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An extension of Wick's theorem

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Abstract

We propose an extension of a result by Repetowicz et al. (Repetowicz 2004) about Wick's theorem and its applications: we first show that Wick's theorem can be extended to the uniform distribution on the sphere and then to the whole class of elliptical distributions. Then, as a special case, we detail this result for distributions that are scale mixtures of Gaussians. Finally, we show that these results allow to recover easily a theorem by Folland (Folland 2001) about integration of polynomials over the sphere.

Key words: Wick's theorem, Gaussian scale mixtures, spherical distribution, uniform distribution on the sphere

1 Wick's theorem

In quantum field theory (Etingof 2002), the determination of the partition function of some systems involves the computation of integrals of the form

$$\int_{\mathbb{R}^n} P(X) \exp\left(-\frac{X^t \Sigma^{-1} X}{2}\right) dX,$$

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where $P(X)$ is a monomial in the components of $X \in \mathbb{R}^n$ i.e., in a probabilistic approach, mixed moments of a Gaussian random vector with covariance matrix Σ .

Wick's theorem (Le Bellac 1991; Etingof 2002) provides a simple formula for these moments: let us denote by X a Gaussian vector in \mathbb{R}^n with zero mean and covariance matrix Σ ; a linear form \mathcal{L}_i on \mathbb{R}^n is identified with the vector l_i of its coefficients

$$\mathcal{L}_i(X) = l_i^t X.$$

A *pairing* σ on the set $I_{2m} = \{1, \dots, 2m\}$ is a partition of I_{2m} into m disjoint pairs; the set Π_{2m} of pairings of I_{2m} is composed of $\frac{2m!}{2^m m!}$ elements. For example, in the case $m = 2$, there are 3 different pairings on $I_4 = \{1, 2, 3, 4\}$, namely $\sigma_1 = \{(1, 2), (3, 4)\}$, $\sigma_2 = \{(1, 3), (2, 4)\}$ and $\sigma_3 = \{(1, 4), (2, 3)\}$. We denote by I_{2m}/σ the set of indices i such that $\sigma = \{(i, \sigma(i))\}$.

Wick's theorem states as follows.

Theorem 1 *Let m be an integer then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = \sum_{\sigma \in \Pi_{2m}} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

If $2m$ is replaced by $2m + 1$, then the expectation above equals zero.

As an example, with $m = 2$ and $\mathcal{L}_i(X) = X_i$, we obtain

$$E[X_1 X_2 X_3 X_4] = E[X_1 X_2] E[X_3 X_4] + E[X_1 X_3] E[X_2 X_4] + E[X_1 X_4] E[X_2 X_3].$$

We note that Wick's theorem can be deduced from the Leonov-Shiryaev formula (Leonov Shiryaev 1959), that relates the moments and the cumulants of a random vector.

2 Spherical and Elliptical distributions

In this paper, we extend Wick's theorem to the class of spherical random vectors: such a vector $X \in \mathbb{R}^n$ is characterized (Fang et al. 1990, Ch.2) by the fact that its characteristic function $\psi(Z) = E \exp(iZ^t X)$ is invariant by

orthogonal transformation; in other words,

$$\psi(Z) = \phi(Z^t Z)$$

for some function $\phi : \mathbb{R}^+ \rightarrow \mathbb{C}$. In this case, a stochastic representation of X writes

$$X = aU$$

where U is a random vector uniformly distributed on the unit sphere $\mathcal{S}_n = \{X \in \mathbb{R}^n; \|X\| = 1\}$ and $a \in \mathbb{R}^+$ is a positive random variable independent of U . If moreover vector X admits a probability density function f_X , then it is necessarily of the form

$$f_X(X) = g(X^t X)$$

for some function $g : \mathbb{R} \rightarrow \mathbb{R}^+$.

An extension of the spherical property is the elliptical property (Fang et al. 1990, Ch.2): a random vector $Y \in \mathbb{R}^n$ is elliptical with $(n \times n)$ positive definite characteristic matrix Σ if

$$Y = \Sigma^{1/2} X = a \Sigma^{1/2} U \quad (1)$$

where X is spherical. We assume without loss of generality that all vectors have zero mean.

3 Wick's theorem for the uniform distribution on the sphere

Wick's theorem can be extended to the case of a vector uniformly distributed on the sphere \mathcal{S}_n as follows.

Theorem 2 *If U is uniformly distributed on the sphere \mathcal{S}_n then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(m + \frac{n}{2}\right)} \sum_{\sigma} \prod_{i \in I_{2m}/\sigma} l_i^t l_{\sigma(i)}$$

and the expectation equals zero if $2m$ is replaced by $2m + 1$.

PROOF. A stochastic representation for vector U is

$$U = \frac{N}{\|N\|}$$

where N is a Gaussian vector with unit covariance matrix. By the polar factorization property, random variable $r = \|N\|$ is independent of U ; we deduce from the linearity of \mathcal{L}_i that

$$E \prod_{i=1}^{2m} \mathcal{L}_i(N) = E \prod_{i=1}^{2m} r \mathcal{L}_i(U) = Er^{2m} E \prod_{i=1}^{2m} \mathcal{L}_i(U)$$

so that

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{E \prod_{i=1}^{2m} \mathcal{L}_i(N)}{Er^{2m}}.$$

Since $r = \sqrt{N^t N}$ is chi-distributed with n degrees of freedom, elementary algebra yields

$$Er^{2m} = \frac{2^m \Gamma\left(m + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \quad (2)$$

□

From the linearity of forms \mathcal{L}_i and by stochastic representation (1), we deduce a version of Wick's theorem for any elliptical vector as follows.

