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Traffic grooming in bidirectional WDM ring networks*

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Abstract

We study the minimization of ADMs (Add-Drop Multiplexers) in optical WDM bidirectional rings considering symmetric shortest path routing and all-to-all unitary requests. We precisely formulate the problem in terms of graph decompositions, and state a general lower bound for all the values of the grooming factor C and N , the size of the ring. We first study exhaustively the cases $C = 1$, $C = 2$, and $C = 3$, providing improved lower bounds, optimal constructions for several infinite families, as well as asymptotically optimal constructions and approximations. We then study the case $C > 3$, focusing specifically on the case $C = k(k + 1)/2$ for some $k \geq 1$. We give optimal decompositions for several congruence classes of N using the existence of some combinatorial designs. We conclude with a comparison of the cost functions in unidirectional and bidirectional WDM rings.

Keywords: Traffic grooming, SONET ADM, optical WDM network, graph decomposition, combinatorial designs.

1 Introduction

1.1 Background and motivation

Optical wavelength division multiplexing (WDM) is today the most promising technology to accommodate the explosive growth of Internet and telecommunication traffic in wide-area, metro-area, and backbone networks. Using WDM, the potential bandwidth of approximately 50 THz of a fiber can be divided into multiple non-overlapping wavelength or frequency channels. Since currently the commercially available optical fibers can support over a hundred frequency channels, such a channel has over one gigabit-per-second transmission speed. However, the network is usually required to support traffic

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connections at rates that are lower than the full wavelength capacity. In order to save equipment cost and improve network performance, it turns out to be very important to aggregate the multiple low-speed traffic connections, namely *requests*, into higher speed streams. Traffic grooming is the term used to carry out this aggregation, while optimizing the equipment cost.

Among possible criteria to minimize the equipment cost, one is to minimize the number of wavelengths used to route all the requests [2, 20]. A better approximation of the true equipment cost is to minimize the number of add/drop locations, namely ADMs using SONET terminology, instead of the number of wavelengths. This leads to the *grooming problem*, that we state formally later in Section 2. These two problems are proved to be different. Indeed, it is known that even for a simple network like the unidirectional ring, the number of wavelengths and the number of ADMs cannot be simultaneously minimized [12, 23].

The SONET ring is the most widely used optical network infrastructure today. In these networks, a communication between a pair of nodes is done via a *lightpath*, and each lightpath uses an Add-Drop Multiplexer (*ADM*), i.e., an electronic termination, at each of its two endpoints (but none in the intermediate nodes). If each request uses $\frac{1}{C}$ of the capacity of a wavelength, then C is said to be the *grooming factor*, i.e., C requests can be aggregated in the same wavelength through the same link. If two or more lightpaths using the same wavelength share a common endpoint, then the same ADM might be used for all lightpaths and therefore the number of ADMs needed could be reduced. Due to this fact, it makes sense to try to minimize the total number of ADMs required.

1.2 Previous work and our contribution

The notion of traffic grooming was introduced in [25] for the ring topology. Since then, traffic grooming has been widely studied in the literature (cf. [22, 29, 35] for some surveys). The problem has been proved to be NP-complete for ring networks and general C [12]. Hardness results for rings and paths have been proved in [1]. Many heuristics have been proposed, but exact solutions have been found only for certain values of C and for the uniform all-to-all traffic case in unidirectional ring and path topologies [8].

Many versions of the problem can be considered, according for example to the routing, the physical graph, and the request graph, among others. For example, in [3, 6] the PATH TRAFFIC GROOMING problem is studied. If the network topology is a *ring* (which is the case of SONET rings), we mainly distinguish two cases depending on the routing. The UNIDIRECTIONAL RING TRAFFIC GROOMING problem has been studied extensively in the literature. In an unidirectional ring, requests are routed following only one direction in the cycle. To date, the all-to-all case has been completely solved for values of the grooming factor up to 8 [4, 5, 8, 16, 17]. Also, recently the unidirectional ring with bounded degree request graph has been studied [28, 30].

