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**On the number of deviations of Geometric Brownian Motion
with drift from its extreme points with applications to transaction costs**

Thomas Poufinas*
University of the Aegean
Department of Statistics and Actuarial-Financial Mathematics

ABSTRACT. The number of deviations of a Geometric Brownian Motion with drifts from its extreme points is considered. The properties of these deviations are studied. As an application based on these results, the time instants at which investors decide to buy or sell are examined, when the price of an asset is assumed to follow a Geometric Brownian Motion. Extensions to the modelling of transactions costs are attempted.

Keywords: Geometric Brownian Motion, deviation, extreme points, stopping times, transaction costs

* Thomas Poufinas, Address: PO Box 76109, Athens 17110, GREECE, Phone Number: 0030-697-250-2454, E-mail: th.poufinas@aegean.gr

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0 Introduction

Our source of inspiration is the movement of the price of a traded risky asset (e.g. a stock) on which an investor, with finite time horizon T , has invested. More specifically we focus at the change in price that leads the investor to transact. A reasonable approach would be to assume that investors choose to buy or sell units of the asset (e.g. shares of stock) when its price deviates by its previous maximum or minimum by a certain amount or percentage.

When such a transaction takes place, transaction costs occur. Thus, modelling these transaction costs is also of great importance. By doing that we know exactly when transaction costs have to be paid and moreover how much has to be paid. Several authors have attempted to model transaction costs, since the Black-Scholes approach, among which Amster [1] and Daamgard [2]. No one has though tried to link them to the time instants at which they occur.

The time instants at which transaction costs are paid are the points of time at which the investor decides to adjust the portfolio as a result of a significant change in the price of the risky asset. The change in price is monitored with respect to the maximum and minimum price achieved since the last transaction took place. When the price deviates by a preset quantity or proportion, then the investor adjusts/ rebalances his or her portfolio by performing a transaction on the (specific risky) asset.

The properties of the time instants where the transaction costs are paid are studied. More precisely we prove that in a time interval $[0, T]$ there are finitely many such time instants almost surely and they are stopping times.

These results though come as consequences of the properties of Geometric Brownian Motions and this is where we turn our attention. What we can essentially achieve is to follow in a time interval $[0, T]$ the change in the value of a Geometric Brownian Motion since its previous extreme point was reached. What we can show is that the number of such deviations by a fixed range, either numerical or proportional, is finite almost surely. In addition we can prove that the time instants on which such deviations occur are stopping times.

To the best of the author's knowledge there has not been a similar approach, both to the study of the properties of deviations of a Geometric Brownian Motion

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from its extreme points as well as the application of these properties to the fluctuations of the prices of risky traded assets and the modelling of transaction costs. The application of such models could assist further in the pricing of options in a continuous-time environment.

In **Section 1** we describe the environment in which we derive our results. In **Section 2** we derive our main results on the number of deviations of a Geometric Brownian Motion from its extreme points and the time instants at which these deviations occur. Next to it the properties of the time instants at which the price of an asset deviates from its extreme points by a given amount or percentage are established. In **Section 3** we deal with the implications to the case of Brownian Motion. In **Section 4** the cost functions are modelled. In **Section 5** we describe some further steps and extensions that could be followed towards the study of the deviations of a Geometric Brownian Motion from its extreme points as well as its applications to the price of a risky asset. In **Section 6** we conclude on the findings of the main body of the paper.

1 Description of the mathematical setting

In the following we consider continuously trading markets with transaction costs (commission fees). Our mathematical setting comprises of a *filtered complete probability space* $(\Omega, \mathcal{F}, Pr, (\mathcal{F}_t)_{0 \leq t \leq \infty})$. We assume that all the stochastic processes that appear are *adapted* (see Protter [9], pp3-4). Also, all the functions and stochastic processes are continuous, unless otherwise stated. (In)equalities that involve random variables are understood to hold almost surely. We assume that we have a Geometric Brownian Motion with a drift - that could be the price of a risky asset (see Merton [7], pp122-124 or Duffie [3], pp80-84) - and whose stochastic differential satisfies

$$\frac{dP}{P} = \alpha(P, t) dt + \sigma(P, t) dW. \quad (1)$$

When P denotes the price of a risky asset, α is the instantaneous conditional expected change in price per unit time, σ^2 is the instantaneous conditional variance per unit time and W is a Brownian Motion. For the rest of our discussion we assume that α, σ do not depend on P, t .

