemannian surface. We consider operators of the form $\Delta + aK - q$, where Δ is the non-negative Laplacian, K the Gaussian curvature, q a non-negative function, and a a positive real number. We address the question "What conclusions on (M, g) and q can one draw from the fact that the operator $\Delta + aK - q$ is non-negative" and we improve earlier results in particular in the cases $a = \frac{1}{4}$ and $a \in (0, \frac{1}{4})$. We also show that the non-negativity is preserved under normal Riemannian covering with amenable covering group.

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1. INTRODUCTION

Let (M, g) be a complete non-compact Riemannian surface. In the sequel, we will always implicitly assume that M is connected and orientable. We denote by Δ the non-negative Laplacian, by K the Gaussian curvature and by μ the measure associated with the metric g.

In this paper, we consider operators of the form $\Delta + aK - q$, where a is a positive parameter and q a non-negative function. Such operators appear naturally when one studies minimal (or constant mean curvature) immersions. Let us mention two examples. The Jacobi (stability) operator of an isometric minimal immersion $M \looparrowright \mathbb{R}^3$ into Euclidean 3-space is $\Delta + 2K$. More generally ([10], Section 3), the Jacobi operator of a minimal immersion $M \looparrowright \widehat{M}^3$ into a 3-manifold with scalar curvature $\widehat{S}$ can be written as $\Delta + K - (\widehat{S} + \frac{1}{2}|A|^2)$, where $|A|$ is the norm of the second fundamental form of the immersion.

More precisely, this paper is concerned with the following question: *What conclusions on the Riemannian surface (M, g), and on the function q, can one draw from the fact that the operator $\Delta + aK - q$ is non-negative on (M, g) ? i.e.* from the fact that the associated quadratic form is non-negative on Lipschitz – or equivalently C^1 – functions with compact support in M,

$$(\bigstar) \qquad 0 \leq \int_M \left(|df|^2 + aKf^2 - qf^2\right) d\mu \qquad \forall f \in \mathrm{Lip}_0(M).$$

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Our first result is the following.

Theorem 1.1. *Let (M,g) be a complete non-compact Riemannian surface, and let q be a non-negative locally integrable function on M. Assume that the operator $\Delta + aK - q$ is non-negative on M, and that either*

(i) $a \in (\frac{1}{4}, \infty)$, *or*
(ii) $a = \frac{1}{4}$ *and (M,g) has sub-exponential volume growth, or*
(iii) $a \in (0, \frac{1}{4})$ *and (M,g) has at most polynomial volume growth of degree k, with $k < 2 + \frac{4a}{1-4a}$.*

Then,

(A) *The surface (M,g) has at most quadratic volume growth and is conformally equivalent to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ with the standard metric.*
(B) *The function q is integrable on (M,g) and $\int_M q\, d\mu \leq 2\pi a \chi(M)$.*
(C) *If M is a cylinder, then (M,g) has at most linear volume growth and $q \equiv 0$.*

Definitions. Let $x \in M$ and let $V(r)$ denote the volume of the geodesic ball $B(x,r)$ for the metric g. We say that (M,g) has *sub-exponential volume growth* if

$$\limsup_{r\to\infty} \frac{\ln V(r)}{r} = 0.$$

We say that (M,g) has *at most polynomial volume growth of degree k* if

$$\limsup_{r\to\infty} \frac{V(r)}{r^k} < \infty.$$

These definitions do not depend on the choice of the point $x \in M$.

Remarks

(1) Theorem 1.1 improves several known results (see the short historical account below). In particular, Case (i) was first treated in [4], with $q \equiv 0$; Cases (ii) and (iii) were first considered in [9], under stronger assumptions on (M,g).
(2) The assumptions / conclusions in the theorem are optimal, see Section 3.2 for more details.
(3) The main new idea in the proof of the theorem is to introduce the function $\widehat{\chi}(t) := \sup\{\chi(B(s)) \mid s \in [t,\infty)\}$, the supremum of the Euler-Poincaré characteristic of open geodesic balls with radius at least t, whose jumps describe the large scale topology of M, see Section 2. We also introduce new functions to test the positivity of the quadratic form ($\bigstar$), see Lemmas 2.4 and 2.5.
(4) Theorem 1.1 can be extended to the case where the operator $\Delta + aK - q$ is only assumed to have finite index. The conclusions, under the assumptions (i), (ii) or (iii), are that (M,g) is conformally equivalent to a closed Riemannian surface with a finite number of points removed, and that q is integrable over (M,g). We refer to Section 4 for a precise statement and its proof.

Short historical account on Theorem 1.1. The ideas behind the proof of Theorem 1.1 go back to [19] in which A. Pogorelov proves that stable minimal

orientable surfaces in $\mathbb{R}^3$ are planes (a different proof of this result appeared at the same time in [3]). For this purpose, he considers the operator $\Delta + 2K$ on a simply-connected surface with non-positive curvature. His results were extended to the case $a > \frac{1}{4}$ by S. Kawai [15]. A more general setting (general topology and curvature, $a > \frac{1}{2}$) was considered by R. Gulliver and B. Lawson in [13]. A. Pogorelov's method was improved by T. Colding and W. Minicozzi [5] and, later on, by Ph. Castillon [4] who first proved Case (i) in Theorem 1.1 (with $q \equiv 0$). Cases (ii) and (iii) were first considered by J. Espinar and H. Rosenberg in [9], under more restrictive assumptions on (M, g). Case (i), with a potential q, was treated in [9] and [18]. These papers also contain applications to constant mean curvature surfaces in 3-manifolds.

In [10], D. Fischer-Colbrie and R. Schoen prove the following result (their Theorem 3 reformulated for our purpose),

Theorem. *Let N be a complete oriented 3-manifold of non-negative scalar curvature. Let M be an oriented complete non-compact stable minimal surface in N. Then M is conformally equivalent to the complex plane or to a cylinder. If M is a cylinder and the absolute total curvature of M is finite, then M is flat and totally geodesic.*

They also point out (Remark 2, p. 207) that *the assumption of finite total curvature* in their theorem *should not be essential*.

Our next results show that this is indeed the case. We first state a corollary of Theorem 1.1 and then an intrinsic theorem of independent interest.

Corollary 1.2. *Let $(M, g) \looparrowright (\widehat{M}, \hat{g})$ be a complete stable minimal cylinder in a 3-manifold $(\widehat{M}, \hat{g})$ with non-negative scalar curvature $\widehat{S}$ (both M and $\widehat{M}$ are assumed to be orientable). Then (M, g) is totally geodesic and $\widehat{S}$ vanishes identically along M.*

Theorem 1.3. *Let (M, g) be a complete non-compact 2-dimensional cylinder. Assume that the operator $\Delta + aK$ is non-negative on (M, g), and that either,*

(i) *$a > \frac{1}{4}$, or*
(ii) *$a = \frac{1}{4}$ and (M, g) has sub-exponential volume growth, or*
(iii) *$a \in (0, \frac{1}{4})$ and (M, g) has at most polynomial growth of degree k, with $k < 2 + \frac{4a}{1-4a}$.*

Then, (M, g) is flat, i.e. *$K \equiv 0$.*

Remark. Clearly, Corollary 1.2 and Theorem 1.3 imply that the answer to the question raised by D. Fischer-Colbrie and R. Schoen is positive. Case (i) in Theorem 1.3 already appears in [20], Section 3.3 (with a more complicated proof), in [9], Theorem 6.3, under the restrictive assumption that $\int_M K_+ \, d\mu$ is finite and in [8] with a different proof.

The next theorem provides an intrinsic version of the optimal length estimate of L. Mazet [16]. Note that this is a local result, we do not need M to be complete. It applies when M is a stable constant mean curvature

surface M, possibly with boundary, isometrically immersed into a simply connected space form $\widehat{M}$, see Corollary 6.2. Our proof follows the same ideas as in Mazet's paper. We clarify the argument by applying a transplantation method.

Theorem 1.4. *Let (M,g) be a Riemannian surface. Assume that the curvature satisfies $K \leq \alpha^2$ for some $\alpha > 0$. Let J be the operator $J = \Delta + aK - c$, with $a \in [\frac{1}{2}, \infty)$ and $c \geq (a+2)\alpha^2$.*

(i) *If J is non-negative in a geodesic ball $B(x,R)$ contained in M, then $R \leq \frac{\pi}{2\alpha}$.*

(ii) *Assume that the geodesic ball $B(x, \frac{\pi}{2\alpha})$ is contained in M. If J is non-negative in this ball, then $c = (a+2)\alpha^2$, $K \equiv \alpha^2$ and $B(x, \frac{\pi}{2\alpha})$ is covered by the hemisphere $S^2_+(\alpha^2)$ in the sphere with constant curvature α^2.*

It is a natural problem to investigate the behaviour of the positivity of the operator $\Delta + aK$ under Riemannian cover. In this direction, we prove the following general theorem which extends Proposition 2.5 in [18], see also [17].