Theorem 3 *If $X \in \mathbb{R}^n$ is an elliptical vector with stochastic representation (1) and if random variable a has moments up to order $2m$, then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = \frac{Ea^{2m}}{Er^{2m}} \sum_{\sigma} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

where Er^{2m} is given by (2).

4 Wick's theorem for Gaussian scale mixtures

A random vector X is a Gaussian scale mixture if its probability density function writes

$$f_X(X) = \int_0^{+\infty} g(x; a\Sigma) dH(a)$$

where $g(x; \Sigma)$ is the Gaussian distribution with $(n \times n)$ covariance matrix Σ on \mathbb{R}^n and H is a cumulative distribution function. A stochastic representation

of such a vector is

$$X = \sqrt{a}N \quad (3)$$

where N is a Gaussian vector with covariance matrix Σ and a is a positive random variable independent of N with cumulative distribution function H . Cauchy, alpha-stable and Student-t random vectors belong to the family of Gaussian scale mixtures.

If X is a Gaussian scale mixture, then it is also elliptical; however, the contrary is not true, as shown by the Pearson II family of n -variate probability density functions (Johnson 1987)

$$f_X(X) = \frac{\Gamma\left(\frac{n}{2} + m + 1\right)}{\Gamma(m + 1) |\pi K|^{1/2}} \left(1 - X^t K^{-1} X\right)_+^m$$

with notation $(x)_+ = \max(x, 0)$, with parameter $m > -1$ and where K is a positive definite matrix.

Using stochastic representation (3), Wick's theorem extends to Gaussian scale mixtures as follows.

Theorem 4 *If X is a Gaussian scale mixture as in (3) with $Ea^m < \infty$, then*

$$E \prod_{i=1}^{2m} \mathcal{L}_i(X) = Ea^m \sum_{\sigma \in \Pi_m} \prod_{i \in I_{2m}/\sigma} l_i^t \Sigma l_{\sigma(i)}$$

We note that Repetowicz et al. (Repetowicz 2004) use Wick's theorem to address a more general problem, namely the determination of mixed moments of linear combinations of i.i.d. Student-t vectors; in this case, random variable a in (3) is inverse Gamma distributed. The above theorem, although restricted to mixed moments of a *single* random vector, holds however for the class of all Gaussian scale mixtures for which $Ea^m < \infty$.

5 Application: a result by Folland

In this section, we show that the results above allow to recover easily the following theorem by Folland (Folland 2001) about integration of a monomial over the sphere \mathcal{S}_n .

Theorem 5 *If*

$$P(X) = \prod_{i=1}^n X_i^{\alpha_i}$$

then

$$\int_{\mathcal{S}_n} P(X) d\mu = \begin{cases} 0 & \text{if some } \alpha_i \text{ is odd} \\ \frac{2\Gamma(\beta_1)\dots\Gamma(\beta_n)}{\Gamma(\beta_1+\dots+\beta_n)} & \text{else} \end{cases}$$

where

$$\beta_i = \frac{\alpha_i + 1}{2}$$

and $d\mu$ is the surface measure on the sphere \mathcal{S}_n .

PROOF. Let us assume first that all α_i are even and define the integer parameter m by $2m = \sum_{i=1}^n \alpha_i$. Defining $\alpha_0 = 0$, we consider the function $T : [1, 2m] \rightarrow [1, n]$ such that

$$T(i) = j \quad \text{if} \quad \alpha_1 + \dots + \alpha_{j-1} < i \leq \alpha_1 + \dots + \alpha_j.$$

We choose forms \mathcal{L}_i according to $l_i = \delta_{T(i)}$ where δ_k is the k -th column vector of the $n \times n$ identity matrix. Since the scalar product $l_i^t l_{\sigma(i)}$ equals zero when $T(i) \neq T(\sigma(i))$, we consider only the $\prod_{i=1}^n \frac{\alpha_i!}{2^{\frac{\alpha_i}{2}} (\frac{\alpha_i}{2})!}$ pairings for which $T(i) = T(\sigma(i))$ and $l_i^t l_{\sigma(i)} = \delta_{T(i)}^t \delta_{T(i)} = 1$. Thus, by theorem (2)

$$E \prod_{i=1}^{2m} \mathcal{L}_i(U) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(m + \frac{n}{2}\right)} \prod_{i=1}^n \frac{\alpha_i!}{2^{\frac{\alpha_i}{2}} \left(\frac{\alpha_i}{2}\right)!} = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m \Gamma\left(\sum_{i=1}^n \beta_i\right)} \frac{2^{\sum_{i=1}^n \beta_i}}{(2\pi)^{n/2}} \prod_{i=1}^n \Gamma(\beta_i).$$

Since $d\mu$ is the unnormalized surface measure and as the surface of the sphere is $S(\mathcal{S}_n) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$,

$$\int_{\mathcal{S}_n} P(X) d\mu = S(\mathcal{S}_n) E \prod_{i=1}^{2m} \mathcal{L}_i(U)$$

which yields the result.

In the case where some α_i is odd, the integral equals zero since the surface measure on the sphere is an even function in each of the variables. \square

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