



## An extension of Wick's theorem

Christophe Vignat, S. Bhatnagar

### ► To cite this version:

Christophe Vignat, S. Bhatnagar. An extension of Wick's theorem. *Statistics and Probability Letters*, 2010, 78 (15), pp.2404. 10.1016/j.spl.2008.03.003 . hal-00644712

**HAL Id: hal-00644712**

**<https://hal.science/hal-00644712>**

Submitted on 25 Nov 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Accepted Manuscript

An extension of Wick's theorem

C. Vignat, S. Bhatnagar

PII: S0167-7152(08)00140-5  
DOI: [10.1016/j.spl.2008.03.003](https://doi.org/10.1016/j.spl.2008.03.003)  
Reference: STAPRO 4990

To appear in: *Statistics and Probability Letters*

Received date: 15 November 2007

Revised date: 5 March 2008

Accepted date: 5 March 2008

Please cite this article as: Vignat, C., Bhatnagar, S., An extension of Wick's theorem. *Statistics and Probability Letters* (2008), doi:10.1016/j.spl.2008.03.003

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



# An extension of Wick's theorem

C. Vignat and S. Bhatnagar

*Institut Gaspard Monge, Université de Marne la Vallée, France  
Department of Computer Science and Automation, Indian Institute of Sciences,  
Bangalore, India*

---

## Abstract

We propose an extension of a result by Repetowicz et al. (Repetowicz 2004) about Wick's theorem and its applications: we first show that Wick's theorem can be extended to the uniform distribution on the sphere and then to the whole class of elliptical distributions. Then, as a special case, we detail this result for distributions that are scale mixtures of Gaussians. Finally, we show that these results allow to recover easily a theorem by Folland (Folland 2001) about integration of polynomials over the sphere.

*Key words:* Wick's theorem, Gaussian scale mixtures, spherical distribution, uniform distribution on the sphere

---

## 1 Wick's theorem

In quantum field theory (Etingof 2002), the determination of the partition function of some systems involves the computation of integrals of the form

$$\int_{\mathbb{R}^n} P(X) \exp\left(-\frac{X^t \Sigma^{-1} X}{2}\right) dX,$$

---

*Email address:* vignat@univ-mlv.fr, shalabh@csa.iisc.ernet.in (C. Vignat and S. Bhatnagar).

where  $P(X)$  is a monomial in the components of  $X \in \mathbb{R}^n$  i.e., in a probabilistic approach, mixed moments of a Gaussian random vector with covariance matrix  $\Sigma$ .

Wick's theorem (Le Bellac 1991; Etingof 2002) provides a simple formula for these moments: let us denote by  $X$  a Gaussian vector in  $\mathbb{R}^n$  with zero mean and covariance matrix  $\Sigma$ ; a linear form  $\mathcal{L}_i$  on  $\mathbb{R}^n$  is identified with the vector  $l_i$  of its coefficients

$$\mathcal{L}_i(X) = l_i^t X.$$

A *pairing*  $\sigma$  on the set  $I_{2m} = \{1, \dots, 2m\}$  is a partition of  $I_{2m}$  into  $m$  disjoint pairs; the set  $\Pi_{2m}$  of pairings of  $I_{2m}$  is composed of  $\frac{2m!}{2^m m!}$  elements. For example, in the case  $m = 2$ , there are 3 different pairings on  $I_4 = \{1, 2, 3, 4\}$ , namely  $\sigma_1 = \{(1, 2), (3, 4)\}$ ,  $\sigma_2 = \{(1, 3), (2, 4)\}$  and  $\sigma_3 = \{(1, 4), (2, 3)\}$ . We denote by  $I_{2m}/\sigma$  the set of indices  $i$  such that  $\sigma = \{(i, \sigma(i))\}$ .

Wick's theorem states as follows.

**Theorem 1** *Let  $m$  be an integer then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = \sum_{\sigma \in \Pi_{2m}} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

*If  $2m$  is replaced by  $2m + 1$ , then the expectation above equals zero.*

As an example, with  $m = 2$  and  $\mathcal{L}_i(X) = X_i$ , we obtain

$$E[X_1 X_2 X_3 X_4] = E[X_1 X_2] E[X_3 X_4] + E[X_1 X_3] E[X_2 X_4] + E[X_1 X_4] E[X_2 X_3].$$

We note that Wick's theorem can be deduced from the Leonov-Shiryaev formula (Leonov Shiryaev 1959), that relates the moments and the cumulants of a random vector.