Theorem 1.5. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 2$. Let $\rho : (\widehat{M}, \hat{g}) \to (M,g)$ be a normal Riemannian covering, with amenable covering group. Let V be a function on M and let $\widehat{V} = V \circ \rho$. Then, the operator $\Delta_g + V$ is non-negative on (M,g) if and only if the operator $\Delta_{\hat{g}} + \widehat{V}$ is non-negative on $\widehat{M}$.*

Remark. For amenable groups, we refer to [12], and to [1], Section 1. See also Section 7.1 below.

The paper is organized as follows. In Section 2, we fix the notations and state some technical lemmas to be used later on. The proof of Theorem 1.1 is given in Section 3.1; the fact that the assumptions / conclusions in the theorem are optimal is explained in Section 3.2. The extension of Theorem 1.1 to finite index operators is given in Section 4. The proofs of Theorems 1.3, 1.4 and 1.5 are given in the subsequent sections.

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2. Notations and preliminary results

In this section, we fix some notations which will be used throughout the paper, and we state some preliminary results.

2.1. **Notations.** In this paper, (M,g) denotes a complete non-compact Riemannian surface. We also assume that M is connected and orientable.

2.1.1. The non-negative Laplacian for the metric g will be denoted by Δ, the Gauss curvature by K and the Riemannian measure by μ.

2.1.2. We choose a reference point x_0 in M. We let $r(x)$ denote the Riemannian distance from the point x to the point x_0. We let $B(s)$ denote the *open* geodesic ball with center x_0 and radius s. For $t < s$, we let $C(t,s)$ denote the open set $C(t,s) = B(s) \setminus \overline{B}(t)$. The volume of the ball $B(s)$ is denoted by $V(s)$, the length of the boundary of $B(s)$ by $L(s)$. As a matter of fact, this length only makes sense for $s \in R_+ \setminus E$, where the set of exceptional values E is closed, of Lebesgue measure zero. This follows from the work of F. Fiala [11], P. Hartman [14] and K. Shiohama and M. Tanaka [21, 22]. More precisely, there exist two real functions H, J defined on $\mathbb{R}_+$, with H absolutely continuous on any compact subset and J non-decreasing, such that the function $L = H - J$ gives the length of the boundary of $B(s)$ when s is not in E. The set E and the function J are defined in terms of the cut locus of the point x_0. The function L is not continuous in general ([14], Figure 1). However, it satisfies,

$$L(t^+) = L(t) \text{ and } L(t^-) \geq L(t), \quad \forall t > 0. \tag{1}$$

Furthermore, the function V is differentiable almost everywhere and $V'(s) = L(s)$ ([22], Theorem 2.2).

From the formula $L = H - J$, one can deduce that

$$L(b) - L(a) \leq \int_a^b L'(t)\, dt, \quad \text{whenever } \ 0 \leq a < b. \tag{2}$$

Remark. In Fiala's paper, $M = \mathbb{R}^2$ and g is real analytic. In this case, the set E is discrete. Hartman's paper considers the case $(\mathbb{R}^2, g)$, with g smooth. The papers of Shiohama and Tanaka deal with the general case in which M may have finite or infinite topology. All these papers rely on a sharp analysis of the cut locus of a simple closed curve, and on a differentieal inequality satisfied by the length function L away from the exceptional set. This was initiated by Fiala and then refined by Hartman and Shiohama-Tanaka to take into account the transitions from a real analytic to a smooth metric, and from $\mathbb{R}^2$ to a general surface M.

2.1.3. We introduce the total curvature of the ball $B(s)$ to be $G(s) = \int_{B(s)} K(x)\, d\mu(x)$.

2.1.4. We denote by $\chi\big(B(s)\big)$ the Euler-Poincaré characteristic of the open ball $B(s)$. We introduce the function

$$\widehat{\chi}(s) = \sup\{\chi(B(t)) \mid t \in [s, \infty)\}. \tag{3}$$

Both functions are continuous to the left. The function $\widehat{\chi}$ is a non-increasing function from $[0, \infty)$ to $\mathbb{Z}$. It has at most countably many discontinuities

$$0 < t_1 < t_2 < \cdots < t_n < \cdots \tag{4}$$

and this sequence is either finite or tends to infinity. At the point t_n, the function $\widehat{\chi}$ has a jump

$$\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+), \text{ with } \omega_n \in \mathbb{N}, \omega_n \geq 1. \tag{5}$$

Therefore,

$$\begin{cases} \widehat{\chi}(s) = 1 \text{ for } s \in [0, t_1], \text{ and} \\ \widehat{\chi}(s) = 1 - (\omega_1 + \cdots + \omega_n) \leq -(n-1) \text{ for } s \in (t_n, t_{n+1}]. \end{cases} \tag{6}$$

The function $\widehat{\chi}$ somehow describes the large scale topology of M as the following lemma shows.

Lemma 2.1. *Let (M,g) be a complete Riemannian surface. Let $\bar{N} \in \mathbb{N} \cup \{\infty\}$ be the number of terms in the sequence $\{t_j\}$ of discontinuities of the function $\widehat{\chi}$. Let $\chi(M)$ be the Euler-Poincaré characteristic of M (with $\chi(M) = -\infty$ if M does not have finite topology). Then*

$$1 - \sum_{n=1}^{\bar{N}} \omega_n \leq \chi(M).$$

Proof. According to Lemma 1.4 in [4],
• If M has finite topology, then there exists a value s_0 such that $\chi(B(s_0)) \leq \chi(M)$ and by (6), we have that $1 - \sum_{n=1}^{\bar{N}} \omega_n \leq \chi(B(s_0)) \leq \chi(M)$.
• Otherwise, $\chi(B(s))$ tends to minus infinity when s tends to infinity and formula (6) implies that $1 - \sum_{n=1}^{\bar{N}} \omega_n = -\infty$. □

2.1.5. The Euler-Poincaré characteristic of balls is related to the length function and to the total curvature of balls. More precisely, we have the inequalities,

$$\text{(7)} \qquad \begin{cases} \text{For all } 0 \leq a < b, \\ L(b) - L(a) \leq 2\pi(b-a) \sup\{\chi(B(s)) \mid s \in [a,b]\} - \int_a^b G(s)\, ds, \\ L(b) - L(a) \leq 2\pi(b-a)\widehat{\chi}(a) - \int_a^b G(s)\, ds. \end{cases}$$

Inequalities (7) follow by integrating the inequality satisfied by $L'(t)$ for $t \in]0,\infty[\backslash E$ ([11], p. 326-328; [14], Proposition 6.1; [21], Proposition 3.7).

2.2. **Technical lemmas. Definition**. Let $0 \leq R < S$. We say that a function $\xi : [R,S] \to \mathbb{R}$ is *admissible* in the interval $[R,S]$ if

$$\text{(8)} \qquad \begin{cases} \xi \text{ is } C^1 \text{ and piecewise } C^2 \text{ in } [R,S], \\ \xi \geq 0,\ \xi' \leq 0 \text{ and } \xi'' \geq 0. \end{cases}$$

The next two lemmas extend Lemma 1.8 in [4] whose proof uses the method of [5].

Lemma 2.2. *For all $0 \leq a < b$, and for all admissible functions ξ on $[a,b]$,*

$$\text{(9)} \qquad \begin{cases} \int_{C(a,b)} K(x)\xi^2(r(x))\, d\mu(x) \leq & \xi^2 G\Big|_a^b - 2\pi\widehat{\chi}(a)\xi^2\Big|_a^b + (\xi^2)' L\Big|_a^b \\ & - \int_{C(a,b)} (\xi^2)''(r(x))\, d\mu(x). \end{cases}$$

Proof. We sketch the proof for completeness. First assume that ξ is C^2. By the co-area formula,

$$\int_{C(a,b)} K\xi^2(r)\, d\mu = \int_a^b \xi^2(t) G'(t)\, dt,$$

where $G(t)$ is the total curvature of the ball $B(t)$. Introduce the function $H(t) := \int_a^t G(s)\, ds$, and integrate the preceding equality by parts twice to

get,

$$\int_{C(a,b)} K\xi^2(r)\,d\mu = \xi^2 G\Big|_a^b - (\xi^2)'(b)H(b) + \int_a^b H(t)(\xi^2)''(t)\,dt.$$

One can estimate H in the right-hand side using (7) and the signs of ξ and its derivatives. After some computations and applying the co-area formula once more, one obtains,

$$\int_{C(a,b)} K\xi^2(r)\,d\mu \leq \{\xi^2 G - 2\pi\widehat{\chi}(a)\xi^2 + (\xi^2)'L\}\Big|_a^b - \int_{C(a,b)} (\xi^2)''(r)\,d\mu.$$

This proves the lemma when ξ is C^2. The fact that the lemma holds for C^1 and piecewise C^2 functions ξ follows by cutting the interval into subintervals in which ξ is C^2, and by estimating the boundary terms using the facts that $\xi' \leq 0$, $\widehat{\chi}$ non-increasing, and the inequality $L(t^-) \geq L(t^+) = L(t)$ (see [22], Theorem 2.2). □

Taking into account the discontinuities $\{t_n\}_{n\geq 1}$ of the function $\widehat{\chi}$, see Section 2.1, formulas (2)-(4), we have the following lemma.