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In the case that $\alpha(P, t) = \alpha(t)$ and $\sigma(P, t) = \sigma(t)$, i.e. they are functions of time only and not the risky asset, then the solution of (1) becomes

$$P_t = P_0 e^{\gamma(t)}, \quad (2)$$

where

$$\gamma(t) = \int_0^t \mu(s) ds + \int_0^t \sigma(s) dW_s \quad (3)$$

and

$$\mu(t) = \alpha(t) - \frac{1}{2} \sigma^2(t). \quad (4)$$

For the remaining of our discussion we assume that μ and σ are continuous or even fixed, hence in any case they are bounded.

2 The deviations of Geometric Brownian Motion from its extreme points

We assume that the deviations of our interest take place on the time instants $\tau_i, i = 1 \dots N^*$, which are determined in the following way (see Poufnas [8], pp374-375, pp392); first let J be a fixed positive real number and $\tau_0 := 0$. Assume that τ_i has already been defined and let $M_t := \max\{P_s : s \in [\tau_i, t]\}$ and $m_t := \min\{P_s : s \in [\tau_i, t]\}$ for $t \in [0, T]$. Let $\phi(t) := \max(M_t - P_t, P_t - m_t)$ and for $\omega \in \Omega$, let $\phi(t, \omega) = \max(M_t(\omega) - P_t(\omega), P_t(\omega) - m_t(\omega))$. Then define τ_{i+1} by

$$\tau_{i+1}(\omega) := \inf\{t \geq \tau_i(\omega) : \phi(t, \omega) = J\}. \quad (5)$$

τ_{i+1} denotes the time instant at which the Geometric Brownian motion has deviated from its maximum or minimum since τ_i by a certain quantity. In the case of the price of a risky asset, it denotes the time instant at which the price has deviated from its maximum or minimum since τ_i by a preset amount J . The rationale behind using such a choice for the time of the transaction lies within the fact that the investor chooses to adjust the risky asset part of his or her portfolio when the price of the risky asset has parted from the maximum or minimum price by a certain amount.

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A different way to define τ_i is given if $J_{i+1} := \nu |P_{\tau_i}|$, where ν is a fixed positive number. Set $\tau_0 := 0$. If τ_i is given we define τ_{i+1} by

$$\tau_{i+1}(\omega) := \inf\{t \geq \tau_i(\omega) : \phi(t, \omega) = J_{i+1}(\omega)\}, \omega \in \Omega. \quad (6)$$

The reasoning behind such a choice is that the investor chooses to perform a transaction and adjust the portfolio when the stock price has increased or decreased compared to its maximum or minimum by a certain multiple of the price at the time of the previous transaction, thus reflecting a "percentage change".

Let I_ω be the index set for the $\tau_i(\omega)$'s. In the next theorem we will prove that for almost every ω , I_ω is finite and in the theorem following it that for each i , τ_i is a stopping time (see Protter [9], pp3-5).

Theorem 1. *Let I_ω be the index set for $\tau_i(\omega)$. Then I_ω is finite for almost every $\omega \in \Omega$.*

Proof: We first give the proof if we choose τ_i to be defined by (5). Recall that with probability 1 a Brownian path is Hölder(ϵ) continuous for every $\epsilon < 1/2$, i.e.

$$|W_t - W_s| \leq C|t - s|^\epsilon \text{ a.s. } \forall t, s \in [0, T] \quad (7)$$

(see Durrett [4], pp336-338). Take $\epsilon = 1/3$. Let $\tilde{\Omega}$ denote the set of $\omega \in \Omega$ such that (7) does not hold. Then $Pr(\tilde{\Omega}) = 0$. Since

$$\gamma(t) = \int_0^t \mu(u) du + \int_0^t \sigma(u) dW_u, \quad (8)$$

we have that

$$\gamma(t) - \gamma(s) = \int_s^t \mu(u) du + \int_s^t \sigma(u) dW_u \quad (9)$$

for all $s, t \in [0, T]$. We take absolute values to see that

$$|\gamma(t) - \gamma(s)| \leq C_1 |t - s| + C_2 |W_t - W_s|. \quad (10)$$

(10) and (7) yield that (see also Poufinas [8], pp382)

