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# An extension of Wick's theorem

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## Abstract

We propose an extension of a result by Repetowicz et al. (Repetowicz 2004) about Wick's theorem and its applications: we first show that Wick's theorem can be extended to the uniform distribution on the sphere and then to the whole class of elliptical distributions. Then, as a special case, we detail this result for distributions that are scale mixtures of Gaussians. Finally, we show that these results allow to recover easily a theorem by Folland (Folland 2001) about integration of polynomials over the sphere.

*Key words:* Wick's theorem, Gaussian scale mixtures, spherical distribution, uniform distribution on the sphere

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## 1 Wick's theorem

In quantum field theory (Etingof 2002), the determination of the partition function of some systems involves the computation of integrals of the form

$$\int_{\mathbb{R}^n} P(X) \exp\left(-\frac{X^t \Sigma^{-1} X}{2}\right) dX,$$

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where  $P(X)$  is a monomial in the components of  $X \in \mathbb{R}^n$  i.e., in a probabilistic approach, mixed moments of a Gaussian random vector with covariance matrix  $\Sigma$ .

Wick's theorem (Le Bellac 1991; Etingof 2002) provides a simple formula for these moments: let us denote by  $X$  a Gaussian vector in  $\mathbb{R}^n$  with zero mean and covariance matrix  $\Sigma$ ; a linear form  $\mathcal{L}_i$  on  $\mathbb{R}^n$  is identified with the vector  $l_i$  of its coefficients

$$\mathcal{L}_i(X) = l_i^t X.$$

A *pairing*  $\sigma$  on the set  $I_{2m} = \{1, \dots, 2m\}$  is a partition of  $I_{2m}$  into  $m$  disjoint pairs; the set  $\Pi_{2m}$  of pairings of  $I_{2m}$  is composed of  $\frac{2m!}{2^m m!}$  elements. For example, in the case  $m = 2$ , there are 3 different pairings on  $I_4 = \{1, 2, 3, 4\}$ , namely  $\sigma_1 = \{(1, 2), (3, 4)\}$ ,  $\sigma_2 = \{(1, 3), (2, 4)\}$  and  $\sigma_3 = \{(1, 4), (2, 3)\}$ . We denote by  $I_{2m}/\sigma$  the set of indices  $i$  such that  $\sigma = \{(i, \sigma(i))\}$ .

Wick's theorem states as follows.

**Theorem 1** *Let  $m$  be an integer then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = \sum_{\sigma \in \Pi_{2m}} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

*If  $2m$  is replaced by  $2m + 1$ , then the expectation above equals zero.*

As an example, with  $m = 2$  and  $\mathcal{L}_i(X) = X_i$ , we obtain

$$E[X_1 X_2 X_3 X_4] = E[X_1 X_2] E[X_3 X_4] + E[X_1 X_3] E[X_2 X_4] + E[X_1 X_4] E[X_2 X_3].$$

We note that Wick's theorem can be deduced from the Leonov-Shiryaev formula (Leonov Shiryaev 1959), that relates the moments and the cumulants of a random vector.

## 2 Spherical and Elliptical distributions

In this paper, we extend Wick's theorem to the class of spherical random vectors: such a vector  $X \in \mathbb{R}^n$  is characterized (Fang et al. 1990, Ch.2) by the fact that its characteristic function  $\psi(Z) = E \exp(iZ^t X)$  is invariant by

orthogonal transformation; in other words,

$$\psi(Z) = \phi(Z^t Z)$$

for some function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{C}$ . In this case, a stochastic representation of  $X$  writes

$$X = aU$$

where  $U$  is a random vector uniformly distributed on the unit sphere  $\mathcal{S}_n = \{X \in \mathbb{R}^n; \|X\| = 1\}$  and  $a \in \mathbb{R}^+$  is a positive random variable independent of  $U$ . If moreover vector  $X$  admits a probability density function  $f_X$ , then it is necessarily of the form

$$f_X(X) = g(X^t X)$$

for some function  $g : \mathbb{R} \rightarrow \mathbb{R}^+$ .