Lemma 2.3. *Let $0 < t_1 < t_2 < \cdots < t_k < \cdots$ be the discontinuities of the function $\widehat{\chi}$. Let $t_0 = 0$. For $0 \leq R < Q$, define the indices $N(R)$ and $N(Q)$ by*

$$t_{N(R)} \leq R < t_{N(R)+1} \quad \text{and} \quad t_{N(Q)} < Q \leq t_{N(Q)+1}. \tag{10}$$

Let ξ be an admissible function in the interval $[R, Q]$. Then

$$\begin{cases} \int_{C(R,Q)} K\xi^2(r)\,d\mu \leq & 2\pi\Big[\xi^2(R)\widehat{\chi}(t_{N(R)}) - \xi^2(Q)\widehat{\chi}(t_{N(Q)}) \\ & -\sum_{n=N(R)+1}^{N(Q)} \omega_n \xi^2(t_n)\Big] \\ & +\xi^2 G\Big|_R^Q + (\xi^2)'L\Big|_R^Q - \int_{C(R,Q)} (\xi^2)''(r)\,d\mu. \end{cases} \tag{11}$$

Taking $R = 0$ and assuming that $\xi(Q) = 0$, we have the inequality

$$\begin{cases} \int_{B(Q)} K\xi^2(r)\,d\mu \leq & 2\pi\left\{\xi^2(0) - \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n)\right\} \\ & -\int_{B(Q)} (\xi^2)''(r)\,d\mu. \end{cases} \tag{12}$$

In particular, assuming that $\xi(Q) = 0$, we have the inequality

$$\int_{B(Q)} K(x)\xi^2(r)\,d\mu \leq 2\pi\xi^2(0) - \int_{B(Q)} (\xi^2)''(r)\,d\mu. \tag{13}$$

Proof. To prove (11), split the integral $\int_{C(R,Q)} K\xi^2(r)\,d\mu$ into a sum,

$$\int_{C(R,Q)} = \int_{C(R,t_{N(R)+1})} + \sum_{n=N(R)+1}^{N(Q)-1} \int_{C(t_n,t_{n+1})} + \int_{C(t_{N(Q)},Q)},$$

apply Lemma 2.2 and use (6). To establish the last two inequalities, use the fact that $\xi(Q) = 0$ and $G(0) = L(0) = 0$. □

The next two lemmas provide admissible functions which we will plug into (★) later on.

Lemma 2.4. *Fix $0 < R < 5R < Q$, and define the function $\xi_{\alpha,\beta,R,Q}$ by*

$$\xi_{\alpha,\beta,R,Q}(t) = \begin{cases} e^{(1-\frac{t}{2R})^2} \text{ for } 0 \le t \le R, \\ \beta\big(e^{-\alpha t} - e^{-\alpha Q}\big) \text{ for } R \le t \le Q. \end{cases}$$

Then, there exists a unique choice $\alpha(R,Q)$, $\beta(R,Q)$ of the parameters α, β such that the corresponding function $\xi_{R,Q}$ is admissible in the interval $[0,Q]$. Furthermore,

$$1 \le 4R\,\alpha(R,Q) \le 2 \text{ and } 1 \le \beta(R,Q) \le 10.$$

Lemma 2.5. *For $a \in (0, \frac{1}{4})$, let $\alpha = \frac{2a}{1-4a}$ and $\beta = \frac{a}{1-4a}$. For $0 < R < Q$ and $0 < \delta, \epsilon$, let $\xi_{R,Q,\delta,\epsilon}$ be the function,*

$$\xi(t) = \begin{cases} (1 + \frac{t}{R})^{-\beta}, \text{ for } t \in [0,R], \\ \delta\Big((1+\epsilon t)^{-\alpha} - (1+\epsilon Q)^{-\alpha}\Big) \text{ for } t \in [R,Q]. \end{cases} \tag{14}$$

For $0 < R \ll Q$, and the above values of α, β, there is a unique choice $\delta(R,Q)$ and $\epsilon(R,Q)$ of the parameters δ, ϵ, such that the function $\xi_{R,Q}$ defined by equation (14) is admissible in the interval $[0,Q]$. Furthermore, there exist positive constants c_1, c_2 such that

$$1 \le 6R\,\epsilon(R,Q) \le 2 \quad \text{and} \quad c_1 \le \delta(R,Q) \le c_2.$$

We leave the proofs of these lemmas to the reader.

3. Proof and optimality of Theorem 1.1

3.1. **Proof of Theorem 1.1.** In Step 1, we make some preparation. In Step 2, we prove that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$, and that q in integrable and satisfies $\int_M q\,d\mu \le 2\pi a\chi(M)$. We can actually finish the proof of the theorem in the Case (i). In Step 3, we prove that (M,g) has at most quadratic volume growth and, under the assumption that M is a cylinder, that it has at most linear volume growth. Steps 2 and 3 both follow from adequate choices of test functions in the stability condition (★), depending on the case at hand (i), (ii) or (iii). We finally conclude the proof of the theorem.

3.1.1. *Step 1.* We choose an admissible function ξ on $[0,Q]$, with $\xi(Q) = 0$, and we apply the stability condition (★) to the Lipschitz function $\xi(r)$, where r is the Riemannian distance to some given point $x_0 \in M$. We obtain,

$$0 \le \int_{B(Q)} q\xi^2(r)\,d\mu \le \int_{B(Q)} \big\{(\xi')^2(r) + aK\xi^2(r)\big\}\,d\mu. \tag{a}$$

Because ξ is admissible in $[0,Q]$ and $\xi(Q) = 0$, we can apply Lemma 2.3, inequality (12), and we obtain.

$$\begin{cases} \int_{B(Q)} q\xi^2(r)\,d\mu + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi^2(t_n) \le 2\pi a \xi^2(0) \\ \qquad\qquad + \int_{B(Q)} \big\{(1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r)\big\}\,d\mu, \end{cases} \tag{b}$$

where we have used the notations of Lemma 2.3. Inequality (b) holds for all admissible functions ξ in $[0,Q]$ which vanish at Q.

The points of discontinuity of the function $\widehat{\chi}$ form an increasing sequence $\{t_n\}$ which is either finite, stopping at some t_N, or infinite, tending to infinity. We let N be either the stopping index, or any fixed integer. We also fix some R, with $0 < R < Q$. If Q is large enough, we will have $Q > t_N$ and inequality (b) implies that

$$(c) \qquad \begin{cases} \int_{B(R)} q\xi^2(r)\, d\mu + 2\pi a \sum_{n=1}^{N} \omega_n \xi^2(t_n) \leq 2\pi a \xi^2(0) \\ \qquad\qquad + \int_{B(Q)} \{(1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r)\}\, d\mu, \end{cases}$$

where this inequality holds for any admissible function ξ in $[0, Q]$ vanishing at Q, and for any fixed N and R as above.

The idea is now to apply (c) to a function ξ which is well adapted to the case at hand, (i), (ii) or (iii), and to the assertion we want to prove.

3.1.2. *Step 2.* We will now show that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ and that q in integrable over (M, g). We consider the cases (i), (ii) and (iii) separately.

Case (i) Here, $a \in (\frac{1}{4}, \infty)$. Choose $\xi(t) = (1 - \frac{t}{Q})^\alpha$ for $t \in [0, Q]$, with $\alpha \geq 1$. Then,

$$(d1) \qquad (1-2a)(\xi')^2 - 2a\xi\xi'' = -\frac{\alpha[(4a-1)\alpha - 2a]}{Q^2}(1 - \frac{t}{Q})^{2\alpha-2}.$$

We now choose some $\alpha > \frac{2a}{4a-1}$ and plug the above equality into inequality (c) and obtain, for all R and N fixed,

$$(e1) \qquad \begin{cases} \int_{B(R)} q\, \xi^2(r)\, d\mu + 2\pi a \sum_{n=1}^{N} \omega_n \xi^2(t_n) \\ \qquad + \frac{\alpha[(4a-1)\alpha-2a]}{Q^2} \int_{B(Q)} (1 - \frac{r}{Q})^{2\alpha-2}\, d\mu \leq 2\pi a. \end{cases}$$

Note that the three terms in the left-hand side of (e1) are non-negative. In particular, we obtain that

$$\sum_{n=1}^{N} \omega_n (1 - \frac{t_n}{Q})^{2\alpha} \leq 1.$$

Letting Q tend to infinity, and using Lemma 2.1, we obtain

$$0 \leq 1 - \sum_{n=1}^{N} \omega_n \leq \chi(M).$$

This proves that M is homeomorphic to either $\mathbb{C}$ or $\mathbb{C}^\bullet$.