$$\begin{aligned} |\gamma(t) - \gamma(s)| &\leq C_1 |t - s| + C_2 C |t - s|^{1/3} \\ \Rightarrow |\gamma(t) - \gamma(s)| &\leq C_1 |t - s|^{1/3} |t - s|^{2/3} + C_2 C |t - s|^{1/3} \\ \Rightarrow |\gamma(t) - \gamma(s)| &\leq C_1 |t - s|^{1/3} T^{2/3} + C_2 C |t - s|^{1/3} \\ \Rightarrow |\gamma(t) - \gamma(s)| &\leq C' |t - s|^{1/3} \end{aligned} \quad (11)$$

$$\Rightarrow |\gamma(t) - \gamma(s)| \leq C' T^{1/3}. \quad (12)$$

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For $\omega \in \Omega - \tilde{\Omega}$ fixed, consider the intervals $\{[\tau_{i-1}, \tau_i] : i \in I\}$, where $I = I_\omega \cup \{0\}$, hence depends on ω . We will prove that $|I|$ is bounded, by a number that does not depend on ω , which proves in addition that there are finitely many τ_i . Let $M_i := \max\{P_t : t \in [\tau_{i-1}, \tau_i]\} \geq m_i := \min\{P_t : t \in [\tau_{i-1}, \tau_i]\}$. Then $M_i = P(t_i^{**})$ and $m_i = P(t_i^*)$ for some $t_i^{**}, t_i^* \in [\tau_{i-1}, \tau_i]$. (5) implies that

$$J \leq |M_i - m_i| = |P(t_i^{**}) - P(t_i^*)|. \quad (13)$$

Recall that $P_t = P_0 e^{\gamma(t)}$ and use (11), (12), (13) to see that

$$J \leq P_0 \left| e^{\gamma(t_i^{**})} - e^{\gamma(t_i^*)} \right| \leq P_0 e^{C' T^{1/3}} \left| e^{\gamma(t_i^{**}) - \gamma(t_i^*)} - 1 \right|. \quad (14)$$

Set $K := P_0 e^{C' T^{1/3}}$. K is a constant, that does not depend on ω . Recall that $\gamma(t_i^{**}) \geq \gamma(t_i^*)$, since $P(t_i^{**}) \geq P(t_i^*)$ and thus we can drop the absolute values in (14) to see that (14) and (11) yield that

$$\frac{J}{K} + 1 \leq e^{\gamma(t_i^{**}) - \gamma(t_i^*)} \Rightarrow \frac{J}{K} + 1 \leq e^{C' |t_i^{**} - t_i^*|^{1/3}}. \quad (15)$$

Since $t_i^{**}, t_i^* \in [\tau_{i-1}, \tau_i]$ we get that $|t_i^{**} - t_i^*| \leq |\tau_i - \tau_{i-1}|$ which along with (15) implies that

$$\frac{J}{K} + 1 \leq e^{C' |\tau_i - \tau_{i-1}|^{1/3}} \Leftrightarrow L' \leq |\tau_i - \tau_{i-1}| \quad (16)$$

where $L' = \{(1/C') \ln(J/K + 1)\}^3$. We sum up (16) for all $i \in I$ to see that since $\tau_i \geq \tau_{i-1}$ and $\tau_i \leq T$,

$$L' |I| \leq \sum_{i \in I} (\tau_i - \tau_{i-1}) \leq T \Leftrightarrow |I| \leq (T/L'). \quad (17)$$

Thus I_ω is finite. T/L' does not depend on ω .

There are a few changes in the proof of the theorem when we choose τ_i to be given by (6). (In)equalities (7) – (12) hold as they are and (13) becomes

$$\nu |P_{\tau_{i-1}}| \leq |M_i - m_i| = |P(t_i^{**}) - P(t_i^*)|, \quad (18)$$

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whereas (14) changes to

$$\nu P_0 \left| e^{\gamma(\tau_{i-1})} \right| \leq P_0 e^{C' T^{1/3}} \left| e^{\gamma(t_i^{**}) - \gamma(t_i^*)} - 1 \right|. \quad (19)$$

Divide both sides by P_0 and use $e^{\gamma(\tau_{i-1})} \geq e^{-|\gamma(\tau_{i-1})|} \geq 0$ to see that

$$\nu e^{-|\gamma(\tau_{i-1})|} \leq e^{C' T^{1/3}} \left| e^{\gamma(t_i^{**}) - \gamma(t_i^*)} - 1 \right|. \quad (20)$$

