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Abstract

We consider the quantum dynamics of many bosons systems in the mean field limit with a singular pair-interaction potential, including the attractive or repulsive Coulombic case in three dimensions. By using a measure transportation technique developed in [3], we show that Wigner measures propagate along the nonlinear Hartree flow. Such property was previously proved only for bounded potentials in our works [5, 6] with a slightly different strategy.

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1 Introduction

The evolution of a system of many quantum non relativistic particles is described by an n-body Schrödinger Hamiltonian. The mean field limit consists in replacing this problem by a non linear 1-particle problem, by considering that one generic particle interacts with the average field of all particles, in the limit when the number of particles is large and the interaction potential is inversely weak. Practically, it is common knowledge that this approximation starts to be very effective when the number of particles exceeds a few tens. In the last decades, many works have been devoted to justify this limit. Most of them considered the mean field dynamics of well prepared quantum states, coherent states or Hermite states, by following and extending the phase-space approach, also known as the Hepp method (see [25, 27, 32, 33, 39, 44, 57]) or by studying the BBGKY hierarchy of reduced density matrices (see [8, 17, 19, 20, 42, 59]). Some of these results achieved to include very singular pair interaction potentials in [9, 19, 20, 44] or considered the rate of convergence (see [7, 57, 44]), sometimes motivated by the modelling of Bose-Einstein condensates (see a.e.[21, 48, 1]). In this article, we continue our program, which consists in deriving the mean field limit, for general initial data in the bosonic framework. Our strategy is inspired by older attempts to give substance to the formal link between bosonic Quantum Field Theory and the finite dimensional microlocal or phase-space analysis (see [10, 23, 24, 45, 47]). With this respect, the small parameter $\varepsilon = \frac{1}{n}$ asymptotics is the infinite dimensional version of semiclassical analysis. And it has been realized in the 90's, that the Wigner (or semiclassical) measures provide a powerful tool in order to obtain the leading term in the semiclassical limit (see [29, 30, 37, 49]), because they flexibly and efficiently incorporate a priori estimates (see [13, 14, 22, 51, 53]).

In [4] Wigner measures were introduced in the infinite dimensional setting and their main properties were studied. This presentation exploited and clarified the intimate relationship between pseudo-differential calculus, phase-space geometry and the probability approach, inherent to bosonic QFT. In [5], the dynamics for well prepared data and bounded interaction potentials was reconsidered within this approach. The general propagation result was obtained in [6] for bounded interaction potentials. In particular, we showed that the BBGKY hierarchy dynamics is a projected picture of the evolution of the Wigner measure, for which there is a closed equation.

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One difficulty which was solved in [6] is concerned with the integration of a weak Liouville equation valid after testing with cylindrical or polynomial observables: Such classes of observables are not preserved by the nonlinear Hamiltonian mean field flow. For bounded interaction potentials, the number conservation allows polynomial approximations of the nonlinear deformation in balls of the phase-space. This done by following a truncated Dyson expansion approach presented in [25, 26, 27]. In applications, an important case is the one of 2-body Coulombic interaction because it models the general non relativistic motion of charged (or gravitational) particles. Again there are results about the mean field problem for specific initial data (see [9, 44]), but the approach that we followed in [6] essentially fails. With a singular pair interaction potential, a solution to this problem is provided by measure transportation techniques developed for optimal transport theory (see [3, 61]). Hence, the dynamical mean field limit relies even more on the fact that Wigner measures are probability measures on the phase-space.

We now expose our main result. The Hamiltonian of an n-body quantum system, with a pair interaction potential, is given by the Schrödinger operator

$$H_\varepsilon^{(n)} = \varepsilon \sum_{i=1}^{n} -\Delta_{x_i} + \varepsilon^2 \sum_{1\leq i<j\leq n} V(x_i - x_j)\,,$$

where ε is a positive parameter and $x_i, x_j \in \mathbb{R}^d$. We assume that the particles obey Bose statistics. So, we consider $H_\varepsilon^{(n)}$ as an operator acting on the space $L^2_s(\mathbb{R}^{dn})$ of symmetric square integrable functions. This means that

$$\Psi \in L^2_s(\mathbb{R}^{dn}) \ \text{ iff } \ \Psi \in L^2(\mathbb{R}^{dn}) \ \text{ and } \ \Psi(x_1, \cdots, x_n) = \Psi(x_{\sigma_1}, \ldots, x_{\sigma_n}) \text{ a.e}$$

for any permutation σ on the symmetric group $\mathfrak{S}_n$. The mean field asymptotics is concerned with the limit as $\varepsilon \to 0$ and $n\varepsilon \to 1$, where $n = \left[\frac{1}{\varepsilon}\right]$ represents the number of particles of the system. Let $\mathcal{H}$ be the direct sum of Hilbert spaces of the form

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} L^2_s(\mathbb{R}^{dn})\,,$$

and consider the Hamiltonian of the many-bosons system (with arbitrary number of particles) as

$$H_\varepsilon = \bigoplus_{n=0}^{\infty} H_\varepsilon^{(n)}\,. \tag{1}$$

An obvious feature of the operator H_ε is the conservation of the number of particles. Hence, it is useful to define the number operator

$$\mathbf{N} = \bigoplus_{n=0}^{\infty} \varepsilon n \, \mathbb{1}_{L^2_s(\mathbb{R}^{dn})}.$$

The free Hamiltonian, corresponding to $V = 0$, will be denoted by H_ε^0:

$$H_\varepsilon^0 = \bigoplus_{n=0}^{\infty} H_\varepsilon^{0,(n)}\,, \quad H_\varepsilon^{0,(n)} = \varepsilon \sum_{i=1}^{n} -\Delta_{x_i}\,.$$

Second quantization is a natural framework for the study of many-body problems and, even more, it helps to understand the mean field limit and the structures behind it. However, the result can be presented without using the language of quantum field theory. We just mention that the operator H_ε can be formally rewritten as

$$H_\varepsilon = \int_{\mathbb{R}^d} \nabla a^*(x).\nabla a(x)\ dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y) a^*(x) a^*(y) a(x) a(y)\ dxdy\,,$$

with the ε-dependent canonical commutation relations $[a(x)\,,\,a^*(y)] = \varepsilon\delta(x-y)$. It is interpreted as the Wick quantization of the classical Hamiltonian

$$h(z,\bar{z}) = \int_{\mathbb{R}^d} |\nabla z(x)|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} |z(x)|^2|z(y)|^2 V(x-y)\, dxdy\,. \tag{2}$$

In our analysis an operator, which violates the number of particles conservation, will play an important role, namely the Weyl operator. Such operators are given for $f \in L^2(\mathbb{R}^d)$ by

$$W(f) = e^{\frac{i}{\sqrt{2}}[a^*(f)+a(f)]}\,,$$

where $a^*(f), a(f)$ are the creation-annihilation operators on $\mathcal{H}$ satisfying the ε-canonical commutation relations (CCR):

$$[a(f_1), a^*(f_2)] = \varepsilon\langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)}\, \mathbb{1}\!\!\mathrm{l}, \quad [a^*(f_1), a^*(f_2)] = 0 = [a(f_1), a(f_2)]\,.$$

Accurate definitions on second quantized operators can be found in Appendix B.

Our approach is based on Wigner measures which are Borel probability measures on the infinite dimensional phase-space $\mathcal{Z}_0 := L^2(\mathbb{R}^d, \mathbb{C})$. The states of the many-bosons system are positive trace-class operators on $\mathcal{H}$ of normalized trace equal to 1 (i.e.: normal states or density operators). To every family of those states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ we asymptotically assign, when $\varepsilon \to 0$, at least one Borel probability measure μ on $\mathcal{Z}_0 := L^2(\mathbb{R}^d, \mathbb{C})$, called Wigner measure, such that there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, $\lim_k \varepsilon_k = 0$ and

$$\lim_{k\to 0} \mathrm{Tr}[\varrho_{\varepsilon_k}\, W(\sqrt{2}\pi\xi)] = \mathcal{F}^{-1}(\mu)(\xi)\,,$$

under the sole uniform estimate $\mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}^\delta\right] \leq C_\delta$ for some $\delta > 0$. Here $\mathcal{F}^{-1}(\mu)$ is the inverse Fourier transform of μ.
The problem of the mean field dynamics questions whether the asymptotic quantities, namely Wigner measures, as $\varepsilon \to 0$ associated with

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,, \quad t\in\mathbb{R}$$

are transported by the flow $\Phi(t,s)$ generated by the classical Hamiltonian $h(z,\bar{z})$ and given, after writing $z_t = \Phi(t,s)(z_s)$, by

$$i\partial_t z_t = (\partial_{\bar{z}} h)(z_t, \bar{z}_t) = -\Delta z_t + V * |z_t|^2 z_t\,. \tag{3}$$

After checking that the Hamiltonian (1) has a self-adjoint realization so that the quantum dynamic is well defined on $\mathcal{H}$ and after checking that the mean field flow is well defined on $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$, our main result is stated below.
Throughout the paper we assume that the real valued potential V satisfies the assumptions

$$V(-x) = V(x) \in \mathbb{R}\,, \tag{A1}$$

$$V(1-\Delta)^{-1/2} \in \mathcal{L}(\mathcal{Z}_0)\,, \tag{A2}$$

$$\text{and} \qquad (1-\Delta)^{-1/2} V (1-\Delta)^{-1/2} \in \mathcal{L}^\infty(\mathcal{Z}_0)\,. \tag{A3}$$

Here we use the notation $\mathcal{L}(\mathfrak{h})$ for the space of bounded operators on the Hilbert space $\mathfrak{h}$ and $\mathcal{L}^p(\mathfrak{h})$, $1 \leq p \leq +\infty$, for the Schatten classes, $\mathcal{L}^\infty(\mathfrak{h})$ being the space of compact operators for $p = +\infty$.

Theorem 1.1. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\mathcal{H}$ with a single Wigner measure μ_0 such that the bound*

$$\mathrm{Tr}[(\mathbf{N} + H_\varepsilon^0)^\delta \varrho_\varepsilon] \leq C_\delta < +\infty\,, \tag{4}$$

holds uniformly w.r.t $\varepsilon \in (0,\bar{\varepsilon})$ for some $\delta > 0$.
Then for all $t\in\mathbb{R}$, the family $(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar{\varepsilon})}$ has a unique Wigner measure μ_t which is a Borel measure on $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$. This measure $\mu_t = \Phi(t,0)_*\mu_0$ is the push forward of the initial measure μ_0 by the flow associated with (3), well defined on $\mathcal{Z}_1$.

In a formal level the proof of the latter theorem is rather simple. We first write an integral formula

$$\mathrm{Tr}[\varrho_\varepsilon(t)W(\xi)] = \mathrm{Tr}[\varrho_\varepsilon W(\xi)] + i\int_0^t \mathrm{Tr}[\varrho_\varepsilon(s)W(\xi)\sum_{j=1}^{4}\varepsilon^{j-1}\mathcal{O}_j]\,ds\,,$$

where $\varrho_\varepsilon(t) = e^{-it/\varepsilon H_\varepsilon}\varrho_\varepsilon e^{it/\varepsilon H_\varepsilon}$ and $\mathcal{O}_j$ are some Wick quantized observables. Then we attempt to take the limit as $\varepsilon \to 0$, where in the r.h.s only the term with $j = 1$ will contribute. We end up in a formal level with a transport equation on the Wigner measures

$$\partial_t\mu + i\,\{h\,,\,\mu\} = 0, \quad \{h,\mu\} = \partial_z h\,\partial_{\bar z}\mu - \partial_z\mu\,\partial_{\bar z}h$$

which we then solve by appealing to the results in [3].

Outline: In Section 2 we prove the self-adjointness of the Hamiltonian H_ε and the existence of a global flow on $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$ for the Hartree equation (3). The derivation of the mean field dynamics is done in Section 3 where we prove Theorem 1.1. Some additional properties are stated in Section 4: in particular, we draw the link with former results on bounded potential and reduced density matrices and provide non trivial examples elucidated by the Wigner measure approach. We finish the article with several appendices dedicated to second quantization, absolutely continuous curves in $\mathrm{Prob}_2(\mathcal{Z})$ as well as some weak L^p conditions for the potential V ensuring the fulfillment of the assumptions (A2) and (A3).

2 Well defined dynamics

In this section we shall prove that:

- the quantum dynamics is well defined, namely H_ε has a natural self-adjoint realization;
- the mean field dynamics is well defined on $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$, with additional useful estimates.

2.1 Self-adjoint realization of H_ε

The Hamiltonian H_ε has a particular structure explained in a general framework in Appendix A. Let V be a real-valued Lebesgue measurable function a.e. finite satisfying assumptions (A1) and (A2). The multiplication operator

$$V_\varepsilon^{(n)} = \varepsilon^2 \sum_{1\le i<j\le n} V(x_i - x_j)$$

with its natural domain $\mathcal{D}(V_\varepsilon^{(n)}) = \{\Psi \in L^2_s(\mathbb{R}^{dn}) : V_\varepsilon^{(n)}\Psi \in L^2_s(\mathbb{R}^{dn})\}$ is self-adjoint on $L^2_s(\mathbb{R}^{dn})$ as well as the differential operator

$$H_\varepsilon^{0,(n)} = \varepsilon\sum_{i=1}^{n} -\Delta_{x_i}\,, \quad \text{with} \quad \mathcal{D}(H_\varepsilon^{0,(n)}) = L^2_s(\mathbb{R}^{dn}) \cap H^2(\mathbb{R}^{dn})\,.$$

Therefore, according to Appendix A

$$V_\varepsilon = \sum_{n=0}^{\infty} V_\varepsilon^{(n)}\,, \quad \text{and} \quad H_\varepsilon^0 = \sum_{n=0}^{\infty} H_\varepsilon^{0,(n)}$$

endowed with their natural domains are self-adjoint on $\mathcal{H}$.

Proposition 2.1. *Under the assumptions (A1) and (A2):*
(i) The operator

$$H_\varepsilon^{(n)} := H_\varepsilon^{0,(n)} + V_\varepsilon^{(n)}$$

is self-adjoint on $\mathcal{D}(H_\varepsilon^{0,(n)}) \subset \mathcal{D}(V_\varepsilon^{(n)})$.
(ii) The operator

$$H_\varepsilon := \sum_{n=0}^{\infty} H_\varepsilon^{(n)}, \quad \mathcal{D}(H_\varepsilon) := \{\Psi \in \mathcal{H}\,, \sum_{n=0}^{\infty} \|H_\varepsilon^{(n)}\Psi^{(n)}\|^2 < \infty\},$$

is self-adjoint and essentially self-adjoint on $\oplus_{n=0}^{alg}\mathcal{D}_n$ *where* $\mathcal{D}_n$ *is any core of* $H_\varepsilon^{(n)}$.

Proof. (i) By assumption (A2), we see that $V_\varepsilon^{(n)}$ is infinitesimally small with respect to $H_\varepsilon^{0,(n)}$. So that, $\mathcal{D}(H_\varepsilon^{0,(n)}) \subset \mathcal{D}(V_\varepsilon^{(n)})$ and the operator $H_\varepsilon^{(n)} = H_\varepsilon^{0,(n)} + V_\varepsilon^{(n)}$ is self-adjoint on the domain of $H_\varepsilon^{0,(n)}$ by Kato-Rellich theorem.
(ii) Applying Proposition A.1, we see that H_ε is self-adjoint and essentially self-adjoint on $\oplus_{n=0}^{alg}\mathcal{D}_n$.
□

Later, it will be useful to use the reference operator

$$S_\varepsilon(\lambda) = \sum_{n=0}^{\infty} H_\varepsilon^{0,(n)} + \varepsilon n + \lambda(\varepsilon n)^3\,. \tag{5}$$

which is self-adjoint by Proposition A.1. Moreover, by functional calculus of strongly commuting self-adjoint operators we observe that $\mathcal{D}(S_\varepsilon(\lambda))$ is invariant with respect to the parameter $\lambda > 0$.

Proposition 2.2. *Under the assumptions (A1) and (A2), for any* $\lambda > 0$*, the operator* V_ε *is* $S_\varepsilon(\lambda)$*-bounded with*

$$\forall\, \Psi \in \mathcal{D}(S_\varepsilon(\lambda)), \quad \|V_\varepsilon\Psi\|_{\mathcal{H}} \leq \lambda \|V(1-\Delta)^{-1/2}\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\, \|S_\varepsilon(\lambda^{-2})\Psi\|_{\mathcal{H}}\,.$$

Therefore H_ε *is essentially self-adjoint on* $\mathcal{D}(S_\varepsilon(\lambda))$.

Proof. The multiplication operator by $V(x_1 - x_2)$, at least defined as a symmetric operator from $\mathcal{S}(\mathbb{R}^{2d})$ into $\mathcal{S}'(\mathbb{R}^{2d})$, satisfies

$$\begin{aligned} e^{ix_2 D_{x_1}} V(x_1-x_2)(1-\Delta_{x_1})^{-1/2} e^{-ix_2 D_{x_1}} &= e^{ix_2 D_{x_1}} V(x_1-x_2) e^{-ix_2 D_{x_1}} (1-\Delta_{x_1})^{-1/2} \\ &= V(x_1)(1-\Delta_{x_1})^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^{2d}))\,. \end{aligned} \tag{6}$$

For $\Psi \in \mathcal{H}, \Phi \in \mathcal{D}(S_\varepsilon(\lambda))$, taking advantage of the symmetry of those wave functions, we compute

$$\begin{aligned} \langle \Psi\,,\, V_\varepsilon \Phi\rangle &= \sum_{n=2}^{\infty} \langle \Psi^{(n)}\,,\, n(n-1)\varepsilon^2 V(x_1-x_2)\Phi^{(n)}\rangle_{L^2_s(\mathbb{R}^{dn})} \\ &= \sum_{n=2}^{\infty} \langle \Psi^{(n)}\,,\, n(n-1)\varepsilon^2 V(x_1-x_2)(1-\Delta_{x_1})^{-1/2}(1-\Delta_{x_1})^{1/2}\Phi^{(n)}\rangle_{L^2_s(\mathbb{R}^{dn})}\,. \end{aligned}$$

By noticing that

$$\begin{aligned} (n^2\varepsilon^2)^2 \|(1-\Delta_{x_1})^{1/2}\Phi^{(n)}\|^2_{L^2(\mathbb{R}^{nd})} &= (\varepsilon n)^3 \langle \Phi^{(n)}, \varepsilon \sum_{i=1}^{n} (1-\Delta_{x_i})\Phi^{(n)}\rangle_{L^2_s(\mathbb{R}^{dn})} \\ &= \|\mathbf{N}^{3/2}(\mathbf{N}+H_\varepsilon^0)^{1/2}\Phi^{(n)}\|^2_{\mathcal{H}}\,, \end{aligned}$$

the Cauchy-Schwarz inequality leads to

$$|\langle \Psi\,,\, V_\varepsilon\Phi\rangle| \leq \|V(1-\Delta)^{-1/2}\|\, \|\Psi\|_{\mathcal{H}}\, \|\mathbf{N}^{3/2}\,(\mathbf{N}+H_\varepsilon^0)^{1/2}\Phi\|_{\mathcal{H}}\,. \tag{7}$$

Now with the inequality $ab \leq (\lambda a)^2 + (b/\lambda)^2$, we see that

$$\begin{aligned} \|\mathbf{N}^{3/2}\,(\mathbf{N}+H_\varepsilon^0)^{1/2}\Phi\|^2_{\mathcal{H}} &= \langle \Phi, \mathbf{N}^3(\mathbf{N}+H_\varepsilon^0)\Phi\rangle \\ &\leq \langle \Phi, \lambda^{-2}\mathbf{N}^6 + \lambda^2(\mathbf{N}+H_\varepsilon^0)^2\Phi\rangle \\ &\leq \lambda^2 \|S_\varepsilon(\lambda^{-2})\Phi\|^2_{\mathcal{H}}\,. \end{aligned} \tag{8}$$

Putting together (7) and (8) yields the estimate. □

We end this section with some invariance properties of the domain $\mathcal{D}(S_\varepsilon(\lambda))$ with respect to the Hamiltonian H_ε and the Weyl operators.

Proposition 2.3. *For any $\lambda > 0$ and $t \in \mathbb{R}$*

$$e^{-i\frac{t}{\varepsilon}H_\varepsilon}\mathcal{D}(S_\varepsilon(\lambda)) \subset \mathcal{D}(S_\varepsilon(\lambda))\,.$$

Moreover there exists $C_\lambda > 0$ such that

$$\|S_\varepsilon(\lambda)e^{-i\frac{t}{\varepsilon}H_\varepsilon}(S_\varepsilon(\lambda)+1)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C_\lambda \quad \text{for all } t \in \mathbb{R}.$$

Proof. For any $\Psi \in \mathcal{D}(S_\varepsilon(\lambda)) \subset \mathcal{D}(H_\varepsilon)$, we observe that $e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi$ belongs to $\mathcal{D}(\mathbf{N}^3)\cap\mathcal{D}(H_\varepsilon)$ since $\mathcal{D}(S_\varepsilon(\lambda)) \subset \mathcal{D}(\mathbf{N}^3)$ and H_ε strongly commute with $\mathbf{N}$. Proposition 2.2 implies $\mathcal{D}(S_\varepsilon(\beta)+V_\varepsilon) = \mathcal{D}(S_\varepsilon(\beta)) = \mathcal{D}(S_\varepsilon(\lambda))$ when $\beta > 0$ is large enough, so that for any $\Phi \in \mathcal{D}(S_\varepsilon(\lambda))$

$$\begin{aligned}\langle (S_\varepsilon(\beta)+V_\varepsilon)\,\Phi, e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi\rangle_{\mathcal{H}} &= \langle (H_\varepsilon+\mathbf{N}+\beta\mathbf{N}^3)\,\Phi, e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi\rangle_{\mathcal{H}}\\ &= \langle \Phi, (H_\varepsilon+\mathbf{N}+\beta\mathbf{N}^3)e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi\rangle_{\mathcal{H}}\end{aligned}$$

and hence $e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi$ belongs to $\mathcal{D}((S_\varepsilon(\beta)+V_\varepsilon)^*) = \mathcal{D}(S_\varepsilon(\beta)) = \mathcal{D}(S_\varepsilon(\lambda))$.
Again for β large enough, we know that

$$1 + S_\varepsilon(\beta) + V_\varepsilon = (1 + V_\varepsilon(S_\varepsilon(\beta)+1)^{-1})(S_\varepsilon(\beta)+1)\,,$$

and

$$(1+\mathbf{N}+\beta\mathbf{N}^3+H_\varepsilon)^{-1} = (S_\varepsilon(\beta)+1)^{-1}(1+V_\varepsilon(S_\varepsilon(\beta)+1)^{-1})^{-1}\,.$$

Therefore the operators $(1+S_\varepsilon(\beta))(1+\mathbf{N}+\beta\mathbf{N}^3+H_\varepsilon)^{-1}$ and $(1+\mathbf{N}+\beta\mathbf{N}^3+H_\varepsilon)(S_\varepsilon(\beta)+1)^{-1}$ are bounded. We conclude with

$$\begin{aligned}S_\varepsilon(\lambda)e^{-i\frac{t}{\varepsilon}H_\varepsilon}(1+S_\varepsilon(\lambda))^{-1} = S_\varepsilon(\lambda)(1+S_\varepsilon(\beta))^{-1}(1+S_\varepsilon(\beta))(1+\mathbf{N}+\beta\mathbf{N}^3+H_\varepsilon)^{-1}\\ \circ\, e^{-i\frac{t}{\varepsilon}H_\varepsilon}(1+\mathbf{N}+\beta\mathbf{N}^3+H_\varepsilon)(1+S_\varepsilon(\beta))^{-1}(1+S_\varepsilon(\beta))(1+S_\varepsilon(\lambda))^{-1}\,.\end{aligned}$$

□

Proposition 2.4. *For any $\xi \in H^2(\mathbb{R}^d)$ and $\lambda > 0$ the domain $\mathcal{D}(S_\varepsilon(\lambda))$ is invariant with respect to Weyl operators $W(\xi)$ with*

$$\|(S_\varepsilon(\lambda)+1)^{-1}W(\xi)S_\varepsilon(\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C_{\lambda,\xi}\,,$$

uniformly w.r.t $\varepsilon \in (0,\bar{\varepsilon})$ for some constant $C_{\lambda,\xi} > 0$.

Proof. For all $\Phi, \Psi \in \mathcal{D}(S_\varepsilon(\lambda))$, we write

$$\langle \Phi, W(\xi)^* S_\varepsilon(\lambda) W(\xi)\Psi\rangle = \langle \Phi, (S_\varepsilon(\lambda) + Q_\varepsilon^{Wick})\Psi\rangle,$$

where Q_ε is the following polynomial

$$Q_\varepsilon(z) = \langle z + \frac{i\varepsilon}{\sqrt{2}}\xi, -\Delta(z+\frac{i\varepsilon}{\sqrt{2}}\xi)\rangle_{\mathcal{Z}_0} - \langle z, -\Delta z\rangle_{\mathcal{Z}_0} + P_\varepsilon(z+\frac{i\varepsilon}{\sqrt{2}}\xi) - P_\varepsilon(z)$$

and $P_\varepsilon(z) = |z|^6_{\mathcal{Z}_0} + 3\varepsilon|z|^4_{\mathcal{Z}_0} + \varepsilon^2|z|^2_{\mathcal{Z}_0}$ is the complete Wick symbol of $\mathbf{N}^3$, according to Proposition B.2 or by direct computation. The assumption $\xi \in H^2(\mathbb{R}^d)$ ensures that Q_ε is uniformly bounded in $\oplus_{p+q\leq 3}\mathcal{P}_{p,q}(\mathcal{Z}_0)$ and the number estimate of Proposition B.3 says that $Q_\varepsilon^{Wick}\langle\mathbf{N}\rangle^{-\frac{3}{2}}$ is a bounded operator and therefore

$$Q_\varepsilon^{Wick}(S_\varepsilon(\lambda)+1)^{-1} \in \mathcal{L}(\mathcal{H})\,.$$

Hence for $\Psi \in \mathcal{D}(S_\varepsilon(\lambda))$,

$$S_\varepsilon(\lambda)W(\xi)\Psi = W(\xi)\left[\frac{S_\varepsilon(\lambda)}{S_\varepsilon(\lambda)+1} + Q_\varepsilon^{Wick}(S_\varepsilon(\lambda)+1)^{-1}\right](S_\varepsilon(\lambda)+1)\Psi$$

and $W(\xi)\Psi$ belongs to $\mathcal{D}(S_\varepsilon(\lambda))$, with $\|S_\varepsilon(\lambda)W(\xi)\Psi\| \leq C_{\lambda,\xi}\|(S_\varepsilon(\lambda)+1)\Psi\|$. □

Proposition 2.5. *For any function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $\lambda > 0$, the operator $\chi(\mathbf{N}, H_\varepsilon)$ satisfies*

$$\forall k \in \mathbb{N}, \quad \|\mathbf{N}^k S_\varepsilon(\lambda)\chi(\mathbf{N}, H_\varepsilon)\|_{\mathcal{L}(\mathcal{H})} \leq C_{\lambda,\chi}^{k+1}$$

for some $C_{\lambda,\chi} > 0$.

Proof. The operators $\mathbf{N}$, H_ε (like $\mathbf{N}$ and $(\mathbf{N} + H_\varepsilon^0)$ are strongly commuting self-adjoint operators so that the functional calculus is well defined for the pair $(\mathbf{N}, H_\varepsilon)$. With a cut-off function $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi_1(x) \equiv 1$ on a neighborhood of supp χ, the operator $\mathbf{N}^k(1 + \mathbf{N} + \beta\mathbf{N}^3 + H_\varepsilon)\chi_1(\mathbf{N})\chi(H_\varepsilon, \mathbf{N})$ is bounded with

$$\|(1 + \mathbf{N} + \beta\mathbf{N}^3 + H_\varepsilon)\mathbf{N}^k\chi_1(\mathbf{N})\chi(H, \mathbf{N})\|_{\mathcal{L}(\mathcal{H})} \leq C_\beta C_\chi^{k+1}\,.$$

For sufficiently large β, Proposition 2.2 says

$$\|S_\varepsilon(\beta)(1 + \mathbf{N} + \beta\mathbf{N}^3 + H_\varepsilon)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C'_\beta\,.$$

This is done with

$$\mathbf{N}^k S_\varepsilon(\lambda)\chi(\mathbf{N}, H_\varepsilon) = S_\varepsilon(\lambda)(1 + S_\varepsilon(\beta))^{-1}(1 + S_\varepsilon(\beta)(1 + \mathbf{N} + \beta\mathbf{N}^3 + H_\varepsilon)^{-1}\chi(\mathbf{N}, H_\varepsilon)\,,$$

and $C_{\lambda,\chi} = \max\left\{C_\chi,\ C_\beta C'_\beta \|S_\varepsilon(\lambda)(1 + S_\varepsilon(\beta))^{-1}\|\right\}$. □

2.1.1 Mean field dynamics

We shall use another more convenient writing of the Cauchy problem

$$\begin{cases} i\partial_t z_t = -\Delta z_t + V * |z_t|^2 z_t \\ z_{t=0} = z_0\,. \end{cases} \tag{9}$$

After setting $\tilde{z}_t = e^{it(-\Delta)}z_t = e^{-it\Delta}z_t$ it becomes

$$\begin{cases} i\partial_t \tilde{z}_t = e^{-it\Delta}\left[V * |e^{it\Delta}\tilde{z}_t|^2(e^{it\Delta}\tilde{z}_t)\right] \\ \tilde{z}_{t=0} = z_0\,. \end{cases} \tag{10}$$

Proposition 2.6. *Assume (A1) and (A2). For any $z_0 \in \mathcal{Z}_1 = H^1(\mathbb{R}^d)$ the Cauchy problem (9) admits a unique solution $(t \to z_t) \in \mathcal{C}^0(\mathbb{R}; H^1(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}; H^{-1}(\mathbb{R}^d))$. More precisely, the Cauchy problem (10) which is equivalent to (9) admits a unique solution in $\mathcal{C}^1(\mathbb{R}; H^1(\mathbb{R}^d))$. Moreover these solutions verify*

$$|z_t|_{L^2} = |\tilde{z}_t|_{L^2} = |z_0|_{L^2} \tag{11}$$

$$\textit{and} \qquad h(z_t, \overline{z_t}) = h(z_0, \overline{z_0})\,, \tag{12}$$

$$\textit{for} \qquad h(z, \overline{z}) = \int_{\mathbb{R}^d} |\nabla z|^2(x)\ dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)|z(x)|^2|z(y)|^2\ dxdy\,.$$

Finally, the time-dependent velocity field defined on $\mathbb{R} \times \mathcal{Z}_1$ by

$$v(t, z) = e^{-it\Delta}([V * |e^{it\Delta}z|^2]e^{it\Delta}z)$$

satisfies the estimates

$$|v(t, z)|_{\mathcal{Z}_0} \leq \|V(1-\Delta)^{-1/2}\||z|^2_{\mathcal{Z}_0}|z|_{\mathcal{Z}_1} \tag{13}$$

$$\textit{and} \qquad |v(t, z)|_{\mathcal{Z}_1} \leq \|V(1-\Delta)^{-1/2}\||z|^2_{\mathcal{Z}_1}|z|_{\mathcal{Z}_0}\,. \tag{14}$$

Proof. The first results are standard (see a.e. [16, 31]) in the analysis on nonlinear evolution equation. Nevertheless, we recall the details of the proof be cause it also contains (13)(14), which is crucial in our analysis.