From (e1) and the previous conclusions, we now obtain,

$$(1 - \frac{R}{Q})^\alpha \int_{B(R)} q\, d\mu \leq 2\pi a (1 - \sum_{n=1}^{N} \omega_n \xi^2(t_n)).$$

Letting Q tend to infinity and using Lemma 2.1, we obtain Assertion (B).

From (e1) and Lemma 2.1, we have

$$Q^{-2} V(\frac{Q}{2}) \leq C_\alpha \begin{cases} 1, & \text{if } M \sim \mathbb{C}, \\ \{1 - (1 - \frac{t_1}{Q})^{2\alpha}\}, & \text{if } M \sim \mathbb{C}^\bullet, \end{cases}$$

where $\sim$ stands for "homeomorphic". Indeed, if $M \sim \mathbb{C}^\bullet$ we have $\omega_1 = 1$. This proves that (M, g) has at most quadratic volume growth in general, and at most linear volume growth when $\chi(M) = 0$.

Note that this completes the proof of Theorem 1.1 in the Case (i). In the next steps, we will concentrate on the cases (ii) and (iii).

Case (ii). Here $a = \frac{1}{4}$ and (M, g) has sub-exponential volume growth. We choose $\xi(t) = e^{-\alpha t} - e^{\alpha Q}$ in $[0, Q]$ for some $\alpha > 0$. Then,

$$(1-2a)(\xi')^2 - 2a\xi\xi'' = \alpha^2 e^{-\alpha Q} e^{-\alpha t}. \tag{$d2$}$$

Plugging (d2) into (c), we obtain, for all R and N fixed,

$$\int_{B(R)} q\xi^2(r)\, d\mu + \frac{\pi}{2}\sum_{n=1}^{N} \omega_n \xi^2(t_n) \leq \frac{\pi}{2}\xi^2(0) + \alpha^2 e^{-\alpha Q} \int_{B(Q)} e^{-\alpha r}\, d\mu. \tag{$e2$}$$

We have the following lemma,

Lemma 3.1. *If (M, g) has sub-exponential volume growth, then for any $\alpha > 0$,*

$$\lim_{Q\to\infty} e^{-\alpha Q} \int_{B(Q)} e^{-\alpha r}\, d\mu = 0.$$

We leave the proof of the lemma to the reader.

Let Q tend to infinity in (e2), and use Lemma 3.1 to obtain,

$$\int_{B(R)} q e^{-\alpha r}\, d\mu + \frac{\pi}{2}\sum_{n=1}^{N} \omega_n e^{-2\alpha t_n} \leq \frac{\pi}{2}, \tag{$f2$}$$

and inequality (f2) holds for all $\alpha > 0$ and N, R as above. Letting α tend to zero, we can conclude as in Case (i) that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$, and that q is integrable, with $\int_M q\, d\mu \leq \frac{\pi}{2}\chi(M)$.

Note that, unlike in Case (i), we have not yet obtained quadratic volume growth (see Step 3).

Case (iii). Here $a \in (0, \frac{1}{4})$ and (M, g) has polynomial volume growth of at most degree k, with $k < 2 + \frac{4a}{1-4a}$. We choose $\xi(t) = (1+\epsilon t)^{-\alpha} - (1+\epsilon Q)^{-\alpha}$ in $[0, Q]$, with $\alpha = \frac{2a}{1-4a}$ and some $\epsilon > 0$. Then,

$$(1-2a)(\xi')^2 - 2a\xi\xi'' = \alpha(\alpha+1)\epsilon^2(1+\epsilon Q)^{-\alpha}(1+\epsilon t)^{-\alpha-2}. \tag{$d3$}$$

Plugging (d3) into (c), we obtain,

$$\begin{aligned} &\textstyle\int_{B(R)} q\xi^2(r)\, d\mu + 2\pi a \sum_{n=1}^{N} \omega_n \xi^2(t_n) \leq 2\pi a \xi^2(0) \\ &\qquad + \alpha(\alpha+1)\epsilon^2(1+\epsilon Q)^{-\alpha} \textstyle\int_{B(Q)} (1+\epsilon r)^{-\alpha-2}\, d\mu. \end{aligned} \tag{$e3$}$$

We have the following lemma,

Lemma 3.2. *Let (M, g) be a Riemannian surface with polynomial growth of degree at most k, with $k < 2 + \frac{4a}{1-4a}$. Then, for $\alpha = \frac{2a}{1-4a}$ and any $\epsilon > 0$,*

$$\lim_{Q\to\infty} (1+\epsilon Q)^{-\alpha} \int_{B(Q)} (1+\epsilon r)^{-\alpha-2}\, d\mu = 0.$$

We leave the proof of this lemma to the reader.

Since both terms in the left-hand side of (e3) are non-negative, letting Q tend to infinity and using Lemma 3.2, we obtain,

$$(f3) \qquad \int_{B(R)} q(1+\epsilon r)^{-2\alpha}\, d\mu + 2\pi a \sum_{n=1}^{N} \omega_n (1+\epsilon t_n)^{-2\alpha} \leq 2\pi a,$$

and inequality (f3) holds for any $\epsilon > 0$. Letting ϵ tend to zero, we obtain,

$$\int_{B(R)} q\, d\mu + 2\pi a \sum_{n=1}^{N} \omega_n \leq 2\pi a,$$

and we can conclude as in the previous cases that M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ and that Assertion (B) holds.

3.1.3. *Step 3.* We now show that (M, g) has at most quadratic volume growth, and at most linear volume growth when $\chi(M) = 0$. We have already dealt with Case (i) in Step 2. We now consider Cases (ii) and (iii).

Case (ii). Here, $a = \frac{1}{4}$ and (M, g) has sub-exponential volume growth. We choose the function ξ to be $\xi_{R,Q}$ as given by Lemma 2.4,

$$\xi(t) = \begin{cases} e^{(1-\frac{t}{2R})^2}, & \text{for } \ t \in [0, R] \\ \beta(e^{-\alpha t} - e^{-\alpha Q}), & \text{for } \ t \in [R, Q], \end{cases}$$

with $0 < R < 5Q$ and α, β given by the lemma, so that ξ is admissible in $[0, Q]$ and vanishes at Q. We apply (c) again (making $q \equiv 0$ which is sufficient for our estimates). For this purpose, we compute,

$$(p2) \qquad (\xi')^2 - \xi\xi'' = \begin{cases} -\frac{1}{2R^2} e^{2(1-\frac{t}{2R})^2}, & \text{for } \ t \in [0, R] \\ \alpha^2\beta^2 e^{-\alpha Q} e^{-\alpha t}, & \text{for } \ t \in [R, Q], \end{cases}$$

and we obtain,

$$(q2) \qquad \begin{cases} \frac{1}{2R^2} \int_{B(R)} e^{2(1-\frac{r}{2R})^2}\, d\mu \leq \frac{\pi}{2}\{e^2 - \omega e^{2(1-\frac{t_1}{2R})^2}\} \\ \qquad\qquad + \alpha^2\beta^2 e^{-\alpha Q} \int_{C(R,Q)} e^{-\alpha r}\, d\mu, \end{cases}$$

where $\omega = 0$ if $M \sim \mathbb{C}$ and $\omega = 1$ if $M \sim \mathbb{C}^\bullet$, and we have chosen $R > t_1$. We fix $R > t_1$ and we let Q tend to infinity, using the fact that α and β remain controlled and that the second term in the right-hand side of (q2) goes to zero when Q tends to infinity because (M, g) has sub-exponential volume growth (Lemmas 2.4 and 3.1). Finally, we obtain,

$$R^{-2} V(R) \leq C\{1 - \omega e^{-\frac{t_1}{R}(2-\frac{t_1}{2R})}\} \leq C,$$

for some constant C independent of R. This gives at most quadratic volume growth in general, and at most linear volume growth when $\chi(M) = 0$.