It follows that

$$\begin{aligned} \nu &\leq e^{C' T^{1/3}} e^{|\gamma(\tau_{i-1})|} \left| e^{\gamma(t_i^{**}) - \gamma(t_i^*)} - 1 \right| \\ &\leq e^{C' T^{1/3}} e^{C' T^{1/3}} \left| e^{\gamma(t_i^{**}) - \gamma(t_i^*)} - 1 \right| \\ &= K' \left| e^{\gamma(t_i^{**}) - \gamma(t_i^*)} - 1 \right|, \end{aligned} \quad (21)$$

where $K' := e^{2C' T^{1/3}}$, a constant that does not depend on ω . We can drop the absolute values, since $\gamma(t_i^{**}) \geq \gamma(t_i^*)$ and see that because of (11)

$$\frac{\nu}{K'} + 1 \leq e^{\gamma(t_i^{**}) - \gamma(t_i^*)} \Rightarrow \frac{\nu}{K'} + 1 \leq e^{C' |\tau_i - \tau_{i-1}|^{1/3}}. \quad (22)$$

Whence,

$$L' \leq |\tau_i - \tau_{i-1}|, \quad (23)$$

where $L' = \{(1/C') \ln(\nu/K + 1)\}^3$. (23) is identical to (16) and thus beyond this point we proceed as we did when τ_i was given by (5). As a result (17) holds once more if we sum up for all $i \in I$. This proves that I_ω is finite. **Q.E.D.**

Let $n_\omega := |I_\omega|$. For a.e. ω , $\tau_i(\omega)$ is defined for $i = 0 \dots n_\omega$. Let $\tilde{\Omega}$ be as defined in the proof of **Theorem 1**. For $\omega \in \Omega - \tilde{\Omega}$, all n_ω 's are bounded by (T/L') , a constant that does not depend on ω . Hence the set $\{n_\omega : \omega \in \Omega - \tilde{\Omega}\}$ is a subset of $\{0 \dots N\}$, where $N := \lceil (T/L') \rceil + 1$. Define $N^* := \max\{n_\omega : \omega \in \Omega - \tilde{\Omega}\} < +\infty$. For $\omega \in \Omega - \tilde{\Omega}$ we define $\tau_i(\omega)$ as before (hence $\tau_i(\omega) \leq T$) if $i \leq n_\omega$ and by $\tau_i(\omega) = +\infty$ (or alternatively by $\tau_i(\omega) = T$) if $N^* \geq i > n_\omega$. This is consistent with the fact that the infimum of the empty set is $+\infty$, and thus τ_i in any case is given by (5) or (6). This also means that if $\tau_i(\omega) = +\infty$ for some ω , then no i -th payment is being made at this particular state ω (for simplicity in the following one may assume that $n_\omega = N^*$ for all $\omega \in \Omega - \tilde{\Omega}$).

We are now in position to state the following theorem.

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Theorem 2. *Let τ_i be as defined by (5) or (6). Then, for any $i = 1 \dots N^*$, τ_i is a stopping time.*

Proof: We first prove the theorem when τ_i is given by (5). Then

$$\tau_1 := \inf\{t \geq 0 : \phi(t) = J\}. \quad (24)$$

It is immediate that

$$\{\tau_1 \leq t\} = \{\phi_t = J\} \cup \bigcap_n \bigcup_{q \in Q \cap [0, t)} \{|\phi_q - J| < \frac{1}{n}\}. \quad (25)$$

The sets at the right-hand side are in \mathcal{F}_t , which yields that $\{\tau_1 \leq t\} \in \mathcal{F}_t$, $\forall t \in [0, T]$, proving that τ_1 is a stopping time.

It suffices to prove that $\phi_t \in \mathcal{F}_t$. This will follow if we prove that $M_t \in \mathcal{F}_t$ and $m_t \in \mathcal{F}_t$. Recall that $M_t := \max\{P_s : s \in [0, t]\}$, which gives that $M_t = \sup\{P_q : q \in [0, t] \cap Q\}$. But $P_q \in \mathcal{F}_q \subseteq \mathcal{F}_t$, hence $M_t \in \mathcal{F}_t$. Similarly $m_t \in \mathcal{F}_t$.