In the BIDIRECTIONAL RING TRAFFIC GROOMING problem, the scenario is quite different. In a bidirectional ring, requests are routed either clockwise or counterclockwise. This case has been much less studied than the unidirectional one, due to its higher complexity. There is important work providing

62 heuristics for the ring traffic grooming [11, 12, 20, 21, 23, 24, 27, 31], but there is still an important lack
 63 of theoretical analysis of the problem. Nevertheless, its study has attracted the interest of numerous
 64 researchers. For instance, in [26] a MILP formulation of the problem can be found. In [33] two lower
 65 bounds are provided for the number of ADMs in a bidirectional ring with traffic grooming, and in [14]
 66 another lower bound is proved, regardless of the routing. In [18, 19, 32, 33] tools from design theory are
 67 applied to the bidirectional ring. Their method is based in the idea of *primitive rings*, which consists
 68 roughly in appropriately generating subgraphs of the request graph inducing unitary load each, and then
 69 packing them into sets of at most C subgraphs. Namely, in [33] several heuristics are proposed, the cases
 70 $C = 2$ and $C = 4$ are studied in [32] (as well as other solutions that do not proceed via primitive rings),
 71 the case $C = 8$ in [19], and the cases $C = 4$ and $C = 8$ in [18]. Nevertheless, they do not provide general
 72 lower bounds and they do not analyze the approximation ratio of the proposed algorithms. Therefore,
 73 the gaps between their solutions and the optimal ones are unknown.

74 In this work we focus on a bidirectional ring with symmetric shortest path routing, and on the all-to-
 75 all case. We begin by formally stating the problem in terms of graph partitioning in Section 2. In Section
 76 3 we provide lower bounds and compare them with those existing in the literature. The remainder of the
 77 article is devoted to finding families of solutions for certain values of C and N . First we solve in Section 4
 78 the case $C = 1$. In Section 5 we study the case $C = 2$, improving the general lower bound and providing
 79 a $\frac{34}{33}$ -approximation. In Section 6 we tackle the case $C = 3$, improving the lower bound when $N \equiv 3$
 80 (mod 4) and giving optimal solutions when $N \equiv 0, 1, 4, 5$ (mod 12). For all other values of N we give
 81 asymptotically optimal solutions. In Section 7 we use design theory to provide optimal solutions when
 82 C is of the form $k(k + 1)/2$, for some congruence classes of values of N . We also give improved lower
 83 bounds when C is not of the form $k(k + 1)/2$. In Section 8 we compare unidirectional and bidirectional
 84 rings in terms of minimizing the cost. We conclude the article in Section 9.

85 **2 Statement of the Problem**

86 **2.1 Load constraint**

87 In a graph-theoretical approach, we are given an optical network represented by a directed graph G on
 88 N vertices (in many cases a symmetric one) – called the *physical graph* – for example a unidirectional
 89 ring \vec{C}_N or a bidirectional symmetric ring C_N^* . We are given also a traffic (or instance) matrix, that is
 90 a family of connection requests represented by an arc-weighted multidigraph I – called the *logical* or
 91 *request graph* – where the number of arcs from i to j corresponds to the number of requests from i to
 92 j , and the weight of each arc corresponds to the amount of bandwidth used by each request. Here we
 93 suppose that there is exactly one request from i to j (all-to-all case) and that each request uses the same
 94 bandwidth. In that case $I = K_N^*$. We also suppose that the bandwidth used by any request is a fraction
 95 $1/C$ of the available bandwidth of a wavelength. Said otherwise, each wavelength ω can carry on a given
 96 arc at most C requests. This positive integer C is called the *grooming factor*. For a wavelength ω , we

107 denote by B_ω the set of requests carried by ω . Satisfying a request r from i to j consists in finding a
 108 dipath $P(r)$ in G and assigning it a wavelength ω . Note that a wavelength ω is directed either clockwise
 109 or counterclockwise, so all the dipaths associated with requests in the same B_ω are directed in the same
 110 way.