By considering the second formulation (10), it suffices to prove that the mapping $z \to (V * |z|^2)z$ is locally Lipschitz in $H^1(\mathbb{R}^d)$. After noticing that the distributional derivative of $(V * |z|^2)z$ is

$$\partial_x[(V * \overline{z_1}z_2)z_3] = V * (\partial_x \overline{z}_1 z_2 + \overline{z}_1 \partial_x z_2)z_3 + (V * \overline{z_1}z_2)(\partial_x z_3)\,, \tag{15}$$

it is reduced to the estimate of $V * (\overline{z_1}z_2)z_3$ in L^2 in terms of the L^2 and H^1- norms of z_1, z_2, z_3. For $\xi \in L^2(\mathbb{R}^d)$, write

$$\langle \xi\,,\, V * (\overline{z_1}z_2)z_3\rangle_{L^2(\mathbb{R}^d)} = \langle z_1 \otimes \xi\,,\, V(x_1 - x_2)z_2 \otimes z_3\rangle_{L^2(\mathbb{R}^{2d})}\,.$$

When $\tilde{b}$ is the multiplication operator by $V(x_1 - x_2)$, the estimate (6) says that $\tilde{b}(1 - \Delta_{x_1})^{-1/2}$ is bounded, with

$$|\langle \xi \otimes z_1\,,\, V(x_1 - x_2)z_2 \otimes z_3\rangle_{L^2(\mathbb{R}^{2d})}| \leq \|V(1-\Delta)^{-1/2}\|\ |\xi|_{L^2}|z_1|_{L^2}|z_2|_{H^1}|z_3|_{L^2}\,.$$

A symmetric argument of (6) says $\tilde{b}(1 - \Delta_{x_2})^{-1/2}$ is bounded, with

$$|\langle \xi \otimes z_1\,,\, V(x_1 - x_2)z_2 \otimes z_3\rangle_{L^2(\mathbb{R}^{2d})}| \leq \|V(1-\Delta)^{-1/2}\|\ |\xi|_{L^2}|z_1|_{L^2}|z_2|_{L^2}|z_3|_{H^1}\,.$$

Finally the symmetry of the expression $V * (\overline{z_1}z_2)z_3$ w.r.t exchanging $\overline{z_1}$ and z_2 gives

$$|\langle \xi \otimes z_1\,,\, V(x_1 - x_2)z_2 \otimes z_3\rangle_{L^2(\mathbb{R}^{2d})}| \leq \|V(1-\Delta)^{-1/2}\|\ |\xi|_{L^2}|z_1|_{H^1}|z_2|_{L^2}|z_3|_{L^2}\,.$$

We have proved, owing to (15),

$$|V * (\overline{z_1}z_2)z_3|_{L^2} \leq \|V(1-\Delta)^{-1/2}\| \min_{\sigma\in\mathfrak{S}_3} |z_{\sigma(1)}|_{H^1}|z_{\sigma(2)}|_{L^2}|z_{\sigma(3)}|_{L^2} \tag{16}$$

from which we deduce

$$|V * (\overline{z_1}z_2)z_3|_{H^1} \leq \|V(1-\Delta)^{-1/2}\| \min_{\sigma\in\mathfrak{S}_3} |z_{\sigma(1)}|_{H^1}|z_{\sigma(2)}|_{H^1}|z_{\sigma(3)}|_{L^2}\,. \tag{17}$$

Since $z \to e^{it\Delta}z$ preserve the L^2 and H^1 norms, the velocity field estimates (13) and (14) are consequences of (16) and (17).
For the sake of completeness, let us finish the proof of the global well-posedness of the Cauchy problem. The estimate (17) provides the Lipschitz property of $z \to V * |z|^2 z$ in $H^1(\mathbb{R}^d)$. This implies the local in time existence and uniqueness of a solution to (10) in $\mathcal{C}^1((-T_{z_0}, T_{z_0}); H^1(\mathbb{R}^d))$, and therefore the local in time existence and uniqueness of a solution to (9) $\mathcal{C}^0((-T_{z_0}, T_{z_0}); H^1(\mathbb{R}^d)) \cap \mathcal{C}^1([-T_{z_0}, T_{z_0}]; H^{-1}(\mathbb{R}^d))$. The global in time existence then comes as usual from the control of $|z_t|_{H^1} = |\tilde{z}_t|_{H^1}$ deduced from the conservations of (11) and (12). For (11), take the real part of the scalar product of each member of (9) with $\overline{z}_t$. This implies $\partial_t |z_t|^2_{L^2} = 0$.
For (12) take the scalar product with $\chi(-R^{-1}\Delta)\partial_t z_t$ where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfies $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of 0, with $R > 0$:

$$\begin{aligned} 0 &= 2\mathrm{Re}\ \langle \partial_t z_t\,,\, \chi(-R^{-1}\Delta) i\partial_t z_t\rangle \\ &= \partial_t \langle z_t\,,\, -\Delta\chi(-R^{-1}\Delta)z_t\rangle + 2\mathrm{Re}\ \langle \partial_t z_t\,,\, \chi(-R^{-1}\Delta)[(V * |z_t|^2)z_t]\rangle \end{aligned}$$

Integrating this identity from 0 to t and taking the limit as $R \to \infty$ with the help of (17) gives

$$\int_{\mathbb{R}^d} |\nabla z_t|^2 dx - \int_{\mathbb{R}^d} |\nabla z_0|^2\, dx + 2\int_0^t \mathrm{Re}\ \langle \partial_s z_s\,,\, (V * |z_s|^2)z_s\rangle\ ds\ = 0\,.$$

Due to the symmetry of $V(x) = V(-x)$, the last integrand equals

$$\begin{aligned} \mathrm{Re}\ \langle \partial_s z_s\,,\, (V * |z_s|^2)z_s\rangle &= \int_{\mathbb{R}^{2d}} \partial_s(|z_s(x)|^2)V(x-y)|z_s(y)|^2\ dxdy \\ &= \frac{1}{2}\partial_s \int_{\mathbb{R}^{2d}} |z_s(x)|^2 V(x-y)|z_s(y)|^2\ dxdy\,. \end{aligned}$$

The conserved quantities (11) and (12) combined with (16) imply $|z_t|_{H^1} \leq C|z_0|_{H^1}$ for some constant independent of $t \in (-T_{z_0}, T_{z_0})$, and hence $T_{z_0} = +\infty$. □

3 Derivation of the mean field dynamics

This section contains the proof of our main Theorem 1.1. Below, we recall from our previous work [4] the notion of infinite dimensional Wigner measures and collect some of their properties. We will often make use of Weyl and Wick quantization throughout this section. So, we suggest first the reading of Appendix B.
Two phase-spaces will be necessary for this analysis: $\mathcal{Z}_0 = L^2(\mathbb{R}^d;\mathbb{C})$ (resp. $\mathcal{Z}_1 = H^1(\mathbb{R}^d;\mathbb{C})$) endowed with its scalar product $\langle\ ,\ \rangle$ (resp. $\langle z_1, z_2\rangle_{\mathcal{Z}_1} = \langle z_1\,,\,(1-\Delta)z_2\rangle$), its norm $|z|^2_{\mathcal{Z}_0} = \langle z\,,\,z\rangle = |z|^2_{L^2}$ (resp. $|z|^2_{\mathcal{Z}_1} = |z|^2_{H^1}$), its real scalar product $\mathrm{Re}\,\langle z\,,\,z\rangle$ (resp. $\mathrm{Re}\,\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_1}$). Only on $\mathcal{Z}_0$, we will use the symplectic structure with $\sigma(z_1,z_2) = \mathrm{Im}\,\langle z_1\,,\,z_2\rangle$. Meanwhile, the real euclidean structure on $\mathcal{Z}_1$ is important especially when referring to Appendix C.

3.1 Wigner measures

The Wigner measures are defined after the next result proved in [4, Theorem 6.2].

Theorem 3.1. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ parametrized by ε. Assume $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\delta] \leq C_\delta$ uniformly w.r.t. $\varepsilon\in(0,\overline{\varepsilon})$ for some fixed $\delta>0$ and $C_\delta\in(0,+\infty)$. Then for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\varepsilon_n = 0$, there exist a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a Borel probability measure μ on $\mathcal{Z}_0$, such that*

$$\lim_{k\to\infty} \mathrm{Tr}[\varrho_{\varepsilon_{n_k}} b^{Weyl}] = \int_{\mathcal{Z}_0} b(z)\,d\mu(z)\,,$$

for all $b\in\mathcal{S}_{cyl}(\mathcal{Z}_0)$.
Moreover this probability measure μ satisfies $\displaystyle\int_{\mathcal{Z}_0} |z|^{2\delta}_{\mathcal{Z}_0}\,d\mu(z) < \infty$.

Definition 3.2.
The set of Wigner measures associated with a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ (resp. a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) which satisfies the assumptions of Theorem 3.1 is denoted by

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon))\,,\quad (\textit{resp. } \mathcal{M}(\varrho_{\varepsilon_n}, n\in\mathbb{N}))\,.$$

Moreover this definition can be extended to any family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ such that

$$\|(1+\mathbf{N})^\delta \varrho_\varepsilon (1+\mathbf{N})^\delta\|_{\mathcal{L}^1(\mathcal{H})} \leq C_\delta$$

for some $\delta>0$ with the decomposition $\varrho_\varepsilon = \lambda^{R,+}_\varepsilon\varrho^{R,+}_\varepsilon - \lambda^{R,-}_\varepsilon\varrho^{R,-}_\varepsilon + i\lambda^{I,+}_\varepsilon\varrho^{I,+}_\varepsilon - i\lambda^{I,-}_\varepsilon\varrho^{I,-}_\varepsilon$.

Wigner measures are in practice identified via their characteristic functions according to the relation

$$\begin{aligned}\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)) = \{\mu\} \quad &\Leftrightarrow \quad \lim_{\varepsilon\to0}\mathrm{Tr}[\varrho_\varepsilon\, W(\sqrt{2}\pi\xi)] = \mathcal{F}^{-1}(\mu)(\xi)\\ &\Leftrightarrow \quad \lim_{\varepsilon\to0}\mathrm{Tr}[\varrho_\varepsilon\, W(\xi)] = \int_{\mathcal{Z}_0} e^{i\sqrt{2}\mathrm{Re}\ \langle\xi,z\rangle} d\mu(z)\,.\end{aligned}$$

The expression $\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)) = \{\mu\}$ simply means that the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ is "pure" in the sense

$$\lim_{\varepsilon\to0}\mathrm{Tr}\left[\varrho_\varepsilon b^{Weyl}\right] = \int_{\mathcal{Z}} b(z)\,d\mu\,,$$

for all cylindrical symbol b without extracting a subsequence. Actually the general case can be reduced to this one, after reducing the range of parameters to $\varepsilon\in\{\varepsilon_{n_k}, k\in\mathbb{N}\}$. For checking properties of the elements of $\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))$, extracting a subsequence in this way allows to suppose without loss of generality $\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)) = \{\mu\}$.
A simple a priori estimate argument allows to extend the convergence to symbols which have a polynomial growth and to test to Wick quantized symbols with compact kernels belonging to $\mathcal{P}^\infty_{alg}(\mathcal{Z}_0) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}^\infty_{p,q}(\mathcal{Z}_0)$ (see [4, Corollary 6.14]).

Proposition 3.3. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{L}(\mathcal{H})$ parametrized by ε such that $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha] \leq C_\alpha$ holds uniformly with respect to $\varepsilon \in (0,\bar\varepsilon)$, for all $\alpha \in \mathbb{N}$,and such that $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)) = \{\mu\}$. Then the convergence*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}[\varrho_\varepsilon b^{Wick}] = \int_{\mathcal{Z}_0} b(z)\, d\mu(z)\,, \tag{18}$$

holds for any $b \in \mathcal{P}^\infty_{alg}(\mathcal{Z}_0)$.

Wigner measures are completely identified via Weyl-quantized observable. One may question whether testing on all the b^{Wick} with $b \in \mathcal{P}^\infty_{alg}(\mathcal{Z}_0)$ also identifies the Wigner measures. Indeed, this is the case according to the result proved in [4, Theorem 6.13].

Proposition 3.4. *Assume that the family of operators $(\varrho^\varepsilon)_{\varepsilon\in(0,\overline{\varepsilon})}$ satisfies*

$$\|(1+\mathbf{N})^\alpha \varrho^\varepsilon (1+\mathbf{N})^\alpha\|_{\mathcal{L}^1(\mathcal{H})} \leq C_\alpha$$

uniformly w.r.t $\varepsilon \in (0,\overline{\varepsilon})$ for all $\alpha \in \mathbb{N}$. For any fixed β belonging to $\mathcal{P}^\infty_{alg}(\mathcal{Z}_0)$ the family $(\beta^{Wick}\varrho^\varepsilon)_{\varepsilon\in(0,\overline{\varepsilon})}$ satisfies the assumptions of Definition 3.2 and

$$\mathcal{M}(\beta^{Wick}\varrho^\varepsilon) = \{\beta\mu\,,\ \mu \in \mathcal{M}(\varrho^\varepsilon)\}\ . \tag{19}$$

3.2 Weak mean field limit of the dynamics in terms of the characteristic function

After some extraction process and for some specific initial data $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$, a family $(\mu_t)_{t\in\mathbb{R}}$ of measures can be defined and solves weakly a transport equation. We consider on $L^2_s(\mathbb{R}^{2d})$ the (unbounded) multiplication operators

$$\tilde V = \frac{1}{2}V(x_1-x_2) \quad \text{and} \quad \tilde V_s = (e^{-is\Delta_{x_1}}\otimes e^{-is\Delta_{x_2}})\tilde V(e^{is\Delta_{x_1}}\otimes e^{is\Delta_{x_2}})\,,$$

and respectively associate with them the polynomials, well defined on $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$,

$$V(z) = \frac{1}{2}\langle z^{\otimes 2}, V(x-y)z^{\otimes 2}\rangle_{L^2_s(\mathbb{R}^{2d})} \quad \text{and} \quad V_s(z) = \langle z^{\otimes 2}, \tilde V_s\, z^{\otimes 2}\rangle_{L^2_s(\mathbb{R}^{2d})}\,, \quad z\in\mathcal{Z}_1\,.$$

Instead of considering

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,$$

we will rather work with

$$\tilde\varrho_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H^0_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H^0_\varepsilon}\,. \tag{20}$$

Our assumptions will be made in terms of the operator $S_\varepsilon(1)$ already introduced in (5) and which can be rewritten with Wick observables.

Definition 3.5. *The operator S_ε is defined by*

$$S_\varepsilon = \sum_{n=0}^\infty H^{0,(n)}_\varepsilon + \varepsilon n + (\varepsilon n)^3 = \mathrm{d}\Gamma(1-\Delta) + \mathbf{N}^3\,,$$

with domain $\mathcal{D}(S_\varepsilon) = \left\{\Psi\in\mathcal{H}\,,\ \sum_{n=0}^\infty \|(H^{0,(n)}_\varepsilon + \varepsilon n + (\varepsilon n)^3)\Psi^{(n)}\|^2_{L^2_s(\mathbb{R}^{dn})} < \infty\right\}$ and $H^{0,(n)}_\varepsilon = \mathrm{d}\Gamma(1-\Delta)\big|_{\bigvee^n \mathcal{Z}_0}$.

Remember that it is self-adjoint with this domain (see (5)). Moreover it can be written $S_\varepsilon = s^{Wick}_\varepsilon$ with

$$s_\varepsilon(z) = \langle z\,,\,(1-\Delta)z\rangle + \left[|z|^6_{\mathcal{Z}_0} + 3\varepsilon|z|^4_{\mathcal{Z}_0} + \varepsilon^2|z|^2_{\mathcal{Z}_0}\right]\,.$$

Proposition 3.6. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ satisfying for some constant $C>0$ the estimate*

$$\mathrm{Tr}[(1+S_\varepsilon)\varrho_\varepsilon(1+S_\varepsilon)]\leq C \quad \text{uniformly w.r.t } \varepsilon\in(0,\bar\varepsilon)\,.$$

The operator S_ε is the one given in Definition 3.5 and $\tilde\varrho_\varepsilon(t)$ is the operator given by (20). Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\bar\varepsilon)$ such that $\lim_{n\to\infty}\varepsilon_n=0$ there exist a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family $(\tilde\mu_t)_{t\in\mathbb{R}}$ of Borel probability measures on $\mathcal{Z}_0$ satisfying for any $t\in\mathbb{R}$

$$\mathcal{M}(\tilde\varrho_{\varepsilon_{n_k}}(t),k\in\mathbb{N})=\{\tilde\mu_t\}\,,$$

with the Liouville equation

$$\begin{aligned}\tilde\mu_t(e^{i\sqrt{2}\mathrm{Re}\ \langle\xi,.\rangle}) &= \tilde\mu_0(e^{i\sqrt{2}\mathrm{Re}\langle\xi,.\rangle})-2\sqrt{2}i\int_0^t\tilde\mu_s(e^{i\sqrt{2}\mathrm{Re}\ \langle\xi,z\rangle}\,\mathrm{Im}\,\langle z^{\otimes 2},\tilde V_s\xi\otimes z\rangle)\,ds\,,\\ &= \tilde\mu_0(e^{i\sqrt{2}\mathrm{Re}\ \langle\xi,.\rangle})+i\int_0^t\tilde\mu_s(\{V_s(.);e^{i\sqrt{2}\mathrm{Re}\ \langle\xi,.\rangle}\})\,ds\,,\quad \text{for all } \xi\in\mathcal{Z}_1\,. \quad (21)\end{aligned}$$

Proof. The proof uses several preliminary lemmas which we defer below. The first step is to prove the existence of Wigner measures defined for all times $t\in\mathbb{R}$. This is done in Proposition 3.9. Let us now prove the Liouville equation.
By Lemma 3.8 we have

$$\mathrm{Tr}[\tilde\varrho_\varepsilon(t)W(\xi)]=\mathrm{Tr}[\varrho_\varepsilon W(\xi)]+i\int_0^t\mathrm{Tr}[\tilde\varrho_\varepsilon(s)W(\xi)\sum_{j=1}^4\varepsilon^{j-1}b_j(s,\xi)^{Wick}]\,ds\,, \quad (22)$$

where b_j are the following polynomials

$$\begin{aligned} b_1(s,\xi)&=-2\sqrt{2}\,\mathrm{Im}\,\langle z^{\otimes 2},\tilde V_s\,\xi\otimes z\rangle \qquad & b_2(s,\xi)&=-\mathrm{Re}\,\langle z^{\otimes 2},\tilde V_s\,\xi^{\otimes 2}\rangle+2\langle\xi\vee z,\tilde V_s\xi\vee z\rangle\\ b_3(s,\xi)&=\sqrt{2}\mathrm{Im}\,\langle\xi^{\otimes 2},\tilde V_s\,\xi\otimes z\rangle \qquad & b_4(s,\xi)&=\frac14\langle\xi^{\otimes 2},\tilde V_s\,\xi^{\otimes 2}\rangle\,.\end{aligned}$$

With the number estimate in Proposition B.1, Lemma 3.7 below will ensure that the sum in the r.h.s over $j=2,\cdots,4$ converges to 0 when $\varepsilon\to 0$. On the other hand, the term with $j=1$ has a limit according to Lemma 3.10 applied with $\tilde\varrho_\varepsilon(t)$ after noticing that $\mathrm{Tr}\,[(1+S_\varepsilon)\tilde\varrho_\varepsilon(t)(1+S_\varepsilon)]\leq C'$ owing to $\|(1+S_\varepsilon)^{\pm1}e^{i\frac{t}{\varepsilon}H_0^\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon}(1+S_\varepsilon)^{\mp1}\|\leq C''$ due to Proposition 2.3. □

The above proof is completed in essentially three steps: 1) The relation (22) is first established by extending Wick-calculus arguments to the case when V is unbounded, and rough estimates for $b_j(s,\xi)^{Wick}$, $j=1,\dots,4$, are given; 2) An Ascoli type argument, relying on these rough estimates allows to make the subsequence extraction $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ uniform for all $t\in\mathbb{R}$; 3) An additional compactness argument is given in order to ensure the convergence of the term with $j=1$ in (22).

3.2.1 Wick calculus with unbounded kernels

The results presented in this paragraph would be direct applications of the Wick calculus given in Proposition B.2 for a bounded potential $V\in L^\infty(\mathbb{R}^d)$. Although the algebra is the same as in the bounded case, justifying the formulas for unbounded potentials fulfilling (A1)(A2)(A3) requires some analysis.

Lemma 3.7. *The identity*

$$\left(V_s^{Wick}W(\xi)-W(\xi)V_s^{Wick}\right)\Psi=W(\xi)\left(\sum_{j=1}^4\varepsilon^j b_j(s,\xi)^{Wick}\right)\Psi\,. \quad (23)$$

holds for any $\xi\in H^2(\mathbb{R}^d)$ and $\Psi\in\mathcal{D}(S_\varepsilon)$, with S_ε given by Definition 3.5. Additionally, for all $\Psi\in\mathcal{D}(S_\varepsilon)\subset\mathcal{D}(S_\varepsilon^{1/2})\subset\mathcal{D}(\mathbf{N}^{\frac32})$, the estimates

$$\begin{aligned}\|b_j(s,\xi)^{Wick}\ \Psi\|\leq C(1+|\xi|_{\mathcal{Z}_1}^4)\|(1+\mathbf{N})^{\frac32}\Psi\|\leq C'(1+|\xi|_{\mathcal{Z}_1}^4)\|(1+S_\varepsilon)^{1/2}\Psi\|\\ \leq C'(1+|\xi|_{\mathcal{Z}_1}^4)\|(1+S_\varepsilon)\Psi\|\,, \quad (24)\end{aligned}$$

hold uniformly w.r.t $j\in\{1,\dots,4\}$, $s\in\mathbb{R}$, when $\xi\in H^1(\mathbb{R}^d)$ and therefore when $\xi\in H^2(\mathbb{R}^d)$.

Proof. We first remark that, owing to the assumption (A2) and the estimate (16), the polynomials $b_j(s,\xi)$, $j=1,\cdots,4$ belong to the set $\oplus_{p,q\leq 3}\mathcal{P}_{p,q}(\mathcal{Z}_0)$, with

$$|b_j|_{\oplus_{p+q\leq 3}\mathcal{P}_{p,q}(\mathcal{Z}_0)} \leq C(1+|\xi|^4_{\mathcal{Z}_1}) .$$

Hence, Proposition B.1 and Proposition 2.2 prove (24) with

$$\mathcal{D}(S_\varepsilon) \subset \mathcal{D}(\mathbf{N}^{3/2}) \subset \mathcal{D}(b_j(s,\xi)^{Wick}) , \ j=1,\ldots,4 , \quad \text{and} \quad \mathcal{D}(S_\varepsilon) \subset \mathcal{D}(V_s^{Wick}) . \tag{25}$$

By Proposition 2.4 the domain $\mathcal{D}(S_\varepsilon)$ is invariant with respect to $W(\xi)$ for all $\xi \in H^2(\mathbb{R}^d)$. A Taylor expansion yields, for all $z\in\mathcal{Z}_1$, the equality

$$V_s(z+\frac{i\varepsilon}{\sqrt{2}}\xi) = V_s(z) + \sum_{j=1}^4 \varepsilon^j b_j(s,\xi)[z] .$$

The formula (23) is standard for bounded $\tilde{V}$ due to $W^*(\xi)b^{Wick}W(\xi) = b(.+\frac{i\varepsilon}{\sqrt{2}}\xi)^{Wick}$ when $b\in\mathcal{P}_{alg}(\mathcal{Z}_0) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z}_0)$. Let us reconsider the proof of this result for our unbounded $\tilde{V}$. With the previous estimates, the quantity $\mathcal{A}(t) = \langle \Phi, W(t\xi)V_s(.+\frac{i\varepsilon}{\sqrt{2}}t\xi)^{Wick}W(t\xi)^*\Psi\rangle$ is well defined for all $\Phi,\Psi\in\mathcal{D}(S_\varepsilon)$ with

$$\mathcal{A}(t) = \sum_{j=1}^4 \varepsilon^j t^j \langle \Phi, W(t\xi)b_j(s,\xi)^{Wick}W(t\xi)^*\Psi\rangle + \langle \Phi, W(t\xi)V_s^{Wick}W(t\xi)^*\Psi\rangle .$$

We first establish in a weak sense the equality (23): Differentiate $\mathcal{A}(t)$ for any $\Psi,\Phi\in\mathcal{D}(S_\varepsilon)$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{A}(t) &= \sum_{j=1}^4 \varepsilon^j \langle \Phi, W(t\xi)\left\{[i\phi(\xi), b_j(s,\xi)^{Wick}]t^j + jt^{j-1}b_j(s,\xi)^{Wick}\right\}W(t\xi)^*\Psi\rangle \\
&\quad + \langle \Phi, W(t\xi)[i\phi(\xi), V_s^{Wick}]W(t\xi)^*\Psi\rangle \\
&= \sum_{j=0}^3 \varepsilon^j t^j \langle \Phi, W(t\xi)\left\{[i\phi(\xi), b_j(s,\xi)^{Wick}] + (j+1)b_{j+1}(s,\xi)^{Wick}\right\}W(t\xi)^*\Psi\rangle
\end{aligned}$$

where $b_0(z) = V_s(z)$. Now, a direct calculation with $\phi(\xi) = \frac{1}{\sqrt{2}}(a(\xi)+a^*(\xi))$ gives

$$[i\phi(\xi), b_j(s,\xi)^{Wick}] = -(j+1)b_{j+1}(s,\xi)^{Wick}$$

for $j=0,\cdots,3$. Therefore $\mathcal{A}(1)=\mathcal{A}(0)$ and, knowing (25), we conclude that

$$W(\xi)\left(V_s^{Wick} + \sum_{j=1}^4 \varepsilon^j b_j(s,\xi)^{Wick}\right)W(\xi)^*\Psi = V_s^{Wick}\Psi . \tag{26}$$

for any $\Psi\in\mathcal{D}(S_\varepsilon)$. Now taking $\Psi = W(\xi)\tilde{\Psi}$ in (26) for any $\tilde{\Psi}\in\mathcal{D}(S_\varepsilon)$, while $\Psi\in\mathcal{D}(S_\varepsilon)$ owing to $\xi\in H^2(\mathbb{R}^d)$, we obtain the claimed equality.