Going to τ_2 we have that

$$\tau_2 := \inf\{t \geq \tau_1 : \phi(t) = J\}, \quad (26)$$

$$M_t(\omega) := \max\{P_s(\omega) : s \in [\tau_1(\omega), t]\}. \quad (27)$$

Define \hat{P}_s by

$$\hat{P}_s(\omega) := \begin{cases} -\infty & 0 \leq s < \tau_1(\omega) \\ P_s(\omega) & \tau_1(\omega) \leq s \leq t \end{cases}. \quad (28)$$

It is obvious that

$$M_t = \sup\{\hat{P}_s : s \in [0, t]\}, \quad (29)$$

and that $\hat{P}_s \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. Thus as before $M_t \in \mathcal{F}_t$. Similarly $m_t \in \mathcal{F}_t$, and $\phi_t \in \mathcal{F}_t$.

To show that τ_2 is a stopping time we extend ϕ by defining

$$\hat{\phi}_s(\omega) := \begin{cases} -\infty & 0 \leq s < \tau_1(\omega) \\ \phi_s(\omega) & \tau_1(\omega) \leq s \leq t \end{cases}. \quad (30)$$

Of course $\hat{\phi}_t \in \mathcal{F}_t$, $\forall t \in [0, T]$. Moreover

$$\tau_2 = \inf\{t \geq 0 : \hat{\phi}_t = J\}, \quad (31)$$

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since J is a positive real. As before

$$\{\tau_2 \leq t\} = \{\hat{\phi}_t = J\} \cup \bigcap_n \bigcup_{q \in Q \cap [0, t)} \left\{ \left| \hat{\phi}_q - J \right| < \frac{1}{n} \right\}. \quad (32)$$

All the sets at the right-hand side are in \mathcal{F}_t , since $\hat{\phi}_q \in \mathcal{F}_q \subseteq \mathcal{F}_t$, whence τ_2 is a stopping time. Working in a similar way we can prove that τ_i is a stopping time for every i .

The proof is similar when for each i , τ_i is given by (6). Recall that

$$\tau_1 := \inf\{t \geq 0 : \phi(t) = \nu |P_0|\}. \quad (33)$$

It is immediate that

$$\{\tau_1 \leq t\} = \{\phi_t = J_1\} \cup \bigcap_n \bigcup_{q \in Q \cap [0, t)} \left\{ |\phi_q - \nu |P_0|| < \frac{1}{n} \right\}. \quad (34)$$

See that $P_0 \in \mathcal{F}_0 \subseteq \mathcal{F}_t, \forall t \in [0, T]$. This is straightforward if P_0 is constant. The result follows since each one of the sets that appear at the right-hand side of (34) is in \mathcal{F}_t . This implies that $\{\tau_1 \leq t\} \in \mathcal{F}_t$, for all t , proving that τ_1 is a stopping time. To see that $\phi_t \in \mathcal{F}_t$ we use the argument we used when τ_1 was given by (24), i.e. $P_q \in \mathcal{F}_q \subseteq \mathcal{F}_t$ and thus $M_t \in \mathcal{F}_t, m_t \in \mathcal{F}_t$ and thus $\phi_t \in \mathcal{F}_t$.

As for τ_2 it is now given by

$$\tau_2 := \inf\{t \geq \tau_1 : \phi(t) = \nu |P_{\tau_1}|\}. \quad (35)$$

We can repeat the discussion following (26) and equations (27) – (35) to extend P_s and ϕ_s on the whole interval $[0, t]$. We denote again their extensions by \hat{P}_s and $\hat{\phi}_s$ respectively. (28) yields that $M_t = \sup\{\hat{P}_s : s \in [0, t]\}$, according to (29) and that $M_t \in \mathcal{F}_t$, since $\hat{P}_s \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. It follows that $\phi_t \in \mathcal{F}_t$. In addition, (30) yields that $\hat{\phi}_t \in \mathcal{F}_t$ for all t . $\nu |P_{\tau_1}|$ is a positive real because $P(t) = P_0 e^{\gamma(t)}$ and $P_0(\omega) \neq 0$ by (1). Therefore,

$$\tau_2 = \{t \geq 0 : \hat{\phi}_t = \nu |P_{\tau_1}|\}. \quad (36)$$

The latter remark is redundant if we assume that P_0 is constant. As a consequence,

$$\{\tau_2 \leq t\} = \{\hat{\phi}_t = J_2\} \cup \bigcap_n \bigcup_{q \in Q \cap [0, t)} \left\{ \left| \hat{\phi}_q - \nu |P_{\tau_1}|\right| < \frac{1}{n} \right\}. \quad (37)$$