Case (iii). Here, $a \in (0, \frac{1}{4})$ and (M, g) has polynomial volume growth of degree at most k, with $k < 2 + \frac{4a}{1-4a}$. We choose the function ξ to be $\xi_{R,Q}$ as given by Lemma 2.5,

$$\xi(t) = \begin{cases} (1+\frac{t}{R})^{-\beta}, & \text{for } \ t \in [0, R], \beta = \frac{a}{1-4a}, \\ \delta\{(1+\epsilon t)^{-\alpha} - (1+\epsilon Q)^{-\alpha}\}, & \text{for } \ t \in [R, Q], \alpha = \frac{2a}{1-4a}, \end{cases}$$

with $0 < R \ll Q$ and δ, ϵ given by the lemma, so that ξ is admissible in $[0, Q]$ and vanishes at Q. We apply (c) again (making $q \equiv 0$ which is sufficient for our estimates). For this purpose, we compute,

$$(p3)\ \ (1-2a)(\xi')^2 - 2a\xi\xi'' = \begin{cases} -\frac{a\beta}{R^2}(1+\frac{t}{R})^{-2\beta-2}, \quad \text{for} \quad t \in [0,R], \\ 2a\alpha(\alpha+1)\delta^2\epsilon^2(1+\epsilon Q)^{-\alpha})(1+\epsilon t)^{-\alpha-2}), \\ \quad \text{for} \quad t \in [R,Q], \end{cases}$$

and we obtain,

$$(q3) \qquad \begin{cases} \frac{a\beta}{R^2}\int_{B(R)}(1+\frac{r}{R})^{-2\beta-2}\,d\mu \le 2\pi a\{1-\omega(1+\frac{t_1}{R})^{-2\beta-2}\} \\ \qquad +2a\alpha(\alpha+1)\delta^2\epsilon^2(1+\epsilon Q)^{-\alpha})\int_{C(R,Q)}(1+\epsilon r)^{-\alpha-2})\,d\mu, \end{cases}$$

where we have chosen $R > t_1$. We fix $R > t_1$ and we let Q tend to infinity, using the fact that δ and ϵ remain controlled and that the second term in the right-hand side of (q3) goes to zero when Q tends to infinity because (M, g) has polynomial volume growth of order k (Lemmas 2.5 and 3.2). Finally, we obtain,

$$R^{-2}V(R) \le C\{1-\omega(1+\frac{t_1}{R})^{-2\beta-2}\} \le C,$$

for some constant C independent of R. This gives at most quadratic volume growth in general and at most linear volume growth when $\chi(M) = 0$.

3.1.4. *Conclusion.* In the three cases (i), (ii) and (iii), we have proved,

- M is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^\bullet$ (Step 2),
- (M, g) has at most quadratic volume growth, and hence ([4], Proposition 2.3) that (M, g) is conformally $\mathbb{C}$ or $\mathbb{C}^\bullet$ (Step 3),
- q is integrable and $\int_M q\,d\mu \le 2\pi a\chi(M)$; in particular, $q \equiv 0$ when $\chi(M) = 0$ (Step 2),
- (M, g) has at most linear volume growth when $\chi(M) = 0$.

The proof of Theorem 1.1 is therefore complete. □

3.2. **The assumptions / conclusions in Theorem 1.1 are optimal.**
• The example of the hyperbolic plane shows that Assertion (A) in Theorem 1.1 cannot hold when $a = \frac{1}{4}$ without an extra assumption on (M, g).

• The examples of the hyperbolic planes of curvature $-c^2, c > 0$, show that the assumptions in Case (ii) are optimal.

• Consider the unit disk $\mathbb{D}$ with the conformal metric $h_\alpha = \big(\frac{2}{1-|z|^2}\big)^{2\alpha}|dz|^2$ for $\alpha \ge 1$. The metric h_1 is the hyperbolic metric with constant curvature -1. When $\alpha > 1$, the metric h_α is a complete conformal metric on $\mathbb{D}$, with negative curvature. A simple computation shows that it has polynomial volume growth of degree $2 + \frac{1}{\alpha-1}$.
Given a complete Riemannian surface (M, g), let $a_+(M, g)$ denote the supremum of the numbers a such that $\Delta + aK \ge 0$. It is proved in [4] that this supremum is achieved (Proposition 1.1) and that $a(\alpha) := a_+(\mathbb{D}, h_\alpha)$ is equal to $\frac{1}{4\alpha}$ (Proposition 4.3). It follows that the volume growth of $(\mathbb{D}, h_\alpha)$ is polynomial with degree equal to $2 + \frac{4a(\alpha)}{1-4a(\alpha)}$. This shows that the assumptions in Case (iii) are optimal.

4. Generalization to finite index

Theorem 4.1. *Let (M,g) be a complete non-compact Riemannian surface. Let q be a locally integrable, non-negative function on M. Assume that the operator $\Delta + aK - q$ has finite index on M. Assume furthermore that either,*

(i) *$a > \frac{1}{4}$, or*
(ii) *$a = \frac{1}{4}$ and (M,g) has sub-exponential volume growth, or*
(iii) *$a \in (0, \frac{1}{4})$, and (M,g) has polynomial volume growth of degree at most k, with $k < 2 + \frac{4a}{1-4a}$.*

Then, (M,g) is conformally equivalent to a closed Riemannian surface with finitely many points removed and the function q is integrable on (M,g).

Proof. We first recall what "finite index" means. Let Ω be a smooth, relatively open compact domain in M. By elliptic theory, the operator $\Delta + aK - q$ has finitely many negative eigenvalues in Ω (with Dirichlet boundary conditions on $\partial\Omega$). Call $\mathrm{Ind}(\Omega)$ the number of these negative eigenvalues. This is an increasing function of Ω with respect to inclusion. If $\sup\{\mathrm{Ind}(\Omega) \mid \Omega \Subset M\}$ is finite, then we say that the operator $\Delta + aK - q$ has finite index in M.

It is a well-known fact that if $\Delta + aK - q$ has finite index on M, then it is stable outside a compact set, *i.e.* that

$$0 \leq \int_M \left(|df|^2 + aKf^2 - qf^2\right) d\mu, \tag{a}$$

for any Lipschitz function f with compact support in $M \setminus \overline{B}(R_0)$, for some positive radius R_0 (see [6] for a proof and for a converse statement). Choose

$$0 < R_0 < R_1 := R_0 + 1 < R_2 := R_0 + 2 < R < Q, \tag{b}$$

and a function ξ which is admissible in the interval $[R_2, Q]$ and vanishes at Q. Define the function η as follows,

$$\eta(t) = \begin{cases} 0, & \text{for } \ t \in [0, R_1], \\ \xi(R_2)(t - R_1), & \text{for } \ t \in [R_1, R_2], \\ \xi(t), & \text{for } \ t \in [R_2, Q]. \end{cases} \tag{c}$$

We shall choose ξ later on, depending on the assumptions we are working with, Cases (i), (ii) or (iii), as in the proof of Theorem 1.1. Consider the function $\eta(r)$. This is a Lipschitz function with compact support in $M \setminus B(R_0)$ and we can therefore plug it into (a). Using (c), we find that,

$$\begin{aligned} 0 \leq \textstyle\int_{C(R_2,Q)} q\xi^2(r)\, d\mu \leq \textstyle\int_{C(R_2,Q)} \left\{(\xi')^2(r) + aK\xi^2(r)\right\} d\mu \\ + C_1\xi^2(R_2), \end{aligned} \tag{d}$$

for some constant C_1 which depends only on a, the geometry of (M,g) and the values of q in the ball $\bar{B}(R_2)$.

We now apply Lemma 2.3 in $[R_2, Q]$ and we define the indices N_2 and $N(Q)$ by

$$t_{N_2} \leq R_2 < t_{N_2+1} \ \text{ and } \ t_{N(Q)} < Q \leq t_{N(Q)+1},$$

so that we can plug (11) into (d) and we obtain the inequality,

$$(e)\qquad \begin{cases} \int_{C(R_2,Q)} q\xi^2(r)\,d\mu + 2\pi a \sum_{n=N_2+1}^{N(Q)} \omega_n \xi^2(t_n) \leq \\ \qquad \int_{C(R_2,Q)} \{(1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r)\,d\mu\} \\ \qquad\qquad +C_2\Big(\xi^2(R_2) + (\xi')^2(R_2)\Big), \end{cases}$$

for some positive constant C_2 which only depends on a, on the geometry of (M,g) and the values of q in $\bar{B}(R_2)$.

We can now proceed from inequality (e) as we proceeded from inequality (b) in the proof of Theorem 1.1. We leave the details to the reader. □

5. Proofs of Corollary 1.2 and Theorem 1.3

Proof of Corollary 1.2. The stability operator of the immersion is $\Delta + K - q$, with $q := \widehat{S} + \frac{1}{2}|A|^2 \geq 0$, where A is the second fundamental form. By assumption this operator is non-negative on M and we can therefore apply Theorem 1.1, Case (i) and conclude that $\int_M (\widehat{S} + \frac{1}{2}|A|^2)\,d\mu \leq 2\pi\chi(M) = 0$. It follows that $A \equiv 0$ and that $\widehat{S} \equiv 0$ on M, as stated. □

Proof of Theorem 1.3. By Theorem 1.1, we already know that (M,g) has at most linear volume growth, so that Theorem 1.3 follows from the following proposition.

Proposition 5.1. *Let (M,g) be a complete cylinder with at most polynomial volume growth of degree $k < 2$. If, for some $a > 0$, the operator $\Delta + aK$ is non-negative on M, then the Gaussian curvature K is non-negative.*

Indeed, assuming the proposition, we can apply Cohn-Vossen's theorem and conclude that $0 \leq \int_M K\,d\mu \leq 2\pi\chi(M)$, so that indeed, $K \equiv 0$.

Proof. For some $a > 0$, we have

$$(a)\qquad 0 \leq \int_M \Big(|df|^2 + aKf^2\Big)\,d\mu, \qquad \forall f \in \mathrm{Lip}_0(M).$$

Choose some $x \in M$. We want to prove that $\kappa := K(x) \geq 0$. We fix this point x and we take the distance function and the geodesic balls with respect to this point.