An extension of the spherical property is the elliptical property (Fang et al. 1990, Ch.2): a random vector  $Y \in \mathbb{R}^n$  is elliptical with  $(n \times n)$  positive definite characteristic matrix  $\Sigma$  if

$$Y = \Sigma^{1/2} X = a \Sigma^{1/2} U \quad (1)$$

where  $X$  is spherical. We assume without loss of generality that all vectors have zero mean.

### 3 Wick's theorem for the uniform distribution on the sphere

Wick's theorem can be extended to the case of a vector uniformly distributed on the sphere  $\mathcal{S}_n$  as follows.

**Theorem 2** *If  $U$  is uniformly distributed on the sphere  $\mathcal{S}_n$  then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(m + \frac{n}{2}\right)} \sum_{\sigma} \prod_{i \in I_{2m}/\sigma} l_i^t l_{\sigma(i)}$$

*and the expectation equals zero if  $2m$  is replaced by  $2m + 1$ .*

**PROOF.** A stochastic representation for vector  $U$  is

$$U = \frac{N}{\|N\|}$$

where  $N$  is a Gaussian vector with unit covariance matrix. By the polar factorization property, random variable  $r = \|N\|$  is independent of  $U$ ; we deduce from the linearity of  $\mathcal{L}_i$  that

$$E \prod_{i=1}^{2m} \mathcal{L}_i(N) = E \prod_{i=1}^{2m} r \mathcal{L}_i(U) = Er^{2m} E \prod_{i=1}^{2m} \mathcal{L}_i(U)$$

so that

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{E \prod_{i=1}^{2m} \mathcal{L}_i(N)}{Er^{2m}}.$$

Since  $r = \sqrt{N^t N}$  is chi-distributed with  $n$  degrees of freedom, elementary algebra yields

$$Er^{2m} = \frac{2^m \Gamma\left(m + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \quad (2)$$

□

From the linearity of forms  $\mathcal{L}_i$  and by stochastic representation (1), we deduce a version of Wick's theorem for any elliptical vector as follows.

**Theorem 3** *If  $X \in \mathbb{R}^n$  is an elliptical vector with stochastic representation (1) and if random variable  $a$  has moments up to order  $2m$ , then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = \frac{Ea^{2m}}{Er^{2m}} \sum_{\sigma} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

where  $Er^{2m}$  is given by (2).

#### 4 Wick's theorem for Gaussian scale mixtures

A random vector  $X$  is a Gaussian scale mixture if its probability density function writes

$$f_X(X) = \int_0^{+\infty} g(x; a\Sigma) dH(a)$$

where  $g(x; \Sigma)$  is the Gaussian distribution with  $(n \times n)$  covariance matrix  $\Sigma$  on  $\mathbb{R}^n$  and  $H$  is a cumulative distribution function. A stochastic representation

of such a vector is

$$X = \sqrt{a}N \quad (3)$$

where  $N$  is a Gaussian vector with covariance matrix  $\Sigma$  and  $a$  is a positive random variable independent of  $N$  with cumulative distribution function  $H$ . Cauchy, alpha-stable and Student-t random vectors belong to the family of Gaussian scale mixtures.

If  $X$  is a Gaussian scale mixture, then it is also elliptical; however, the contrary is not true, as shown by the Pearson II family of  $n$ -variate probability density functions (Johnson 1987)

$$f_X(X) = \frac{\Gamma\left(\frac{n}{2} + m + 1\right)}{\Gamma(m + 1) |\pi K|^{1/2}} \left(1 - X^t K^{-1} X\right)_+^m$$

with notation  $(x)_+ = \max(x, 0)$ , with parameter  $m > -1$  and where  $K$  is a positive definite matrix.

Using stochastic representation (3), Wick's theorem extends to Gaussian scale mixtures as follows.

**Theorem 4** *If  $X$  is a Gaussian scale mixture as in (3) with  $Ea^m < \infty$ , then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = Ea^m \sum_{\sigma \in \Pi_m} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

We note that Repetowicz et al. (Repetowicz 2004) use Wick's theorem to address a more general problem, namely the determination of mixed moments of linear combinations of i.i.d. Student-t vectors; in this case, random variable  $a$  in (3) is inverse Gamma distributed. The above theorem, although restricted to mixed moments of a *single* random vector, holds however for the class of all Gaussian scale mixtures for which  $Ea^m < \infty$ .