□

Lemma 3.8. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ a family of normal states on $\mathcal{H}$. Assume that $\varrho_\varepsilon(S_\varepsilon+1)\in\mathcal{L}^1(\mathcal{H})$ for all $\varepsilon\in(0,\bar{\varepsilon})$, with S_ε given by Definition 3.5 and $\tilde{\varrho}_\varepsilon(t)$ by (20). Then for any $\xi\in H^2(\mathbb{R}^d)$, the map $s\mapsto \mathrm{Tr}[\tilde{\varrho}_\varepsilon(s)\,W(\xi)]$ belongs to $C^1(\mathbb{R})$ and the following integral formula holds true*

$$\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t)W(\xi)] = \mathrm{Tr}[\varrho_\varepsilon W(\xi)] + \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}[\tilde{\varrho}_\varepsilon(s)W(\xi)\sum_{j=1}^4 \varepsilon^j b_j(s,\xi)^{Wick}]\,ds .$$

Proof. We write

$$\begin{aligned}
&\mathrm{Tr}[(\tilde{\varrho}_\varepsilon(s)-\tilde{\varrho}_\varepsilon(s))W(\xi)]\\
&\quad=\mathrm{Tr}\left[\varrho_\varepsilon(S_\varepsilon+1)(S_\varepsilon+1)^{-1}\left(e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}-e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}\right)W(\xi)e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon}\right]\\
&\qquad+\mathrm{Tr}\left[\varrho_\varepsilon e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}W(\xi)(S_\varepsilon+1)(S_\varepsilon+1)^{-1}\left(e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon}-e^{i\frac{s}{\varepsilon}H_\varepsilon^0}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}\right)\right].
\end{aligned}$$

The following limits hold true on $\mathcal{D}(S_\varepsilon)$

$$\begin{aligned}
&\lim_{s\to t}\frac{1}{t-s}(S_\varepsilon+1)^{-1}\left(e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}-e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}\right)=\frac{i}{\varepsilon}(S_\varepsilon+1)^{-1}e^{i\frac{t}{\varepsilon}H_\varepsilon}(H_\varepsilon-H_\varepsilon^0)e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}\\
&\lim_{s\to t}\frac{1}{t-s}(S_\varepsilon+1)^{-1}\left(e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon}-e^{i\frac{s}{\varepsilon}H_\varepsilon^0}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}\right)=\frac{i}{\varepsilon}(S_\varepsilon+1)^{-1}e^{i\frac{t}{\varepsilon}H_\varepsilon^0}(H_\varepsilon^0-H_\varepsilon)e^{-i\frac{t}{\varepsilon}H_\varepsilon},
\end{aligned}$$

by Stone's theorem and the invariance of $\mathcal{D}(S_\varepsilon)$ w.r.t $e^{itH_\varepsilon^0}$ and e^{itH_ε}. By using the estimate in Proposition 2.3 we see that the latter limits are limits in $\mathcal{L}(\mathcal{H})$ w.r.t the strong convergence topology. After noticing that $\varrho_\varepsilon e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}W(\xi)(S_\varepsilon+1)$ is trace class when $\xi\in H^2(\mathbb{R})$ owing to Proposition 2.4 and Proposition 2.3, we are done by taking the trace and letting $s\to t$.
Now it suffices to take the integral to get

$$\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t)W(\xi)]=\mathrm{Tr}[\varrho_\varepsilon W(\xi)]+\frac{i}{\varepsilon}\int_0^t\mathrm{Tr}\left[\tilde{\varrho}_\varepsilon(s)\,\left(V_s^{Wick}W(\xi)-W(\xi)V_s^{Wick}\right)\right]\,ds\,.$$

When $\xi\in H^2(\mathbb{R}^d)$, the equality

$$\mathrm{Tr}\left[(1+S_\varepsilon)\tilde{\varrho}_\varepsilon(s)\left(V_s^{Wick}W(\xi)-W(\xi)V_s^{Wick}\right)(1+S_\varepsilon)^{-1}\right]=\mathrm{Tr}[\tilde{\varrho}_\varepsilon(s)W(\xi)\sum_{j=1}^4\varepsilon^j b_j(s,\xi)^{Wick}]\,,$$

makes sense, since $(1+S_\varepsilon)\tilde{\varrho}_\varepsilon(s)\in\mathcal{L}^1(\mathcal{H})$ and

$$\left(V_s^{Wick}W(\xi)-W(\xi)V_s^{Wick}\right)(1+S_\varepsilon)^{-1}=W(\xi)\left[\sum_{j=1}^4\varepsilon^j b_j(s,\xi)^{Wick}\right](1+S_\varepsilon)^{-1}\quad\text{in }\mathcal{L}(\mathcal{H})\,.$$

by Lemma 3.7. □

3.2.2 Subsequence extraction for all times

The first step in the proof of Proposition 3.6 is to show the existence of Wigner measures for all times. This is accomplished below following merely the same lines as [6, Proposition 3.3].

Proposition 3.9. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\mathcal{H}$ satisfying for some constant $C>0$ the estimate*

$$\mathrm{Tr}[\varrho_\varepsilon(1+S_\varepsilon)]\le C\quad\text{ uniformly w.r.t }\varepsilon\in(0,\bar{\varepsilon})\,.$$

The operator S_ε and $\tilde{\varrho}_\varepsilon(t)$ are respectively given by Definition 3.5 and (20). Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\bar{\varepsilon})$ such that $\lim_{n\to\infty}\varepsilon_n=0$ there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family of Borel probability measures on $\mathcal{Z}_0$, $(\tilde{\mu}_t)_{t\in\mathbb{R}}$, satisfying

$$\mathcal{M}(\tilde{\varrho}_{\varepsilon_{n_k}}(t),\ k\in\mathbb{N})=\{\tilde{\mu}_t\}\,,$$

for any $t\in\mathbb{R}$.

Proof. We only sketch the proof and essentially indicate the points which differ from [6, Proposition 3.3]. Let us write

$$G_\varepsilon(t,\xi)=\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t)W(\xi)]\,.$$

By using Proposition B.1 and $(1+\mathbf{N}) \leq 2(1+\mathbf{N}^3) \leq 2(1+S_\varepsilon)$, we can prove like in [6] that

$$|G_\varepsilon(s,\xi) - G_\varepsilon(s,\eta)| \leq C|\xi-\eta|_{\mathcal{Z}_0}^{\frac{1}{2}} \left(|\xi|_{\mathcal{Z}_0}^2 + |\eta|_{\mathcal{Z}_0}^2 + 1\right)^{\frac{1}{4}} \tag{27}$$

for some constant $C > 0$. We have

$$|G_\varepsilon(t,\eta) - G_\varepsilon(s,\xi)| \leq |\mathrm{Tr}\left[(\tilde{\varrho}_\varepsilon(t) - \tilde{\varrho}_\varepsilon(s))\, W(\xi)\right]| + |G_\varepsilon(s,\xi) - G_\varepsilon(s,\eta)| \ .$$

On the other hand, by making use of Lemma 3.8 we know

$$\begin{aligned}
|\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t) - \tilde{\varrho}_\varepsilon(s)]W(\xi)]| &\leq \left| \int_s^t \mathrm{Tr}[\tilde{\varrho}_\varepsilon(w) \sum_{j=1}^4 \varepsilon^{j-1} b_j(w,\xi)^{wick}]\, dw \right| \\
&\leq C_0|t-s| \| (1+S_\varepsilon)^{1/2} \varrho_\varepsilon (1+S_\varepsilon)^{1/2} \|_{\mathcal{L}^1(\mathcal{H})} \\
&\quad \times \sup_{w\in[t,s]} \| (1+S_\varepsilon)^{-1/2} \left[\sum_{j=1}^4 \varepsilon^{j-1} b_j(w,\xi)^{wick} \right] (1+S_\varepsilon)^{-1/2} \|_{\mathcal{L}(\mathcal{H})} \\
&\leq C_1|t-s|(1+|\xi|_{\mathcal{Z}_1})^4 \, ,
\end{aligned}$$

when $\xi \in H^2(\mathbb{R}^d)$. Taking an approximation $\xi_n \in H^2(\mathbb{R})$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \|\xi - \xi_n\|_{\mathcal{Z}_1} = 0$, $\mathcal{Z}_1 = H^1(\mathbb{R}^n)$, and taking the limit as $n \to \infty$ of the left-hand side with the help of (27), allows first to extend the previous inequality to any $\xi \in \mathcal{Z}_1$.
Thus, we conclude that

$$|G_\varepsilon(t,\eta) - G_\varepsilon(s,\xi)| \leq \tilde{C} \left(|t-s|(|\xi|_{\mathcal{Z}_1}+1)^4 + |\eta-\xi|_{\mathcal{Z}_0} \sqrt{|\eta|_{\mathcal{Z}_0}^2 + |\xi|_{\mathcal{Z}_0}^2} \right) , \tag{28}$$

holds for all $(s,\xi),(t,\eta) \in \mathbb{R}\times\mathcal{Z}_1$, uniformly w.r.t. $\varepsilon \in (0,\bar{\varepsilon})$. Remember also the uniform estimate $|G_\varepsilon(s,\xi)| \leq 1$.
Now, we apply the same Ascoli type argument used in [6, Proposition 3.3] in order to prove the existence of a subsequence $(\varepsilon_{n_k})_k$ and a continuous function $G(.,.) : \mathbb{R}\times\mathcal{Z}_1 \to \mathbb{C}$ such that $G_{\varepsilon_k}(t,\xi)$ converges to $G(t,\xi)$ for any $t \in \mathbb{R}$ and $\xi \in \mathcal{Z}_1$. Further from (27) we can extend $G(.,.)$ to a continuous function on $\mathbb{R}\times\mathcal{Z}_0$. An "$\delta/3$"-argument shows that for any $(t,\xi) \in \mathbb{R}\times\mathcal{Z}_0$, $\lim_{n\to\infty} G_{\varepsilon_n}(t,\xi)$ exists and equals $G(t,\xi)$, so that $G(t,.)$ is a norm continuous normalized function of positive type. Therefore, for any $t \in \mathbb{R}$, $G(t,.)$ is a characteristic function of weak distribution (or projective family of probability measures) $\tilde{\mu}_t$ on $\mathcal{Z}_0$. Finally we end the proof as in [6, Proposition 3.3]. □

3.2.3 An additional compactness argument

Here, the compactness assumptions (A3) is translated into some compactness property of the Wick symbol b_1. It allows to refer indirectly to Proposition 3.3 and to take the limit as $\varepsilon \to 0$ in the term with $j = 1$ in (22). With the rough estimates used in Proposition 3.9, the terms in (22) corresponding to $j > 1$ with a factor ε^{j-1} will vanish as $\varepsilon \to 0$. The next Lemma applied with $\tilde{\varrho}_\varepsilon(s)$ in the integral term of (22), will end the proof of Proposition 3.6.

Lemma 3.10. *Let ϱ_ε be a family of normal states on $\mathcal{H}$ satisfying for some constant $C > 0$ the estimate*

$$\mathrm{Tr}\left[(1+S_\varepsilon)\varrho_\varepsilon(1+S_\varepsilon)\right] \leq C \quad \textit{uniformly w.r.t } \varepsilon \in (0,\bar{\varepsilon}) \, .$$

Here S_ε is given by Definition 3.5. Assume that $\mathcal{M}(\varrho_\varepsilon) = \{\mu\}$, then for any $\xi \in \mathcal{Z}_1$,

$$\lim_{\varepsilon\to 0} \mathrm{Tr}[\varrho_\varepsilon W(\xi) b_1(s,\xi)^{Wick}] = \int_{\mathcal{Z}_0} e^{\sqrt{2}i\mathrm{Re}\ \langle \xi, z\rangle} b_1(s,\xi)[z]\, d\mu(z) \, .$$

Proof. The polynomial $b_1(s,\xi) \in \mathcal{P}_{1,2} + \mathcal{P}_{2,1}$ splits into two similar terms, namely

$$B_1(z) = \langle \xi \otimes z, \tilde{V}_s z^{\otimes 2} \rangle \quad \text{and} \quad B_2(z) = \langle z^{\otimes 2}, \tilde{V}_s(z\otimes\xi) \rangle \, .$$

with their associated operators

$$\tilde{B}_1 = (\langle\xi| \otimes 1\!\!1)\,\tilde{V}_s \in \mathcal{L}(L^2_s(\mathbb{R}^{2d}), L^2(\mathbb{R}^d)) \quad \text{and} \quad \tilde{B}_2 = \mathcal{S}_2\tilde{V}_s\,(1\!\!1 \otimes |\xi\rangle) \in \mathcal{L}(L^2(\mathbb{R}^d), L^2_s(\mathbb{R}^{2d}))\,.$$

Let $\chi \in C^\infty_0(\mathbb{R})$ with $\chi(x) = 1$ if $|x| \leq 1$, $\chi(x) = 0$ if $|x| \geq 2$ and $0 \leq \chi \leq 1$. For $m \in \mathbb{N}^*$ we put $\chi_m(x) = \chi(\frac{x}{m})$ and define

$$\tilde{B}_{1,m} = \chi_m(|D_x|)\tilde{B}_1\,(1\!\!1 \otimes \chi_m(|D_x|))\mathcal{S}_2 \quad \text{and} \quad \tilde{B}_{2,m} = \mathcal{S}_2(1\!\!1 \otimes \chi_m(|D_x|))\,\tilde{B}_2\chi_m(|D_x|)$$

as bounded operators in $\mathcal{L}(L^2_s(\mathbb{R}^{2d}), L^2(\mathbb{R}^d))$ and $\mathcal{L}(L^2(\mathbb{R}^d), L^2_s(\mathbb{R}^{2d}))$ respectively. We claim that both operators $\tilde{B}_{1,m}$ and $\tilde{B}_{2,m}$ are compact. Actually, $\tilde{B}_{2,m} = \tilde{B}^*_{1,m}$ and

$$\tilde{B}_{1,m} = \frac{1}{2}(1\!\!1 \otimes e^{-is\Delta_{x_2}})(\langle e^{is\Delta}\xi| \otimes 1\!\!1)(e^{-ix_1D_{x_2}}\chi_m(|D_{x_2}|)V(x_2)\chi_m(|D_{x_2}|)e^{ix_1D_{x_2}})(e^{is\Delta_{x_1}} \otimes e^{is\Delta_{x_2}})\mathcal{S}_2\,.$$

Moreover, the linear norm continuous application

$$A \in \mathcal{L}(L^2(\mathbb{R}^d)) \quad \longmapsto \quad (\langle e^{is\Delta}\xi| \otimes 1\!\!1)e^{-ix_1D_{x_2}}(1 \otimes A) \in \mathcal{L}(L^2_s(\mathbb{R}^{2d}), L^2(\mathbb{R}^d))$$

maps Hilbert-Schmidt operators into Hilbert-Schmidt since

$$\|(\langle e^{is\Delta}\xi| \otimes 1\!\!1)e^{-ix_1D_{x_2}}(1 \otimes A)\|_{\mathcal{L}^2(L^2_s(\mathbb{R}^{2d}), L^2(\mathbb{R}^d))} = |\xi|_{\mathcal{Z}_0}\|A\|_{\mathcal{L}^2(\mathcal{Z}_0)}$$

comes by computing the Schwartz kernel with $\|K\|^2_{\mathcal{L}^2(L^2(\nu))} = \int |K(x,y)|^2\; d\nu(x)d\nu(y)$. Hence it maps compact operators into compact operators, because the space of compact operators, $\mathcal{L}^\infty$, is the norm closure of $\mathcal{L}^2$ in $\mathcal{L}$. Therefore, by taking $A = \chi_m(|D_x|)\,V(x)\,\chi_m(|D_x|)$ which is compact by assumption (A3), we conclude that $\tilde{B}_{1,m}$ and $\tilde{B}_{2,m}$ are compact.
Now, we write for $j = 1, 2$

$$|\mathrm{Tr}[\varrho_\varepsilon W(\xi)B^{Wick}_j] - \mu(e^{\sqrt{2}i\mathrm{Re}\;\langle\xi,z\rangle}B_j(z))| \leq |\mathrm{Tr}[\varrho_\varepsilon W(\xi)(B^{Wick}_j - B^{Wick}_{j,m})]| \tag{29}$$

$$+|\mathrm{Tr}[\varrho_\varepsilon W(\xi)B^{Wick}_{j,m}] - \mu(e^{\sqrt{2}i\mathrm{Re}\;\langle\xi,z\rangle}B_{j,m}(z))| \tag{30}$$

$$+|\mu(e^{\sqrt{2}i\mathrm{Re}\;\langle\xi,z\rangle}B_{j,m}(z)) - \mu(e^{\sqrt{2}i\mathrm{Re}\;\langle\xi,z\rangle}B_j(z))|\,, \tag{31}$$

with $B_{j,m} \in \mathcal{P}^\infty_{alg}(\mathcal{Z}_0)$. The right hand side (30) converges to 0 owing to Proposition 3.4. Since $s - \lim_{m\to\infty}\chi_m(|D_x|) = 1\!\!1$, the polynomials $B_{j,m}(z)$ converge to $B_j(z)$ for any $z \in \mathcal{Z}_0$, while the estimate

$$|B_{j,m}(z)| \leq c|\xi|_{\mathcal{Z}_1}\,\|(1-\Delta)^{-1/2}V\|\,|z|^3_{\mathcal{Z}_0}$$

holds true uniformly in m for some constant c. In addition $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^{3/2}] \leq C$ implies that

$$\int_{\mathcal{Z}_0} |z|^3_{\mathcal{Z}_0}\,d\mu \leq C\,.$$

Therefore by dominated convergence theorem the right hand side (31) tends to 0 when $m \to \infty$. It still remains to prove the convergence of the r.h.s of (29). Indeed, writing

$$\begin{aligned}\mathrm{Tr}[\varrho_\varepsilon W(\xi)(B^{Wick}_j - B^{Wick}_{j,m})] \;&=\; \mathrm{Tr}[(S_\varepsilon+1)\varrho_\varepsilon(S_\varepsilon+1)(S_\varepsilon+1)^{-1}W(\xi)(S_\varepsilon+1)\\ &\qquad\times(S_\varepsilon+1)^{-1}(B^{Wick}_j - B^{Wick}_{j,m})(S_\varepsilon+1)^{-1}]\end{aligned}$$

and then by using Proposition 2.4, we obtain the estimate

$$\|\mathrm{Tr}[\varrho_\varepsilon W(\xi)(B^{Wick}_j - B^{Wick}_{j,m})]\| \leq c\|(S_\varepsilon+1)^{-1}(B^{Wick}_j - B^{Wick}_{j,m})(S_\varepsilon+1)^{-1}\|_{\mathcal{L}(\mathcal{H})}\,.$$

By functional calculus of strongly commuting self-adjoint operators we see that $(S_\varepsilon+1)^{-1}(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(1-\Delta)+1)$ is uniformly bounded with respect to $\varepsilon \in (0, \bar{\varepsilon})$. Now, applying Lemma B.4 (with $A = 1-\Delta$), we conclude that

$$\begin{aligned}\|\mathrm{Tr}[\varrho_\varepsilon W(\xi)(B^{Wick}_j - B^{Wick}_{j,m})]\| \;&\lesssim\; \|(1-\Delta_{x_2})^{-1/2}(\tilde{B}_j - \tilde{B}_{j,m})(1-\Delta_{x_2})^{-1/2}\|\\ &\lesssim\; |\xi|_{\mathcal{Z}_0}\,\|(1-\Delta)^{-1/2}V\|_{\mathcal{L}(\mathcal{Z}_0)}\,\|(1-\Delta_x)^{-1/2}(1-\chi_m(|D_x|))\|_{\mathcal{L}(\mathcal{Z}_0)}\end{aligned}$$

Since by functional calculus $\|(1-\Delta_x)^{-1/2}(1-\chi_m(|D_x|))\| \lesssim \frac{1}{m}$, the r.h.s of (29) goes to 0 when $m \to \infty$ uniformly in $\varepsilon \in (0, \bar{\varepsilon})$. Finally, a "$\delta/3$-argument" with the established convergence of (29), (30) and (31) yields the result. □

3.3 Asymptotic a priori estimates

In this section, a priori information on Wigner measures are derived from a priori estimates on the state ϱ_ε. In particular, we shall prove the next result.

Proposition 3.11. *Let S_ε be the operator given by Definition 3.5 and assume that the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies*

$$\forall\alpha\in\mathbb{N},\exists C_\alpha>0,\forall\varepsilon\in(0,\bar\varepsilon),\quad \mathrm{Tr}\,[(1+S_\varepsilon)\varrho_\varepsilon(1+S_\varepsilon)(1+\mathbf{N})^\alpha]\leq C_\alpha\,,$$

and $\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))=\{\mu\}$. Then the measure μ is carried by $\mathcal{Z}_1$, its restriction to $\mathcal{Z}_1$ is a Borel probability measure on $\mathcal{Z}_1$ and

$$\int_{\mathcal{Z}_0}|z|^4_{\mathcal{Z}_1}|z|^2_{\mathcal{Z}_0}\,d\mu(z)=\int_{\mathcal{Z}_1}|z|^4_{\mathcal{Z}_1}|z|^2_{\mathcal{Z}_0}\,d\mu(z)<+\infty\,. \tag{32}$$

We will need the two next Lemmas.

Lemma 3.12. *Let $\tilde b$ be a non negative (self-adjoint) operator on $\bigvee^p\mathcal{Z}_0$ and assume that the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$, with $\mathrm{Tr}\,[\varrho_\varepsilon\mathbf{N}^\alpha]\leq C_\alpha$ for all $\alpha\in\mathbb{N}$, satisfies*

$$\mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right]\leq C\quad\textit{and}\quad\mathcal{M}(\varrho_\varepsilon,\,\varepsilon\in(0,\bar\varepsilon))=\{\mu\}\,.$$

Then $\mathcal{Z}_0\ni z\to b(z)=\langle z^{\otimes p}\,,\,\tilde b z^{\otimes p}\rangle\in[0,+\infty]$ is a Borel function on $\mathcal{Z}_0$ and $\int_{\mathcal{Z}_0}b(z)\,d\mu(z)\leq C$.

Proof. When $b\in\mathcal{P}^\infty_{p,p}(\mathcal{Z}_0)$ has a compact kernel $\tilde b$ we know after Proposition 3.3 (see [4, Corollary 6.14] for a complete proof) that

$$C\geq\lim_{\varepsilon\to0}\mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right]=\int_{\mathcal{Z}_0}b(z)\,d\mu(z)\,.$$

We use the fact that $\tilde b\to b^{Wick}$ is operator monotone, in the following sense: if the (possibly unbounded) non negative operators $\tilde b_1,\tilde b_2$ in $\bigvee^p\mathcal{Z}_0$ satisfy $\tilde b_2\geq\tilde b_1\geq0$, then $b_2^{Wick}\geq b_1^{Wick}\geq0$ in $\mathcal{H}$.
By taking $\tilde b\in\mathcal{L}(\bigvee^p\mathcal{Z})$, for $b\in\mathcal{P}_{p,p}(\mathcal{Z})$, as the supremum of $\tilde b_n$ with $\tilde b_n$ compact, we obtain firstly for all $n\in\mathbb{N}$

$$C\geq\liminf_{\varepsilon\to0}\ \mathrm{Tr}\left[\varrho_\varepsilon^{1/2}b^{Wick}\varrho_\varepsilon^{1/2}\right]\geq\lim_{\varepsilon\to0}\ \mathrm{Tr}\left[\varrho_\varepsilon^{1/2}b_n^{Wick}\varrho_\varepsilon^{1/2}\right]=\int_{\mathcal{Z}_0}b_n(z)\,d\mu(z)\,.$$

Secondly, the monotone convergence yields

$$C\geq\sup_{n\in\mathbb{N}}\int_{\mathcal{Z}_0}\langle z^{\otimes p}\,,\,\tilde b_n z^{\otimes p}\rangle\,d\mu(z)=\int_{\mathcal{Z}_0}b(z)\,d\mu(z)\,.$$

When $\tilde b$ is unbounded, it can be approximated by $\tilde b_n=\frac{\tilde b}{1+\frac{\tilde b}{n}}\in\mathcal{L}(\bigvee^p\mathcal{Z}_0)$, for $n\geq1$. Set $b_n(z)=\langle z^{\otimes p}\,,\,\tilde b_n z^{\otimes p}\rangle$. The function $b(z)=\langle z^{\otimes p}\,,\,\tilde b z^{\otimes p}\rangle=\sup_{n\in\mathbb{N}}b_n(z)$ is a Borel function on $\mathcal{Z}_0$ as a supremum of a sequence of continuous functions. The uniform estimate

$$\mathrm{Tr}\left[\varrho_\varepsilon b_n^{Wick}\right]\leq\mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right]\leq C$$

with the result for $\tilde b_n\in\mathcal{L}(\bigvee^p\mathcal{Z}_0)$ gives $\int_{\mathcal{Z}_0}b_n(z)\,d\mu(z)\leq C$, for all $n\in\mathbb{N}^*$. Again by monotone convergence, we get

$$\int_{\mathcal{Z}_0}b(z)\,d\mu(z)=\sup_{n\in\mathbb{N}^*}\int_{\mathcal{Z}_0}b_n(z)\,d\mu(z)\leq C\,.$$

□

Lemma 3.13. *Let A be a non negative, self-adjoint with domain $\mathcal{D}(A)$, operator in $\mathcal{Z}_0$. Assume that the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies the uniform estimate $\mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\alpha] \leq C_\alpha$, for all $\alpha \in \mathbb{N}$, and $\mathcal{M}(\varrho_\varepsilon,\, \varepsilon\in(0,\bar\varepsilon)) = \{\mu\}$. Then the following implication hold:*

$$\begin{aligned}
(\forall \varepsilon\in(0,\bar\varepsilon),\ \mathrm{Tr}\,[\varrho_\varepsilon \mathrm{d}\Gamma(A)] \leq C) \quad &\Rightarrow \quad \left(\int_{\mathcal{Z}_0} \langle z\,,\, Az\rangle\ d\mu(z) \leq C\right),\\
\left(\forall \varepsilon\in(0,\bar\varepsilon),\ \mathrm{Tr}\,\left[\varrho_\varepsilon \mathrm{d}\Gamma(A)^2\right] \leq C\right) \quad &\Rightarrow \quad \left(\int_{\mathcal{Z}_0} \langle z\,,\, Az\rangle^2\ d\mu(z) \leq C\right),\\
\left(\forall \varepsilon\in(0,\bar\varepsilon),\ \mathrm{Tr}\,\left[\varrho_\varepsilon \mathrm{d}\Gamma(A)^2\mathbf{N}\right] \leq C\right) \quad &\Rightarrow \quad \left(\int_{\mathcal{Z}_0} \langle z\,,\, Az\rangle^2 |z|^2_{\mathcal{Z}_0}\ d\mu(z) \leq C\right).
\end{aligned}$$

In all the three cases, the measure μ is carried by the form domain $Q(A)$ of A.