## 2 Spherical and Elliptical distributions

In this paper, we extend Wick's theorem to the class of spherical random vectors: such a vector  $X \in \mathbb{R}^n$  is characterized (Fang et al. 1990, Ch.2) by the fact that its characteristic function  $\psi(Z) = E \exp(iZ^t X)$  is invariant by

orthogonal transformation; in other words,

$$\psi(Z) = \phi(Z^t Z)$$

for some function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{C}$ . In this case, a stochastic representation of  $X$  writes

$$X = aU$$

where  $U$  is a random vector uniformly distributed on the unit sphere  $\mathcal{S}_n = \{X \in \mathbb{R}^n; \|X\| = 1\}$  and  $a \in \mathbb{R}^+$  is a positive random variable independent of  $U$ . If moreover vector  $X$  admits a probability density function  $f_X$ , then it is necessarily of the form

$$f_X(X) = g(X^t X)$$

for some function  $g : \mathbb{R} \rightarrow \mathbb{R}^+$ .

An extension of the spherical property is the elliptical property (Fang et al. 1990, Ch.2): a random vector  $Y \in \mathbb{R}^n$  is elliptical with  $(n \times n)$  positive definite characteristic matrix  $\Sigma$  if

$$Y = \Sigma^{1/2} X = a \Sigma^{1/2} U \quad (1)$$

where  $X$  is spherical. We assume without loss of generality that all vectors have zero mean.

### 3 Wick's theorem for the uniform distribution on the sphere

Wick's theorem can be extended to the case of a vector uniformly distributed on the sphere  $\mathcal{S}_n$  as follows.

**Theorem 2** *If  $U$  is uniformly distributed on the sphere  $\mathcal{S}_n$  then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{\Gamma(\frac{n}{2})}{2^m \Gamma(m + \frac{n}{2})} \sum_{\sigma} \prod_{i \in I_{2m}/\sigma} l_i^t l_{\sigma(i)}$$

*and the expectation equals zero if  $2m$  is replaced by  $2m + 1$ .*

**PROOF.** A stochastic representation for vector  $U$  is

$$U = \frac{N}{\|N\|}$$

where  $N$  is a Gaussian vector with unit covariance matrix. By the polar factorization property, random variable  $r = \|N\|$  is independent of  $U$ ; we deduce from the linearity of  $\mathcal{L}_i$  that

$$E \prod_{i=1}^{2m} \mathcal{L}_i(N) = E \prod_{i=1}^{2m} r \mathcal{L}_i(U) = E r^{2m} E \prod_{i=1}^{2m} \mathcal{L}_i(U)$$

so that

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{E \prod_{i=1}^{2m} \mathcal{L}_i(N)}{E r^{2m}}.$$

Since  $r = \sqrt{N^t N}$  is chi-distributed with  $n$  degrees of freedom, elementary algebra yields

$$E r^{2m} = \frac{2^m \Gamma\left(m + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \quad (2)$$

□

From the linearity of forms  $\mathcal{L}_i$  and by stochastic representation (1), we deduce a version of Wick's theorem for any elliptical vector as follows.

**Theorem 3** *If  $X \in \mathbb{R}^n$  is an elliptical vector with stochastic representation (1) and if random variable  $a$  has moments up to order  $2m$ , then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = \frac{E a^{2m}}{E r^{2m}} \sum_{\sigma} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

where  $E r^{2m}$  is given by (2).

#### 4 Wick's theorem for Gaussian scale mixtures

A random vector  $X$  is a Gaussian scale mixture if its probability density function writes

$$f_X(X) = \int_0^{+\infty} g(x; a\Sigma) dH(a)$$

where  $g(x; \Sigma)$  is the Gaussian distribution with  $(n \times n)$  covariance matrix  $\Sigma$  on  $\mathbb{R}^n$  and  $H$  is a cumulative distribution function. A stochastic representation

of such a vector is

$$X = \sqrt{a}N \quad (3)$$

where  $N$  is a Gaussian vector with covariance matrix  $\Sigma$  and  $a$  is a positive random variable independent of  $N$  with cumulative distribution function  $H$ . Cauchy, alpha-stable and Student-t random vectors belong to the family of Gaussian scale mixtures.

If  $X$  is a Gaussian scale mixture, then it is also elliptical; however, the contrary is not true, as shown by the Pearson II family of  $n$ -variate probability density functions (Johnson 1987)

$$f_X(X) = \frac{\Gamma\left(\frac{n}{2} + m + 1\right)}{\Gamma(m + 1) |\pi K|^{1/2}} \left(1 - X^t K^{-1} X\right)_+^m$$

with notation  $(x)_+ = \max(x, 0)$ , with parameter  $m > -1$  and where  $K$  is a positive definite matrix.

Using stochastic representation (3), Wick's theorem extends to Gaussian scale mixtures as follows.

