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PERTURBATIONS OF FLEXIBLE LATTÈS MAPS

XAVIER BUFF AND THOMAS GAUTHIER

Abstract. We prove that any Lattès map can be approximated by strictly postcritically finite rational maps which are not Lattès maps.

Introduction

A rational map of degree $D \geq 2$ is *strictly postcritically finite* if the orbit of each critical point intersects a repelling cycle. Among those, flexible Latts maps (the definition is given below) play a special role. The following result answers a question raised in [BE].

Theorem. *Every flexible Lattès map can be approximated by strictly postcritically finite rational maps which are not Lattès maps.*

Given $D \geq 2$, denote by Rat_D the space of rational maps of degree D. A rational map $f \in \mathrm{Rat}_D$ has a *Julia set* J_f which may be defined as the closure of the set of repelling cycles of f. The Julia set is the support of a measure μ_f which may be defined as the unique invariant measure of (maximal) entropy $\log D$.

The *bifurcation locus* in Rat_D is the closure of the set of discontinuity of the map $f \mapsto J_f$. Laura DeMarco [DeM] proved that the bifurcation locus is the support of a positive closed $(1,1)$-current $T_{\mathrm{bif}}{:=}dd^c\mathfrak{L}$, where $\mathfrak{L}$ is the plurisubharmonic function which sends a rational map f to its Lyapunov exponent with respect to μ_f.

Möbius transformations act by conjugacy on Rat_D and the quotient space is an orbifold known as the *moduli space* $\mathcal{M}_D$ of rational maps of degree D. Giovanni Bassanelli and François Berteloot [BB1] introduced a measure μ_{bif} on this moduli space, which may be obtained by pushing forward $T_{\mathrm{bif}}^{\wedge(2D-2)}$.

In [BE], the first author and Adam Epstein, using a transversality result in Rat_D, proved that the conjugacy class of a strictly postcritically finite map which is not a flexible Lattès map is in the support of μ_{bif}. Since the support of μ_{bif} is closed and since the conjugacy class of a strictly postcritically finite rational maps which is not a Lattès maps is in this support, our result has the following consequence.

Corollary 1. *The classes of flexible Lattès maps in $\mathcal{M}_D$ lie in the support of the bifurcation measure μ_{bif}.*

In [G1], the second author proved that the support of the bifurcation measure has maximal Hausdorff dimension, i.e. has dimension $2(2D-2)$, and that it is homogeneous (the support of μ_{bif} has maximal dimension in any neighborhood of its points). Corollary 1 thus yields the following result.

Corollary 2. *Let $f \in \mathrm{Rat}_D$ be a flexible Lattès map and let $V \subset \mathcal{M}_D$ be an open neighborhood of the conjugacy class of f. Then,* $\dim_H(\mathrm{supp}(\mu_{\mathrm{bif}}) \cap V) = 2(2D-2)$.

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Bassanelli and Berteloot [BB2] proved that every point in the support of $\mu_{\rm bif}$ can be approximated by rational maps having $2D-2$ distinct neutral cycles. Their argument can be adapted to prove that the support of $\mu_{\rm bif}$ can be approximated by hyperbolic maps having $2D-2$ distinct attracting cycles (see [B] Section 5.2). By Corollary 1, we have the following result.

Corollary 3. *Any flexible Lattès map $f \in \mathrm{Rat}_D$ can be approximated by hyperbolic rational maps having $2D-2$ distinct attracting cycles.*

The approach for solving this problem was suggested by John Milnor. We wish to express our gratitude. We also wish to thank the Banff International Research Station for hosting the workshop "Frontiers in Complex Dynamics (Celebrating John Milnor's 80th birthday)" during which we developed our proof.