Proof. The first implication is a direct application of Lemma 3.12 applied with

$$b(z) = \langle z\,,\, Az\rangle \quad , \quad \tilde b = A \quad , \quad b^{Wick} = \mathrm{d}\Gamma(A)\,.$$

For the second one we use

$$\mathrm{d}\Gamma(A)^2 = \left(\langle z^{\otimes 2}\,,\, (A\otimes A) z^{\otimes 2}\rangle\right)^{Wick} + \varepsilon \mathrm{d}\Gamma(A^2) \geq \left(\langle z^{\otimes 2}\,,\, (A\otimes A) z^{\otimes 2}\rangle\right)^{Wick}\,.$$

We apply Lemma 3.12 with

$$b(z) = \langle z\,,\, Az\rangle^2 \quad \text{and} \quad \tilde b = A\otimes A\,.$$

For the last one we notice that $\mathbf{N} = \mathrm{d}\Gamma(1)$ and $\mathrm{d}\Gamma(A)$ commute so that

$$\mathrm{d}\Gamma(A)^2\mathbf{N} \geq \mathbf{N}\left(\langle z^{\otimes 2}\,,\, (A\otimes A) z^{\otimes 2}\rangle\right)^{Wick}\,.$$

With $\mathbf{N} = (|z|^2_{\mathcal{Z}_0})^{Wick}$, the composition formula of Proposition B.2 (extended to an unbounded A) says that $\mathbf{N}\left(\langle z^{\otimes 2}\,,\, (A\otimes A) z^{\otimes 2}\rangle\right)^{Wick} = b_\varepsilon^{Wick}$ with

$$b_\varepsilon(z) = |z|^2_{\mathcal{Z}_0}\langle z\,,\, Az\rangle^2 + 2\varepsilon \langle z\,,\, Az\rangle^2\,.$$

Hence we get

$$\mathrm{d}\Gamma(A)^2\mathbf{N} \geq \left(|z|^2_{\mathcal{Z}_0}\langle z\,,\, Az\rangle^2\right)^{Wick}\,,$$

and we apply Lemma 3.12 with

$$b(z) = |z|^2_{\mathcal{Z}_0}\langle z\,,\, Az\rangle^2 \quad , \quad \tilde b = \frac{1}{3}(\mathbb{1}\otimes A\otimes A + A\otimes \mathbb{1}\otimes A + A\otimes A\otimes \mathbb{1})\,.$$

For the last statement it suffices to notice that the integrand is infinite in the Borel subset of $\mathcal{Z}_0$, $\mathcal{Z}_0 \setminus Q(A) = \{z\in\mathcal{Z}_0, \langle z\,, Az\rangle = +\infty\}$. □

Proof of Proposition 3.11: With $S_\varepsilon = \mathrm{d}\Gamma(1-\Delta) + \mathbf{N}^3$, while $\mathrm{d}\Gamma(1-\Delta)$ and $\mathbf{N}$ commute, we know

$$(1+S_\varepsilon)(1+\mathbf{N}) \geq \mathrm{d}\Gamma(1-\Delta)\,.$$

Hence Lemma 3.13 says that the measure μ is carried by $Q(1-\Delta) = \mathcal{Z}_1$ with

$$\int_{\mathcal{Z}_0} |z|^2_{\mathcal{Z}_1}\ d\mu(z) = \int_{\mathcal{Z}_1} |z|^2_{\mathcal{Z}_1}\ d\mu(z) \leq C\,.$$

Let us check that μ is a Borel measure on $(\mathcal{Z}_1, |\ |_{\mathcal{Z}_1})$. The tightness property is given by the above inequality. According to [4][54][58], it suffices to check that

$$\begin{aligned}
&G_1(\xi) = \int_{\mathcal{Z}_1} e^{-2i\pi \mathrm{Re}\ \langle \xi\,,\, z\rangle_{\mathcal{Z}_1}}\ d\mu(z)\\
\text{with} \qquad &\langle u\,, v\rangle_{\mathcal{Z}_1} = \langle u\,,\, (1-\Delta)u\rangle_{\mathcal{Z}_0}\,,
\end{aligned}$$

is a positive type function which is continuous w.r.t ξ restricted to any finite dimensional subspace of $\mathcal{Z}_1$.
Consider the regularized version

$$G_{1,n}(\xi) = \int_{\mathcal{Z}_1} e^{-2i\pi \mathrm{Re}\ \langle \frac{A}{1+\frac{A}{n}}\xi\,,\,z\rangle_{\mathcal{Z}_0}}\ d\mu(z) = \int_{\mathcal{Z}_0} e^{-2i\pi \mathrm{Re}\ \langle \frac{A}{1+\frac{A}{n}}\xi\,,\,z\rangle_{\mathcal{Z}_0}}\ d\mu(z)$$

with $A = (1-\Delta)$. For all $\xi \in \mathcal{Z}_1$ the pointwise convergence

$$\forall z \in \mathcal{Z}_1\,, \quad \lim_{n\to\infty} \langle \frac{A}{1+\frac{A}{n}}\xi\,,\,z\rangle_{\mathcal{Z}_0} = \langle \xi\,,\,z\rangle_{\mathcal{Z}_1}$$

and the uniform bound

$$|e^{-2i\pi \mathrm{Re}\ \langle \frac{A}{1+\frac{A}{n}}\xi\,,\,z\rangle_{\mathcal{Z}_0}}| \leq 1$$

imply the pointwise convergence of the integrals

$$\forall \xi \in \mathcal{Z}_1, \quad \lim_{n\to\infty} G_{1,n}(\xi) = G_1(\xi)\,.$$

But $G_{1,n}(\xi)$ equals $G((1+\frac{A}{n})^{-1}A\xi)$, where G is the characteristic function of μ in $\mathcal{Z}_0$:

$$G(\eta) = \int_{\mathcal{Z}_0} e^{-2i\pi \mathrm{Re}\ \langle \eta\,,\,z\rangle_{\mathcal{Z}_0}}\ d\mu(z)\,.$$

Hence for every $n \in \mathbb{N}$, the function $G_{1,n}(\xi)$ is a positive type function. As a pointwise limit of $G_{1,n}$, the function G_1 is also a positive type function.
For the continuity, the equality

$$G_1(\xi) - G_1(\xi') = \int_{\mathcal{Z}_1} \left(e^{-i\pi \mathrm{Re}\ \langle \xi-\xi',z\rangle_{\mathcal{Z}_1}} - e^{i\pi \mathrm{Re}\ \langle \xi-\xi',z\rangle_{\mathcal{Z}_1}}\right) e^{-i\pi \mathrm{Re}\ \langle \xi+\xi'\,,\,z\rangle_{\mathcal{Z}_1}}\ d\mu(z)\,.$$

implies

$$|G_1(\xi) - G_1(\xi')| \leq 2\pi|\xi-\xi'|_{\mathcal{Z}_1} \int_{\mathcal{Z}_1} |z|_{\mathcal{Z}_1}\ d\mu(z) \leq 2\pi \left(\int_{\mathcal{Z}_1} 1 + |z|^2_{\mathcal{Z}_1}\ d\mu(z)\right) |\xi-\xi'|_{\mathcal{Z}_1}\,,$$

and the function G_1 is a Lipschitz function on $\mathcal{Z}_1$. This finishes the proof that μ is a Borel probability measure on $\mathcal{Z}_1$.
For the inequality (32), it suffices to notice the inequality of (commuting) operators

$$(1+S_\varepsilon)^2(1+\mathbf{N}) \geq (\mathrm{d}\Gamma(1-\Delta)^2)\mathbf{N}\,.$$

Applying Lemma 3.13 yields

$$\int_{\mathcal{Z}_1} |z|^4_{\mathcal{Z}_1}|z|^2_{\mathcal{Z}_0}\ d\mu(z) \leq C\,.$$

□

3.4 Uniqueness of the mean field dynamics via measure transportation technique

Now we are in position to prove Theorem 1.1. This will be done in three steps: 1) Writing a transport equation, in a weak sense in $\mathcal{Z}_1$ for μ_t ; 2) Solving this equation as $\mu_t = \Phi(t,0)_*\mu_0$ when the initial state ϱ_ε fulfills strong decay estimates; 3) Relaxing the strong decay estimates.

3.5 The transport equation on $\mathcal{Z}_1$

We shall need similar notions about cylindrical functions, as the one used in $\mathcal{Z}_0$ and recalled in Appendix B.1. Let $\mathbb{P}_1$ denote the set of all finite rank orthogonal projections on $\mathcal{Z}_1$ and for a given $\wp \in \mathbb{P}_1$ let $L_{\wp,1}(dz)$ denote the Lebesgue measure on the finite dimensional subspace $\wp\mathcal{Z}_1$, with volume 1 for a $\mathcal{Z}_1$-orthonormal hypercube. A function $f : \mathcal{Z}_1 \to \mathbb{C}$ is said cylindrical if there exists $\wp \in \mathbb{P}_1$ and a function g on $\wp\mathcal{Z}_1$ such that $f(z) = g(\wp z)$, for all $z \in \mathcal{Z}_1$. In this case we say that f is based on the subspace $\wp\mathcal{Z}_1$. The set of $\mathcal{C}_0^\infty$ (resp. $\mathcal{S}$) cylindrical functions on $\mathcal{Z}_1$, is denoted by $\mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1)$ (resp. $\mathcal{S}_{cyl}(\mathcal{Z}_1)$). We shall also need $\mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1 \times \mathbb{R})$, in which the algebraic tensor product $\mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1) \overset{alg}{\otimes} \mathcal{C}_0^\infty(\mathbb{R})$ is dense. Finally the Fourier transform of elements of $\mathcal{S}_{cyl}(\mathcal{Z}_1)$ is given by

$$\mathcal{F}_1[f](\xi) = \int_{\wp\mathcal{Z}_1} f(\xi)\ e^{-2\pi i \,\mathrm{Re}\ \langle z,\xi\rangle_{\mathcal{Z}_1}}\ L_{\wp,1}(dz)\,,$$
$$f(z) = \int_{\wp\mathcal{Z}_1} \mathcal{F}_1[f](\xi)\ e^{2\pi i \,\mathrm{Re}\ \langle z,\xi\rangle_{\mathcal{Z}_1}}\ L_{\wp,1}(d\xi)\,.$$

According to the notations of [3] and Appendix C, we consider the space $\mathrm{Prob}_2(\mathcal{Z}_1)$ of Borel probability measures μ such that

$$\int_{\mathcal{Z}_1} |z|^2_{\mathcal{Z}_1}\ d\mu(z) < +\infty\,.$$

On this space, we introduce the Wasserstein distance

$$W^2(\mu_1,\mu_2) = \inf_{\mu\in\Gamma(\mu_1,\mu_2)} \int_{\mathcal{Z}_1^2} |z_1 - z_2|^2_{\mathcal{Z}_1}\ d\mu(z_1,z_2) \tag{33}$$

where $\Gamma(\mu_1,\mu_2)$ is the set of Borel probability measures μ on $\mathcal{Z}_1 \times \mathcal{Z}_1$ such that the marginals $(\Pi_1)_*\mu = \mu_1$ and $(\Pi_2)_*\mu = \mu_2$.

Proposition 3.14. *Let S_ε and $\tilde\varrho_\varepsilon(t)$ be the operators given by Definition 3.5 and* (20)*. Assume that the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies*

$$\forall\alpha\in\mathbb{N},\ \exists C_\alpha > 0,\ \forall\varepsilon\in(0,\bar\varepsilon),\quad \mathrm{Tr}\,[(1+S_\varepsilon)\varrho_\varepsilon(1+S_\varepsilon)(1+\mathbf{N})^\alpha] \le C_\alpha\,,$$

and consider a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \overset{k\to\infty}{\to} 0$ such that

$$\mathcal{M}(\tilde\varrho_{\varepsilon_k}(t),\ k\in\mathbb{N}) = \{\tilde\mu_t\}$$

according to Proposition 3.6. Then the measure $\tilde\mu_t$ is a Borel probability measure on $\mathcal{Z}_1$ which satisfies

- *$\int_{\mathcal{Z}_1} |z|^4_{\mathcal{Z}_1}|z|^2_{\mathcal{Z}_0}\ d\tilde\mu_t(z) \le C'$ for some C' independent of $t\in\mathbb{R}$.*
- *It is a Lipschitz function of $t\in\mathbb{R}$ in $Prob_2(\mathcal{Z}_1)$ endowed with the Wasserstein distance* (33)*.*
- *It is a weak solution to the Liouville equation*

$$\partial_t\tilde\mu + i\,\{V_s,\tilde\mu_t\} = 0\,,$$

in the sense that for $f\in\mathcal{C}^\infty_{cyl,0}(\mathcal{Z}_1\times\mathbb{R})$

$$\int_{\mathbb{R}}\int_{\mathcal{Z}_1} (\partial_t f + i\,\{V_s,f\})\ d\tilde\mu_t(x)dt = 0 \tag{34}$$

Proof. The Proposition 2.3 as well as the commutations $[e^{itH_\varepsilon},\mathbf{N}] = [e^{itH^0_\varepsilon},\mathbf{N}]$ ensure

$$\forall\alpha\in\mathbb{N},\exists C'_\alpha>0,\forall t\in\mathbb{R},\forall\varepsilon\in(0,\bar\varepsilon)\,,\quad \mathrm{Tr}\,[(1+S_\varepsilon)\tilde\varrho_\varepsilon(t)(1+S_\varepsilon)(1+\mathbf{N})^\alpha] \le C'_\alpha\,.$$

The Proposition 3.11 for any $t \in \mathbb{R}$, provides the first results, which also contain $\tilde{\mu}(t) \in \mathrm{Prob}_2(\mathcal{Z}_1)$. It remains to check the continuity with respect to the Wasserstein distance W^2 and the Liouville equation. Replace $\xi \in \mathcal{Z}_0$, by $\sqrt{2}\pi\xi$ in the equation (21) in Proposition 3.6 and integrate with $\mathcal{F}[f](\xi)\ dL_{\wp}(z)$ with

$$f(z) = \int_{\wp\mathcal{Z}_0} \mathcal{F}[f](z) e^{2i\pi \mathrm{Re}\ \langle z,\xi\rangle}\ dL_{\wp}(z)$$

when $f \in \mathcal{S}_{cyl}(\mathcal{Z}_0)$ based on $\wp\mathcal{Z}_0$. This gives for any $t, t' \in \mathbb{R}$ and all $f \in \mathcal{S}_{cyl}(\mathcal{Z}_0)$:

$$\int_{\mathcal{Z}_0} f(z)\ d\tilde{\mu}_{t'}(z) = \int_{\mathcal{Z}_0} f(z)\ d\tilde{\mu}_t(z) + i\int_t^{t'} \int_{\mathcal{Z}_0} \{V_s, f\}(z)\ d\tilde{\mu}_s(z)\ ds\,.$$

With the uniform estimate

$$\int_{\mathcal{Z}_0} |z|^6_{\mathcal{Z}_0}\ d\tilde{\mu}_s(z) \leq \int_{\mathcal{Z}_0} |z|^4_{\mathcal{Z}_1} |z|^2_{\mathcal{Z}_0}\ d\tilde{\mu}_s(z) \leq C_0\,,$$

while V_s is quartic, it can be extended by dominated convergence to any cylindrical polynomial f of total degree less than 4.
Consider now an increasing sequence of finite rank positive operators A_n on $\mathcal{Z}_0$ such that

$$\sup_{n\in\mathbb{N}} A_n = (1-\Delta)\,.$$

Since $\tilde{\mu}_t$ and $\tilde{\mu}_{t'}$ belong to $\mathrm{Prob}_2(\mathcal{Z}_1)$ and are Borel measures on $\mathcal{Z}_0$ carried by $\mathcal{Z}_1$, we have

$$W^2(\tilde{\mu}_{t'}, \tilde{\mu}_t) \leq \int_{\mathcal{Z}_1\times\mathcal{Z}_1} |z_1 - z_2|^2_{\mathcal{Z}_1}\ d\tilde{\mu}_t(z_1) d\tilde{\mu}_{t'}(z_2) = \int_{\mathcal{Z}_0\times\mathcal{Z}_0} |z_1 - z_2|^2_{\mathcal{Z}_1}\ d\tilde{\mu}_t(z_1) d\tilde{\mu}_{t'}(z_2)$$

and the monotone convergence ensures

$$W^2(\tilde{\mu}_{t'}, \tilde{\mu}_t) \leq \sup_{n\in\mathbb{N}} \int_{\mathcal{Z}_0\times\mathcal{Z}_0} \langle z_1 - z_2\,,\, A_n(z_1 - z_2)\rangle\ d\tilde{\mu}_t(z_1) d\tilde{\mu}_{t'}(z_2)\,.$$

For a given $n \in \mathbb{N}$, and a fixed $z_1 \in \mathcal{Z}_0$, the function $z_2 \to f_n(z_1, z_2) = \langle z_1 - z_2\,,\, A_n(z_1 - z_2)\rangle$ is a cylindrical polynomial of total degree 2. Hence we get

$$\int_{\mathcal{Z}_0\times\mathcal{Z}_0} \langle z_1 - z_2\,,\, A_n(z_1 - z_2)\rangle\ d\tilde{\mu}_t(z_1) d\tilde{\mu}_{t'}(z_2) \leq \int_{[t,t']} ds \int_{\mathcal{Z}_0\times\mathcal{Z}_0} |\{V_s, f_n(z_1,.)\}|\ d\tilde{\mu}_s(z_2) d\tilde{\mu}_t(z_1)\,.$$

We compute

$$\{V_s\,,\, f_n(z_1,.)\}(z_2) = 4i\ \mathrm{Im}\ \langle z_2^{\otimes 2}\,,\, \tilde{V}_s(z_2 \otimes (A_n z_2))\rangle - 4i\ \mathrm{Im}\ \langle z_2^{\otimes 2}\,,\, \tilde{V}_s(z_2 \otimes (A_n z_1))\rangle\,.$$

From (17) and with $|A_n^{1/2} u|_{\mathcal{Z}_0} \leq |u|_{\mathcal{Z}_1}$, we deduce

$$|\{V_s\,,\, f_n(z_1,.)\}(z_2)| \leq C(|A_n^{1/2} z_2|_{\mathcal{Z}_0} + |A_n^{1/2} z_1|_{\mathcal{Z}_0}) |z_2|^2_{\mathcal{Z}_1} |z_2|_{\mathcal{Z}_0} \leq C\,(|z_2|_{\mathcal{Z}_1} + |z_1|_{\mathcal{Z}_1})\,|z_2|^2_{\mathcal{Z}_1} |z_2|_{\mathcal{Z}_0}\,.$$

The uniform estimate $\int_{\mathcal{Z}_1} |z|^4_{\mathcal{Z}_1} |z|^2_{\mathcal{Z}_0}\ d\tilde{\mu}_\tau(z) \leq C_0$ valid for all times, thus leads to

$$W^2(\mu_{t'}, \mu_t) \leq C'|t' - t|\,.$$

In order to prove the last point, let us come back to the equation (21). Again the estimate (17) implies

$$|\mathrm{Im}\ \langle z^{\otimes}\,,\, \tilde{V}_s(\xi \otimes z)\rangle| \leq C|z|^2_{\mathcal{Z}_1} |z|_{\mathcal{Z}_0} |\xi|_{H^{-1}(\mathbb{R}^d)}\,.$$

Since the measure $\tilde{\mu}_t$ is carried by $\mathcal{Z}_1$ for all times with moment estimates, the first line of (21) shows that it still makes sense for $\xi \in H^{-1}(\mathbb{R}^d)$. By taking $\xi = \sqrt{2}\pi(1-\Delta)\eta$ with $\eta \in \mathcal{Z}_1$, we obtain

$$\tilde{\mu}_t(e^{2i\pi \mathrm{Re}\ \langle \eta\,,\, z\rangle_{\mathcal{Z}_1}}) = \tilde{\mu}_0(e^{2i\pi \mathrm{Re}\ \langle \eta\,,\, z\rangle_{\mathcal{Z}_1}}) + i\int_0^t \tilde{\mu}_s(\Big\{V_s(.)\,,\, e^{2i\pi \mathrm{Re}\langle \eta\,,\, z\rangle_{\mathcal{Z}_1}}\Big\})\ ds\,.$$

By integrating with $\mathcal{F}_1(g)(\eta)\ dL_{\wp,1}(z)$ for a cylindrical function $g \in \mathcal{S}_{cyl}(\mathcal{Z}_1)$ based on $\wp\mathcal{Z}_1$, we get

$$\int_{\mathcal{Z}_1} g(z)\ d\tilde{\mu}_t(z) = \int_{\mathcal{Z}_1} g(z)\ d\tilde{\mu}_0(z) + i\int_0^t \int_{\mathcal{Z}_1} \{V_s\,,\, g\}(z)\ d\tilde{\mu}_s(z)\ ds\,.$$

Hence for any $f \in \mathcal{S}_{cyl}(\mathcal{Z}_1)$, the function $I_g : t \to \int_{\mathcal{Z}_1} g(z)\ d\tilde{\mu}_t(z)$ belongs to $\mathcal{C}^1(\mathbb{R})$ with

$$\partial_t I_g(t) = i\int_{\mathcal{Z}_1} \{V_t\,,\, g\}(z)\ d\tilde{\mu}_t(z)\,.$$

By multiplying the above relation by $\varphi(t)$, with $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and integrating by part proves (34) when $f(t,z) = \varphi(t)g(z)$. We conclude by the density of $\mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1) \overset{alg}{\otimes} \mathcal{C}_0^\infty(\mathbb{R})$ in $\mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1 \times \mathbb{R})$. □

3.6 Uniqueness of the measure for regular initial data

Proposition 3.15. *Let S_ε and $\tilde{\varrho}_\varepsilon(t)$ be the operators given by Definition 3.5 and* (20)*. Assume that the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies*

$$\forall \alpha \in \mathbb{N}, \exists C_\alpha > 0, \forall \varepsilon \in (0,\bar{\varepsilon}), \quad \mathrm{Tr}\,[(1+S_\varepsilon)\varrho_\varepsilon(1+S_\varepsilon)(1+\mathbf{N})^\alpha] \leq C_\alpha$$
$$\textit{and} \qquad \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu_0\}\ .$$

Then for any time $t \in \mathbb{R}$, the family $(\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar{\varepsilon})}$ admits a unique Wigner measure $\mu_t = \Phi(t,0)_*\mu_0$, where Φ is the Hartree flow defined by (3) *on $\mathcal{Z}_1$. It is a Borel probability measure on $\mathcal{Z}_1$ which belongs to $Prob_2(\mathcal{Z}_1)$, which is Lipschitz continuous w.r.t time for the Wasserstein distance W^2 and which satisfies*

$$\forall t \in \mathbb{R}\,, \quad \int_{\mathcal{Z}_1} |z|^4_{\mathcal{Z}_1}|z|^2_{\mathcal{Z}_0}\ d\mu_t(z) \leq C\,.$$

Proof. We still start with the state $\tilde{\varrho}_\varepsilon(t)$ defined in (20). Proposition 2.6 says that the group Φ associated with (3) and the dynamical system $\tilde{\Phi}$ associated with

$$i\partial_t z = v(t,z) \quad , \quad v(t,z) = e^{-it\Delta}([V * |e^{it\Delta}z|^2]e^{it\Delta}z)$$

are well defined on $\mathcal{Z}_1$. Further it gives the estimate for the velocity field

$$|v(t,z)|_{\mathcal{Z}_1} \leq \|V(1-\Delta)^{-1/2}\||z|^2_{\mathcal{Z}_1}|z|_{\mathcal{Z}_0}\,.$$

When $\tilde{\mu}_t$ is the Wigner measure defined for all times and associated with a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$, we obtain

$$\forall t \in \mathbb{R}, \quad \int_{\mathcal{Z}_1} |v(t,z)|^2_{\mathcal{Z}_1}\ d\tilde{\mu}_t(z) \leq C\int_{\mathcal{Z}_1} |z|^4_{\mathcal{Z}_1}|z|^2_{\mathcal{Z}_0}\ d\tilde{\mu}_t(z) \leq C'\,.$$

With Proposition 3.14, this means that $(\tilde{\mu}_t)$ fulfills all the conditions of Proposition C.8 and we deduce that $\tilde{\mu}_t = \tilde{\Phi}(t,0)_*\mu_0$. But this uniqueness implies $\mathcal{M}(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})) = \{\tilde{\mu}_t\}$ for the whole family $(\tilde{\varrho}_\varepsilon(t))_{\varepsilon\in(0,\bar{\varepsilon})}$ and all times $t \in \mathbb{R}$.
Going back to $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H^0_\varepsilon}\tilde{\varrho}_\varepsilon(t)e^{i\frac{t}{\varepsilon}H^0_\varepsilon}$, it gives $\mu_t = \Phi(t,0)_*\mu_0$.
The additional properties are the ones coming from Proposition 3.14. □

3.7 Evolution of the Wigner measure for general data

We follow the truncation scheme used in [6]. When the initial data satisfies only

$$\|(\mathbf{N}+H^0_\varepsilon)^{\delta/2}\varrho_\varepsilon(\mathbf{N}+H^0_\varepsilon)^{\delta/2}\| \leq C_\delta$$

for some $\delta > 0$, we approximate ϱ_ε by

$$\varrho_{\varepsilon,R} = \frac{1}{\mathrm{Tr}\,[\chi_R(\mathbf{N},H^0_\varepsilon)\varrho_\varepsilon\chi_R(\mathbf{N},H^0_\varepsilon)]}\chi_R(\mathbf{N},H^0_\varepsilon)\varrho_\varepsilon\chi_R(\mathbf{N},H^0_\varepsilon)$$

with $\chi_R(n,h) = \chi(\frac{n}{R}, \frac{h}{R})$ with $R \to +\infty$ and $0 \leq \chi \leq 1$, $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $\chi \equiv 1$ in a neighborhood of 0. The time evolved state is defined by

$$\varrho_{\varepsilon,R}(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\, \varrho_{\varepsilon,R}\, e^{i\frac{t}{\varepsilon}H_\varepsilon}\,.$$

The assumptions ensure that for all times

$$\|\varrho_\varepsilon(t) - \varrho_{\varepsilon,R}(t)\|_{\mathcal{L}^1} \leq \nu(R)$$

with ν independent of (t,ε) and $\lim_{R\to\infty}\nu(R) = 0$. We recall the Proposition 2.10 of [6].

Proposition 3.16. *Let $(\varrho_\varepsilon^j)_{\varepsilon\in(0,\bar\varepsilon)}$, $j = 1,2$, be two families (or sequences) of normal states on $\mathcal{H}$ such that $\mathrm{Tr}\left[\varrho_\varepsilon^j \mathbf{N}^\delta\right] \leq C_\delta$ uniformly w.r.t. $\varepsilon \in (0,\bar\varepsilon)$ for some $\delta > 0$ and $C_\delta \in (0,+\infty)$. Assume further $\mathcal{M}(\varrho_\varepsilon^j, \varepsilon \in (0,\bar\varepsilon)) = \{\mu_j\}$ for $j = 1,2$. Then*

$$\int |\mu_1 - \mu_2| \leq \liminf_{\varepsilon\to 0} \|\varrho_\varepsilon^1 - \varrho_\varepsilon^2\|_{\mathcal{L}^1(\mathcal{H})}\,.$$

Proof of Proposition 3.16, end of the proof of Theorem 1.1: For $R \in (0,+\infty)$, the state $\varrho_{\varepsilon,R}$ fulfills the conditions of Proposition 3.15 except the uniqueness of the Wigner measure at time $t = 0$. Out of any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ can be extracted in order to ensure

$$\mathcal{M}(\varrho_{\varepsilon_{n_k},R}, k \in \mathbb{N}) = \{\mu_{0,R}\}\,.$$

Thus after this extraction we obtain

$$\forall t \in \mathbb{R}\,, \quad \mathcal{M}(\varrho_{\varepsilon_{n_k},R}(t), k \in \mathbb{N}) = \{\Phi(t,0)_*\mu_{0,R}\}\,.$$

Take $t \in \mathbb{R}$ and let μ belong to $\mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon))$. There exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{M}(\varrho_{\varepsilon_n}(t), n \in \mathbb{N}) = \{\mu\}\,.$$

After extracting a subsequence like above and by using Proposition 3.16, we obtain

$$\int |\mu - \Phi(t,0)_*\mu_0| \leq \int |\mu - \Phi(t,0)_*\mu_{0,R}| + \int |\mu_{0,R} - \mu_0| \leq 2\nu(R)\,.$$

Taking the limit as $R \to \infty$ implies $\mu = \Phi(t,0)_*\mu_0$ and therefore

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)) = \{\Phi(t,0)_*\mu_0\}\,.$$

This also proves that $\lim_{R\to\infty}\int_{\mathcal{Z}_0} |\mu_t - \Phi(t,0)_*\mu_{0,R}| = 0$, while all the measures $\Phi(t,0)_*\mu_{0,R}$ are Borel probability measures carried by, and on, $\mathcal{Z}_1$. This implies that μ_t is carried by $\mathcal{Z}_1$ and is also a Borel measure on $\mathcal{Z}_1$. This ends the proof of Theorem 1.1 □

4 Complements

Additional results are given in the three first paragraphs, concerned with the BBGKY hierarchy or the propagation of energy. The fourth one shows some examples and the last one is an informal discussion about the classical mean field problem.

4.1 BBGKY hierarchy

Although the analysis here is different from our previous work [6] it is possible to combine them, in order to strengthen the result of Theorem 1.1. It is also interesting to reformulate our result in terms of reduced density matrices since, in the literature, several mathematical results on mean field limit use the BBGKY hierarchy method ([8, 9, 42] for example). For a family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ on $\mathcal{H}$ and $r \in \mathbb{N}$, the reduced density matrices $\gamma_\varepsilon^{(p)} \in \mathcal{L}^1(L^2_s(\mathbb{R}^{dp}))$ is defined according to

$$\mathrm{Tr}\left[\gamma_\varepsilon^{(p)}\tilde b\right] = \frac{\mathrm{Tr}\left[\varrho_\varepsilon\right]}{\mathrm{Tr}\left[\varrho_\varepsilon(|z|^{2p})^{Wick}\right]}\mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right]\,, \quad \forall \tilde b \in \mathcal{L}(L^2_s(\mathbb{R}^{dp}))\,, \tag{35}$$

with the convention that the right-hand side is 0 when $\mathrm{Tr}\left[\varrho_\varepsilon(|z|^{2p})^{Wick}\right] = 0$.

Proposition 4.1. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$, satisfying the hypothesis of Theorem 1.1, with a single Wigner measure μ_0 such that*

$$\forall\alpha\in\mathbb{N},\quad \lim_{\varepsilon\to 0}\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^\alpha]=\int_{\mathcal{Z}_0}|z|^{2\alpha}\,d\mu_0(z)<+\infty\,. \tag{36}$$

Then for all $t\in\mathbb{R}$, the convergence

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\left[\varrho_\varepsilon(t)b^{Wick}\right]=\int_{\mathcal{Z}_0}b(\Phi(t,0)z)\,d\mu_0(z)=\int_{\mathcal{Z}_0}b(z)\,d\mu_t(z)$$

holds for any $b\in\mathcal{P}_{alg}(\mathcal{Z}_0)=\oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z}_0)$, with $\mu_t=\Phi(t,0)_*\mu_0$.
Finally, the convergence of the reduced density matrices

$$\lim_{\varepsilon\to 0}\gamma^{(p)}_\varepsilon(t)=\frac{1}{\int_{\mathcal{Z}_0}|z|^{2p}\,d\mu_t(z)}\int_{\mathcal{Z}_0}|z^{\otimes p}\rangle\langle z^{\otimes p}|\,d\mu_t(z)\,,$$

holds in the $\mathcal{L}^1(L^2_s(\mathbb{R}^{dp}))$-norm for all $p\in\mathbb{N}$.

Proof. We know from Theorem 1.1 that the the family of normal states $(\varrho_\varepsilon(t))_{\varepsilon\in(0,\bar\varepsilon)}$ admits a single Wigner measure μ_t equal to $\Phi(t,0)_*\mu_0$. Since the quantum and classical flows preserve the total number, the state $\varrho_\varepsilon(t)$ satisfies as well the condition (36) for any time $t\in\mathbb{R}$. The [6, Proposition 2.11, 2.13] provides the claimed results. □

4.2 Moment upper bounds

In [4], it was proved that the sole a priori estimate $\mathrm{Tr}\left[\varrho_\varepsilon\mathbf{N}^\delta\right]\leq C_\delta$ for a given $\delta>0$ (possibly small), with $\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))=\{\mu\}$ leads to

$$\int_{\mathcal{Z}_0}|z|^{2\delta}\,d\mu(z)<+\infty\,.$$

The a priori estimate, assumed in Theorem 1.1 at time $t=0$, leads to

$$\int_{\mathcal{Z}_1}|z|^{2\delta}_{\mathcal{Z}_1}\,d\mu(z)<+\infty\,,$$

according to the following result which is a variation of Lemma 3.12.