For a subgraph B_ω of requests of I , we define the *load* of an arc e of G , $L(B_\omega, e)$, as the number of requests which are routed through e , that is

$$L(B_\omega, e) := |\{P(r) : r \in E(B_\omega), e \in P(r)\}|.$$

111 Note that if B_ω is associated with a clockwise (resp. counterclockwise) wavelength ω , only the
 112 clockwise (resp. counterclockwise) arcs of the ring are loaded by B_ω . The constraint given by the
 113 grooming factor C means that for each subgraph B_ω and each arc e , $L(B_\omega, e)$ is at most C . In this article
 114 we focus on the bidirectional ring topology with all-to-all unitary requests. Therefore, our problem
 115 consists of finding a partition of K_N^* into subdigraphs B_ω satisfying the load constraint for C_N^* and such
 116 that the total number of vertices is minimized. We have two choices for routing a request (i, j) : either
 117 clockwise or counterclockwise. Although there is no physical constraint imposing it, it is common for
 118 the operators to consider symmetric routings. That is, if the request (i, j) is routed clockwise, then the
 119 request (j, i) is routed counterclockwise. Furthermore it is also common for the sake of simplicity to use
 120 shortest path routing. Therefore we will restrict ourselves to symmetric shortest path routings. Let us see
 121 how the restrictions on the routing affect the solutions.

112 2.2 Constraints on the routing

113 In a ring C_N^* with an odd number of vertices, shortest path routing implies symmetric routing. But in a
 114 ring with an even number of vertices this is not necessarily the case, as a request of the form $(i, i + \frac{N}{2})$
 115 can be routed via a shortest path in both directions. Consider for example $N = 4$ and $C = 2$. If we do
 116 not impose symmetric routing, we can have a solution consisting of the two subdigraphs B_{ω_1} with the
 117 requests $(0, 1), (1, 2), (2, 3), (3, 0), (0, 2),$ and $(2, 0)$ routed clockwise, and B_{ω_2} with the requests $(1, 0),$
 118 $(0, 3), (3, 2), (2, 1), (1, 3),$ and $(3, 1)$ routed counterclockwise. Altogether we use 8 ADMs. Suppose now
 119 that we further impose symmetric routing, and assume without loss of generality that the requests $(0, 2)$
 120 and $(1, 3)$ are routed clockwise. The best we can do for a B_ω with 4 vertices is to put 5 requests if ω is
 121 clockwise, namely $(0, 1), (1, 2), (2, 3), (3, 0),$ and at most one of $(0, 2)$ and $(1, 3)$. The other request out
 122 of $(0, 2)$ and $(1, 3)$ will need 2 ADMs, so we use a total of 12 ADMs. If we do not use any B_ω with 4
 123 vertices, note that a subdigraph with 3 (resp. 2) vertices contains at most 3 requests (resp. 1 request).
 124 Therefore to route all the requests we need at least 12 ADMs.

125 Imposing shortest path routing might increase the number of ADMs of an optimal solution. Consider
 126 for example $N = 3$ and $C = 3$. With shortest path routing, we need two subdigraphs B_{ω_1} with the requests
 127 $(0, 1), (1, 2), (2, 0)$ and B_{ω_2} with the requests $(1, 0), (2, 1), (0, 2),$ for a total of 6 ADMs (each arc of C_3^* is
 128 loaded once). Without the constraint of shortest path routing, we can do it with 3 ADMs, namely with

129 all the requests routed clockwise. In that case, the requests $(1, 0)$, $(2, 1)$, and $(0, 2)$ are routed via dipaths
 130 of length 2 (for instance, the request $(1, 0)$ uses the arcs $(1, 2)$ and $(2, 0)$). In that case the load of the arcs
 131 (in the clockwise direction) is 3.