1. Flexible Lattès maps

Following Milnor [M], we define a *flexible Lattès map* of degree $D \geq 2$ to be a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which there is a commutative diagram :

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "L"] \arrow[d, "\Theta"'] & \mathbb{C}/\Lambda \arrow[d, "\Theta"] \\ \widehat{\mathbb{C}} \arrow[r, "f"'] & \widehat{\mathbb{C}} \end{tikzcd}$$

where

- $\Lambda \subset \mathbb{C}$ is a lattice of rank 2;
- $\mathcal{T}{:=}\mathbb{C}/\Lambda$ is the quotient torus;
- $L : \mathcal{T} \ni \tau \mapsto a\tau + b \in \mathcal{T}$ with $a \in \mathbb{Z}$, $a^2 = D$, and $2b \in \Lambda/\big(2\Lambda + (a-1)\Lambda\big)$;
- $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$ is a 2-to-1 holomorphic map ramifying at points in $\Lambda/2$.

In the rest of the article, the lattice Λ will be of the form $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$, where γ is a complex number which is allowed to vary in the upper half-plane $\mathbb{H}$ of complex numbers with positive imaginary part. We shall use the following normalization for $\Theta_\gamma : \mathcal{T} \to \widehat{\mathbb{C}}$:

$$\Theta_\gamma(0) = 0, \quad \Theta_\gamma\left(\frac{\gamma+1}{2}\right) = \infty, \quad \Theta_\gamma\left(\frac{1}{2}\right) = v(\gamma) \quad \text{and} \quad \Theta_\gamma\left(\frac{\gamma}{2}\right) = w(\gamma)$$

where $v : \mathbb{H} \to \mathbb{C} - \{0\}$ and $w : \mathbb{H} \to \mathbb{C} - \{0\}$ are holomorphic functions (this may be achieved for example by requiring that $v \equiv 1$). Then, for all $\gamma \in \mathbb{H}$, $v(\gamma) \neq w(\gamma)$.

The derivative of the torus endomorphism $L_\gamma : \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ will be a fixed integer a which does not depend on γ. When $a = \pm 2$, the critical value set of the Lattès map $f_\gamma : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is $\big\{\infty, v(\gamma), w(\gamma)\big\}$ and if $|a| \geq 3$, the critical value set is $\big\{0, \infty, v(\gamma), w(\gamma)\big\}$. In all cases, the postcritical set of f_γ is $\big\{0, \infty, v(\gamma), w(\gamma)\big\}$. More precisely, we will have to study three cases :

- Case 1 : a is even and $L_\gamma(\tau) = a\tau$. In that case, all the critical values are mapped to 0 which is a fixed point of f.
- Case 2 : a is odd and $L_\gamma(\tau) = a\tau$. In that case, all the critical values are fixed with multiplier a^2.

- Case 3 : a is odd and $L_\gamma(\tau) = a\tau + (\gamma+1)/2$. In that case, the Lattès map permutes the critical value at 0 with that at infinity. It also permutes the critical value at $v(\gamma)$ with the critical value at $w(\gamma)$. The multiplier of each cycle is a^4.

From now on, we assume that we are in one of those three cases, and we consider the analytic family of Lattès maps

$$\mathbb{H} \ni \gamma \mapsto f_\gamma \in \mathrm{Rat}_D,$$

where Rat_D is the space of rational maps of degree D. We shall use the notation f, v, w, ... in place of f_γ, $v(\gamma)$, $w(\gamma)$, ... when γ is assumed to be fixed and there is no confusion.

2. Estimates for Θ

Lemma 1. *As $\tau \to 0$, we have the following expansion*

$$\Theta(1/2+\tau) = v + \lambda\tau^2 + o(\tau^2) \quad \textit{and} \quad \Theta(\gamma/2+\tau) = w + \mu\tau^2 + o(\tau^2)$$

with

$$\frac{\lambda}{v} = -\frac{\mu}{w} \neq 0.$$

Proof. Since Θ has simple critical points at $1/2$ and $\gamma/2$, we have an expansion as in the statement with $\lambda \neq 0$ and $\mu \neq 0$. Our work consists in proving the relation between λ/v and μ/w. Let q be the meromorphic quadratic differential on $\widehat{\mathbb{C}}$ defined by :

$$q := \frac{dz^2}{z(z-v)(z-w)}.$$

Since q has simple poles and since Θ is totally ramified above the polar set of q, the quadratic differential $\Theta^* q$ is holomorphic on $\mathcal{T}$, whence

$$\Theta^* q = \kappa \cdot d\tau^2 \quad \text{with} \quad \kappa \in \mathbb{C} - \{0\}.$$