Proposition 4.2. *Let $(A,\mathcal{D}(A))$ be a self-adjoint operator on $\mathcal{Z}_0$ such that $A\geq 1$. If the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies $\mathrm{Tr}\left[\varrho_\varepsilon(\mathrm{d}\Gamma(A))^\delta\right]\leq C_\delta$ for some $\delta>0$ and $\mathcal{M}\left\{\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\right\}=\{\mu\}$, then*

$$\int_{\mathcal{Z}_0}\langle z\,,\,Az\rangle^\delta\,d\mu(z)<+\infty\,.$$

Proof. By Wick calculus (see Proposition B.2 when A is bounded), one gets

$$\mathrm{d}\Gamma(A)^k\geq\left(\langle z\,,\,Az\rangle^k\right)^{Wick}\,,\forall k\in\mathbb{N}\,.$$

Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{Z}_0$ such that $e_j\in\mathcal{D}(A)$ for all $j\in\mathbb{N}$, and set

$$A_J=\sum_{j=0}^J A^{1/2}|e_j\rangle\langle e_j|A^{1/2}=\sum_{j=0}^J|A^{1/2}e_j\rangle\langle A^{1/2}e_j|\,.$$

The inequality $A^{\otimes k}\geq A_J^{\otimes k}$ holds for all $J\in\mathbb{N}$, while $\tilde b\to b^{Wick}$ is operator monotone when restricted to operators $\tilde b$ acting in $\bigvee^{2k}\mathcal{Z}_0$. Therefore, we obtain

$$\begin{aligned}(1+\mathrm{d}\Gamma(A))^n &= \sum_{k=0}^n C_n^k\mathrm{d}\Gamma(A)^k\geq\left(\sum_{k=0}^n C_n^k\langle z\,,\,Az\rangle^k\right)^{Wick}\\ &\geq \left(\sum_{k=0}^n C_n^k\langle z\,,\,A_Jz\rangle^k\right)^{Wick}=\left[(1+\langle z\,,\,A_Jz\rangle)^n\right]^{Wick}\,.\end{aligned}\tag{37}$$

We now use the same argument as the one used in [4] when $A = \mathrm{Id}$, relying on the semiclassical calculus in finite dimension (see [12][40][50][52][56]).
Let $\wp_J$ be the orthogonal projection from $\mathcal{Z}_0$ onto $\oplus_{j=0}^{J} A^{1/2} e_j$. The symbol $1 + \langle z\,,\, A_J z\rangle$ is a cylindrical symbol based on $\wp_J \mathcal{Z}_0$. Since $\ker A^{1/2} = \{0\}$ and $e_0, \ldots, e_J$ are linearly independent, the symbol

$$(1 + \langle z\,,\, A_J z\rangle) = 1 + \sum_{j=0}^{J} |\langle A^{1/2} e_j, z\rangle|^2$$

is an elliptic symbol on $\wp_J \mathcal{Z}_0 \sim \mathbb{C}^{J+1}$ in the Hörmander class $\mathcal{S}(1 + |z|^2_{\mathbb{C}^{J+1}}, \frac{|dz|^2_{\mathbb{C}^{J+1}}}{1+|z|^2_{\mathbb{C}^{J+1}}})$. The functional calculus of Weyl ε-quantized elliptic operators in finite dimensions gives

$$\forall s \in \mathbb{R}, \left[(1 + \langle z\,,\, A_J z\rangle)^{Weyl}\right]^s \geq (1 - C_{J,s}\varepsilon)\left[(1 + \langle z\,,\, A_J z\rangle)^s\right]^{Weyl}. \tag{38}$$

The finite dimensional comparison of Wick and Weyl quantization, also gives

$$\forall n \in \mathbb{N}, \left[(1 + \langle z\,,\, A_J z\rangle)^n\right]^{Wick} \geq (1 - C_{J,n}\varepsilon)\left[(1 + \langle z\,, A_J z\rangle)^{Weyl}\right]^n. \tag{39}$$

From (37)(39) and the operator monotonicity of $B \to B^t$ for $t \in (0, 1]$, we deduce

$$\forall s \in \mathbb{R}\,, \quad (1 + \mathrm{d}\Gamma(A))^s \geq (1 - C'_{J,s}\varepsilon)\left[(1 + \langle z\,,\, A_J z\rangle)^{Weyl}\right]^s,$$

and (38) gives

$$\forall s \in \mathbb{R}\,, \quad (1 + \mathrm{d}\Gamma(A))^s \geq (1 - C''_{J,s}\varepsilon)\left[(1 + \langle z\,,\, A_J z\rangle)^s\right]^{Weyl}. \tag{40}$$

The definition of Wigner measures, recalled in Theorem 3.1, says

$$\lim_{\varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon b^{Weyl}\right] = \int_{\mathcal{Z}_0} b(z)\, d\mu(z)\,,$$

for all $b \in \mathcal{S}_{cyl}(\mathcal{Z}_0)$, in particular the b's based on $\wp_J \mathcal{Z}_0$. Take now $s = \delta$ in (40). The a priori estimate

$$\mathrm{Tr}\left[\varrho_\varepsilon \left[(1 + \langle z\,,\, A_J z\rangle)^\delta\right]^{Weyl}\right] \leq (1 + C_{J,\delta}\varepsilon)\mathrm{Tr}\left[\varrho_\varepsilon (1 + \mathrm{d}\Gamma(A))^\delta\right] \leq C_\delta(1 + C_{J,\delta}\varepsilon)\,,$$

and the ellipticity of $(1 + \langle z\,,\, A_J z\rangle)^\delta$ allows to extend the above convergence to any cylindrical $b = f \circ \wp_J$ with $f \in \mathcal{S}((1 + |z|^2_{\mathbb{C}^{J+1}})^\delta, \frac{|dz|^2_{\mathbb{C}^{J+1}}}{1+|z|^2_{\mathbb{C}^{J+1}}})$. In particular, this leads to

$$C'_\delta \geq \limsup_{\varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon (1 + \mathrm{d}\Gamma(A))^\delta\right] \geq \lim_{\varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon \left[(1 + \langle z\,,\, A_J z\rangle)^\delta\right]^{Weyl}\right] = \int_{\mathcal{Z}_0} (1 + \langle z\,, A_J z\rangle)^\delta\, d\mu(z)\,.$$

Since $A = \sup_J A_J$ with $A_{J'} \geq A_J$ for $J' \geq J$, the monotone convergence implies

$$C'_\delta \geq \int_{\mathcal{Z}_0} (1 + \langle z\,,\, Az\rangle)^\delta\, d\mu(z) \geq \int_{\mathcal{Z}_0} \langle z\,,\, Az\rangle^\delta\, d\mu(z)\,.$$

□

Proposition 4.3. *Within the framework of Theorem 1.1 with the assumption* $\mathrm{Tr}[(\mathbf{N} + H_\varepsilon^0)^\delta \varrho_\varepsilon] \leq C_\delta$ *for* $\delta \leq 6$*, the measure* μ_t *satisfies the additional estimate*

$$\int_{\mathcal{Z}_1} |z|_{\mathcal{Z}_1}^{\frac{2\delta}{3}}\, d\mu_t(z) \leq C_\delta$$

for all times $t \in \mathbb{R}$*.*

Proof. The functional calculus of commuting operators implies

$$c_\delta(1+S_\varepsilon)^{\delta/3} \leq (1+\mathbf{N}+H_\varepsilon^0)^\delta \leq (1+S_\varepsilon)^\delta\,.$$

Thus the initial state ϱ_ε, satisfies

$$\mathrm{Tr}\left[\varrho_\varepsilon(1+S_\varepsilon)^{\delta/3}\right] \leq C'_\delta\,.$$

From Proposition 2.3, we deduce

$$e^{i\frac{t}{\varepsilon}H_\varepsilon}(1+S_\varepsilon)^2 e^{-i\frac{t}{\varepsilon}H_\varepsilon} \leq C(1+S_\varepsilon)^2\,.$$

Since $B \to B^s$ is operator monotone for $s \in (0,1]$, this implies

$$e^{i\frac{t}{\varepsilon}H_\varepsilon}(1+S_\varepsilon)^{\delta/3} e^{-i\frac{t}{\varepsilon}H_\varepsilon} \leq C^{\delta/6}(1+S_\varepsilon)^{\delta/3}$$

as soon as $\frac{\delta}{3} \leq 2$. The inequality

$$\mathrm{Tr}\left[\varrho_\varepsilon(t)(\mathrm{d}\Gamma(1-\Delta))^{\delta/3}\right] \leq \mathrm{Tr}\left[\varrho_\varepsilon(t)(1+S_\varepsilon)^{\delta/3}\right] \leq C'_\delta$$

and the previous Proposition 4.2 applied with $A = (1-\Delta)$, yields the result. □

A more accurate version of this last result is given below by making use of the conservation of energy.

4.3 Convergence of moments and energy conservation

For a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ of normal states with a single Wigner measure μ_0 the condition (36) is an important and non trivial assumption. Indeed, we proved in [6] the following equivalence

$$\left(\forall\alpha\in\mathbb{N},\ \lim_{\varepsilon\to0}\mathrm{Tr}\left[\varrho_\varepsilon\mathbf{N}^\alpha\right] = \int_{\mathcal{Z}}|z|^{2\alpha}\ d\mu_0(z)\right) \Leftrightarrow \left(\forall b\in\mathcal{P}_{alg}(\mathcal{Z}_0),\ \lim_{\varepsilon\to0}\mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right] = \int_{\mathcal{Z}} b(z)\ d\mu_0\right). \quad (41)$$

Hence the condition (36), although it involves only the number operator, is exactly the one which leads to a good asymptotic behaviour of the reduced density matrices.

Proposition 4.4. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$, satisfying the hypothesis of Theorem 1.1, with a single Wigner measure μ_0. Assume $\mathrm{Tr}[\varrho_\varepsilon\ \mathbf{N}^\alpha] \leq C_\alpha$ uniformly in $\varepsilon\in(0,\bar\varepsilon)$, for all $\alpha\in\mathbb{N}$.*
Then for every $\alpha\in\mathbb{N}$, the quantity

$$\liminf_{\varepsilon\to0}\ \mathrm{Tr}[\varrho_\varepsilon(t)\,\mathbf{N}^\alpha] - \int_{\mathcal{Z}_0}|z|^{2\alpha}_{\mathcal{Z}_0}\ d\mu_t(z),$$

does not depend on time when $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}$ and $\mathcal{M}(\varrho_\varepsilon(t),\varepsilon\in(0,\bar\varepsilon)) = \{\mu_t\}$.
The condition (36) is satisfied by $(\varrho_\varepsilon(t))_{\varepsilon\in(0,\bar\varepsilon)}$ and μ_t, for all times $t\in\mathbb{R}$, as soon as it is true for one $t_0\in\mathbb{R}$.

Proof. According to Theorem 1.1, we know that $\mathcal{M}(\varrho_\varepsilon(t),\varepsilon\in(0,\bar\varepsilon)) = \{\mu_t\}$ with $\mu_t = \Phi(t,0)_*\mu_0$. The conservation of the $|.|_{\mathcal{Z}_0}$-norm by the nonlinear flow $\Phi(t,0)$ yields

$$\int_{\mathcal{Z}_0}|z|^{2\alpha}_{\mathcal{Z}_0}\ d\mu_t(z) = \int_{\mathcal{Z}_0}|z|^{2\alpha}_{\mathcal{Z}_0}\ d\mu_0(z)$$

for any $t\in\mathbb{R}$. On the other hand, H_ε and $\mathbf{N}$ are strongly commuting self-adjoint operators and therefore $\mathrm{Tr}[\varrho_\varepsilon(t)\,\mathbf{N}^\alpha] = \mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^\alpha]$ for every $\alpha\in\mathbb{N}$. □

Proposition 4.5. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ with a single Wigner measure μ_0 satisfying the hypothesis of Theorem 1.1 with $\delta = 2$ and condition* (36)*. Then for any $t \in \mathbb{R}$*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}[\varrho_\varepsilon(t)\, H_\varepsilon] = \int_{\mathcal{Z}_1} h(z,\bar z)\; d\mu_t(z) \quad \in (-\infty,\infty) \tag{42}$$

where $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}$, $\mu_t = \Phi(t,0)_*\mu_0$ and $h(z,\bar z)$ is the classical energy given in (2)*, and both sides of the identity do not depend on time.*

Proof. With the energy conservation, it suffices to prove (42) for $t = 0$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(s) = 1$ if $|s| \le 1$ and $\chi(s) = 0$ if $|s| \ge 2$. For $m \in \mathbb{N}^*$, set $\chi_m(x) = \chi(\frac{x}{m})$. Let $B_1(z)$ and $B_2(z)$ be respectively the polynomial $\langle z, -\Delta z\rangle$ and $B_2(z) = V(z) = \frac{1}{2}\langle z^{\otimes 2}, V(x-y)z^{\otimes 2}\rangle$ well defined for $z \in \mathcal{Z}_1$. Remember that although the kernels of B_1 and B_2 are unbounded operators their Wick quantization still have a meaning as densely defined operators on $\mathcal{H}$ (see Appendix B). We write for $j = 1, 2$

$$\begin{aligned} |\mathrm{Tr}[\varrho_\varepsilon\, B_j^{Wick}] - \int_{\mathcal{Z}_1} B_j(z)\, d\mu_0(z)| \;\le\; & |\mathrm{Tr}[\varrho_\varepsilon\, (B_j^{Wick} - B_{j,m}^{Wick})]| && (43)\\ & + |\mathrm{Tr}[\varrho_\varepsilon\, B_{j,m}^{Wick}] - \int_{\mathcal{Z}_0} B_{j,m}(z)\, d\mu_0(z)| && (44)\\ & + |\int_{\mathcal{Z}_1} B_{j,m}(z)\, d\mu_0(z) - \int_{\mathcal{Z}_1} B_j(z)\, d\mu_0(z)|\,, && (45) \end{aligned}$$

where $B_{1,m}(z) = \langle z, -\Delta\chi_m(-\Delta)z\rangle$ and $B_{2,m}(z) = \frac{1}{2}\langle z^{\otimes 2}, V(x-y)\chi_m(-\Delta_x)z^{\otimes 2}\rangle$. We observe that Lemma B.6 leads to

$$\|(\mathrm{d}\Gamma(-\Delta)+\mathbf{N}+1)^{-1}\,\mathrm{d}\Gamma(\Delta(1-\chi_m)(-\Delta))\,(\mathrm{d}\Gamma(-\Delta)+\mathbf{N}+1)^{-1}\| \le \|\frac{-\Delta}{(1-\Delta)^2}(1-\chi_m)(-\Delta)\| \overset{m\to\infty}{\to} 0$$

and

$$\begin{aligned} &\|(\mathrm{d}\Gamma(-\Delta)+\mathbf{N}^2+1)^{-1/2}\,(B_2^{Wick} - B_{2,m}^{Wick})\,(\mathrm{d}\Gamma(-\Delta)+\mathbf{N}^2+1)^{-1/2}\| \\ &\qquad\qquad \le C_V\|(1-\Delta)^{-1/2}(1-\chi_m)(-\Delta)\| \overset{m\to\infty}{\to} 0\,. \end{aligned}$$

Therefore the r.h.s (43) tends to 0 when $m \to \infty$ thanks to the regularity of ϱ_ε and by noticing that $(\mathrm{d}\Gamma(-\Delta)+\mathbf{N}^2+1)^{1/2}(\mathrm{d}\Gamma(-\Delta)+\mathbf{N}+1)^{-1}$ is bounded. Now, since $B_{j,m}, j = 1, 2$ belong to $\mathcal{P}_{alg}(\mathcal{Z}_0)$ then by the statement (41), proved in [6, Proposition 2.12], the r.h.s (44) converges to 0 when $\varepsilon \to 0$. Further, by the dominated convergence theorem and with the help of Lemma 3.13, we see that the r.h.s (45) vanishes too as $m \to \infty$. Hence an $\delta/3$-argument gives

$$\lim_{\varepsilon\to 0} \mathrm{Tr}[\varrho_\varepsilon\, B_j^{Wick}] = \int_{\mathcal{Z}_1} B_j(z)\, d\mu_0(z) \quad \text{for} \quad j = 1, 2\,.$$

Thus we have proved (42). □

4.4 Examples

We give here two examples, other can be seen in our previous articles [4][5][6]. The first one recalls that the transport of the Wigner measure takes into account some correlations. The second one is about the mean field dynamics of states, which do not satisfy (36) and makes a connection with Bose-Einstein condensation.

4.5 Deformed torii

For two elements $\psi_1, \psi_2 \in \mathcal{Z}_1 \subset \mathcal{Z}_0$ such that $\|\psi_1\| = \|\psi_2\| = 1$ and $\langle\psi_1\,,\,\psi_2\rangle = 0$, the space $\mathcal{Z}_0$ can be decomposed into

$$\mathcal{Z}_0 = \mathbb{C}\psi_1 \overset{\perp}{\oplus} \mathbb{C}\psi_1 \overset{\perp}{\oplus} \psi^\perp\,.$$

This decomposition is second-quantized into the Hilbert tensor product

$$\mathcal{H} = \Gamma_s(\mathcal{Z}_0) = \Gamma_s(\mathbb{C}\psi_1) \otimes \Gamma_s(\mathbb{C}\psi_2) \otimes \Gamma_s(\psi^\perp)\,,$$

which allows an analysis by separating the variables. The number observable is now

$$\mathbf{N} = (\mathbf{N}_1 \otimes 1\!\!1 \otimes 1\!\!1) \oplus (1\!\!1 \otimes \mathbf{N}_2 \otimes 1\!\!1) \oplus (1\!\!1 \otimes 1\!\!1 \otimes \mathbf{N}')\,,$$

simply written as $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}'$ and where $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}'$ are respectively the number operators on $\Gamma_s(\mathbb{C}\psi_1)$, $\Gamma_s(\mathbb{C}\psi_2)$ and $\Gamma_s(\psi^\perp)$. Consider in this decomposition, the state

$$\varrho_\varepsilon = \varrho_\varepsilon^1 \otimes \varrho_\varepsilon^2 \otimes (|\Omega'\rangle\langle\Omega'|)$$

where $|\Omega'\rangle$ is the vacuum state of $\Gamma_s(\psi^\perp)$ and

$$\varrho_\varepsilon^1 = |\psi_1^{\otimes n_1}\rangle\langle\psi_1^{\otimes n_1}| \quad , \quad \varrho_\varepsilon^2 = |\psi_2^{\otimes n_2}\rangle\langle\psi_2^{\otimes n_2}|\,,$$
$$\text{with} \qquad \lim_{\varepsilon\to 0} \varepsilon n_1 = \lim_{\varepsilon\to 0} \varepsilon n_2 = \frac{1}{2}\,.$$

In $\mathcal{H} = \Gamma_s(\mathcal{Z}_0)$, this state is explicitly written (see [6]) as

$$\varrho_\varepsilon = |\psi^{\vee(n_1,n_2)}\rangle\langle\psi^{\vee(n_1,n_2)}| \tag{46}$$

$$\text{with} \qquad \psi^{\vee(n_1,n_2)} = \frac{1}{\sqrt{\varepsilon^{n_1+n_2} n_1! n_2!}} \overbrace{a^*(\psi_1)\dots a^*(\psi_1)}^{n_1 \text{ times}} \overbrace{a^*(\psi_2)\dots a^*(\psi_2)}^{n_2 \text{ times}} |\Omega\rangle\,. \tag{47}$$

The state satisfies

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\mathbf{N}^k \varrho_\varepsilon\right] = (\frac{1}{2} + \frac{1}{2})^k = 1\,,$$

owing to $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}'$. Moreover, with (46)(47), $\mathbf{N} + H_\varepsilon^0 = \mathrm{d}\Gamma(1-\Delta)$ and the help of Wick calculus, it also fulfills

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[(\mathbf{N} + H_\varepsilon^0)\varrho_\varepsilon\right] = \frac{|\psi_1|^2_{\mathcal{Z}_1} + |\psi_2|^2_{\mathcal{Z}_1}}{2}\,.$$

Meanwhile the separation of variables allows to compute explicitly the (it is unique) Wigner measure of $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$

$$\mu_0 = \delta^{S^1}_{\frac{\sqrt{2}}{2}\psi_1} \otimes \delta^{S^1}_{\frac{\sqrt{2}}{2}\psi_1} \otimes \delta_0 \quad \text{on} \quad \mathcal{Z}_1 = (\mathbb{C}\psi_1) \times (\mathbb{C}\psi_2) \times \psi^\perp\,,$$
$$\text{with} \qquad \delta^{S^1}_u = \frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}u}\, d\theta\,.$$

We get

$$\int_{\mathcal{Z}_1} |z|^{2k}\, d\mu_0(z) = \int_{\mathcal{Z}_1} \left(|z_1|^2 + |z_2|^2 + |z'|^2\right)^k\, d\mu_0(z) = 1 = \lim_{\varepsilon\to\infty} \mathrm{Tr}\left[\mathbf{N}^k \varrho_\varepsilon\right]\,.$$

Hence all the assumptions of Theorem 1.1 and Proposition 4.1 are fulfilled.
This measure is carried by a torus in $\mathcal{Z}_1$ better described by using an other orthonormal basis of $\mathbb{C}\psi_1 \oplus \mathbb{C}\psi_2$:

$$\psi_0 = \frac{\sqrt{2}}{2}(\psi_1 + \psi_2) \quad , \quad \psi_{\frac{\pi}{2}} = i\frac{\sqrt{2}}{2}(\psi_1 - \psi_2)\,,$$
$$\psi_\varphi = \cos(\varphi)\psi_0 + \sin(\varphi)\psi_{\frac{\pi}{2}}\,,$$
$$\frac{\sqrt{2}}{2}(e^{i\theta}\psi_1 + e^{i\theta'}\psi_2) = e^{i\frac{\theta+\theta'}{2}}\psi_{\frac{\theta-\theta'}{2}}\,,$$
$$\mu_0 = \frac{1}{2\pi}\int_0^{2\pi} \delta^{S^1}_{\psi_\varphi}\, d\varphi\,.$$

Two elements $e^{i\theta}\psi_\varphi$ and $e^{i\theta'}\psi_{\varphi'}$ in the support of μ_0 are equal when

$$(\theta' = \theta \text{ and } \varphi' = \varphi) \quad \text{or} \quad (\theta' = \theta + \pi \text{ and } \varphi' = \varphi + \pi)\ .$$

Hence a one to one parametrization of the torus can be done by $\varphi \in [0, 2\pi)$ and $\theta \in [\varphi, \varphi + \pi)$.

Let $\psi_\varphi(t) = \Phi(t, 0)\psi_\varphi$, be the solution to the Hartree equation

$$\begin{cases} i\partial_t\psi_\varphi(t) = -\Delta\psi_\varphi(t) + (V * |\psi_\varphi(t)|^2)\psi_\varphi(t) \\ \psi_\varphi(t=0) = \psi_\varphi = e^{i\frac{\pi}{4}}\cos(\varphi)\psi_1 + e^{-i\frac{\pi}{4}}\sin(\varphi)\psi_2 \end{cases},$$

The gauge invariance of the equation says that for any $\theta \in [0, 2\pi]$, $e^{i\theta}\psi_\varphi(t) = \Phi(t, 0)\left[e^{i\theta}\psi_\varphi\right]$. By applying the result of Theorem 1.1 and Proposition 4.1 we get

$$\mu_t = \frac{1}{2\pi}\int_0^{2\pi} \delta^{S^1}_{\psi_\varphi(t)}\ d\varphi = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}\psi_\varphi(t)}\ d\varphi d\theta$$

$$\forall p \in \mathbb{N}, \quad \lim_{\varepsilon\to 0}\gamma^{(p)}_\varepsilon(t) = \frac{1}{2\pi}\int_0^{2\pi} |[\psi_\varphi(t)]^{\otimes p}\rangle\langle[\psi_\varphi(t)]^{\otimes p}|\ d\varphi\,.$$

Since the Hartree flow is non linear, the complete hierarchy of reduced density matrices have to be taken into account if one wants to write evolution equation for them. More simply, they can be computed after solving an autonomous equation for the Wigner measure. Due to the nonlinear term the dynamics of correlations is by far nontrivial. This can also be thought geometrically: The initial measure is initially supported by a torus which lies in a 2-dimensional complex vector space (think of the circle in the plane $\mathbb{R}\psi_0 \oplus \mathbb{R}\psi_{\frac{\pi}{2}}$); along the time evolution, the measure μ_t is still carried by a torus in $\mathcal{Z}_1$, which nevertheless, is a priori not embedded in any finite dimensional subspace.

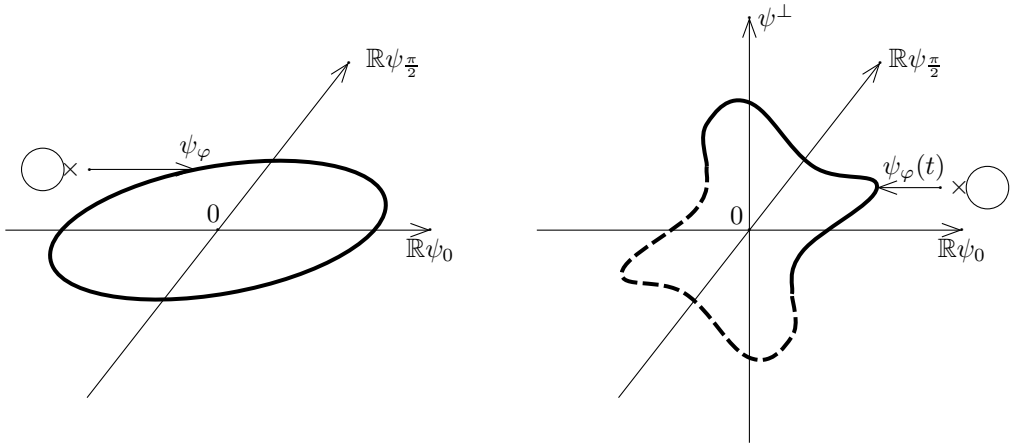

Fig.1: Evolution of the measure initially carried by a torus in $\mathbb{C}\psi_0 \oplus \mathbb{C}\psi_{\frac{\pi}{2}}$. The complex gauge parameter $e^{i\theta}$ is represented by the small circle.

In Figure 1, the deformed torus for time $t \neq 0$, has to be imagined in the infinite-dimensional phase-space $\mathcal{Z}_1 \subset \mathcal{Z}_0$. Contrary to the picture, there might be no intersection with the real plane $\mathbb{R}\psi_0 \oplus \mathbb{R}\psi_{\frac{\pi}{2}}$.

This discussion can also be extended to higher dimensional torii after taking a finite (or countable) orthornormal family $(\psi_n)_{1\leq n\leq N}$ for building the initial states ϱ_ε with a measure $\prod_{j=1}^N \delta^{S^1}_{\lambda_j\psi_j}$ (see [6]).

4.6 Propagation without the convergence of moments

In [4] we considered the thermodynamic limit of a free Bose gas on a torus with the one particle energy given by $-\Delta$. We showed that when there is Bose condensation the condition (36) fails and illustrates what we called a dimensional defect of compactness, in opposition to the phase

space or microlocal defect of compactness (see [28][60]). Others examples were given. In [6] the propagation result for bounded interactions but without any compactness condition, cannot be applied for such initial states. With Theorem 1.1 the propagation holds for these kind of initial states. Since our analysis is valid on $\mathbb{R}^d$ the analysis for the torus does not apply directly and we adapt the presentation of the Bose-Einstein condensation.
Moreover the dimensional defect of compactness which plays with all the directions of the phase-space $\mathcal{Z}_0 = L^2(\mathbb{R}^d)$, can be geometrically thought in the one particle phase-space $T^*\mathbb{R}^d$. The condition (4), which leads to estimates of $\int_{\mathcal{Z}_0} |z|^{2\delta}_{H^1}\, d\mu$, suggests that the dimensional defect of compactness is due to mass going to ∞ in the position variable rather than in the momentum variable, in $T^*\mathbb{R}^d$. The mean field limit that we consider here, can be tested by using the harmonic oscillator Hamiltonian $A = -\partial_x^2 + \frac{x^2}{4} - \frac{d}{4}$. The motivated reader will then see that the dimensional defect of compactness ϱ_ε is incompatible with the condition (4).