• According to the assumption that M is a cylinder, the function $\widehat{\chi}(t)$ (considering balls centered at x) has at least one discontinuity at t_1 (which depends on the choice of x), with $\omega_1 \geq 1$.

• Let $0 < \alpha < 1$ and $0 < R < t_1 < Q$. Define the function ξ (with parameters α, R, Q) to be

$$(b)\qquad \xi(t) = \begin{cases} 1 - \alpha\frac{t}{R} & \text{for } \ t \in [0,R], \\ (1-\alpha)\frac{Q-t}{Q-R} & \text{for } \ t \in [R,Q]. \end{cases}$$

• Use the function $\xi(r)$ to test the positivity condition (a). Straightforward computations give

$$(c) \qquad \int_{B(Q)} (\xi')^2(r)\, d\mu = \frac{\alpha^2}{R^2} V(R) + \Big(\frac{1-\alpha}{Q-R}\Big)^2 \Big(V(Q) - V(R)\Big).$$

Applying Lemma 2.3 to the ball $B(R)$ and to the set $C(R,Q)$, another computation yields

$$\begin{cases} \int_{B(Q)} K\xi^2(r)\, d\mu \leq & -\frac{2\alpha^2}{R^2} V(R) + 2(1-\alpha)\frac{R-\alpha Q}{R(Q-R)} L(R) \\ & +2\pi\Big(1 - \omega_1(1-\alpha)^2(\frac{Q-t_1}{Q-R})^2\Big) \\ & -2(\frac{1-\alpha}{Q-R})^2\Big(V(Q) - V(R)\Big), \end{cases}$$

and hence, since $\omega_1 \geq 1$,

$$(d) \qquad \begin{cases} \int_{B(Q)} K\xi^2(r)\, d\mu \leq & -\frac{2\alpha^2}{R^2} V(R) + 2(1-\alpha)\frac{R-\alpha Q}{R(Q-R)} L(R) \\ & +2\pi\Big(1 - (1-\alpha)^2(\frac{Q-t_1}{Q-R})^2\Big) \\ & -2(\frac{1-\alpha}{Q-R})^2\Big(V(Q) - V(R)\Big). \end{cases}$$

Finally, we obtain that for $a > 0$ and the above choice (b) of ξ,

$$(e) \qquad \begin{cases} 0 \leq \int_{B(Q)} \Big((\xi')^2(r) + aK\xi^2(r)\Big)\, d\mu \leq (1-2a)\alpha^2 \frac{V(R)}{R^2} \\ \qquad +(1-2a)(\frac{1-\alpha}{Q-R})^2\Big(V(Q) - V(R)\Big) + 2a(1-\alpha)\frac{R-\alpha Q}{Q-R}\frac{L(R)}{R} \\ \qquad +2\pi a\Big(1 - (1-\alpha)^2(\frac{Q-t_1}{Q-R})^2\Big). \end{cases}$$

• Inequality (e) holds for all choices of $\alpha \in (0,1)$ and $0 < R < t_1 < Q$. Letting Q tend to infinity, we first find that

$$(f) \qquad \begin{cases} 0 \leq \int_{B(Q)} \Big((\xi')^2(r) + aK\xi^2(r)\Big)\, d\mu \leq \\ \qquad (1-2a)\alpha^2 \frac{V(R)}{R^2} - 2a\alpha(1-\alpha)\frac{L(R)}{R} \\ \qquad +2\pi a\alpha(2-\alpha), \end{cases}$$

for all $\alpha \in (0,1)$ and $R \in (0, t_1)$ (where we have used that the area growth is less than quadratic).

• We now use the classical expansions for the volume of small geodesic circles centered at the point x,

$$(g) \qquad \begin{cases} L(r) = 2\pi r\Big(1 - \frac{\kappa}{6} r^2 + r^2\epsilon_1(r)\Big), \\ V(r) = \pi r^2\Big(1 - \frac{\kappa}{12} r^2 + r^2\epsilon_2(r)\Big), \end{cases}$$

where κ is the value of the curvature at x.

• Finally, we find that

$$(h) \qquad 0 \leq \pi\alpha^2 + \frac{\pi\kappa R^2}{12}\alpha\Big(8a - (1+6a)\alpha\Big) + \alpha R^2\epsilon(R).$$

Dividing by α and letting α tend to zero gives and asymptotic expansion whose first term is non-negative, and hence that $\kappa := K(x) \geq 0$. □

6. Proof of Theorem 1.4

Let us first consider the case of the sphere with constant curvature α^2, $M_0 = S^2(\alpha^2)$. In the sequel, the subscript 0 refers to M_0. Let $J_0 = \Delta_0 + a\alpha^2 - c$. The operator J_0 is non-negative in the ball $B_0(R)$ if and only if the first Dirichlet eigenvalue of the Laplacian Δ_0 in this ball satisfies $\lambda_1(B_0(R)) \geq c - a\alpha^2$. Since $c \geq (a+2)\alpha^2$, it follows that J_0 non-negative in the ball $B_0(R)$ implies that $\lambda_1(B_0(R)) \geq \lambda_1(B_0(\frac{\pi}{2\alpha}))$, and hence that $R \leq \frac{\pi}{2\alpha}$, because $\lambda_1(B_0(R))$ is a decreasing function of R. If $J_0 \geq 0$ in $B(\frac{\pi}{2\alpha})$, then $c = (a+2)\alpha^2$, since all previous inequalities become equalities. Recall that the first Dirichlet eigenfunction for the Laplacian Δ_0 in the hemisphere $B(\frac{\pi}{2\alpha})$ is $\cos(\alpha r_0)$, up to a scaling factor, where r_0 is the distance function to a point on the sphere.

Proof of Theorem 1.4. Recall that this theorem is of a local nature. We first state a lemma.

Lemma 6.1. *Let (M, g) be a Riemannian surface. Assume that the curvature satisfies $K \leq \alpha^2$ for some $\alpha > 0$. Let J be the operator $J = \Delta + aK - c$, with $a \in [\frac{1}{2}, \infty)$ and $c \geq (a+2)\alpha^2$. Assume furthermore that the ball $B(x, \frac{\pi}{2\alpha})$ is contained in M, for some $x \in M$. Then the least eigenvalue of the operator J with Dirichlet boundary conditions in this ball is non-positive. If J has least Dirichlet eigenvalue 0 in the ball $B(x, \frac{\pi}{2\alpha})$, then $c = (a+2)\alpha^2$, $K \equiv \alpha^2$ and $B(x, \frac{\pi}{2\alpha})$ is covered by the hemisphere $S^2_+(\alpha^2)$ in the sphere with constant curvature α^2.*

Clearly, the lemma implies the theorem. Indeed, Assertion (i) follows from the lemma and from the monotonicity of eigenvalues with respect to domain inclusion. Assertion (ii) follows immediately.

Proof of the lemma. First observe that we can assume that $a \in [\frac{1}{2}, 2]$. Indeed, if $a > 2$, then for any $a' \in [\frac{1}{2}, 2]$, we can write

$$\Delta + aK - c = \Delta + a'K + (a - a')K - c \leq \Delta + a'K - c',$$

where $c' = c + (a' - a)\alpha^2 \geq (a' + 2)\alpha^2$. Moreover, if $c' = (a' + 2)\alpha^2$, then $c = (a + 2)\alpha^2$.

Let $A := \frac{\pi}{2\alpha}$. Because $K \leq \alpha^2$, the map $\exp_x : T_xM \to M$ is a local diffeomorphism on the ball $D(0, A)$. Let $\tilde{g} = \exp_x^* g$ be the pulled-back metric to T_xM. Let μ_1 be the least Dirichlet eigenvalue of $\Delta + aK - c$ in $B(x, A)$. Then, $\Delta + aK - c - \mu_1 \geq 0$ in $B(x, A)$ and hence, there exists a positive function $u : B(x, A) \to \mathbb{R}$ such that $(\Delta + aK - c - \lambda_1)u = 0$ (see [10]). Let $\tilde{u} = u \circ \exp_x$. Because $\exp_x$ is a local isometry, we have $(\tilde{\Delta} + a\tilde{K} - c - \mu_1)\tilde{u} = 0$ and hence the least Dirichlet eigenvalue of $\tilde{\Delta} + a\tilde{K} - c$ is at least μ_1. To show that μ_1 is non-positive, it suffice to show that the least Dirichlet eigenvalue of $\tilde{\Delta} + a\tilde{K} - c$ in $D(0, A)$ is non-positive. We have reduced to the simply-connected case.