Let τ tend to 0 and set $z = \Theta(1/2+\tau)$. Then, $z - v \sim \lambda\tau^2$ and $dz^2 \sim 4\lambda^2\tau^2 d\tau^2$. It follows that

$$\frac{dz^2}{z(z-v)(z-w)} \sim \frac{4\lambda^2\tau^2 d\tau^2}{v \cdot (v-w) \cdot \lambda\tau^2} = \frac{4\lambda d\tau^2}{v(v-w)}.$$

This show that

$$\kappa = \frac{4\lambda}{v(v-w)}.$$

Using the expansion of Θ at $\gamma/2$ yields similarly

$$\kappa = \frac{4\mu}{w(w-v)}.$$

The result now follows easily. $\square$

3. Hyperbolic sets

From now on, we let α, α', β and β' be rational numbers. We assume that the denominators of α and β are coprime with a and 2. We let σ and τ be the points in the torus $\mathcal{T}$ defined by

$$\sigma{:=}\alpha+\alpha'\gamma \quad \text{and} \quad \tau{:=}\beta+\beta'\gamma.$$

Remark that, as σ and τ have rational coordinates, they are (pre)periodic under multiplication by a and that σ, τ, $a\sigma$ and $a\tau$ are (pre)periodic under multiplication by a^2. For each integer $k\geq 1$, let τ_k and σ_k be the points in $\mathcal{T}$ defined by

$$\sigma_k{:=}\frac{1}{2}+\frac{\sigma}{a^k} \quad \text{and} \quad \tau_k{:=}\frac{\gamma}{2}+\frac{\tau}{a^k}$$

and let x_k and y_k be the points in $\widehat{\mathbb{C}}$ defined by

$$x_k{:=}\Theta(\sigma_k) \quad \text{and} \quad y_k{:=}\Theta(\tau_k).$$

We shall denote by X, Y, and Z the following subsets of $\widehat{\mathbb{C}}$:

$$X{:=}\bigcup_{k\geq 1}\bigcup_{n\geq 0} f^{\circ n}(x_k), \quad Y{:=}\bigcup_{k\geq 1}\bigcup_{n\geq 0} f^{\circ n}(y_k) \quad \text{and} \quad Z{:=}X\cup Y\cup\{0,\infty,v,w\}.$$

Note that Z is an invariant set.

Lemma 2. *There is a neighborhood $\mathbf{G}$ of f in Rat_D and a dynamical holomorphic motion $\phi:\mathbf{G}\times Z\to\widehat{\mathbb{C}}$ such that for all $g\in\mathbf{G}$ and all $z\in Z$, we have*

$$g\big(\phi(g,z)\big)=\phi\big(g,g(z)\big).$$

Proof. **Case 1 :** a is even. In that case, $f^{\circ k}$ maps x_k to $\Theta(\sigma)$ and y_k to $\Theta(\tau)$. Since σ and τ are (pre)periodic under the action of L, the points x_k and y_k are (pre)periodic.

Case 2 : a is odd and $b=0$. In that case, $f^{\circ k}$ maps x_k to $\Theta(\frac{1}{2}+\sigma)$ and y_k to $\Theta(\frac{\gamma}{2}+\tau)$. As $L(\frac{1}{2}+\sigma)=\frac{1}{2}+a\sigma$ and $L(\frac{\gamma}{2}+\tau)=\frac{\gamma}{2}+a\tau$ and as σ and τ are (pre)periodic under multiplication by a, the points $\frac{1}{2}+\sigma$ and $\frac{\gamma}{2}+\tau$ are (pre)periodic under the action of L, which means that the points x_k and y_k are (pre)periodic.