We work in dimension $d \geq 2$. Let e_0 be an L^2-normalized $\mathcal{C}_0^\infty(\mathbb{R}^d)$ supported in the hypercube $(-\frac{1}{2}, \frac{1}{2})^d$ and set

$$\forall k \in \mathbb{N}^d\,, \quad e_k(x) = e_0(x-k)\,.$$

The family $(e_k)_{k\in\mathbb{N}}$ is orthonormal in $\mathcal{Z}_0 = L^2(\mathbb{R}^d)$. The spanned Hilbert subspace and the corresponding orthogonal projection are respectively denoted by $\mathcal{Z}_e$ and Π_e, $\mathrm{Ran}\,\Pi_e = \mathcal{Z}_e$. Note that

$$\Pi_e(-\Delta)\Pi_e = \left(\int_{\mathbb{R}} |\nabla e_0|^2\right)\Pi_e\,.$$

Consider now the self-adjoint operator defined on $\mathcal{Z}_0 = L^2(\mathbb{R}^d)$ by

$$A = \sum_{k\in\mathbb{N}^d} |k||e_k\rangle\langle e_k|\,, \quad |k| = \sum_{j=1}^d k_j\,,$$

which restricted to $\mathcal{Z}_e$ is unitarily equivalent to the harmonic oscillator Hamiltonian $A = -\partial_x^2 + \frac{x^2}{4} - \frac{d}{4}$ on $\mathbb{R}^d$. We use the tensor decomposition

$$\begin{aligned}
&\mathcal{Z}_0 = \mathcal{Z}_e \overset{\perp}{\oplus} \mathcal{Z}_e^\perp \quad , \quad \mathcal{H} = \Gamma_s(\mathcal{Z}_0) = \Gamma_s(\mathcal{Z}_e)\otimes\Gamma_s(\mathcal{Z}_e^\perp)\,,\\
&\mathbf{N} = \mathbf{N}_e\otimes 1\!\!1 + 1\!\!1\otimes\mathbf{N}_e^\perp = \mathbf{N}_e + \mathbf{N}_e^\perp \quad , \quad |\Omega\rangle = |\Omega_e\rangle\otimes|\Omega_e^\perp\rangle\,,\\
&\forall B_e \in \mathcal{L}(\mathcal{Z}_e)\,, \|B_e\| \leq 1, \quad \Gamma(B_e)\otimes(|\Omega_e^\perp\rangle\langle\Omega_e^\perp|) = \Gamma(\Pi_e B_e \Pi_e)\,,\\
&\forall B \in \mathcal{L}(\mathcal{Z}_0)\,, \|B\| \leq 1, \quad \Gamma(\Pi_e B \Pi_e) = \Gamma(\Pi_e \circ B|_{\mathcal{Z}_e})\otimes(|\Omega_e^\perp\rangle\langle\Omega_e^\perp|)\,.
\end{aligned}$$

In particular the last relation with $B = e^{i\varepsilon t\Delta}$ differentiated at time $t = 0$ gives

$$\Gamma(\Pi_e)\mathrm{d}\Gamma(-\Delta)\Gamma(\Pi_e) = \left(\int_{\mathbb{R}^d} |\nabla e_0|^2\right)\mathbf{N}_e\,.$$

Consider on $\mathcal{H}$, the ε-dependent gauge invariant (tensorized) quasi-free state

$$\begin{aligned}
\varrho_\varepsilon &= \frac{1}{\mathrm{Tr}\left[\Gamma(\Pi_e e^{-\beta_\varepsilon(A-\mu_\varepsilon)}\Pi_e)\right]}\Gamma(\Pi_e e^{-\beta_\varepsilon(A-\mu_\varepsilon)}\Pi_e) \qquad (48)\\
&= \frac{1}{\mathrm{Tr}\left[\Gamma(\Pi_e Z_\varepsilon e^{-\beta_\varepsilon A}\Pi_e)\right]}\Gamma(\Pi_e Z_\varepsilon e^{-\beta_\varepsilon A}\Pi_e)\,,\\
&= \frac{1}{\mathrm{Tr}\left[\Gamma(Z_\varepsilon e^{-\beta_\varepsilon A|_{\mathcal{Z}_e\cap\mathcal{D}(A)}})\right]}\Gamma(\Pi_e)\Gamma(Z_\varepsilon e^{-\beta_\varepsilon A|_{\mathcal{Z}_e\cap\mathcal{D}(A)}})\Gamma(\Pi_e)\,.
\end{aligned}$$

The chemical potential μ_ε is negative of order $\varepsilon^{1-1/d}$ and the temperature is large according to

$$Z_\varepsilon = e^{\beta_\varepsilon\mu_\varepsilon} = 1 - \frac{\varepsilon}{\nu_C} \quad , \quad \beta_\varepsilon = \varepsilon^{1/d}\,.$$

With the ε-dependent definition of $a(f)$, $a^*(f)$, $[a(g), a^*(f)] = \varepsilon\langle g, f\rangle$, and $W(f)$, this quasi-free state is characterized by the two-point function

$$\mathrm{Tr}\left[\varrho_\varepsilon a^*(f)a(g))\right] = \varepsilon\left\langle \Pi_e g\,, Z_\varepsilon e^{-\beta_\varepsilon A_\varepsilon}(1 - Z_\varepsilon e^{-\beta_\varepsilon A_\varepsilon})^{-1}\Pi_e f\right\rangle\,. \qquad (49)$$

$$\text{or} \qquad \mathrm{Tr}\left[\varrho_\varepsilon W(f)\right] = \exp\left[-\varepsilon\langle\Pi_e f\,, (1 + Z_\varepsilon e^{-\beta_\varepsilon A_\varepsilon})(1 - Z_\varepsilon e^{-\beta_\varepsilon A_\varepsilon})^{-1}\Pi_e f\rangle/4\right]\,. \qquad (50)$$

In particular the total number (multiplied by ε) is given by

$$\begin{aligned}\mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}\right] &= \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_e\right] \sum_{k\in\mathbb{N}^d} \mathrm{Tr}\left[\varrho_\varepsilon a^*(e_k)a(e_k)\right] \\ &= \nu_C + \nu + r(\varepsilon) \quad \text{with } \lim_{\varepsilon\to 0} r(\varepsilon) = 0\,, \qquad (51)\end{aligned}$$

$$\text{and} \qquad \nu = \int_{\mathbb{R}^d} \frac{e^{-|u|}}{1-e^{-|u|}}\, du = |S^{d-1}| \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} t^{d-1}\, dt\,, \quad (d\geq 2)\,. \qquad (52)$$

We deduce

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}\right] = \nu_C + \nu \quad ,$$

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon H_\varepsilon^0\right] = \lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \Gamma(\Pi_e) \mathrm{d}\Gamma(-\Delta)\Gamma(\Pi_e)\right] = \left(\int_{\mathbb{R}^d} |\nabla e_0|^2\right)(\nu_C+\nu)\,,$$

and the condition (4) of Theorem 1.1 is satisfied.
Actually $\nu_C > 0$ corresponds, in the analysis of the free Bose gas (see [4]), to the density associated with the condensate phase. In the scaling that we consider, it is the other part which produces the dimensional defect of compactness. Let us compute the Wigner measure, by considering the limit of $\mathrm{Tr}\left[\varrho_\varepsilon W(\sqrt{2}\pi f)\right]$ as $\varepsilon\to 0$. With

$$f = \sum_{k\in\mathbb{N}^d} f_k e_k + f^\perp\,, \quad |f|^2 = \sum_{k\in\mathbb{N}^d} |f_k|^2 + |f^\perp|^2 = |\Pi_e f|^2 + |f^\perp|^2\,,$$

the expression (50) gives

$$\mathrm{Tr}\left[\varrho^\varepsilon W(\sqrt{2}\pi f)\right] = e^{-\varepsilon\pi^2|\Pi_e f|^2/2} \times \exp\left[-\varepsilon\pi^2 \sum_{k\in\mathbb{N}^d} |f_k|^2 \frac{Z_\varepsilon e^{-\varepsilon^{1/d}|k|}}{(1-Z_\varepsilon e^{-\varepsilon^{1/d}|k|})}\right] \overset{\varepsilon\to 0}{\to} e^{-\pi^2\nu_C|f_0|^2}\,. \quad (53)$$

The family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ admits the unique Wigner measure

$$\mu_0 = \frac{e^{-\frac{|z_0|^2}{\nu_C}}}{\pi\nu_C} \otimes \delta_0 \quad \text{on } \mathcal{Z}_0 = (\mathbb{C}e_0)\times e_0^\perp\,,$$

which is carried by $\mathbb{C}e_0 \subset \mathcal{Z}_1$ and which can also be written

$$\mu_0 = \int_{\mathbb{C}e_0} \frac{e^{-\frac{|z|^2}{\nu_C}}}{\pi\nu_C} \delta_{ze_0}\, L_{\mathbb{C}e_0}(dz) = \int_0^{+\infty} \frac{e^{-\frac{u}{\nu_C}}}{\nu_C} \delta^{S^1}_{\sqrt{u}e_0}\, du\,.$$

In particular, we get

$$\int_{\mathcal{Z}_1} |z|^2_{\mathcal{Z}_0} d\mu_0(z) = \nu_C < \nu_C + \nu = \lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}\right]$$

$$\text{and} \qquad \int_{\mathcal{Z}_1} |z|^2_{\mathcal{Z}_1} d\mu_0(z) = \nu_C \left(\int_{\mathbb{R}^d} |\nabla e_0|^2\right) < (\nu_C+\nu)\left(\int_{\mathbb{R}^d} |\nabla e_0|^2\right) = \lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathrm{d}\Gamma(1-\Delta)\right]\,,$$

and the condition (36) does not hold. Even at time $t=0$, no formula is available for the reduced density matrices in terms of the Wigner measure. Nevertheless the time-dependent Wigner measure of $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}$ is given by Theorem 1.1, since the condition (4) is verified. Consider the solutions to the Hartree initial value problems

$$\begin{cases} i\partial_t\psi_u = -\Delta\psi_u + (V*|\psi_u|^2)\psi_u \\ \psi_u(t=0) = \sqrt{u}e_0\,, \quad u\in(0,+\infty)\,. \end{cases}$$

Then the Wigner measure of $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}$ is given by

$$\mu_t = \int_0^{+\infty} \frac{e^{-\frac{u}{\nu_C}}}{\nu_C} \delta^{S^1}_{\psi_u(t)}\, du\,.$$

Again like in the example of the previous section, the measure μ_t is carried by surface containing 0 and topologically equivalent to $\mathbb{C}$, but this 2-dimensional surface does not remain a priori in any finite dimensional subspace of $\mathcal{Z}_1$ for $t\neq 0$.

4.7 About the classical mean field problem

The classical analogue of our analysis is the derivation of the Vlasov equation

$$\left\{\begin{array}{l} \partial_t f + v.\partial_x f - \frac{1}{m}(\partial_x V_f(x,t)).\partial_v f = 0 \\ f(t,x,v) = f_0(x,v) \\ V_f(x,t) = V * \varrho_f(x,t) \end{array}\right. , \quad \varrho_f(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\, dv$$

where $f(x,v,t)$ represents the particle density in the 1-particle phase space $\mathbb{R}^{2d}_{x,v}$, from the classical Hamilton many body system

$$\left\{\begin{array}{l} \dot{x}_i = v_i\,, \\ \dot{v}_i = -\frac{1}{mN}(\sum_{j=1}^N \partial_{x_i} V(x_i - x_j)) \end{array}\right. , \quad i = 1, \ldots, N\,,$$

in the limit $N \to \infty$. This problems is still open for singular potential and C. Villani, in a recent survey article about the Landau damping [62] quotes the work of Hauray-Jabin [36] as the most advanced one in this direction. It works for a potential such that $|\nabla V| = \mathcal{O}(|x|^{-s})$, $s \in (0,1)$, and does not include the Coulomb interaction.
Indirectly our result, justifies the mean field model up to Coulomb interaction in dimension $d = 3$. In [49] and more recently [2], the Vlasov equation is proved to be the semiclassical limit of the semiclassical Hartree equation. This means that there are two "semiclassical" limits, one in the phase-space $L^2(\mathbb{R}^d;\mathbb{C})$ with the small parameter $1/N$, another one on the phase-space $T^*\mathbb{R}^d \sim \mathbb{R}^{2d}$ for the one particle nonlinear problem. This double asymptotic regime is well presented in [25, 27, 35].

A possible strategy, for deriving directly the classical mean field limit from the classical many body problem, consists in adapting our approach by, as usual, replacing traces by integrals. For information, we refer the reader to the presentation [18] by J. Derezinski of the classical analogue of second quantization. Of course classical mechanics, although living in the commutative world, is often more singular than quantum mechanics, from the analysis point of view. With the Coulomb interaction, the Kustaanheimo-Stiefel desingularization of the hamiltonian flow may be useful (see a.e. [15, 38, 41, 43, 46]).

Appendix

A Commuting self-adjoint operators on a graded Hilbert space

We briefly study the general structure of self-adjoint operators on a graded Hilbert space. Properties collected in this section are useful for the analysis of the quantum Hamiltonian (1). In this appendix, the small parameter is not required and we work with $\varepsilon = 1$.
Remember that a graded Hilbert space $\mathcal{H}$ is a direct sum of Hilbert spaces $\mathcal{H}_n, n \in \mathbb{N}$, of the form

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n\,.$$

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of self-adjoint operators where each A_n acts on $\mathcal{H}_n$. We define the operator

$$\mathcal{D}(A) = \{\Psi \in \mathcal{H} : \sum_{n=0}^{\infty} \|A_n \Psi^{(n)}\|^2_{\mathcal{H}_n} < \infty\}, \quad A\,\Psi = \sum_{n=0}^{\infty} A_n\,\Psi^{(n)}\,, \text{ for all } \Psi \in \mathcal{D}(A)\,. \tag{54}$$

Taking in particular $A_n = n\mathbb{1}_{\mathcal{H}_n}$ for $n \in \mathbb{N}$, we obtain the *number* operator

$$N = \sum_{n=0}^{\infty} n\mathbb{1}_{\mathcal{H}_n}\,. \tag{55}$$

We say that two self-adjoint operators B and C on a Hilbert space *strongly commute* if their spectral projections mutually commute. This is equivalent to the commutation of their resolvents for some $z \in \mathbb{C} \setminus \mathbb{R}$ and also to the commutation of their associated unitary groups. More precisely, B and C strongly commute if and only if for all $t, s \in \mathbb{R}$

$$e^{itC} e^{isB} = e^{isB} e^{itC} .$$

Proposition A.1. *Let A and N be the operators given by (54)-(55). The following assertions hold:*
(i) A and N are self-adjoint.
(ii) For any bounded Borel function on $\mathbb{R}$,

$$f(A) = \sum_{n=0}^{\infty} f(A_n) .$$

(iii) The operators A and N strongly commute.
(iv) If $\mathcal{D}_n$ is a core for A_n for each $n \in \mathbb{N}$ then $\oplus_{n=0}^{alg} \mathcal{D}_n$ is a core for A.
(v) For any real polynomial p the operator $A + p(N)_{|\mathcal{D}(A) \cap \mathcal{D}(p(N))}$ is essentially self-adjoint and

$$\overline{A + p(N)_{|\mathcal{D}(A) \cap \mathcal{D}(p(N))}} = \sum_{n=0}^{\infty} A_n + p(n) \mathbb{1}_{\mathcal{H}_n} .$$

Proof. (i) Clearly, A is a densely defined operator. It is also symmetric, since for any $\Psi, \Phi \in \mathcal{D}(A)$

$$\langle \Phi, A\Psi \rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, A_n \Psi^{(n)} \rangle_{\mathcal{H}_n} = \sum_{n=0}^{\infty} \langle A_n \Phi^{(n)}, \Psi^{(n)} \rangle_{\mathcal{H}_n} = \langle A\Phi, \Psi \rangle_{\mathcal{H}} .$$

For any $\Psi \in \mathcal{D}(A)$ and $\Phi \in \mathcal{D}(A^*)$,

$$\langle A^* \Phi, \Psi \rangle = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, A_n \Psi^{(n)} \rangle_{\mathcal{H}_n}$$

Hence the inequality holds

$$\left| \sum_{n=0}^{\infty} \langle \Phi^{(n)}, A_n \Psi^{(n)} \rangle_{\mathcal{H}_n} \right| \leq \|A^* \Phi\|_{\mathcal{H}} \, \|\Psi\|_{\mathcal{H}} .$$

By taking any $\Psi^{(n)} \in \mathcal{D}(A_n)$, this means $\Phi^{(n)} \in \mathcal{D}(A_n^*) = \mathcal{D}(A_n^*) = \mathcal{D}(A_n)$. The extension to any $\Psi \in \mathcal{H}$ gives $\Phi \in \mathcal{D}(A)$. This proves that A and N are self-adjoint.
(ii) For each $n \in \mathbb{N}$, the map $t \mapsto e^{itA} e^{-itA_n} \Psi^{(n)}$ is of class C^1 for any $\Psi^{(n)} \in \mathcal{D}(A_n)$ by the Stone's theorem with the derivative

$$\frac{d}{dt} e^{itA} e^{-itA_n} \Psi^{(n)} = i e^{itA} (A - A_n) e^{-itA_n} \Psi^{(n)} = 0 .$$

Hence, for any $\Psi \in \oplus_{n=0}^{alg} \mathcal{D}(A_n)$ (and then for any $\Psi \in \mathcal{H}$ since $\oplus_{n=0}^{alg} \mathcal{D}(A_n)$ is dense in $\mathcal{H}$) we see that for all $t \in \mathbb{R}$

$$e^{itA} \Psi = \sum_{n=0}^{\infty} e^{itA_n} \Psi^{(n)} . \tag{56}$$

By functional calculus we extend the identity (56) to any bounded Borel function f on $\mathbb{R}$.
(iii) By using (ii), we get for all $s, t \in \mathbb{R}$ and $\Psi \in \mathcal{H}$

$$e^{itN} e^{isA} \Psi = e^{itN} \sum_{n=0}^{\infty} e^{isA_n} \Psi^{(n)} = \sum_{n=0}^{\infty} e^{itn} e^{isA_n} \Psi^{(n)} = e^{isA} e^{itN} \Psi .$$

(iv) The algebraic direct sum $\mathcal{D}_{fin} = \oplus_{n=0}^{alg} \mathcal{D}(A_n) \subset \mathcal{D}(A)$ is dense in $\mathcal{H}$ and invariant with respect to the group $(e^{itA})_{t\in\mathbb{R}}$. Therefore, $\mathcal{D}_{fin}$ is a core for A. In the other hand, the subspace $\mathcal{D}^0_{fin} = \oplus_{n=0}^{alg} \mathcal{D}_n$ satisfies

$$A_{|\mathcal{D}_{fin}} \subset \overline{A_{|\mathcal{D}^0_{fin}}} \subset A_{|\mathcal{D}(A)}\,.$$

Hence $\mathcal{D}^0_{fin}$ is also a core for A since $\overline{A_{|\mathcal{D}_{fin}}} = A_{|\mathcal{D}(A)}$.
(v) The operator $B = \sum_{n=0}^{\infty} A_n + p(n)\mathbb{1}_{\mathcal{H}_n}$ (with its natural domain) is self-adjoint by assertion (i). It is clear that

$$\mathcal{D}_{fin} = \oplus_{n=0}^{alg} \mathcal{D}(A_n) \subset \mathcal{D}(A) \cap \mathcal{D}(p(N)) \subset \mathcal{D}(B)\,,$$

and furthermore

$$B_{|\mathcal{D}_{fin}} = A + p(N)_{|\mathcal{D}_{fin}} \subset A + p(N)_{|\mathcal{D}(A)\cap\mathcal{D}(p(N))} \subset B_{|\mathcal{D}(B)}\,.$$

Therefore, the operator $A + p(N)_{|\mathcal{D}(A)\cap\mathcal{D}(p(N))}$ is essentially self-adjoint since $\overline{B_{|\mathcal{D}_{fin}}} = B_{|\mathcal{D}(B)}$. □

B Second quantization

For reader convenience we recall here the general framework of second quantization and put some related notations.
The phase-space, a complex separable Hilbert space, is denoted by $\mathcal{Z}$ with the scalar product $\langle .,. \rangle$. The symmetric Fock space over $\mathcal{Z}$ is defined as the following direct Hilbert sum

$$\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \bigvee^n \mathcal{Z}\,,$$

where $\bigvee^n \mathcal{Z}$ is the n-fold symmetric tensor product. The orthogonal projection from $\mathcal{Z}^{\otimes n}$ onto the closed subspace $\bigvee^n \mathcal{Z}$ is given by

$$\mathcal{S}_n(\xi_1 \otimes \xi_2 \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \cdots \otimes \xi_{\sigma(n)}\,.$$

Algebraic direct sums or tensor products are denoted with a ^{alg} superscript. Hence

$$\mathcal{H}_{fin} = \bigoplus_{n\in\mathbb{N}}^{alg} \bigvee^n \mathcal{Z}$$

denotes the subspace of vectors with a finite number of particles. The creation and annihilation operators $a^*(z)$ and $a(z)$, parameterized by $\varepsilon > 0$, are then defined by :

$$\begin{aligned} a(z)\varphi^{\otimes n} &= \sqrt{\varepsilon n}\ \langle z, \varphi\rangle \varphi^{\otimes(n-1)} \\ a^*(z)\varphi^{\otimes n} &= \sqrt{\varepsilon(n+1)}\ \mathcal{S}_{n+1}(\, z \otimes \varphi^{\otimes n})\,, \quad \forall \varphi, z \in \mathcal{Z}. \end{aligned}$$

They extend to closed operators and they are adjoint of one another. They also satisfy the ε-canonical commutation relations (CCR):

$$[a(z_1), a^*(z_2)] = \varepsilon\langle z_1, z_2\rangle \mathbb{1}, \quad [a^*(z_1), a^*(z_2)] = 0 = [a(z_1), a(z_2)]\,. \tag{57}$$

The Weyl operators are given for $z \in \mathcal{Z}$ by

$$W(z) = e^{\frac{i}{\sqrt{2}}[a^*(z)+a(z)]}\,,$$

and they satisfy Weyl commutation relations in the Fock space

$$W(z_1)W(z_2) = e^{-\frac{i\varepsilon}{2}\mathrm{Im}\ \langle z_1, z_2\rangle}\ W(z_1 + z_2),\ \ z_1, z_2 \in \mathcal{Z}\,. \tag{58}$$

The number operator is also parametrized by $\varepsilon > 0$,

$$\mathbf{N}_{|\bigvee^n \mathcal{Z}} = \varepsilon n \mathbb{1}_{|\bigvee^n \mathcal{Z}} .$$

For any self-adjoint operator $A : \mathcal{Z} \supset \mathcal{D}(A) \to \mathcal{Z}$, the operator $\mathrm{d}\Gamma(A)$ is the self-adjoint operator given by

$$\mathrm{d}\Gamma(A)_{|\bigvee^{n,\mathrm{alg}} \mathcal{D}(A)} = \varepsilon \left[\sum_{k=1}^{n} \mathbb{1} \otimes \cdots \otimes \underbrace{A}_{k} \otimes \cdots \otimes \mathbb{1} \right] .$$

B.1 Weyl, Anti-Wick quantized operators

Let $\mathbb{P}$ denote the set of all finite rank orthogonal projections on $\mathcal{Z}$ and for a given $\wp \in \mathbb{P}$ let $L_\wp(dz)$ denote the Lebesgue measure on the finite dimensional subspace $\wp\mathcal{Z}$, with volume 1 for an orthonormal hypercube in $\wp\mathcal{Z}$. A function $f : \mathcal{Z} \to \mathbb{C}$ is said *cylindrical* if there exists $\wp \in \mathbb{P}$ and a function g on $\wp\mathcal{Z}$ such that $f(z) = g(\wp z)$, for all $z \in \mathcal{Z}$. In this case we say that f is based on the subspace $\wp\mathcal{Z}$. We set $\mathcal{S}_{cyl}(\mathcal{Z})$ to be the cylindrical Schwartz space:

$$(f \in \mathcal{S}_{cyl}(\mathcal{Z})) \Leftrightarrow (\exists \wp \in \mathbb{P}, \exists g \in \mathcal{S}(\wp\mathcal{Z}), \quad f(z) = g(\wp z)) .$$

The Fourier transform of a function $f \in \mathcal{S}_{cyl}(\mathcal{Z})$ based on the subspace $\wp\mathcal{Z}$ is defined as

$$\mathcal{F}[f](\xi) = \int_{\wp\mathcal{Z}} f(z) \;\; e^{-2\pi i \,\mathrm{Re}\, \langle z,\xi \rangle} \; L_\wp(dz)$$

and its inverse Fourier transform as

$$f(z) = \int_{\wp\mathcal{Z}} \mathcal{F}[f](\xi) \;\; e^{2\pi i \,\mathrm{Re}\, \langle z,\xi \rangle} \; L_\wp(d\xi) .$$

With any symbol $b \in \mathcal{S}_{cyl}(\mathcal{Z})$ based on $\wp\mathcal{Z}$, a *Weyl observable* can be associated according to

$$b^{Weyl} = \int_{\wp\mathcal{Z}} \mathcal{F}[b](z) \;\; W(\sqrt{2}\pi z) \; L_\wp(dz) . \tag{59}$$

Notice that b^{Weyl} is a well defined bounded operator on $\mathcal{H}$ for all $b \in \mathcal{S}_{cyl}(\mathcal{Z})$ and that this quantization of cylindrical symbols depends on the parameter ε.

We also recall the Anti-Wick quantization thought its usual finite dimensional relation to Weyl operators. In fact, we have

$$\begin{aligned} b^{A-Wick} &= \left(b \underset{\wp\mathcal{Z}}{*} \frac{e^{-\frac{|z|^2_{\wp\mathcal{Z}}}{\varepsilon/2}}}{(\pi\varepsilon/2)^{\dim \wp\mathcal{Z}}} \right)^{Weyl} && (60) \\ &= \int_{\wp\mathcal{Z}} \mathcal{F}[b](\xi) \;\; W(\sqrt{2}\pi\xi) \;\; e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2_{\wp\mathcal{Z}}} \; L_\wp(d\xi) , && (61) \end{aligned}$$

for any $b \in \mathcal{S}(\wp\mathcal{Z})$ by setting

$$b \underset{\wp\mathcal{Z}}{*} \gamma(z) = \int_{\wp\mathcal{Z}} b(z)\gamma(z - z') \; L_\wp(dz') .$$

B.2 Wick quantized operators

For any $p, q \in \mathbb{N}$, the space $\mathcal{P}_{p,q}(\mathcal{Z})$ of complex-valued polynomials on $\mathcal{Z}$ is defined with the following continuity condition:

$$b \in \mathcal{P}_{p,q}(\mathcal{Z}) \text{ iff there exists } \tilde{b} \in \mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$$

such that:

$$b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle \, .$$

On these spaces the norms are given by $|b|_{p,q} = \|\tilde{b}\|_{\mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})}$.
The subspace of $\mathcal{P}_{p,q}(\mathcal{Z})$ made of polynomials b such that $\tilde{b}$ is a compact operator is denoted by $\mathcal{P}^{\infty}_{p,q}(\mathcal{Z})$.
The *Wick monomial* of a 'symbol' $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ is the linear operator

$$b^{Wick} : \mathcal{H}_{fin} \to \mathcal{H}_{fin}$$

defined as :

$$b^{Wick}_{|\bigvee^n \mathcal{Z}} = 1_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} \, \varepsilon^{\frac{p+q}{2}} \, \mathcal{S}_{n-p+q} \left(\tilde{b} \otimes 1\!\!1^{(n-p)} \right) . \tag{62}$$

where $\tilde{b} \otimes 1\!\!1^{(n-p)}$ is the operator with the action $(\tilde{b} \otimes 1\!\!1^{(n-p)} \varphi^{\otimes n} = (\tilde{b} \varphi^{\otimes p}) \otimes \varphi^{\otimes (n-p)}$. Notice that b^{Wick} depends on the scaling parameter ε . When $\tilde{b}$ is an unbounded operator with domain $\mathcal{D}(\tilde{b})$ containing $\bigvee^{p,alg} \mathcal{D}$, the formula (62) makes sense when applied to $\Psi \in \bigvee^{n,alg} \mathcal{D}$.

Proposition B.1. *For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the following number estimate holds*

$$\left| \langle \mathbf{N} \rangle^{-\frac{q}{2}} \, b^{Wick} \, \langle \mathbf{N} \rangle^{-\frac{p}{2}} \right|_{\mathcal{L}(\mathcal{H})} \leq |b|_{\mathcal{P}_{p,q}} \, . \tag{63}$$

An important property of our class of Wick polynomials is that a composition of $b_1^{Wick} \circ b_2^{Wick}$ with $b_1, b_2 \in \oplus^{alg}_{p,q} \mathcal{P}_{p,q}(\mathcal{Z})$ is a Wick polynomial with symbol in $\oplus^{\text{alg}}_{p,q} \mathcal{P}_{p,q}(\mathcal{Z})$. This was checked with a convenient writing in [4] and widely used also in [5][6].
We need some notations: For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the k-th differential is well defined according to

$$\partial_z^k b(z) \in (\bigvee^k \mathcal{Z})^* \quad \text{and} \quad \partial_{\bar{z}}^k b(z) \in \bigvee^k \mathcal{Z} \, ,$$

for any fixed $z \in \mathcal{Z}$. Actually $(\bigvee^k \mathcal{Z})^*$ is the dual of $(\bigvee^k \mathcal{Z})$ with a $\mathbb{C}$-bilinear duality bracket. For two symbols $b_i \in \mathcal{P}_{p_i,q_i}(\mathcal{Z})$, $i = 1, 2$, and any $k \in \mathbb{N}$, the new symbol $\partial_z^k b_1 . \partial_{\bar{z}}^k b_2$ is now defined by

$$\partial_z^k b_1 \, . \, \partial_{\bar{z}}^k b_2(z) = \langle \partial_z^k b_1(z), \partial_{\bar{z}}^k b_2(z) \rangle_{(\bigvee^k \mathcal{Z})^*, \bigvee^k \mathcal{Z}} \quad . \tag{64}$$

We also use the following notation for multiple Poisson brackets:

$$\begin{aligned} \{b_1, b_2\}^{(k)} &= \partial_z^k b_1 . \partial_{\bar{z}}^k b_2 - \partial_z^k b_2 . \partial_{\bar{z}}^k b_1, \\ \{b_1, b_2\} &= \{b_1, b_2\}^{(1)}. \end{aligned}$$

With these notations, the composition formula of Wick symbols has a very familiar form.