We now work in $D(0, A)$ with a metric (also denoted) g such that $K \leq \alpha^2$. We denote by $L(r)$ the length of $\partial D(0, r)$ for this metric and we let $L_0(r)$

be the corresponding length on the sphere, $L_0(r) = 2\pi\frac{\sin(\alpha r)}{\alpha}$. By Bishop's comparison theorem, we have that

$$(a) \qquad L(r) \geq \hat{L}(r).$$

We now use Pogorelov's trick. Let $\xi : [0, A] \to \mathbb{R}$ be a C^2 function such that $\xi(0) = 1$ and $\xi(A) = 0$. We compute the quadratic form Q associated with $J = \Delta + aK - c$ on the function $\xi(r)$, where r is the geodesic distance to 0 in $D(0, A)$. We also introduce the total curvature of $D(0, r)$,

$$(b) \qquad G(r) = \int_{D(r)} K\, d\mu,$$

with respect to the Riemannian measure in $D(0, A)$. Applying the co-area formula, d,

$$\begin{aligned} Q(\xi(r)) &= \textstyle\int_{D(0,A)} \left(|d\xi(r)|^2 + (aK - c)\xi^2(r)\right) d\mu \\ &= \textstyle\int_0^A \left((\xi')^2 - c\xi^2\right) L\, dt + a \int_0^A G'\xi^2\, dt, \end{aligned}$$

and we can compute the second integral in the right-hand side by integration by parts,

$$(c) \qquad Q(\xi(r)) = \int_0^A \Big((\xi')^2 - c\xi^2\Big) L\, dt - a \int_0^A G(\xi^2)'\, dt.$$

By the Gauss-Bonnet formula, we have $G(t) = 2\pi - L'(t)$ so that (c) becomes, after another integration by parts,

$$Q(\xi(r)) = \int_0^A \Big((\xi')^2 - c\xi^2 - a(\xi^2)''\Big) L\, dt + 2\pi a.$$

Finally, we obtain

$$(d) \qquad Q(\xi(r)) = \int_0^A \Big((1 - 2a)(\xi')^2 - 2a\xi\xi'' - c\xi^2\Big) L\, dt + 2\pi a,$$

for any function $\xi : [0, A] \to \mathbb{R}$ which is C^2 and such that $\xi(0) = 1$ and $\xi(A) = 0$.

We now use the test function $\eta(r) = \cos(\alpha r)$ in the quadratic form (d), where r is the Riemannian distance to the center of the ball, *i.e.* we transplant the first eigenfunction of the hemisphere to a function on the ball $D(0, A)$.

$$Q(\eta(r)) = \int_0^A \Big((1 - 2a)\alpha^2 \sin^2(\alpha t) + (2a\alpha^2 - c)\cos^2(\alpha t)\Big) L(t)\, dt + 2\pi a,$$

and hence

$$(e) \quad Q(\eta(r)) \leq \int_0^A \Big((1 - 2a)\alpha^2 \sin^2(\alpha t) + (a - 2)\alpha^2 \cos^2(\alpha t)\Big) L(t)\, dt + 2\pi a,$$

where we have used the fact that $c \geq (2 + a)\alpha^2$. Because of our assumptions on a, using (a), we find that

$$(f) \qquad \begin{cases} Q(\eta(r)) \leq 2\pi a \\ \qquad + \int_0^A \left((1 - 2a)\alpha^2 \sin^2(\alpha t) + (a - 2)\alpha^2 \cos^2(\alpha t)\right) L_0(t)\, dt. \end{cases}$$

The right-hand side of (f) is zero because this is the value of the quadratic form of the operator $J_0 = \Delta - 2\alpha^2$ on the hemisphere $S^2_+(\alpha^2) = B_0(\frac{\pi}{2\alpha})$. We conclude that $Q(\eta(r)) \leq 0$ and hence that the least Dirichlet eigenvalue

of $\Delta + aK - c$ in $D(0, A)$ is non-positive, as stated in the lemma. If this eigenvalue is zero, then $Q(\eta(r)) = 0$, and we must have equality in both (e) and (f), *i.e.* $c = (2 + a)\alpha^2$ and $L(t) \equiv 2\pi\frac{\sin(\alpha t)}{\alpha}$. We then deduce that $G(t) \equiv 2\pi(1 - \cos(\alpha t))$. Since $K \leq \alpha^2$, integrating K we find that $K \equiv \alpha^2$ and hence we conclude that $D(0, A) = S^2_+(\alpha^2)$. This proves the lemma. □

From Theorem 1.4, we obtain the following corollary.

Corollary 6.2. *Let $(M, g) \looparrowright (\widehat{M}, \hat{g})$ be an isometric immersion with constant mean curvature H in a simply connected space form with constant sectional curvature κ. Assume furthermore that $H^2 + \kappa > 0$ and that the immersion is (strongly) stable. Then,*

$$d_g(x, \partial M) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}},$$

where $d_g(x, \partial M)$ is the distance from $x \in M$ with respect to the metric g to the boundary of M, with equality if and only if M is the hemisphere of a sphere of mean curvature H in $\widehat{M}$.

Proof. The Jacobi operator of the immersion is $J = \Delta - |A|^2 - \widehat{\mathrm{Ric}}(n)$, where A is the second fundamental form of the immersion and n the unit normal along the immersion. By the Gauss equation, we find that $J = \Delta + 2K - 4(H^2 + \kappa)$ and that $K = H^2 + \kappa - \frac{1}{4}(k_1 - k_2)^2$, where k_i are the principal curvatures. We can apply Theorem 1.4 with $a = 2$ and $\alpha^2 = H^2 + \kappa$. For the equality case, note that equality implies that M is totally umbilic. □

Remarks.

(1) This corollary provides a unified proof of Theorem 3.1 and Corollary 3.2 in Mazet's paper [16], without using Lawson's correspondence.
(2) The proof of Theorem 1.4 is simpler than that of Theorem 3.1 in [16], but it uses the same idea which goes back to A. Pogorelov [19].

7. Proof of Theorem 1.5

Let (M, g) be a complete Riemannian manifold. We define the set

$$I(M, g) = \{a \in \mathbb{R} \mid \Delta + aK \geq 0\}.$$

This set is a closed interval which contains 0 (see [4]). It is easy to see that $I(M, g) \subset I(\widehat{M}, \hat{g})$ for any Riemannian covering $p : (\widehat{M}, \hat{g}) \to (M, g)$. Indeed, if $a \in I(M, g)$, then there exists a positive function u on M which satisfies $(\Delta + aK)u = 0$ ([10], Theorem 1). Clearly, with obvious notations, the function $\hat{u} = u \circ p$ satisfies $(\widehat{\Delta} + a\widehat{K})\hat{u} = 0$ on $\widehat{M}$ and hence $a \in I(\widehat{M}, \hat{g})$. It is a natural question to ask when the reverse inclusion is true. Proposition 2.5 in [18], tells us that this is the case when the inverse image $p^{-1}(\Omega)$ of any relatively compact subset Ω in M has subexponential volume growth in $\widehat{M}$. Theorem 1.5 tells us that if $(\widehat{M}, \hat{g}) \to (M, g)$ is a normal covering with *amenable* covering group G, then $I(M, g) = I(\widehat{M}, \hat{g})$.

In the first sub-section below, we give some basic facts on amenable groups. We also state a corollary of Theorem 1.5. We give the proof of the theorem

in the next sub-section. Theorem 1.5 is very much inspired by the work of R. Brooks, [1, 2].

7.1. **Amenable groups.** For this section, see [12], Sections 1 and 3.6. A discrete group G is amenable if there exists an invariant mean on G *i.e.* a bounded linear functional $\mu : L^\infty(G) \to \mathbb{R}$ which satisfies

$$\begin{cases} \mu(1) = 1, \\ \mu(f) \geq 0 \ \text{ if } \ f \geq 0, \\ \mu(\gamma \cdot f) = \mu(f) \ \text{ for all } \ \gamma \in G, \ \text{ where } \ (\gamma \cdot f)(x) := f(\gamma^{-1} x). \end{cases}$$

A finite group G is amenable. If suffices to take $\mu(f) = \frac{1}{|G|} \sum_{\gamma \in G} f(\gamma)$, where $|G|$ is the cardinal of G. The group $\mathbb{Z}$ is amenable. To see this, consider the means $\mu_n, n \in \mathbb{N}^\bullet$ defined by $\mu_n(f) = \frac{1}{2n+1} \sum_{k=-n}^{n} f(k)$ and take their weak-$*$ limit. More generally $\mathbb{Z}^n$ is amenable for all n. The basic example of a non-amenable group is the free group on two generators $\mathbb{F}_2$, and it can be shown that any group containing a subgroup isomorphic to $\mathbb{F}_2$ is not amenable.

There are several criteria for proving that a group is amenable. We will use Følner's criterion, [12], Section 3.6.

Theorem [Følner] *A discrete group G is amenable if and only if, for any $\eta \in (0, 1)$ and for any finite set $A \subset G$, there exists a finite set $B \subset G$ such that*

$$\eta \, \sharp(B) \leq \sharp(B \cap \alpha \cdot B) \ \ \forall \alpha \in A,$$

where $\sharp(B)$ is the number of elements of the set B.

As a consequence of this criterion, on can show ([1], Proposition 1) that groups with sub-exponential growth are amenable. This fact relates our Theorem 1.5 to Proposition 2.5 in [18].