Case 3 : a is odd and $b=(\gamma+1)/2$. In that case, we use that

$$L^{\circ 2}\Big(\frac{1}{2}+z\Big)=\frac{1}{2}+a^2z \quad \text{and} \quad L^{\circ 2}\Big(\frac{\gamma}{2}+z\Big)=\frac{\gamma}{2}+a^2z$$

for any $z\in\mathcal{T}$. If k is even, $f^{\circ k}$ maps x_k to $\Theta(\frac{1}{2}+\sigma)$ and y_k to $\Theta(\frac{\gamma}{2}+\tau)$. Since σ and τ are (pre)periodic under multiplication by a^2, $\frac{1}{2}+\sigma$ and $\frac{\gamma}{2}+\tau$ are (pre)periodic under iteration of $L^{\circ 2}$ and the points x_k and y_k are (pre)periodic.

Now, if k is odd, $f^{\circ k}$ maps x_k to $\Theta(\frac{\gamma}{2}+\sigma)$ and y_k to $\Theta(\frac{1}{2}+\tau)$. Since $L(\frac{\gamma}{2}+\sigma)=\frac{1}{2}+a\sigma$ and $L(\frac{1}{2}+\tau)=\frac{\gamma}{2}+a\tau$ and since $a\sigma$ and $a\tau$ are (pre)periodic under multiplication by a^2, $\frac{1}{2}+a\sigma$ and $\frac{\gamma}{2}+a\tau$ are (pre)periodic under the action of $L^{\circ 2}$, which implies that the points x_k and y_k are (pre)periodic.

In all cases, the set Z is closed and each point in Z is (pre)periodic to a repelling cycle of f. Since the denominators of α and β are coprime with a and 2, the cycles capturing points in X and Y are disjoint from the postcritical set of f. In particular, Z does not contain critical points. As a consequence, Z is a hyperbolic

invariant compact set (see [ST] Theorem 1.2 page 266) and the result follows (see [S] (1.2) section 2 for a sketch of the proof, or [G2] section 2.1 for a proof). □

For all $g \in \mathbf{G}$, the map $\phi_g : z \mapsto \phi(g,z)$ is a homeomorphism from Z to $Z_g{:=}\phi_g(Z)$ which conjugates $f : Z \to Z$ to $g : Z_g \to Z_g$. Every point $z \in Z_g$ is (pre)periodic under iteration of g to a repelling cycle of g. For $k \geq 1$, we shall use the notation $\mathbf{x}_k(g){:=}\phi_g(x_k)$ and $\mathbf{y}_k(g){:=}\phi_g(y_k)$. As $k \to +\infty$, the sequence x_k tends to $x_\infty{:=}v$. We set $\mathbf{x}_\infty(g){:=}\phi_g(v)$. Similarly, as $k \to +\infty$, the sequence y_k tends to $y_\infty{:=}w$. We set $\mathbf{y}_\infty(g){:=}\phi_g(w)$.

4. A family of perturbations

Let us now restrict to the family of rational maps $t \mapsto f_t$ which is defined for $t \in \mathbb{C}$ by

$$f_t{:=}(1+t)\cdot f.$$

If $t \neq -1$, the rational map f_t has degree D and $f_0 = f$. The critical values of the rational map f_t are contained in the set $\{0, \infty, (1+t)\cdot v, (1+t)\cdot w\}$ and for t sufficiently close to 0, the point 0 is either a repelling fixed point or a repelling periodic point of period 2, and ∞ is either mapped to a repelling fixed point at 0, or a repelling fixed point, or a repelling periodic point of period 2. We will now study the relative motion of the critical values $\mathbf{v}(f_t){:=}(1+t)\cdot v$ and $\mathbf{w}(f_t){:=}(1+t)\cdot w$ with respect to the points $\mathbf{x}_\infty(f_t)$ and $\mathbf{y}_\infty(f_t)$. We will use the notations