132 We cannot always use shortest path routing and have a minimum load. Indeed, consider the case
 133 $C = 1$ and a set of 3 requests (i, j) , (j, k) , and (k, i) forming a triangle. The subdigraph formed by the
 134 3 requests routed in the same direction has load 1, but there is no reason that the associated routes are
 135 shortest paths. For example, let $N = 5$ and $(0, 1)$, $(1, 2)$, $(2, 0)$ be the three mentioned requests, which we
 136 assume to be routed clockwise. If we want a valid solution, then the request $(2, 0)$ is routed via the path
 137 $[2, 3, 4, 0]$ of length 3 (and not 2). If we want to use shortest paths, then these three requests induce load
 138 2, hence they cannot fit together in the same wavelength. Summarizing, in this example either we use
 139 shortest paths and the load is 2 or we get a solution with load one but not using shortest paths.

140 2.3 Symmetric shortest path routing

141 In the sequel we will only consider **symmetric shortest path routings**. Besides being a common sce-
 142 nario in telecommunication networks, this assumption also simplifies the problem, as we can split it into
 143 two separate problems, half of the requests being routed clockwise and half counterclockwise. Each of
 144 these two subproblems can be viewed as a grooming problem where $G = \vec{C}_N$ (the unidirectional cycle)
 145 and $I = T_N$, where T_N is a tournament on N vertices, that is, a complete oriented graph (for each pair of
 146 vertices $\{i, j\}$ there is exactly one of the arcs (i, j) or (j, i)).

147 As we consider shortest path routing, for N odd T_N is unique. But for N even we have two possibili-
 148 ties for the pairs of the form $\{i, i + \frac{N}{2}\}$: either the arc $(i, i + \frac{N}{2})$ or $(i + \frac{N}{2}, i)$. So the choice of these arcs has
 149 to be made. We are now ready to state precisely our problem.

TRAFFIC GROOMING IN BIDIRECTIONAL WDM RING NETWORKS

Input: A unidirectional cycle \vec{C}_N with vertices $0, \dots, N-1$, a grooming factor C and a digraph
 of requests consisting of the tournament T_N with arcs $(i, i+1)$ for $0 \leq i \leq N-1$ and $1 \leq q \leq \frac{N-1}{2}$,
 plus if N is even $\frac{N}{2}$ arcs of the form $(i, i + \frac{N}{2})$, where we cannot have both $(i, i + \frac{N}{2})$ and $(i + \frac{N}{2}, i)$
 150 (or said otherwise, for N even we have one of the two arcs $(i, i + \frac{N}{2})$ or $(i + \frac{N}{2}, i)$ for $0 \leq i \leq \frac{N}{2} - 1$).

Output: A partition of T_N into digraphs B_ω , $1 \leq \omega \leq W$, such that for each arc $e \in E(\vec{C}_N)$,
 $L(B_\omega, e) \leq C$.

Objective: Minimize $\sum_{\omega=1}^W |V(B_\omega)|$. The minimum will be denoted $A(C, N)$.

151 Note that for N even we do not specify a particular orientation of the arcs of the form $(i, i + \frac{N}{2})$.

152 **Remark 2.1** *Solutions to the original problem can be found by solving the above problem and using*
 153 *the solution for the counterclockwise requests by reversing the orientation of the arcs of \vec{C}_N and T_N .*
 154 *Therefore, the total number of ADMs for the original problem – under the constraints of symmetric*
 155 *shortest path routing – is $2A(C, N)$.*

156 Let us see an example for $N = 5$ and $C = 1$. Then the following three subdigraphs form a solution
 157 with 10 ADMs: one with arcs $(0, 1), (1, 3), (3, 0)$, another with arcs $(1, 2), (2, 4), (4, 1)$, and another with
 158 arcs $(0, 2), (2, 3), (3, 4), (4, 0)$. Thus, a solution for the bidirectional ring C_5^* and $I = K_5^*$ needs 20 ADMs.