Let G be a finitely generated group and let $A = \{\alpha_1, \cdots, \alpha_n\}$ be a system of generators for G. Given a finite set $E \subset G$, we define the boundary ∂E of E – interpreted as a subset of the Cayley graph of (G, A) – to be,

$$\partial E = \{\gamma \in E \mid \alpha_i \cdot \gamma \notin E \ \text{ for some } \ i\}.$$

Another consequence of Følner's theorem is the following corollary which states that there is no linear isoperimetric inequality on the Cayley graph of a finitely generated amenable group.

Proposition 7.1. *Let G be a finitely generated group. If G is amenable then, for any $\epsilon > 0$, there exists a finite set $E \subset G$ such that $\sharp(\partial E) \leq \epsilon \, \sharp(E)$.*

Proof. Let E be a finite subset in G. Then

$$\partial E = \cup_{i=1}^{n} \{\gamma \in E \mid \alpha_i \cdot \gamma \notin E\}.$$

Then,

$$\begin{aligned} \sharp(\partial E) &\leq \textstyle\sum_{i=1}^{n} \sharp(\{\gamma \in E \mid \alpha_i \cdot \gamma \notin E\}) \\ &\leq \textstyle\sum_{i=1}^{n} \Big(\sharp(E) - \sharp(\{\gamma \in E \mid \alpha_i \cdot \gamma \in E\})\Big) \\ &\leq \textstyle\sum_{i=1}^{n} \Big(\sharp(E) - \sharp(E \cap \alpha_i^{-1} \cdot E)\Big). \end{aligned}$$

Choose $\epsilon > 0$ and $\eta \in (0,1)$ such that $n(1-\eta) \leq \epsilon$ and apply Følner's theorem: there exists a finite set $E \subset G$ such that $\eta\, \sharp(E) \leq \sharp(E \cap \alpha_i^{-1} \cdot E)$ for $i = 1, \cdots, n$. Using the above inequalities, we obtain for this subset,

$$\sharp(\partial E) \leq \sum_{i=1}^{n} \big(\sharp(E) - \eta \sharp(E)\big) = n(1-\eta)\sharp(E) \leq \epsilon \sharp(E).$$

This proves the proposition. □

Proposition 7.1 has the following geometric interpretation. Let $(\widehat{M}, \hat{g}) \to (M, g)$ be a normal Riemannian covering with group $G = \pi_1(M)/\pi_1(\widehat{M})$. Assume that G is a finitely generated amenable group. Let F be a fundamental domain for the action of G on $\widehat{M}$ (choose for example a Dirichlet fundamental domain). Let $\beta_1, \cdots, \beta_m$ be the elements of G such that $\beta_i \cdot F$ intersects F (along ∂F). Then the elements $\beta_1^{-1}, \cdots, \beta_m^{-1}$ form a system of generators of G. Let $E \subset G$ be a finite subset, and define the set $\Omega = \cup_{\gamma \in E} \gamma \cdot F \subset \widehat{M}$. Then,

$$\partial E = \{\gamma \in E \mid \partial(\gamma \cdot F) \cap \partial\Omega \neq \emptyset\}, \quad E \setminus \partial E = \{\gamma \in E \mid \gamma \cdot F \subset \mathrm{int}(\Omega)\}.$$

The set Ω is a union of fundamental domains and the proposition tells us that most of them are in the interior of Ω.

To prove Theorem 1.5, we want to test the quadratic form $\int_M |df|^2 + Vf^2$ on compactly supported functions. Take $f \in C_0^1(M)$ and lift it to a function $\hat{f}$ on $\widehat{M}$, which is not compactly supported, but behaves like f on fundamental domains. We can multiply the function $\hat{f}$ by a cut-off function which is 1 in the interior of Ω and the proposition tells us that the effect of the cut-off function will be negligible. These ideas are developed in the next section.

7.2. **Proof of Theorem 1.5.** In view of the introduction to this section, we only need to prove that $\Delta_{\hat{g}} + \widehat{V} \geq 0$ implies that $\Delta_g + V \geq 0$. Assume that $\Delta_{\hat{g}} + \widehat{V} \geq 0$ and let $f \in C_0^1(M)$. We want to prove that $0 \leq \int_M |df|^2 + Vf^2$.

• Fix some $\epsilon > 0$. Let $F \subset \widehat{M}$ be a fundamental domain for the action of the covering group G, and let $\beta_1, \cdots, \beta_n$ be the elements of G such that $\beta_i \cdot F \cap F \neq \emptyset$. As G is amenable, there exists a finite set $E \subset G$ such that $\sharp(\partial E) \leq \epsilon\, \sharp(E)$.

• Lift $\mathrm{supp}(f)$ to F. As it is compact, there exists some $\alpha > 0$ such that the α-neighborhood of $\partial\big(F \bigcup \cup_{i=1}^n \beta_i \cdot F_i\big)$ does not intersect le lift of $\mathrm{supp}(f)$. Let $\Omega = \cup_{\gamma \in E} \gamma \cdot F$ and consider the cut-off function $\xi : \widehat{M} \to \mathbb{R}$ defined by

$$\xi(x) = \begin{cases} 0 \ \text{ if } \ x \notin \Omega, \\ \frac{1}{\alpha} d(x, \partial\Omega) \ \text{ if } \ d(x, \partial\Omega) < \alpha, \\ 1 \ \text{ if } \ d(x, \partial\Omega) \geq \alpha. \end{cases}$$

• Still call f the lift of f to $\widehat{M}$ and consider the test function ξf on $\widehat{M}$. From the assumption $\Delta_{\hat{g}} + \widehat{V} \geq 0$, we have that

$$\int_{\widehat{M}} |d(\xi f)|^2 + \widehat{V} f^2 \xi^2 \geq 0.$$

• Let $c := \sharp(E)$ and $b := \sharp(\partial E)$. Then, $b \leq \epsilon\, c$. On the other-hand, we have the following estimates.

$$(a) \qquad |d(f\xi)|^2 \leq f^2|d\xi|^2 + 2|f\xi||df||d\xi| + \xi^2|df|^2,$$

with $|\xi| \leq 1$ on Ω and $|d\xi| \leq \frac{1}{\alpha}$ on $\gamma \cdot F$ for $\gamma \in \partial E$ and 0 otherwise. It follows that,

$$(b) \qquad \int_{\widehat{M}} |d(f\xi)|^2 \leq \frac{b}{\alpha^2}\int_M f^2 + \frac{2b}{\alpha}\Big(\int_M f^2\Big)^{1/2}\Big(\int_M |df|^2\Big)^{1/2} + c\int_M |df|^2.$$

Consider the positive and negative parts of $\widehat{V}$, $\widehat{V} = \widehat{V}_+ - \widehat{V}_-$. Then,

$$\begin{array}{rcl} \int_{\widehat{M}} \widehat{V} f^2\xi^2 & = & \int_{\widehat{M}} \widehat{V}_+ f^2\xi^2 - \int_{\widehat{M}} \widehat{V}_- f^2\xi^2 \\ (c) & \leq & c\int_M V_+ f^2 - (c-b)\int_M V_- f^2 \\ & \leq & c\int_M V f^2 + b\int_M V_- f^2. \end{array}$$

• The inequalities (b)–(c) yield,

$$0 \leq \int_M |df|^2 + Vf^2 + \frac{b}{c}\Big[\frac{1}{\alpha^2}\int_M f^2 + \frac{2}{\alpha}\Big(\int_M |df|^2 \int_M f^2\Big)^{1/2} + \int_M V_- f^2\Big].$$

Since $0 \leq b/c \leq \epsilon$ and letting ϵ tend to zero, we find that $0 \leq \int_M |df|^2 + Vf^2$. □

As a corollary, we have the following result.

Corollary 7.2. *If $M = \mathbb{R}^2$ and g is a $\mathbb{Z}^2$-invariant metric, or if $M = S^1 \times \mathbb{R}$ and g is a $\mathbb{Z}$-invariant metric, then*

- *$I(M,g) = \{0\}$ or $I(M,g) = \mathbb{R}$.*
- *Furthermore, $I(M,g) = \mathbb{R}$ if and only if $K \equiv 0$.*

Proof. Since the groups $\mathbb{Z}$ and $\mathbb{Z}^2$ are amenable [12], it suffices to look at the quotient, *i.e.* at the torus T^2. Assume that $\Delta + aK \geq 0$ on T^2, for some $a \neq 0$. As in [10], taking u to be the constant function 1, we find that $\int_{T^2} |du|^2 + aKu^2 = 0$ because $\int_{T^2} K = 0$. Since $\Delta + aK \geq 0$, the function u realizes the minimum of the Rayleigh quotient, so that it is an eigenfunction associated with the eigenvalue 0 and we have $(\Delta + aK)u = 0$ and hence $K \equiv 0$ (because $a \neq 0$). Hence, if $I(g) \neq \{0\}$ then $K \equiv 0$ and $I(g) = \mathbb{R}$. If $K \not\equiv 0$, then $I(g) = \{0\}$. □

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