$$\dot{\mathbf{v}}{:=}\frac{d\mathbf{v}(f_t)}{dt}\Big|_{t=0} \in T_v\widehat{\mathbb{C}}, \quad \dot{\mathbf{w}}{:=}\frac{d\mathbf{w}(f_t)}{dt}\Big|_{t=0} \in T_w\widehat{\mathbb{C}},$$

$$\dot{\mathbf{x}}_\infty{:=}\frac{d\mathbf{x}_\infty(f_t)}{dt}\Big|_{t=0} \in T_v\widehat{\mathbb{C}}, \quad \text{and} \quad \dot{\mathbf{y}}_\infty{:=}\frac{d\mathbf{y}_\infty(f_t)}{dt}\Big|_{t=0} \in T_w\widehat{\mathbb{C}}.$$

We shall also use the notation

$$\dot{f}{:=}\frac{df_t}{dt}\Big|_{t=0}$$

which is a section of the pullback bundle $f^\star T\widehat{\mathbb{C}}$: for all $z \in \widehat{\mathbb{C}}$, $\dot{f}(z) \in T_{f(z)}\widehat{\mathbb{C}}$.

Lemma 3. *We have*

$$\frac{\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}}{v} = \frac{\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}}}{w} \neq 0.$$

Proof. By construction,

$$\dot{\mathbf{v}} = v\frac{d}{dz} \quad \text{and} \quad \dot{\mathbf{w}} = w\frac{d}{dz}.$$

In addition, for all $z \in \widehat{\mathbb{C}}$,

$$\dot{f}(z) = f(z)\frac{d}{dz}.$$

Case 1 : a is even. In that case, f maps $x_\infty = v$ to 0 which is a fixed point. As t varies, f_t still maps v to 0 and still fixes 0. Thus, $\mathbf{x}_\infty \equiv v$ and $\dot{\mathbf{x}}_\infty = 0$, whence

$$\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}} = -\dot{\mathbf{v}}.$$

Similarly

$$\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}} = -\dot{\mathbf{w}}.$$

Case 2 : a is odd and $b=0$. In that case, f fixes $x_\infty = v$. So, $\mathbf{x}_\infty(f_t)$ is the fixed point of f_t which is close to v. Derivating $f_t\big(\mathbf{x}_\infty(f_t)\big) = \mathbf{x}_\infty(f_t)$ with respect to t, and evaluating at $t=0$, we get :

$$\dot f(v) + D_v f(\dot{\mathbf{x}}_\infty) = \dot{\mathbf{x}}_\infty.$$

Since $\dot f(v) = v\frac{d}{dz} = \dot{\mathbf{v}}$ and $D_v f$ is multiplication by a^2, we get

$$\dot{\mathbf{x}}_\infty = \frac{\dot{\mathbf{v}}}{1-a^2}$$

whence

$$\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}} = \left(\frac{1}{1-a^2} - 1\right)\dot{\mathbf{v}} = \frac{a^2}{1-a^2}\dot{\mathbf{v}}.$$

Similarly

$$\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}} = \frac{a^2}{1-a^2}\dot{\mathbf{w}}.$$

Case 3 : a is odd and $b=(\gamma+1)/2$. In that case f permutes $x_\infty = v$ and $y_\infty = w$. Derivating $f_t\big(\mathbf{x}_\infty(f_t)\big) = \mathbf{y}_\infty(f_t)$ and $f_t\big(\mathbf{y}_\infty(f_t)\big) = \mathbf{x}_\infty(f_t)$ with respect to t, and evaluating at $t=0$, we get :

$$\dot f(v) + D_v f(\dot{\mathbf{x}}_\infty) = \dot{\mathbf{y}}_\infty \quad \text{and} \quad \dot f(w) + D_w f(\dot{\mathbf{y}}_\infty) = \dot{\mathbf{x}}_\infty. \tag{1}$$