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Consider any of the sets $\{\omega : |\hat{\phi}_q(\omega) - \nu |P_{\tau_1(\omega)}(\omega)| < \frac{1}{n}\}$. If ω is in such a set, then $\tau_1(\omega) \leq q \leq t$, since $\nu |P_{\tau_1(\omega)}(\omega)| > 0$, for every ω such that $\tau_1(\omega) < \infty$, and $|\hat{\phi}_q(\omega) - \nu |P_{\tau_1(\omega)}(\omega)| < \frac{1}{n} < \infty$. This is true, because if $q < \tau_1(\omega)$, then $\hat{\phi}_q = -\infty$ and thus the latter could not hold. Each one of the above sets is in \mathcal{F}_t . To see that it suffices to show that $P_{\tau_1}|_{\{\tau_1 \leq q\}} \in \mathcal{F}_q$. Indeed, let $\pi : \{\tau_1 \leq q\} \rightarrow \Omega$ be the inclusion map. Then

$$\{\omega : (P_{\tau_1} \circ \pi)(\omega) \leq b\} = \{\omega : P(\tau_1(\omega), \omega) \leq b\} \cap \{\omega : \tau_1(\omega) \leq q\}, \quad (38)$$

where $b \in \mathfrak{R}$. The right-hand side is in $\mathcal{F}_q \subseteq \mathcal{F}_t$, since τ_1 is a stopping time. Conclude that $P_{\tau_1}|_{\{\tau_1 \leq q\}} \in \mathcal{F}_q$. It thus follows that all the sets at the right-hand side of (37) are in \mathcal{F}_t , since $\hat{\phi}_q \in \mathcal{F}_q \subseteq \mathcal{F}_t$, which proves that τ_2 is a stopping time. Working similarly we can prove that τ_i is a stopping time for every i . This completes the proof of the theorem. **Q.E.D.**

Theorem 1 and **Theorem 2** readily yield the following corollaries.

Corollary 1. *The price of a risky asset can deviate finitely many times almost surely from its extreme points by a given amount or percentage.*

Corollary 2. *The time instants at which the price of a risky asset deviates from its extreme points by a given amount or percentage are stopping times.*

3 The case of Brownian Motion with drifts

One can relatively easily conclude that similar results hold true in the case of Brownian Motion with drifts. To the best of our knowledge, there has not been a similar approach in this case either. In this case equation (1) becomes

$$dP = \alpha(P, t) dt + \sigma(P, t) dW. \quad (39)$$

We once more assume that α and σ are functions of time only and not of the risky asset. Equations (2), (3) and (4) change to

$$P_t = P_0 + \gamma(t), \quad (40)$$

$$\gamma(t) = \int_0^t \mu(s) ds + \int_0^t \sigma(s) dW_s \quad (41)$$

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and

$$\mu(t) = \alpha(t). \quad (42)$$

We keep the same notation so we can refer to our previous results. We also assume that μ and σ are continuous or fixed, hence bounded. The time instants τ_i are defined as in (5). When we examine the case of Brownian Motion we feel that absolute changes are more appropriate than proportional changes as given by (6). In incorporating proportional changes, i.e. allowing τ_i to be defined as in (6), we would put the additional requirement that $\min |P_t| = m_P > 0$, or that $|P_t|$ is bounded from below by a positive number. This is a natural restriction; namely that in a finite time interval the stock price cannot reach zero. We introduce this so as to avoid the complications that a price equal to zero would result into our structure.

We can now state the following corollaries that map the implications to the case of a Brownian Motion with drift.