Proposition B.2. *Let $b_1 \in \mathcal{P}_{p_1,q_1}(\mathcal{Z})$ and $b_2 \in \mathcal{P}_{p_2,q_2}(\mathcal{Z})$.*
For any $k \in \{0, \ldots, \min\{p_1, q_2\}\}$, $\partial_z^k b_1 . \partial_{\bar{z}}^k b_2$ belongs to $\mathcal{P}_{p_2-k,q_1-k}(\mathcal{Z})$ with the estimate

$$|\partial_z^k b_1 . \partial_{\bar{z}}^k b_2|_{\mathcal{P}_{p_2,q_1}} \leq \frac{p_1!}{(p_1-k)!} \frac{q_2!}{(q_2-k)!} |b_1|_{\mathcal{P}_{p_1,q_1}} |b_2|_{\mathcal{P}_{p_2,q_2}} \, .$$

The formulas

$$\begin{aligned} (i) \quad b_1^{Wick} \circ b_2^{Wick} &= \left(\sum_{k=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^k}{k!} \, \partial_z^k b_1 . \partial_{\bar{z}}^k b_2 \right)^{Wick} \\ &= \left(e^{\varepsilon \langle \partial_z, \partial_{\bar{\omega}} \rangle} b_1(z) b_2(\omega) \, |_{z=\omega} \right)^{Wick} , \\ (ii) \quad [b_1^{Wick}, b_2^{Wick}] &= \left(\sum_{k=1}^{\max\{\min\{p_1,q_2\}, \min\{p_2,q_1\}\}} \frac{\varepsilon^k}{k!} \, \{b_1, b_2\}^{(k)} \right)^{Wick} , \end{aligned}$$

hold as identities on $\mathcal{H}_{fin}$.

Combined with Proposition B.1 and $(b^{Wick})^* = \left(\overline{b(z)}\right)^{Wick}$ this also gives the

Proposition B.3. *For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, $\langle \mathbf{N}\rangle^{-\frac{(p+q)}{2}} b^{Wick}$ and $b^{Wick}\langle \mathbf{N}\rangle^{-\frac{(p+q)}{2}}$ extend as bounded operators on $\mathcal{H}$ with norm smaller that $C_{p,q}\|\tilde{b}\|_{\mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})}$, for all $\varepsilon \in (0, \bar{\varepsilon})$.*

We will also need some more particular estimates stated in the following two lemmata.

Lemma B.4. *Let A be a self-adjoint operator on $\mathcal{Z}$ with $A \geq \mathbb{1}$. For any polynomials $b_1 \in \mathcal{P}_{1,2}(\mathcal{Z})$ and $b_2 \in \mathcal{P}_{2,1}(\mathcal{Z})$ the estimates below hold true:*

$$(i) \qquad \|(\mathrm{d}\Gamma(A) + \sqrt{\mathbf{N}} + 1)^{-1}\, b_1^{Wick}\, (\mathrm{d}\Gamma(A) + \sqrt{\mathbf{N}} + 1)^{-1}\| \leq \|A^{-1/2}\, \tilde{b}_1\, (\mathbb{1} \otimes A^{-1/2})\|_{\mathcal{L}(\bigvee^2 \mathcal{Z}, \mathcal{Z})}\,,$$

$$(ii) \qquad \|(\mathrm{d}\Gamma(A) + \sqrt{\mathbf{N}} + 1)^{-1}\, b_2^{Wick}\, (\mathrm{d}\Gamma(A) + \sqrt{\mathbf{N}} + 1)^{-1}\| \leq \|(\mathbb{1} \otimes A^{-1/2})\, \tilde{b}_2\, A^{-1/2}\|_{\mathcal{L}(\mathcal{Z}, \bigvee^2 \mathcal{Z})}\,.$$

Remark B.5. *The term $\sqrt{\mathbf{N}}$ can be absorbed in $\mathrm{d}\Gamma(A) + 1$, if one accepts constants larger than 1 as factors of the right-hand sides of (i) and (ii).*

Proof. The estimate (ii) follows from (i) by taking the adjoint. Let us prove (i).
For $\Phi, \Psi \in \oplus_{n=0}^{alg} \bigvee^{n,alg} \mathcal{D}(A)$, we write

$$\begin{aligned}
\langle \Psi, b_1^{Wick}\Phi\rangle &= \sum_{n=2}^{\infty} \varepsilon^{3/2} \sqrt{n(n-1)^2}\, \langle \Psi^{(n-1)}, (\tilde{b}_1 \otimes \mathbb{1}^{(n-2)})\Phi^{(n)}\rangle \\
&= \sum_{n=2}^{\infty} \varepsilon^{3/2} \sqrt{n(n-1)^2} \langle (A^{1/2} \otimes \mathbb{1}^{(n-2)})\Psi^{(n-1)}\,,\, B_A(\mathbb{1} \otimes A^{1/2} \otimes \mathbb{1}^{(n-2)})\, \Phi^{(n)}\rangle\,,
\end{aligned}$$

with

$$B_A = [(A^{-1/2}\tilde{b}_1)(\mathbb{1} \otimes A^{-1/2})] \otimes \mathbb{1}^{(n-2)}\,.$$

Hence, by Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|\langle \Psi, b_1^{Wick}\Phi\rangle| \;\leq\; & \|A^{-1/2}\tilde{b}_1\, (\mathbb{1} \otimes A^{-1/2})\|_{\mathcal{L}(\bigvee^2 \mathcal{Z}, \mathcal{Z})} \\
& \times \left(\sum_{n=2}^{\infty} \varepsilon^{3/2} \sqrt{n(n-1)^2} \|(A^{1/2} \otimes \mathbb{1}^{(n-2)})\Psi^{(n-1)}\|^2 \right)^{1/2} \\
& \times \left(\sum_{n=2}^{\infty} \varepsilon^{3/2} \sqrt{n(n-1)^2} \|(\mathbb{1} \otimes A^{1/2} \otimes \mathbb{1}^{(n-2)})\Phi^{(n)}\|^2 \right)^{1/2}.
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
\varepsilon^{3/2} \sqrt{n(n-1)^2} \|(A^{1/2} \otimes \mathbb{1}^{(n-2)})\Psi^{(n-1)}\|^2 &\leq 2\sqrt{\varepsilon(n-1)} \langle \Psi^{(n-1)}, (\varepsilon(n-1))(A \otimes \mathbb{1}^{(n-2)})\Psi^{(n-1)}\rangle \\
&\leq 2\langle \Psi^{(n-1)}, \sqrt{\mathbf{N}}\mathrm{d}\Gamma(A)\Psi^{(n-1)}\rangle
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{3/2} \sqrt{n(n-1)^2} \|(\mathbb{1} \otimes A^{1/2} \otimes \mathbb{1}^{(n-2)})\Phi^{(n)}\|^2 &\leq \sqrt{n\varepsilon} \langle \Phi^{(n)}, n\varepsilon\, (\mathbb{1} \otimes A \otimes \mathbb{1}^{(n-2)})\Phi^{(n)}\rangle \\
&\leq \sqrt{n\varepsilon} \langle \Phi^{(n)}, n\varepsilon\, (A \otimes \mathbb{1}^{(n-1)})\Phi^{(n)}\rangle \\
&\leq \langle \Phi^{(n)}, \sqrt{\mathbf{N}}\mathrm{d}\Gamma(A)\Phi^{(n)}\rangle\,.
\end{aligned}$$

On the other hand, with the inequality $2ab \leq a^2 + b^2$, we see that

$$2\langle \Psi^{(n-1)}, \sqrt{\mathbf{N}}\mathrm{d}\Gamma(A)\Psi^{(n-1)}\rangle \leq \langle \Psi^{(n-1)}, (\mathbf{N} + \mathrm{d}\Gamma(A)^2)\Psi^{(n-1)}\rangle \leq \|(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(A))\Psi^{(n-1)}\|^2$$

and $\quad 2\langle \Phi^{(n)}, \sqrt{\mathbf{N}}\mathrm{d}\Gamma(A)\Phi^{(n)}\rangle \leq \langle \Phi^{(n)}, (\mathbf{N} + \mathrm{d}\Gamma(A)^2)\Psi^{(n)}\rangle \leq \|(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(A))\Phi^{(n)}\|^2\,,$

where the last inequalities come from $2\sqrt{N}\mathrm{d}\Gamma(A) \geq 0$. Therefore, we obtain

$$|\langle \Psi, b_1^{Wick}\Phi\rangle| \leq \|A^{-1/2}\tilde{b}_1\, (\mathbb{1} \otimes A^{-1/2})\|\;\; \|(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(A))\Psi\|\;\; \|(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(A))\Phi\|$$

and hence the estimate extends to $\Phi, \Psi \in \mathcal{D}(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(A)) \cap \mathcal{H}_{fin}$. This means that the operator $(\mathrm{d}\Gamma(A) + \sqrt{\mathbf{N}} + 1)^{-1} b_1^{Wick} (\mathrm{d}\Gamma(A) + \sqrt{\mathbf{N}} + 1)^{-1}_{|\mathcal{D}(\sqrt{\mathbf{N}} + \mathrm{d}\Gamma(A)) \cap \mathcal{H}_{fin}}$ extends to a bounded operator satisfying (i). □

Lemma B.6. *Let A, B two self-adjoint operators on $\mathcal{Z}$ with $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B \geq 0$. Let C be a self-adjoint operator on $\bigvee^2 \mathcal{Z}$ such that $\mathcal{D}(C) \subset \mathcal{D}(B_2)$ where $B_2 = B \otimes \mathbb{1} + \mathbb{1} \otimes B$. Then the estimates below hold true:*

$$(i) \qquad \|(\mathrm{d}\Gamma(B) + \mathbf{N} + 1)^{-1}\, \mathrm{d}\Gamma(A)\, (\mathrm{d}\Gamma(B) + \mathbf{N} + 1)^{-1}\| \leq \|(1+B)^{-1}\, A\, (1+B)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\,,$$

$$(ii) \qquad \|(\mathrm{d}\Gamma(B) + \mathbf{N}^2 + 1)^{-1/2}\, C^{Wick}\, (\mathrm{d}\Gamma(B) + \mathbf{N}^2 + 1)^{-1/2}\| \leq \|(1+B_2)^{-1/2}\, C\, (1+B_2)^{-1/2}\|_{\mathcal{L}(\bigvee^2 \mathcal{Z})}\,.$$

Proof. We follow a similar argument as in the proof of Lemma B.4. Indeed, Cauchy-Schwarz inequality gives for every $\Psi, \Phi \in \oplus_{n=0}^{alg} \bigvee^{n,alg} \mathcal{D}(B)$

$$\begin{aligned} |\langle \Psi, \mathrm{d}\Gamma(A)\, \Phi\rangle| \quad &\leq \quad \|(1+B)^{-1} A\, (1+B)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \left(\sum_{n=1}^{\infty} \varepsilon n \|((1+B) \otimes \mathbb{1}^{(n-1)})\Psi^{(n)}\|^2\right)^{1/2} \\ &\quad \times \left(\sum_{n=1}^{\infty} \varepsilon n \|((1+B) \otimes \mathbb{1}^{(n-1)})\Phi^{(n)}\|^2\right)^{1/2}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} \varepsilon n \|((1+B) \otimes \mathbb{1}^{(n-1)})\Psi^{(n)}\|^2 \quad &= \quad \langle \Psi^{(n)}, \mathrm{d}\Gamma((1+B)^2)\Psi^{(n)}\rangle \\ &\leq \quad \|\mathrm{d}\Gamma(1+B)\Psi^{(n)}\|^2_{\bigvee^n \mathcal{Z}}\,, \end{aligned}$$

since in the sense of quadratic forms $\mathrm{d}\Gamma((1+B)^2) \leq \mathrm{d}\Gamma(1+B)^2$. Hence we obtain

$$|\langle \Psi, \mathrm{d}\Gamma(A)\, \Phi\rangle| \leq \|(1+B)^{-1} A\, (1+B)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\, \|\mathrm{d}\Gamma(1+B)\Psi\|\, \|\mathrm{d}\Gamma(1+B)\Phi\|\,.$$

This proves (i).
Expressing C^{Wick} as a quadratic form for $\Psi, \Phi \in \oplus_{n=0}^{alg} \bigvee^{n,alg} \mathcal{D}(B)$ and then applying Cauchy-Schwarz inequality yield

$$\begin{aligned} |\langle \Psi, C^{Wick}\, \Phi\rangle| \quad &\leq \quad \|(\mathbb{1} + B \otimes \mathbb{1} + \mathbb{1} \otimes B)^{-1/2} C\, (\mathbb{1} + B \otimes \mathbb{1} + \mathbb{1} \otimes B)^{-1/2}\|_{\mathcal{L}(\bigvee^2 \mathcal{Z})} \\ &\quad \times \left(\sum_{n=2}^{\infty} \varepsilon^2 n(n-1) \|[(\mathbb{1} + B \otimes \mathbb{1} + \mathbb{1} \otimes B)^{1/2} \otimes \mathbb{1}^{(n-2)}]\Psi^{(n)}\|^2\right)^{1/2} \\ &\quad \times \left(\sum_{n=2}^{\infty} \varepsilon^2 n(n-1) \|[(\mathbb{1} + B \otimes \mathbb{1} + \mathbb{1} \otimes B)^{1/2} \otimes \mathbb{1}^{(n-2)}]\Phi^{(n)}\|^2\right)^{1/2}. \end{aligned}$$

Due to the symmetry of the vector $\Phi^{(n)}$ we remark that

$$\begin{aligned} \varepsilon^2 n^2 \|[(\mathbb{1} + B \otimes \mathbb{1} + \mathbb{1} \otimes B)^{1/2} \otimes \mathbb{1}^{(n-2)}]\Phi^{(n)}\|^2 \quad &= \quad \varepsilon^2 n^2 \langle \Phi^{(n)}, [(\mathbb{1} + B \otimes \mathbb{1} + \mathbb{1} \otimes B) \otimes \mathbb{1}^{(n-2)}]\Phi^{(n)}\rangle \\ &= \quad 2\langle \Phi^{(n)}, (1 + \mathrm{d}\Gamma(B))\, \mathbf{N}^2\, \Phi^{(n)}\rangle \\ &\leq \quad \|(1 + \mathrm{d}\Gamma(B) + \mathbf{N}^2)\, \Phi^{(n)}\|^2 \end{aligned}$$

So we obtain

$$\begin{aligned} |\langle \Psi, C^{Wick}\, \Phi\rangle| \quad &\leq \quad \|(1+B_2)^{-1/2} C\, (1+B_2)^{-1/2}\|_{\mathcal{L}(\bigvee^2 \mathcal{Z})}\, \|(1 + \mathrm{d}\Gamma(B) + \mathbf{N}^2)\, \Psi\| \\ &\quad \times \|(1 + \mathrm{d}\Gamma(B) + \mathbf{N}^2)\, \Phi\|\,. \end{aligned}$$

This proves (ii). □

C Absolutely continuous curves in $\mathrm{Prob}_2(\mathcal{Z}_{1,\mathbb{R}})$

This section firstly gathers results presented in [3] about Borel probability measures on a separable real Hilbert space which are weak solutions to continuity equations. In a second step, we shall adapt it to a complex Hilbert space $\mathcal{Z}_1$ endowed with its real euclidean structure.

C.1 Absolutely continuous curves in $\mathrm{Prob}_2(E)$

Let E be a real Hilbert space with scalar product $\langle\,,\,\rangle$ and norm $|\,|$. The symbol $\mathrm{Prob}_p(E)$ (resp. $\mathrm{Prob}(E)$) refers to the set of Borel probability measures μ on E such that $\int_E |x|^p\ d\mu(x) < +\infty$ (resp. with no momentum condition), and we simply work with $p = 2$. On $\mathrm{Prob}_2(E)$, the 2-Wasserstein distance is defined by

$$W^2(\mu^1,\mu^2) := \min\left\{\int_{E^2} |x_1 - x_2|_E^2\ d\mu(x_1,x_2)\,;\ \Pi_{j,*}\mu = \mu_j\right\}\,,$$

where $\Pi_j : E^2 \to E$ is the natural projection, $j = 1, 2$. Narrow convergence of a sequence $(\mu_n)_{n\in\mathbb{N}}$ of $\mathrm{Prob}_2(E)$, with a uniform control of $\int_E |x|^2\ d\mu_n$ is equivalent to the W^2 convergence on $\mathrm{Prob}_2(E)$ (see Proposition 7.1.5 in [3]). Remember also that the tightness property of subsets of $\mathrm{Prob}_2(E)$ can be checked in the infinite dimensional case with the weak topology, or after introducing a Hilbert basis $(e_n)_{n\in\mathbb{N}^*}$, with the distance $d_\omega(x_1,x_2) = \sum_{n\in\mathbb{N}^*} \frac{|\langle x_1-x_2,e_n)|^2}{(1+n)^2}$. This use of weak or d_ω topology, is done also when considering probability measures on set of absolutely continuous curves in E.

This tightness property is called the weak tightness property in [3] since it refers to the weak topology on E. Especially when one considers the narrow convergence in $\mathrm{Prob}_2(E)$, there is a weak narrow convergence and a strong narrow convergence (see the discussions about this in Chapter 5 and 7 of [3]). The terms "narrow convergence" or "narrow continuity" refer **to the strong ones** and we shall specify "weak narrow convergence" and "weak narrow continuity" when necessary.

We recall two results of [3] and give a complete proof in the infinite dimensional case of the second one, for the sake of completeness (it is left as an exercise to the reader in [3]).

The following result is the second part of Theorem 8.3.1 in [3] with $p = 2$.

Proposition C.1. *Let I be an open interval in $\mathbb{R}$. If a W^2-continuous curve $\mu_t : I \to \mathrm{Prob}_2(E)$ satisfies the continuity equation*

$$\partial_t\mu_t + \nabla^T(v_t\mu_t) = 0 \tag{65}$$

in the weak sense

$$\int_I\int_E (\partial_t\varphi(x,t) + \langle v_t(x)\,,\,\nabla_x\varphi(x,t)\rangle_E)\ d\mu_t(x)dt = 0\,,\quad \forall\varphi\in\mathcal{C}^\infty_{0,cyl}(E\times I)\,, \tag{66}$$

for some Borel velocity field v_t, with $|v_t|_{L^2(E,\mu_t)} \in L^1(I)$, then $\mu_t : I \to \mathrm{Prob}_2(E)$ is absolutely continuous and $|\mu'|(t) \le |v_t|_{L^2(E,\mu_t)}$ for Lebesgue almost every $t \in I$. Moreover for Lebesgue almost every $t \in I$, v_t belongs to the closure in $L^2(E,\mu_t)$ of the subspace spanned by $\left\{\nabla\varphi, \varphi\in\mathcal{C}^\infty_{0,cyl}(E)\right\}$.

The previous result concerns non regular (non Lipschitz) vector fields for which there is no uniqueness result for the Cauchy problem. Remember that the infinite dimensional case, which relies on the cylindrical integration of v_t and cylindrical disintegration of the measure μ_t, requires the introduction of such singular vector fields (see the proof of Theorem 8.3.1 in [3]). Nevertheless an interpretation of the continuity equation (65)(66) in terms of characteristic curves can be done via a probabilistic representation. For the sake of completeness, we adapt the proof of Theorem 8.2.1 stated in [3] for the finite dimensional case, to our infinite dimensional case. For $T \in (0,+\infty)$, consider the set $\Gamma_T = \mathcal{C}^0([-T,T];E)$ endowed with the norm $|\gamma|_{\infty,T} = \max_{t\in[-T,T]}|\gamma(t)|_E$. For a Borel probability measure $\boldsymbol{\eta}$ defined on $E\times\Gamma_T$, consider the time dependent Borel probability measure $\mu_t^{\boldsymbol{\eta}}$ defined by

$$\int_E \varphi\ d\mu_t^{\boldsymbol{\eta}} = \int_{E\times\Gamma_T} \varphi(\gamma(t))\ d\boldsymbol{\eta}(x,\gamma)\,,\quad \forall\varphi\in\mathcal{C}^0_{b,cyl}(E),\, t\in[-T,T]\,. \tag{67}$$

The measure $\mu_t^{\boldsymbol{\eta}}$ is the push-forward of $\boldsymbol{\eta}$ by the evaluation map

$$e_t : (x,\gamma)\in E\times\Gamma_T \to \gamma(t)\in E\,,\quad \text{for } t\in[-T,T]\,.$$

Proposition C.2. *Let $\mu_t : [-T,T] \to \mathrm{Prob}_2(E)$ be a W^2-continuous solution to the continuity equation (65)(66), with $I = (-T,T)$, for a suitable Borel vector field $v(t,x) = v_t(x)$ such that $|v_t|_{L^2(E,\mu_t)} \in L^1([-T,T])$. Then there exists a Borel probability measure $\boldsymbol{\eta}$ in $E \times \Gamma_T$ such that*

(i) *$\boldsymbol{\eta}$ is concentrated on the set of pairs (x,γ) such that $\gamma \in AC^2([-T,T];E)$ is a solution to the ODE $\dot\gamma(t) = v_t(\gamma(t))$ for Lebesgue almost every $t \in (-T,T)$ with $\gamma(0) = x$;*

(ii) *$\mu_t = \mu_t^{\boldsymbol{\eta}}$ for any $t \in [-T,T]$, with $\mu_t^{\boldsymbol{\eta}}$ defined as in (67).*

Conversely, any $\boldsymbol{\eta}$ satisfying (i) and

$$\int_0^T \int_{E\times\Gamma_T} |v_t(\gamma(t))|_E \; d\boldsymbol{\eta}(x,\gamma)dt < +\infty\,,$$

induces via (67) a solution to the continuity equation, with $\mu_0 = \gamma(0)_*\boldsymbol{\eta}$.

Remark C.3. *The notation $AC^2([-T,T];E)$ refers to the set of absolutely continuous curves in E with $L^2([-T,T];E)$ derivative. We keep the notation $\Phi_*\mu$ of differential geometry, for the push-forward or direct image of a measure μ, by the Borel map Φ.*

Proof. The result is proved in [3] when E is finite dimensional. The proof of the second (converse) part of the statement, is exactly the same as in finite dimension, after replacing regular (Lipschitz) test functions by cylindrical ones. We now show, for the first part, how the infinite dimensional case is deduced from the finite dimensional result, following an approximation scheme like in the proof of [3, Theorem 8.3.1]. After introducing an Hilbert basis $(e_n)_{n\in\mathbb{N}^*}$ of E, the maps $\pi^d : E \to \mathbb{R}^d$, $\pi^{d,T} : \mathbb{R}^d \to E$ and $\hat\pi^d : E \to E$ are defined according to

$$\begin{aligned}
\pi^d(x) &= (\langle e_1,x\rangle,\dots,\langle e_d,x\rangle)\,,\\
\pi^{d,T}(y_1,\dots,y_d) &= \sum_{j=1}^d y_j e_j\,,\\
\hat\pi^d &= \pi^{d,T}\circ\pi^d\,.
\end{aligned}$$

With the measure $\mu_t \in \mathrm{Prob}_2(E)$, the measure $\mu_t^d \in \mathrm{Prob}_2(\mathbb{R}^d)$ is defined by $\mu_t^d = \pi^d_*\mu_t$ and $\{\mu_{t,y}, y \in \mathbb{R}^d\}$ denotes the disintegration of μ_t w.r.t μ_t^d. Within the space E endowed with the basis $(e_n)_{n\in\mathbb{N}^*}$, $\hat\mu_t^d$ is nothing but $\mu_t^d \otimes \delta_0$ in the decomposition $Z = F_d \times F_d^\perp$ with $F_d = \mathrm{span}(e_1,\dots,e_d)$. The vector field v_t^d (resp. $\hat v_t^d$) is defined on $\mathbb{R}^d$ (resp. on E) by

$$\begin{aligned}
v_t^d(y) &= \int_{(\pi^d)^{-1}(y)} \pi^d v_t(x)\; d\mu_{t,y}(x)\,, \quad y \in \mathbb{R}^d\\
\textit{resp.}\qquad \hat v_t^d(y) &= \int_{(\hat\pi^d)^{-1}(\hat\pi^d y)} \hat\pi^d v_t(x)\; d\mu_{t,y}(x)\,, \quad y \in E\,.
\end{aligned}$$

Within the proof of Theorem 8.3.1 in [3], it was checked that μ_t^d (resp. $\hat\mu_t^d$) is a weak solution to the continuity equation

$$\begin{aligned}
&\partial_t\mu_t^d + \nabla^T(v_t^d\mu_t^d) = 0\,,\\
\text{resp.}\qquad &\partial_t\hat\mu_t^d + \nabla^T(\hat v_t^d\hat\mu_t^d) = 0\,.
\end{aligned}$$

with the following properties:

1) $|\hat v_t^d|_{L^2(E,\hat\mu_t^d)} = |v_t^d|_{L^2(\mathbb{R}^d,\mu_t^d)} \le \|v_t\|_{L^2(E,d\mu_t)}$;

2) $W^2(\mu_{t_1}^d,\mu_{t_2}^d) \le \int_{t_1}^{t_2} |v_t^d|_{L^2(\mathbb{R}^d,\mu_t^d)}\; dt \le \int_{t_1}^{t_2} |v_t|_{L^2(E,\mu_t)}\; dt$, for $-T < t_1 \le t_2 < T$;

3) the sequence $(\hat\mu_t^d)_{d\in\mathbb{N}^*}$ converges weakly narrowly to μ_t with the estimate

$$W^2(\mu_{t_1},\mu_{t_2}) \le \liminf_{d\to\infty} W^2(\mu_{t_1}^d,\mu_{t_2}^d) \le \int_{t_1}^{t_2} |v_t|_{L^2(E,\mu_t)}\; dt\,, \quad -T < t_1 \le t_2 < T\,.$$

The set of continuous maps from $[-T,T]$ to $\mathbb{R}^d$ is denoted by Γ_T^d. By using the finite dimensional result, stated in Theorem 8.2.1 of [3], there exists for any $d \in \mathbb{N}$ a probability measure, $\boldsymbol{\eta}^d$, on $\mathbb{R}^d \times \Gamma_T^d$ such that the properties (i) and (ii) hold when (μ_t, v_t, E) is replaced by $(\mu_t^d, v_t^d, \mathbb{R}^d)$. Equivalently the result can be formulated in E after using $(\hat{\mu}_t^d, \hat{v}_t^d, E)$ instead of $(\mu_t^d, v_t^d, \mathbb{R}^d)$ and using $\hat{\boldsymbol{\eta}}^d = (\pi^{d,T} \times \pi^{d,T})_* \boldsymbol{\eta}_t^d$. Hence we have a sequence $(\hat{\boldsymbol{\eta}}^d)_{d\in\mathbb{N}}$ of probability measures on $E \times \Gamma_T$ which satisfy

$$\begin{aligned}\int_E \varphi \circ \pi^d \; d\hat{\mu}_t^d &= \int_{\mathbb{R}^d} \varphi \; d\mu_t^d = \int_{\mathbb{R}^d \times \Gamma_T^d} \varphi(\gamma(t)) \; d\boldsymbol{\eta}^d(x,\gamma) \\ &= \int_{E\times\Gamma_T} \varphi \circ \pi^d(\gamma(t)) \; d\hat{\boldsymbol{\eta}}^d(x,\gamma)\,, \; \forall \varphi \in \mathcal{C}_b^0(\mathbb{R}^d),\, t \in [-T,T]\,, \end{aligned} \tag{68}$$

where $\varphi \circ \pi^d$ can be replaced by $\varphi \circ \hat{\pi}^d$ with $\varphi \in \mathcal{C}_b^0(F_d)$.
After some regularization done [3] pp179-180, it is proved that any measure $\hat{\boldsymbol{\eta}}^d$ satisfies

$$\int_{\Gamma_T} \int_{-T}^T |\dot{\gamma}(t)|^2 \; dt d\hat{\boldsymbol{\eta}}^d \le \int_{-T}^T \int_E |v_t^d(x)|^2 \; d\hat{\mu}_t^d dt \le \int_{-T}^T \int_E |v_t(x)|^2 \; d\hat{\mu}_t dt\,.$$