On the one hand $\dot f(v) = w\frac{d}{dz} = \dot{\mathbf{w}}$ and $\dot f(w) = v\frac{d}{dz} = \dot{\mathbf{v}}$. On the other hand,

$$L\left(\frac{1}{2}+\tau\right) = \frac{\gamma}{2} + a\tau \quad \text{and} \quad L\left(\frac{\gamma}{2}+\tau\right) = \frac{1}{2} + a\tau.$$

Thus,

$$f\big(v+\lambda\tau^2+o(\tau^2)\big) = w+\mu a^2\tau^2+o(\tau^2) \quad \text{and} \quad f\big(w+\mu\tau^2+o(\tau^2)\big) = v+\lambda a^2\tau^2+o(\tau^2)$$

which implies

$$\lambda f'(v) = \mu a^2 \quad \text{and} \quad \mu f'(w) = \lambda a^2.$$

By Lemma 1, it follows that

$$D_v f(\dot{\mathbf{x}}_\infty) = \frac{\mu a^2}{\lambda}\dot{\mathbf{x}}_\infty = -\frac{a^2 w}{v}\dot{\mathbf{x}}_\infty.$$

Similarly,

$$D_w f(\dot{\mathbf{y}}_\infty) = -\frac{a^2 v}{w}\dot{\mathbf{x}}_\infty.$$

So, equation (1) yields

$$\dot{\mathbf{w}} - \frac{a^2 w}{v}\dot{\mathbf{x}}_\infty = \dot{\mathbf{y}}_\infty \quad \text{and} \quad \dot{\mathbf{v}} - \frac{a^2 v}{w}\dot{\mathbf{y}}_\infty = \dot{\mathbf{x}}_\infty$$

which boils down to

$$\dot{\mathbf{x}}_\infty = \frac{1-a^2}{1-a^4}\dot{\mathbf{v}} \quad \text{and} \quad \dot{\mathbf{y}}_\infty = \frac{1-a^2}{1-a^4}\dot{\mathbf{w}}.$$

As a consequence

$$\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}} = \left(\frac{1-a^2}{1-a^4} - 1\right)\dot{\mathbf{v}} = -\frac{a^2}{1+a^2}\dot{\mathbf{v}}.$$

Similarly

$$\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}} = -\frac{a^2}{1+a^2}\dot{\mathbf{w}}.$$

In all cases, there is a constant $c \neq 0$ ($c = -1$ in the first case, $c = a^2/(1-a^2)$ in the second case and $c = -a^2/(1+a^2)$ in the third case) such that

$$\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}} = c\dot{\mathbf{v}} = cv\frac{d}{dz} \quad \text{and} \quad \dot{\mathbf{y}}_\infty - \dot{\mathbf{w}} = c\dot{\mathbf{w}} = cw\frac{d}{dz}.$$

This shows that

$$\frac{\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}}{v} = \frac{\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}}}{w} = c\frac{d}{dz} \neq 0. \qquad \square$$

5. Proof of the Theorem

Let $\gamma_0 := x_0 + iy_0 \in \mathbb{H}$ be a point with rational real part x_0 and rational imaginary part y_0 which are coprime with a and 2. Set $\alpha := -x_0$, $\alpha' := 1$, $\beta := y_0$ and $\beta' := 0$. We will now allow γ to vary in $\mathbb{H}$. So, σ, τ, v, w, λ, μ, σ_k, τ_k, x_k, y_k, x_∞, y_∞ are functions of γ. For example,

$$\sigma : \mathbb{H} \ni \gamma \mapsto \alpha + \alpha'\gamma \in \mathbb{H} \quad \text{and} \quad \tau : \mathbb{H} \ni \gamma \mapsto \beta + \beta'\gamma \in \mathbb{H}.$$

Note that the Möbius transformation σ/τ sends γ_0 to i.