Corollary 3. *Let I_ω be the index set for $\tau_i(\omega)$. Then I_ω is finite for almost every $\omega \in \Omega$.*

Proof: When τ_i is defined by (5) the proof is almost identical to the one of **Theorem 1**. Simply observe that

$$J \leq |M_i - m_i| = |P(t_i^{**}) - P(t_i^*)| = |\gamma(t_i^{**}) - \gamma(t_i^*)| \quad (43)$$

$$\Rightarrow J \leq |\gamma(t_i^{**}) - \gamma(t_i^*)| \leq C' |t_i^{**} - t_i^*|^{1/3} \leq C' |\tau_i - \tau_{i-1}|^{1/3} \quad (44)$$

$$\Rightarrow L' := (J/C')^3 \leq |\tau_i - \tau_{i-1}|. \quad (45)$$

We sum up (45) for all $i \in I$ to see that since $\tau_i \geq \tau_{i-1}$ and $\tau_i \leq T$,

$$L' |I| \leq \sum_{i \in I} (\tau_i - \tau_{i-1}) \leq T \Leftrightarrow |I| \leq (T/L'). \quad (46)$$

Thus I_ω is finite. T/L' does not depend on ω .

When τ_i is given by (6), we receive that

$$\nu m_P \leq \nu |P_{\tau_{i-1}}| \leq |M_i - m_i| = |P(t_i^{**}) - P(t_i^*)| = |\gamma(t_i^{**}) - \gamma(t_i^*)| \quad (47)$$

$$\Rightarrow L' := (\nu m_P / C')^3 \leq |\tau_i - \tau_{i-1}|. \quad (48)$$

This is identical to (46), thus summing up for all i we conclude once more that I_ω is finite. **Q.E.D.**

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Corollary 4. *Let τ_i be as defined by (5) or (6). Then, for any $i = 1 \dots N^*$, τ_i is a stopping time.*

Proof: The proof is identical to the proof of **Theorem 2**. **Q.E.D.**

4 The cost functions

As mentioned earlier, the time instants at which the investor decides to buy or sell units of the risky asset, transaction costs have to be paid. The natural question is how to model these transaction costs.

The amount that the individual has to pay at τ_i , $i = 1 \dots N^*$ may be given by different cost functions. Let β_i be this amount. Let $c(t)$ denote the cost function. Then

$$c(t) dt = \sum_{i=1}^{N^*} \beta_i \delta_0(\tau_i - t) dt. \quad (49)$$

$c(t)$ is essentially the individual's "consumption" at time t , since his or her expenses are only the commission fees. On the other hand the individual's consumption in terms of his or her shares of stock is given by $dN(dP + P)$. Conclude that

$$-c(t)dt = PdN + dNdP. \quad (50)$$

5 Future Research

By looking at our results we realize that we limited ourselves to a finite time horizon T . This is reasonable even when applied to the case of an investor that holds a risky asset. It is of interest to see whether a similar result holds for an infinite time horizon or prove that it does not hold.

The next step would be to consider that cost is given as

$$\beta_i := f(\Delta Y).$$

A reasonable assumption is that f is a nonnegative concave function. Of particular interest is the case where the transaction costs are built as an expression of the form

$$\beta_i = \alpha_0(\Delta Y) + \lambda(\Delta Y)^\Gamma,$$

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where ΔY is the total change in the risky asset position since the last payment of transaction costs, for λ and Γ constants in $[0, 1]$ and $\alpha_0(x)$ a step function that becomes equal to a constant if x is not 0 and equals to 0 otherwise. The above transaction costs have a fixed and a proportional part but are not necessarily linear.

Moving towards another direction we could try to retrieve what conditions on our fee functions would allow for our results to be readily used to the pricing of (European) options. Such a result appears to be very interesting as it would incorporate transactions only when the risky asset price moves from its maximum or minimum within a certain time interval by a certain amount or percentage.

Another step in our analysis could be the study of portfolios with more than one risky asset. Such portfolios allow for the trading of more than one risky assets and give more choices. This means that there is not only one strategy that can be followed, but the investor would have to choose among the available ones. This hints that we need to move towards the use of utility functions and control theory so as to derive the strategy that is optimal for the investor.

6 Conclusion

We proved that given a Geometric Brownian Motion with a drift, the number of times it deviates from its extreme points by a certain quantity or proportion are finitely many almost surely. In addition, the time instants on which this occurs are stopping times. The latter was applied to the time instants where the price of a risky asset departs from its maximum or minimum either by a certain amount or a certain percentage. We proved that these time instants are finitely many and they are stopping times. We deemed reasonable that when this occurs the investor chooses to buy or sell and thus encounters transaction costs, which we attempted to model.

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