Since the functional $g \to \int_{-T}^T |\dot{g}(t)|^2 \; dt$, defined on $\{g \in \Gamma_T\,, \quad g(0)=0\}$ and set to $+\infty$ if $g \notin AC^2([-T,T];E)$, has compact sublevel sets in Γ_T, the two mappings

$$r^1 : (x,\gamma) \in E \times \Gamma_T \to x \in E \quad , \quad r^2 : (x,\gamma) \in E \times \Gamma_T \to g_{\gamma,x} = \gamma - x \in \Gamma_T$$

give rise to (weakly) tight families of marginals $(r^1_* \hat{\boldsymbol{\eta}}^d)_{d\in\mathbb{N}} = (\hat{\mu}_0^d)_{d\in\mathbb{N}}$ and $(r^2_* \hat{\boldsymbol{\eta}}^d)_{d\in\mathbb{N}}$. Remember that the compactness of subsets of E or Γ_T is considered with the weak topology on E or the distance d_ω. Hence the family $(\hat{\boldsymbol{\eta}}^d)_{d\in\mathbb{N}}$ is (weakly) tight in $\mathrm{Prob}(E\times\Gamma_T)$ and we take for $\boldsymbol{\eta}$ a weak narrow limit point of $\hat{\boldsymbol{\eta}}^d$. By assuming the test function φ in (68) to depend only on d' coordinates with $d' \le d$, and by taking the limit $d \to +\infty$ while d' and φ are fixed, we get

$$\int_E (\varphi \circ \pi^{d'}) \; d\mu_t = \int_{E\times\Gamma} (\varphi \circ \pi^{d'})(\gamma(t)) \; d\boldsymbol{\eta}(x,\gamma)\,,$$

for all $\varphi \in \mathcal{C}_b^0(\mathbb{R}^{d'})$ and $t \in [-T,T]$, where $\varphi \circ \pi^{d'}$ can then be replaced by any cylindrical function or Borel bounded function on E. It remains to prove the condition (i) for $\boldsymbol{\eta}$, namely that this measure is concentrated on curves verifying $\dot{\gamma} = v(\gamma(t))$ for Lebesgue almost every $t \in (-T,T)$ ($\gamma(0) = x$ is already known).
The estimate (8.2.6) used in [3] for the finite dimensional case, provides the inequality

$$\int_{\mathbb{R}^d \times \Gamma_T^d} |\gamma(t) - x - \int_0^t w_s(\gamma(s)) \; ds|^2 \; d\boldsymbol{\eta}^d(x,\gamma) \le (2T) \int_{-T}^T \int_{\mathbb{R}^d} |v_t^d - w_t|^2 d\mu_t^d \; dt\,,$$

for any family $w_s(x) = w(s,x)$ uniformly bounded continuous function from $[-T,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$. After assuming that w actually belongs to $\mathcal{C}_b^0([-T,T]\times\mathbb{R}^{d'};\mathbb{R}^{d'})$ with $d' \le d$ fixed and, by using $\hat{w}_t = \pi^{d',T} \circ w_t \circ \pi_{d'} \in \mathcal{C}_b^0([-T,T]\times E;E)$, taking the limit as $d \to \infty$ gives

$$\int_{E\times\Gamma_T} |\gamma(t) - x - \int_0^t \hat{w}_s(\gamma(s)) \; ds|^2 \; d\boldsymbol{\eta}(x,\gamma) \le (2T) \limsup_{d\to\infty} \int_{-T}^T \int_{\mathbb{R}^d} |\hat{v}_t^d - \hat{w}_t|^2 d\hat{\mu}_t^d \; dt\,.$$

But the condition 1) for $|\hat{v}_t^d|_{L^2(E,\hat{\mu}_t^d)}$ is easily extended to

$$|\hat{v}_t^d - \hat{w}_t|_{L^2(E,\hat{\mu}_t^d)} \le |v_t - \hat{w}_t|_{L^2(E,\mu_t)}\,,$$

by the same argument, relying on

$$\begin{aligned}\left|\int_E \langle \hat{v}_t^d - \hat{w}_t\,,\, \chi\rangle \; d\hat{\mu}_t^d\right| &\le \left|\int_E \langle \hat{\pi}_d(v_t(x) - \hat{w}_t(x))\,,\, \chi(\hat{\pi}^d(x))\rangle \; d\mu_t(x)\right| \\ &\le |v_t - \hat{w}_t|_{L^2(E;\mu_t)} |\chi|_{L^2(E,\hat{\mu}_t^d)}\,, \quad \forall \chi \in L^2(E,\hat{\mu}_t^d)\,.\end{aligned}$$

This uniform upper bound leads to

$$\int_{E\times\Gamma_T} |\gamma(t) - x - \int_0^t \hat{w}_s(\gamma(s))\ ds|^2\ d\boldsymbol{\eta}(x,\gamma) \leq (2T) \int_0^T \int_E |v_t(x) - \hat{w}_t(x)|^2\ d\mu_t dt\,.$$

According to the last statement of Proposition C.1, v_t can be approximated in $L^2(E,\mu_t)$ by a sequence of bounded regular cylindrical functions, $(\hat{w}_{t,n})_{n\in\mathbb{N}}$. By possibly truncating with respect to times $t \to \hat{w}_{t,n}$ so that $|v_t - \hat{w}_{t,n}|_{L^2(E,\mu_t)} \leq 1$ for a.e. $t \in (-T,T)$ and all $n \in \mathbb{N}^*$, Lebesgue's Theorem implies

$$\int_{E\times\Gamma_T} |\gamma(t) - x - \int_0^t v_s(\gamma(s))\ ds|^2\ d\hat{\boldsymbol{\eta}}^d(x,\gamma) = 0\,,$$

which ends the proof. □

Below is a consequence of the above probabilistic interpretation when the Cauchy problem $\dot{\gamma}(t) = v_t(\gamma(t))$, $\gamma(0) = x$ admits a unique solution for all $x \in E$. The fact that we have to pass by the probabilistic representation is a real question. Contrary to the finite dimensional case, the well-posedness of the Cauchy problem, even with the standard Picard's contraction argument, defining a flow on the whole space E, does not give a representation formula for observables. The point is that the natural observables, or test functions, are cylindrical functions, a property which is not preserved a nonlinear flow.

Proposition C.4. *Let $\mu_t : \mathbb{R} \to \mathrm{Prob}_2(E)$ be a W^2-continuous solution to the continuity equation (65)(66) for a suitable Borel velocity field $v(t,x) = v_t(x)$ such that $|v_t|_{L^2(E,\mu_t)} \in L^1([-T,T])$ for all $T > 0$. Assume additionally that the Cauchy problem*

$$\dot{\gamma}(t) = v_t(\gamma(t))\,, \quad \gamma(s) = x$$

or

$$\gamma(t) = x + \int_s^t v_s(\gamma(s))\ ds\,,$$

admits a unique global continuous solution on $\mathbb{R}$ for all $s \in \mathbb{R}$ and all $x \in E$, such that $\gamma(t) = \Phi(t,s)\gamma(s)$ defines a Borel flow on E (i.e. $\Phi(t,s) : E \to E$ is a Borel function for all $t,s \in \mathbb{R}$). Then the measure μ_t satisfies

$$\forall t,s \in \mathbb{R}, \quad \mu_t = \Phi(t,s)_*\mu_s\,.$$

Proof. It suffices to work with $t \in [-T,T]$ as in Proposition C.2. Since the evaluation map $e_t : E \times (x,\gamma)\Gamma_T \to \gamma(t) \in E$ is a continuous, thus Borel, map. The relation $\mu_t = \mu_t^{\boldsymbol{\eta}}$ defined according to (67) extends to any bounded Borel function φ on E:

$$\int_E \varphi\ d\mu_t = \int_{E\times\Gamma_T} \varphi(\gamma(t))\ d\boldsymbol{\eta}(x,\gamma)\,.$$

By using $\gamma(t) = \Phi(t,s)\gamma(s)$, with $\Phi(t,s)$ Borel, we deduce

$$\int_E \varphi\ d\mu_t = \int_{E\times\Gamma_T} [\varphi\circ\Phi(t,s)](\gamma(s))\ d\boldsymbol{\eta}(x,\gamma) = \int_E [\varphi\circ\Phi(t,s)]\ d\mu_s$$

which is nothing but $\mu_t = \Phi(t,s)_*\mu_s$. □

C.2 Application to Hamiltonian fields

We finally specify how these results apply to our case, when the phase-space $\mathcal{Z}_0$ is a complex Hilbert space and the velocity field is associated with a (singular) Hamiltonian vector field, only defined on $\mathcal{Z}_1 \subset \mathcal{Z}_0$.

Consider a complex Hilbert triple $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}_{-1}$, with $\mathcal{Z}_1$ densely continuously embedded in $\mathcal{Z}_0$ and $\mathcal{Z}_{-1}$ being the dual of $\mathcal{Z}_1$ for the duality bracket extending $\langle z_1\,,\, z_2\rangle_{\mathcal{Z}_0}$. The dual of a complex Hilbert space $\mathcal{Z}$ while keeping the $\mathbb{C}$-bilinear duality bracket, written $u\,.\,v$ in (64), is still denoted by $\mathcal{Z}^*$. In the case treated in the article $\mathcal{Z}_0 = L^2(\mathbb{R}^d,dx)$, $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$ and $\mathcal{Z}_{-1} = H^{-1}(\mathbb{R}^d)$. The

space $\mathcal{Z}_0$ is endowed with its scalar product $\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_0}$, real euclidean structure with Re $\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_0}$ and its symplectic structure $\sigma(z_1,z_2) = \text{Im}\ \langle z_1\,,\,z_2\rangle_{\mathcal{Z}_0}$. On $\mathcal{Z}_1$ we will use the hermitian $\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_1}$ and euclidean scalar product

$$\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_1,\mathbb{R}} = \text{Re}\ \langle z_1\,,\,z_2\rangle_{\mathcal{Z}_1}\,.$$

For a cylindrical function $f \in \mathcal{S}_{cyl}(\mathcal{Z}_0)$, based on $\wp\mathcal{Z}_0$, the differentials $\partial_z f(z)$ and $\partial_{\overline{z}} f(z)$ are defined

$$\partial_z f(z) = \int_{\wp\mathcal{Z}_0} i\pi\langle\xi| e^{2i\pi\text{Re}\ \langle z\,,\,\xi\rangle}\mathcal{F}[f](\xi)\ L_\wp(d\xi)$$
$$\partial_{\overline{z}} f(z) = \int_{\wp\mathcal{Z}_0} i\pi|\xi\rangle e^{2i\pi\text{Re}\ \langle z\,,\,\xi\rangle}\mathcal{F}[f](\xi)\ L_\wp(d\xi)\,.$$

Hence $\partial_z f(z)$ is a continuous $\mathbb{C}$-linear form on $\mathcal{Z}_0$ while $\partial_{\overline{z}} f(z) \in \mathcal{Z}_0$. This notation is coherent with the definition of $\partial_z b(z)$ and $\partial_{\overline{z}} b(z)$ when b is a Wick symbol in $\oplus^{alg}_{p,q}\mathcal{P}_{p,q}(\mathcal{Z})$.
A function $f \in \mathcal{S}_{cyl}(\mathcal{Z}_1)$ is given by

$$f(z) = \varphi(\langle\xi_1\,,\,z\rangle_{\mathcal{Z}_1},\dots,\langle\xi_N\,,\,z\rangle_{\mathcal{Z}_1}) = \varphi(\langle\eta_1\,,\,z\rangle,\dots,\langle\eta_N\,,\,z\rangle)$$

with $\varphi(w_1,\dots w_N) \in \mathcal{S}(\mathbb{R}^N)$ and $\xi_1,\dots,\xi_N \in \mathcal{Z}_1$ and $\eta_1,\dots,\eta_N \in \mathcal{Z}_{-1}$, such that $\langle\xi_j\,,\,z\rangle_{\mathcal{Z}_1} = \langle\eta_j\,,\,z\rangle$ for all $z \in \mathcal{Z}_1$. The derivatives $\partial_z f$ and $\partial_{\overline{z}} f$ are thus given by

$$\forall z \in \mathcal{Z}_1\,,\qquad \partial_z f(z) = \sum_{j=1}^N \partial_{w_j}\varphi(\langle\eta_1\,,\,z\rangle,\dots,\langle\eta_N\,,\,z\rangle)\langle\eta_j| \quad \in \mathcal{Z}_{-1}^*\,,$$
$$\forall z \in \mathcal{Z}_1,\qquad \partial_{\overline{z}} f(z) = \sum_{j=1}^N \partial_{\overline{w_j}}\varphi(\langle\eta_1\,,\,z\rangle,\dots,\langle\eta_N\,,\,z\rangle)|\eta_j\rangle \quad \in \mathcal{Z}_{-1}\,.$$

When $h(z)$ is an unbounded polynomial on $\mathcal{Z}_0$ but which happens to be a real-valued Fréchet $\mathcal{C}^1$-function on $\mathcal{Z}_1$, the derivatives $\partial_z h(z)$ and $\partial_{\overline{z}} h(z)$ are defined only for $z \in \mathcal{Z}_1$ and we have

$$\forall z \in \mathcal{Z}_1\,,\quad \partial_{\overline{z}} h(z) \in \mathcal{Z}_1 \quad,\quad \partial_z h(z) \in \mathcal{Z}_{-1}^*\,.$$

When $f \in \mathcal{S}_{cyl}(\mathcal{Z}_0)$ (resp. $g \in \mathcal{S}_{cyl}(\mathcal{Z}_1)$ or h) is real valued differentiating $f(z+te)$ at $t=0$, $t \in \mathbb{R}$, for any $e \in \mathcal{Z}_0$ (resp. any $e \in \mathcal{Z}_1$) leads to

$$\overline{\partial_z f(z).u} = \langle u\,,\,\partial_{\overline{z}} f(z)\rangle\,,\quad z \in \mathcal{Z}_0\,,\,u \in \mathcal{Z}_0 \tag{69}$$
$$\overline{\partial_z g(z).u} = \langle u\,,\,\partial_{\overline{z}} g(z)\rangle\,,\quad z \in \mathcal{Z}_1\,,\,u \in \mathcal{Z}_1\,. \tag{70}$$
$$\overline{\partial_z h(z).u} = \langle u\,,\,\partial_{\overline{z}} h(z)\rangle\,,\quad z \in \mathcal{Z}_1, u \in \mathcal{Z}_{-1}\,. \tag{71}$$

Note that the Poisson bracket

$$i\,\{h,b\}\,(z) = i\,(\partial_z h.\partial_{\overline{z}} b - \partial_z b.\partial_{\overline{z}} h)\,(z)\,,\quad z \in \mathcal{Z}_1$$

is well defined for $b \in \mathcal{S}_{cyl}(\mathcal{Z}_1;\mathbb{R})$ and our aim is to write it as the real scalar product

$$\langle v(z)\,,\,(\nabla b)(z)\rangle_{\mathcal{Z}_1,\mathbb{R}}\,,\quad z \in \mathcal{Z}_1\,.$$

Definition C.5. *For a cylindrical function on $\mathcal{Z}_1$, $f \in \mathcal{S}_{cyl}(\mathcal{Z}_1)$, the gradients $\nabla_{\overline{z}}$ and ∇ are defined by*

$$\forall z \in \mathcal{Z}_1, u \in \mathcal{Z}_1,\quad \langle u\,,\,\nabla_{\overline{z}} f(z)\rangle_{\mathcal{Z}_1} = \langle u\,,\,\partial_{\overline{z}} f(z)\rangle\,,$$
$$\nabla = 2\nabla_{\overline{z}}.$$

Remark C.6. • *Although it is not necessary, these definitions can be justified by introducing a complex conjugation $u \to \bar{u}$ on $\mathcal{Z}_0$, which remains a conjugation on $\mathcal{Z}_1$, that is an isometric $\mathbb{C}$-antilinear application such that $\overline{\langle u,v\rangle_{\mathcal{Z}_{0,1}}} = \langle\overline{u}\,,\,\overline{v}\rangle_{\mathcal{Z}_{0,1}}$. When $\mathcal{Z}_0 = L^2(\mathbb{R}^d;\mathbb{C})$ and $\mathcal{Z}_1 =$*

$H^1(\mathbb{R}^d)$ this is the usual pointwise complex conjugation.
For real valued function one then sets

$$\nabla_R f = \nabla_{\overline{z}} f + \nabla_z f \quad \textit{and} \quad \nabla_I f = \frac{1}{i}\left(\nabla_{\overline{z}} f - \nabla_z f\right) .$$

so that

$$\nabla_{\overline{z}} f = \frac{1}{2}\left(\nabla_R f + i\nabla_I f\right) .$$

Similarly, an element X of $\mathcal{Z}_1$ can be written $X = X_R + iX_I$ with $\overline{X_{R,I}} = X_{R,I}$ or $X = \begin{pmatrix} X_R \\ X_I \end{pmatrix}$ and the real scalar product

$$\langle X\,,\,Y\rangle_{\mathcal{Z}_1,\mathbb{R}} = \mathrm{Re}\,\langle X\,,\,Y\rangle_{\mathcal{Z}_1} = \langle X_R\,,Y_R\rangle_{\mathcal{Z}_1} + \langle X_I\,,\,Y_I\rangle_{\mathcal{Z}_1} .$$

Then the definition of the gradient of a real cylindrical function f becomes

$$\nabla f = \begin{pmatrix} \nabla_R f \\ \nabla_I f \end{pmatrix} .$$

- *It is important to notice that we do not use the $\mathcal{Z}_1$-gradient for the real valued function $h(z)$, but keep the derivative, $\partial_{\overline{z}} h(z)$ modeled on the duality bracket $\langle\ ,\ \rangle$. With a complex conjugation and since h is real valued, it can be decomposed into $\partial_{\overline{z}} h = \frac{1}{2}\left(\partial_R h + i\partial_I h\right)$ and*

$$-i\partial_{\overline{z}} h = \frac{1}{2}\partial_I h - \frac{i}{2}\partial_R h .$$

Lemma C.7. *With the above notations and assumptions the equality*

$$\forall z \in \mathcal{Z}_1\,, \quad i\left\{h, b\right\}(z) = 2\mathrm{Re}\,\langle -i\partial_{\overline{z}} h(z)\,,\,\nabla_{\overline{z}} b(z)\rangle_{\mathcal{Z}_1} = \langle v(z)\,,\,\nabla b(z)\rangle_{\mathcal{Z}_1,\mathbb{R}}\,,$$

holds for any $b \in \mathcal{S}_{cyl}(\mathcal{Z}_1;\mathbb{R})$ with $v(z) = -i\partial_{\overline{z}} h(z)$.

Proof. It suffices to compute

$$\begin{aligned} i\left\{h\,,\,b\right\} &= i\left[\partial_z h.\partial_{\overline{z}} b - \partial_z b.\partial_{\overline{z}} h\right] \\ &= i\left[\overline{\langle \partial_{\overline{z}} b\,,\,\partial_{\overline{z}} h\rangle_{\mathcal{Z}_0}} - \overline{\langle \partial_{\overline{z}} h\,,\,\partial_{\overline{z}} b\rangle_{\mathcal{Z}_0}}\right] \\ &= -2\mathrm{Im}\,\langle \partial_{\overline{z}} h\,,\,\partial_{\overline{z}} b\rangle = 2\mathrm{Re}\,\langle -i\partial_{\overline{z}} h\,,\,\partial_{\overline{z}} b\rangle = 2\mathrm{Re}\,\langle -i\partial_{\overline{z}} h\,,\,\nabla_{\overline{z}} b\rangle_{\mathcal{Z}_1} \\ &= \langle -i\partial_{\overline{z}} h\,,\,\nabla b\rangle_{\mathcal{Z}_1,\mathbb{R}}\,. \end{aligned}$$

□

Proposition C.8. *Let $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}_{-1}$ be a Hilbert triple of separable complex Hilbert spaces. Consider a time dependent real sesquilinear form $z \to h(z,t)$ on $\mathcal{Z}_1$ which is Fréchet-$\mathcal{C}^1$ and such that $\mathcal{Z}_1 \times \mathbb{R} \ni (z,t) \to (\partial_{\overline{z}} h(z,t)\,,\,\partial_z h(z,t)) \in \mathcal{Z}_1 \times \mathcal{Z}_{-1}^*$ is strongly continuous. Assume also that the time-dependent Hamilton equation*

$$i\partial_t z_t = \partial_{\bar{z}} h(z_t, \bar{z}_t, t) \quad , \quad z_{t=s} = z$$

admits a unique continuous solution $z_t = \Phi(t,s)z$ for all $t, s \in \mathbb{R}$ and all $z \in \mathcal{Z}_1$, with $\Phi(t,s) : \mathcal{Z}_1 \to \mathcal{Z}_1$ Borel.
Consider a time dependent measure $\mu(t) \in \mathrm{Prob}_2(\mathcal{Z}_1)$ which satisfies

- *$t \to \mu_t \in \mathrm{Prob}_2(\mathcal{Z}_1)$ is W^2-continuous.*
- *For all $T > 0$, $|\partial_{\bar{z}} h(t)|_{L^2(\mathcal{Z},\mu_t)} \in L^1([-T,T])$.*

- *The time-dependent probability measure μ_t is a weak solution to*

$$\partial_t \mu + i\,\{h(t)\,,\,\mu\} = 0\,,$$

namely for all $\varphi \in \mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1 \times \mathbb{R}; \mathbb{R})$,

$$\int_{\mathbb{R}} \int_{\mathcal{Z}_1} \left(\partial_t \varphi(z,t) + i\,\{h, \varphi\}\,(z,t)\right)\, d\mu_t(z) dt = 0\,.$$

Then the measure μ_t satisfies

$$\forall t, s \in \mathbb{R}, \quad \mu_t = \Phi(t,s)_* \mu_s$$

and it is unique when μ_0 is fixed.

Proof. We apply Proposition C.4 while $E = \mathcal{Z}_1$ is endowed with its euclidean structure $\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_1,\mathbb{R}} = \mathrm{Re}\,\langle z_1\,,\,z_2\rangle_{\mathcal{Z}_1}$. Lemma C.7 says that the weak Liouville equation is

$$\forall \varphi \in \mathcal{C}^\infty_{0,cyl}(\mathcal{Z}_1 \times \mathbb{R})\,, \quad \int_{\mathbb{R}} \int_{\mathcal{Z}_1} \left(\partial_t \varphi(z,t) + \langle v\,,\,\nabla \varphi\rangle(t,z)\right)\, d\mu_t(z) dt = 0\,,$$

with $v(z,t) = -i\partial_{\overline{z}} h(z,t)$. The measure μ_t is a weak solution to

$$\partial_t \mu + \nabla^T (v\mu) = 0$$

where ∇ and ∇^T are defined according to the real structure on $\mathcal{Z}_1$. Our hypothesis on μ and h cover all the assumptions of Proposition C.4 □

D Weak L^p conditions for the potential V

Let $L_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$. Let $0 < p < \infty$, a Lebesgue measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be weak L^p integrable, or shortly in $L^{p,\infty}(\mathbb{R}^d)$, if there exists a constant $c > 0$ such that for all $t > 0$

$$L_{\mathbb{R}^d}\{x : |f(x)| > t\} \leq c^p / t^p\,.$$

Two functions in $L^{p,\infty}(\mathbb{R}^d)$ are equal if they are equal $L_{\mathbb{R}^d}$-almost everywhere. The quantity

$$\begin{aligned} \|f\|_{p,\infty} &= \inf\{c : L_{\mathbb{R}^d}\{x : |f(x)| > t\} \leq c^p/t^p, \forall t > 0\} \\ &= \sup_{t>0}\{t L_{\mathbb{R}^d}\{x : |f(x)| > t\}^{1/p}\} \end{aligned}$$

defines a complete quasi-norm on $L^{p,\infty}(\mathbb{R}^d)$ with $\|f\|_{p,\infty} \leq \|f\|_p$.
By combining Hunt and Marcinkiewicz interpolation theorems according to [34, 55, 11]), the Young and Hölder inequalities can be extended to weak L^p spaces.

Proposition D.1 (generalized Young's inequality)**.** *Let $1 < p, q, r < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. There exists a constant $c_{p,q} > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^{q,\infty}(\mathbb{R}^d)$*

$$\|f * g\|_r \leq c_{p,q}\, \|f\|_p\, \|g\|_{q,\infty}\,.$$

Proposition D.2 (generalized Hölder inequality)**.** *Let $1 < p, q, r < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. There exists a constant $c_{p,q}$ such that for all $f \in L^{p,\infty}(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$*

$$\|f.g\|_r \leq c_{p,q}\, \|f\|_{p,\infty}\, \|g\|_q\,.$$

Proposition D.3 (Hardy inequality)**.** *Suppose that $d \geq 3$ and $V \in L^{d,\infty}(\mathbb{R}^d)$. There exists a constant $c > 0$ such that for all $u \in H^1(\mathbb{R}^d)$*

$$\|Vu\|_2 \leq c \|u\|_{H^1(\mathbb{R}^d)}\,.$$

Proof. For $u \in L^2(\mathbb{R}^d)$, we can write $(1-\Delta)^{-1/2}u(x) = G * u(x)$ with G the inverse Fourier transform of $(1+|x|^2)^{-1/2}$. It is not difficult to prove that $G \in L^{\frac{d}{d-1},\infty}$ (see [55, Exercice 50]). Hence, we conclude that

$$\begin{aligned} \|V\,(1-\Delta)^{-1/2}u\|_2 = \|V\,G*u\|_2 &\overset{\text{Hölder}}{\leq} C_1\,\|V\|_{d,\infty}\,\|G*u\|_{\frac{2d}{d-2}} \\ &\overset{\text{Young}}{\leq} C_2\,\|V\|_{d,\infty}\,\|G\|_{\frac{d}{d-1},\infty}\,\|u\|_2 \end{aligned}$$

□

The above proposition provides a class of potentials which are bounded multiplication operators from $H^1(\mathbb{R}^d)$ into $L^2(\mathbb{R})$ when the dimension $d \geq 3$. For lower dimension, the Sobolev embeddings give at once:

- if $d=1$, $V \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ then $V \in \mathcal{L}(H^1(\mathbb{R}), L^2(\mathbb{R}))$.
- if $d=2$, $V \in L^p(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ for $p>2$, then $V \in \mathcal{L}(H^1(\mathbb{R}^2), L^2(\mathbb{R}^2))$.

We denote by $L^p(\mathbb{R}^d) + L^\infty_0(\mathbb{R}^d)$ the space of Lebesgue measurable functions f such that there exists $(f_n)_{n\in\mathbb{N}} \in L^p(\mathbb{R}^d)^{\mathbb{N}}$ satisfying $\lim_{n\to\infty}\|f-f_n\|_\infty = 0$.

Lemma D.4. *For $0<p<q$,*

$$L^{q,\infty}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) + L^\infty_0(\mathbb{R}^d)$$

Proof. For $\epsilon>0$, we decompose each $f \in L^{q,\infty}(\mathbb{R}^d)$ into a sum $f = f_\epsilon + f^\epsilon$ such that $f_\epsilon = f1_{|f|>\epsilon}$ and $f^\epsilon = f1_{|f|\leq\epsilon}$. We observe that for any $\epsilon>0$

$$\begin{aligned} \|f_\epsilon\|_p^p &= p\int_0^\infty t^{p-1} L_{\mathbb{R}^d}\{x : |f_\epsilon(x)| > t\}\,dt \\ &= p\int_\epsilon^\infty t^{p-1} L_{\mathbb{R}^d}\{x : |f(x)| > t\}\,dt + \epsilon^p L_{\mathbb{R}^d}\{x : |f(x)|>\epsilon\} \\ &\leq c\int_\epsilon^\infty \frac{t^{p-1}}{t^q}\,dt + \epsilon^p L_{\mathbb{R}^d}\{x : |f(x)|>\epsilon\} < \infty \end{aligned}$$

Moreover, when $\epsilon \to 0$

$$\|f^\epsilon\|_\infty = \|f1_{|f|\leq\epsilon}\|_\infty \leq \epsilon \to 0\,.$$

Therefore, each $f \in L^{q,\infty}(\mathbb{R}^d)$ belongs to the space $L^p(\mathbb{R}^d) + L^\infty_0(\mathbb{R}^d)$. □

Proposition D.5. *For any $V \in L^2(\mathbb{R}^d) + L^\infty_0(\mathbb{R}^d)$ such that $V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$ the operator $(1-\Delta)^{-1/2}V(1-\Delta)^{-1/2}$ is compact (i.e.: V is a relatively compact perturbation of $-\Delta$).*

Proof. Let $g(\xi) = (1+|\xi|^2)^{-1/2}$ and $g_m(\xi) = 1_{[0,m]}(|\xi|)g(\xi)$. The following norm convergence holds

$$\lim_{m\to\infty} g_m(D)\,V\,g_m(D) = g(D)\,V\,g(D)\,,$$

using the fact that $\lim_{m\to\infty}\|g_m(D)-g(D)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = 0$ and $\|V\,g(D)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} < \infty$.
By Lemma D.4, there exist $V_n \in L^2(\mathbb{R}^d)$ such that $\lim_{n\to\infty}\|V-V_n\|_\infty = 0$. We observe now that the Hilbert-Schmidt norm of $g_m(D)V_n(x)g_m(D)$ is

$$\|g_m(D)\,V_n\,g_m(D)\|_{\mathcal{L}^2(L^2(\mathbb{R}^d))} = \|V_n\|_2\,\|g_m\|_4^2 < \infty.$$

Therefore by norm convergence, the operator $g(D)\,V\,g(D)$ is compact. □

Corollary D.6. *The potential V satisfies the assumptions (A2)-(A3) in the following cases:*

- *if $d=1$ and $V \in L^2(\mathbb{R}) + L^\infty_0(\mathbb{R})$,*

- *if* $d = 2$ *and* $V \in L^p(\mathbb{R}^2) + L^\infty_0(\mathbb{R}^2)$ *with* $p > 2$,
- *if* $d \geq 3$ *and* $V \in L^{d,\infty}(\mathbb{R}^d)$.

Proof. Combine Proposition D.3, Lemma D.4 and Proposition D.5, with the observation in dimension $d = 2$

$$L^p(\mathbb{R}^2) + L^\infty_0(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) + L^\infty_0(\mathbb{R}^2) \quad \text{for } p > 2\,.$$

□

In particular, in dimension $d = 3$ the Coulomb potential $V(x) = \pm\frac{1}{|x|}$ satisfies the assumptions (A1), (A2) and (A3) because $\frac{1}{|x|} \in L^{3,\infty}(\mathbb{R}^3)$.

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