We still consider the family of flexible Lattès maps $\mathbb{H} \ni \gamma \mapsto f_\gamma \in \mathrm{Rat}_D$ and we introduce the family of perturbed maps

$$\mathbb{H} \times (\mathbb{C} - \{-1\}) \ni (\gamma, t) \longmapsto f_{\gamma,t} := (1+t) \cdot f_\gamma \in \mathrm{Rat}_D.$$

As in the previous section, we will consider the critical value functions

$$\mathbf{v} : f_{\gamma,t} \mapsto (1+t) \cdot v(\gamma) \quad \text{and} \quad \mathbf{w} : f_{\gamma,t} \mapsto (1+t) \cdot w(\gamma).$$

The notation $\dot{\mathbf{v}}$, $\dot{\mathbf{w}}$, $\dot{\mathbf{x}}_\infty$, $\dot{\mathbf{y}}_\infty$ refers to the derivative with respect to t evaluated at $t = 0$. In particular, those are functions of $\gamma \in \mathbb{H}$.

Lemma 4. *There are a neighborhood Γ of γ_0 in $\mathbb{H}$, an integer k_0 and for $k \geq k_0$, holomorphic functions $s_k : \Gamma \to \mathbb{C}$ and $t_k : \Gamma \to \mathbb{C}$ such that :*

- *the sequences (s_k) and (t_k) converge uniformly on Γ to 0,*
- *$\mathbf{x}_k(f_{\gamma,s_k(\gamma)}) = \mathbf{v}(f_{\gamma,s_k(\gamma)})$, $\mathbf{y}_k(f_{\gamma,t_k(\gamma)}) = \mathbf{w}(f_{\gamma,t_k(\gamma)})$ and*
- *the sequence (s_k/t_k) converges uniformly on Γ to $-\sigma^2/\tau^2$.*

Proof. First, note that as $k \to +\infty$, the holomorphic functions $x_k : \mathbb{H} \to \widehat{\mathbb{C}}$ and $v : \mathbb{H} \to \widehat{\mathbb{C}}$ satisfy

$$x_k - v \sim \lambda \cdot \left(\sigma_k - \frac{1}{2}\right)^2 = \frac{\lambda\sigma^2}{a^{2k}}.$$

Second, let us consider the sequence of functions

$$\mathfrak{X}_k := (\gamma, u) \mapsto a^{2k} \cdot \big(\mathbf{x}_k(f_{\gamma,u/a^{2k}}) - \mathbf{v}(f_{\gamma,u/a^{2k}})\big).$$

Note that as $k \to +\infty$,

$$\begin{aligned}\mathfrak{X}_k(\gamma, u) &= a^{2k} \cdot \Big(x_k(\gamma) + \frac{u}{a^{2k}} \cdot \dot{\mathbf{x}}_\infty(\gamma) - v(\gamma) - \frac{u}{a^{2k}} \cdot \dot{\mathbf{v}}(\gamma) + o(1/a^{2k})\Big) \\ &= \lambda(\gamma)\sigma^2(\gamma) + u \cdot \big(\dot{\mathbf{x}}_\infty(\gamma) - \dot{\mathbf{v}}(\gamma)\big) + o(1).\end{aligned}$$

So, as $k \to +\infty$, the sequence $(\mathfrak{X}_k)$ converges uniformly on every compact subset of $\mathbb{H} \times \mathbb{C}$ to the function

$$\mathfrak{X}_\infty : (\gamma, u) \mapsto \lambda\sigma^2 + u \cdot (\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}).$$

According to Lemma 3, $\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}$ does not vanish on $\mathbb{H}$. Thus, the function $\mathfrak{X}_\infty$ vanishes along the graph of a holomorphic function $u_\infty : \mathbb{H} \to \mathbb{C}$ which satisfies

$$u_\infty \cdot (\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}) = -\lambda\sigma^2.$$