**Theorem 4** *If  $X$  is a Gaussian scale mixture as in (3) with  $Ea^m < \infty$ , then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = Ea^m \sum_{\sigma \in \Pi_m} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

We note that Repetowicz et al. (Repetowicz 2004) use Wick's theorem to address a more general problem, namely the determination of mixed moments of linear combinations of i.i.d. Student-t vectors; in this case, random variable  $a$  in (3) is inverse Gamma distributed. The above theorem, although restricted to mixed moments of a *single* random vector, holds however for the class of all Gaussian scale mixtures for which  $Ea^m < \infty$ .

## 5 Application: a result by Folland

In this section, we show that the results above allow to recover easily the following theorem by Folland (Folland 2001) about integration of a monomial over the sphere  $\mathcal{S}_n$ .

**Theorem 5** *If*

$$P(X) = \prod_{i=1}^n X_i^{\alpha_i}$$

*then*

$$\int_{\mathcal{S}_n} P(X) d\mu = \begin{cases} 0 & \text{if some } \alpha_i \text{ is odd} \\ \frac{2\Gamma(\beta_1)\dots\Gamma(\beta_n)}{\Gamma(\beta_1+\dots+\beta_n)} & \text{else} \end{cases}$$

*where*

$$\beta_i = \frac{\alpha_i + 1}{2}$$

*and  $d\mu$  is the surface measure on the sphere  $\mathcal{S}_n$ .*

**PROOF.** Let us assume first that all  $\alpha_i$  are even and define the integer parameter  $m$  by  $2m = \sum_{i=1}^n \alpha_i$ . Defining  $\alpha_0 = 0$ , we consider the function  $T : [1, 2m] \rightarrow [1, n]$  such that

$$T(i) = j \quad \text{if} \quad \alpha_1 + \dots + \alpha_{j-1} < i \leq \alpha_1 + \dots + \alpha_j.$$

We choose forms  $\mathcal{L}_i$  according to  $l_i = \delta_{T(i)}$  where  $\delta_k$  is the  $k$ -th column vector of the  $n \times n$  identity matrix. Since the scalar product  $l_i^t l_{\sigma(i)}$  equals zero when  $T(i) \neq T(\sigma(i))$ , we consider only the  $\prod_{i=1}^n \frac{\alpha_i!}{2^{\frac{\alpha_i}{2}} (\frac{\alpha_i}{2})!}$  pairings for which  $T(i) = T(\sigma(i))$  and  $l_i^t l_{\sigma(i)} = \delta_{T(i)}^t \delta_{T(i)} = 1$ . Thus, by theorem (2)

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(m + \frac{n}{2}\right)} \prod_{i=1}^n \frac{\alpha_i!}{2^{\frac{\alpha_i}{2}} \left(\frac{\alpha_i}{2}\right)!} = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(\sum_{i=1}^n \beta_i\right)} \frac{2^{\sum_{i=1}^n \beta_i}}{(2\pi)^{n/2}} \prod_{i=1}^n \Gamma(\beta_i).$$

Since  $d\mu$  is the unnormalized surface measure and as the surface of the sphere is  $S(\mathcal{S}_n) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ ,

$$\int_{\mathcal{S}_n} P(X) d\mu = S(\mathcal{S}_n) E \prod_{i=1}^{2m} \mathcal{L}_i(U)$$

which yields the result.

In the case where some  $\alpha_i$  is odd, the integral equals zero since the surface measure on the sphere is an even function in each of the variables.  $\square$



## Acknowledgements

This work was performed during a visit by C. Vignat to S. Bhatnagar to the Computer Science and Automation Department of the Indian Institute of Sciences, Bangalore. C.V. thanks S. B. for his nice welcome. The authors thank the reviewer for his interesting remarks.

## References

- [Repetowicz 2004] P. Repetowicz and P. Richmond, The Wick theorem for non-Gaussian distributions and its application for noise filtering of correlated  $q$ -Exponentially distributed random variables, unpublished, arXiv:math-ph/0411020 v1, Nov 2004
- [Etingof 2002] P. Etingof, Lecture Notes, Geometry and Quantum Field Theory, Fall 2002, MIT Open Courseware, <http://ocw.mit.edu/OcwWeb/Mathematics/18-238Fall2002/LectureNotes/index.htm>
- [Fang et al. 1990] K.-T. Fang, , S. Kotz & K.W. Ng, Symmetric Multivariate and Related Distributions, Chapman and Hall, London, 1990
- [Folland 2001] G. B. Folland, How to Integrate a Polynomial over a Sphere, The American Mathematical Monthly, Vol. 108-5, May 2001, pp. 446-448
- [Leonov Shiryaev 1959] V. P. Leonov and A. N. Shiryaev, On a method of calculation of semi-invariants, Theor. Prob. Appl., Vol.4, pp.319-329
- [Le Bellac 1991] M. Le Bellac, Quantum and Statistical Field Theory, Clarendon Press, Oxford, 1991
- [Johnson 1987] M. E. Johnson, Multivariate Statistical Simulation, Wiley, New York, 1987.