## 5 Application: a result by Folland

In this section, we show that the results above allow to recover easily the following theorem by Folland (Folland 2001) about integration of a monomial over the sphere  $\mathcal{S}_n$ .

**Theorem 5** *If*

$$P(X) = \prod_{i=1}^n X_i^{\alpha_i}$$

*then*

$$\int_{\mathcal{S}_n} P(X) d\mu = \begin{cases} 0 & \text{if some } \alpha_i \text{ is odd} \\ \frac{2\Gamma(\beta_1)\dots\Gamma(\beta_n)}{\Gamma(\beta_1+\dots+\beta_n)} & \text{else} \end{cases}$$

*where*

$$\beta_i = \frac{\alpha_i + 1}{2}$$

*and*  $d\mu$  *is the surface measure on the sphere*  $\mathcal{S}_n$ .

**PROOF.** Let us assume first that all  $\alpha_i$  are even and define the integer parameter  $m$  by  $2m = \sum_{i=1}^n \alpha_i$ . Defining  $\alpha_0 = 0$ , we consider the function  $T : [1, 2m] \rightarrow [1, n]$  such that

$$T(i) = j \quad \text{if} \quad \alpha_1 + \dots + \alpha_{j-1} < i \leq \alpha_1 + \dots + \alpha_j.$$

We choose forms  $\mathcal{L}_i$  according to  $l_i = \delta_{T(i)}$  where  $\delta_k$  is the  $k$ -th column vector of the  $n \times n$  identity matrix. Since the scalar product  $l_i^t l_{\sigma(i)}$  equals zero when  $T(i) \neq T(\sigma(i))$ , we consider only the  $\prod_{i=1}^n \frac{\alpha_i!}{2^{\frac{\alpha_i}{2}} (\frac{\alpha_i}{2})!}$  pairings for which  $T(i) = T(\sigma(i))$  and  $l_i^t l_{\sigma(i)} = \delta_{T(i)}^t \delta_{T(i)} = 1$ . Thus, by theorem (2)

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(m + \frac{n}{2}\right)} \prod_{i=1}^n \frac{\alpha_i!}{2^{\frac{\alpha_i}{2}} \left(\frac{\alpha_i}{2}\right)!} = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(\sum_{i=1}^n \beta_i\right)} \frac{2^{\sum_{i=1}^n \beta_i}}{(2\pi)^{n/2}} \prod_{i=1}^n \Gamma(\beta_i).$$

Since  $d\mu$  is the unnormalized surface measure and as the surface of the sphere is  $S(\mathcal{S}_n) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$ ,

$$\int_{\mathcal{S}_n} P(X) d\mu = S(\mathcal{S}_n) E \prod_{i=1}^{2m} \mathcal{L}_i(U)$$

which yields the result.

In the case where some  $\alpha_i$  is odd, the integral equals zero since the surface measure on the sphere is an even function in each of the variables.  $\square$



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## References

- [Repetowicz 2004] P. Repetowicz and P. Richmond, The Wick theorem for non-Gaussian distributions and its application for noise filtering of correlated  $q$ -Exponentially distributed random variables, unpublished, arXiv:math-ph/0411020 v1, Nov 2004
- [Etingof 2002] P. Etingof, Lecture Notes, Geometry and Quantum Field Theory, Fall 2002, MIT Open Courseware, <http://ocw.mit.edu/OcwWeb/Mathematics/18-238Fall2002/LectureNotes/index.htm>
- [Fang et al. 1990] K.-T. Fang, , S. Kotz & K.W. Ng, Symmetric Multivariate and Related Distributions, Chapman and Hall, London, 1990
- [Folland 2001] G. B. Folland, How to Integrate a Polynomial over a Sphere, The American Mathematical Monthly, Vol. 108-5, May 2001, pp. 446-448
- [Leonov Shiryaev 1959] V. P. Leonov and A. N. Shiryaev, On a method of calculation of semi-invariants, Theor. Prob. Appl., Vol.4, pp.319-329
- [Le Bellac 1991] M. Le Bellac, Quantum and Statistical Field Theory, Clarendon Press, Oxford, 1991
- [Johnson 1987] M. E. Johnson, Multivariate Statistical Simulation, Wiley, New York, 1987.