Since $(\mathfrak{X}_k)$ converges locally uniformly to $\mathfrak{X}_\infty$, given any neighborhood Γ of γ_0 compactly contained in $\mathbb{H}$, there are, for k large enough, holomorphic functions $u_k : \Gamma \to \mathbb{C}$ which satisfy

$$(\forall \gamma \in \Gamma) \quad \mathfrak{X}_k\big(\gamma, u_k(\gamma)\big) = 0,$$

the sequence u_k converging uniformly on Γ to u_∞. Now, set

$$s_k := \frac{u_k}{a^{2k}} : \Gamma \to \mathbb{C}.$$

Then the sequence (s_k) obviously converges uniformly to 0 on Γ and

$$\mathbf{x}_k(f_{\gamma,s_k(\gamma)}) = \mathbf{v}(f_{\gamma,s_k(\gamma)}) \quad \text{and} \quad a^{2k} \cdot s_k \cdot (\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}) \underset{k\to+\infty}{\longrightarrow} -\lambda\sigma^2.$$

Similarly, there are, for k large enough, holomorphic functions $t_k : \Gamma \to \mathbb{C}$ which converge uniformly on Γ to 0 and satisfy

$$\mathbf{y}_k(f_{\gamma,t_k(\gamma)}) = \mathbf{w}(f_{\gamma,t_k(\gamma)}) \quad \text{and} \quad a^{2k} \cdot t_k \cdot (\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}}) \underset{k\to+\infty}{\longrightarrow} -\mu\tau^2.$$

Those uniform convergences on Γ may be reformulated as

$$a^{2k} \cdot \frac{s_k}{\sigma^2} \underset{k\to+\infty}{\longrightarrow} -\frac{\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}}{\lambda} \quad \text{and} \quad a^{2k} \cdot \frac{t_k}{\tau^2} \underset{k\to+\infty}{\longrightarrow} -\frac{\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}}}{\mu}.$$

According to Lemma 1 and Lemma 3,

$$\frac{\dot{\mathbf{x}}_\infty - \dot{\mathbf{v}}}{\lambda} = -\frac{\dot{\mathbf{y}}_\infty - \dot{\mathbf{w}}}{\mu}$$

are non-vanishing functions. This shows that the sequence (s_k/t_k) converges uniformly on Γ to $-\sigma^2/\tau^2$. □

The previous construction yields, for k large enough, functions s_k and t_k which are defined and holomorphic near γ_0. As $k \to +\infty$, the sequence of functions $(1 - s_k/t_k)$ converges to $1 + \sigma^2/\tau^2$ which has a simple zero at γ_0. Therefore, for k large enough, the function s_k/t_k takes the value 1 at a point γ_k close to γ_0 :

$$s_k(\gamma_k) = t_k(\gamma_k) \quad \text{with} \quad \gamma_k \underset{k\to+\infty}{\longrightarrow} \gamma_0.$$

Let $g_k \in \mathrm{Rat}_D$ be the rational map

$$g_k := (1 + r_k) \cdot f_{\gamma_k} \quad \text{with} \quad r_k := s_k(\gamma_k) = t_k(\gamma_k).$$

Note that r_k tends to 0 and that g_k converges to f_{γ_0} as k tends to $+\infty$. The critical values of g_k are contained in the set $\big\{0, \infty, (1+r_k)v, (1+r_k)w\big\}$ (if $a = 2$, then 0 is not a critical value). By construction, 0 and ∞ are (pre)periodic to repelling cycles. In addition, $(1 + r_k)v$ coincides with $x_k(r_k)$ which is (pre)periodic to a repelling cycle and $(1 + r_k)w$ coincides with $y_k(r_k)$ which is also (pre)periodic to a repelling cycle. Thus, g_k is strictly postcritically finite but is not a Lattès map.

We have just proved that f_{γ_0} can be approximated by rational maps g_k which are strictly postcritically finite but which are not Lattès maps. Since the set of allowable γ_0 is dense in the upper-half plane, this shows that any flexible Lattès map can be approximated by rational maps which are strictly postcritically finite but which are not Lattès maps.

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