159 Let now $N = 5$ and $C = 2$. We can use the preceding solution or another one with also 10 ADMs
 160 with only two \vec{C}_5 's with arcs $(0, 2), (1, 2), (2, 3), (3, 4), (4, 5)$ and $(0, 2), (2, 4), (4, 1), (1, 3), (3, 0)$, the sec-
 161 ond one inducing load 2. But we can do better, with only 8 ADMs, with one subdigraph with arcs
 162 $(1, 3), (3, 4), (4, 1)$, and another one with arcs $(0, 1), (1, 2), (0, 2), (2, 3), (2, 4), (3, 0), (4, 0)$. This latter par-
 163 tition is optimal. In that case, we need 16 ADMs for the bidirectional ring.

164 To tackle our problem we will use tools from design theory, similar to those used for the unidirec-
 165 tional ring and $I = K_N$ [7, 8]. In particular, it is helpful to use, for a given C , digraphs having a maximum
 166 ratio of the number of arcs to the number of vertices (see Section 3.2).

167 2.4 Admissible digraphs

168 Let $B_\omega = (V_\omega, E_\omega)$ be a digraph with $V_\omega = \{a_0, \dots, a_{p-1}\}$ involved in a partition of the tournament T_N .
 169 Note that the edges of B_ω belong to T_N , so $(a_i, a_j) \in E_\omega$ if and only if $d_{\vec{C}_N}(a_i, a_j) \leq \frac{N}{2}$, where $d_{\vec{C}_N}(a_i, a_j)$ is
 170 the distance between a_i and a_j in \vec{C}_N .

171 A digraph B_ω is said to be *admissible* if it satisfies the load constraint, that is, $L(B_\omega, e) \leq C$ for each
 172 arc $e \in E(\vec{C}_N)$. A partition of T_N into admissible subdigraphs is called *valid*. As the paths associated
 173 with an arc of B_ω form a dipath (an interval) in \vec{C}_N , the load is exactly the same as if we consider B_ω
 174 embedded in a cycle \vec{C}_p with vertex set $0, 1, \dots, p-1$. More precisely, we associate with B_ω the digraph
 175 B_ω^p having vertices $0, 1, \dots, p-1$ and with $(i, j) \in E(B_\omega^p)$ if and only if $(a_i, a_j) \in E(B_\omega)$. Hence, to
 176 compute the load we will consider digraphs with p vertices and their load in the associated \vec{C}_p . Note that
 177 it can happen that $d_{\vec{C}_N}(a_i, a_j) \leq \frac{N}{2}$ but $d_{\vec{C}_p}(i, j) > \frac{p}{2}$, and vice versa.

178 Figure 1(a) illustrates a digraph B_ω that is admissible for $N = 8$ and $C = 2$, as it induces load 2 in
 179 \vec{C}_8 . Its associated digraph B_ω^4 is shown in Figure 1(b). Figure 1(c) shows a digraph B'_ω which has also
 180 B_ω as associated digraph, but it is not admissible as (a_3, a_0) is not an arc of T_8 .

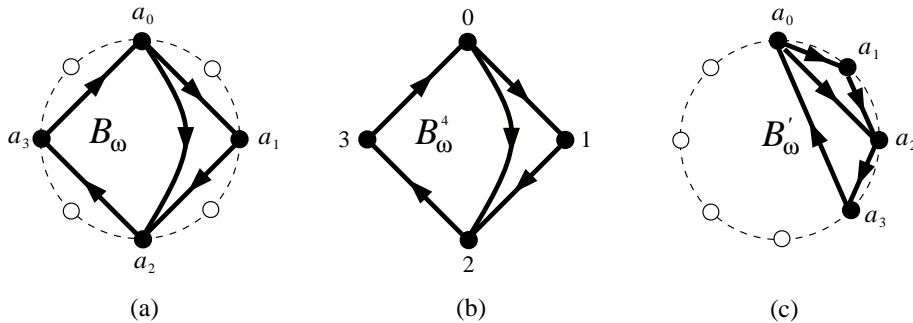


Figure 1: (a) Digraph B_ω admissible for $N = 8$ and $C = 2$; (b) Its associated digraph B_ω^4 ; (c) Non-admissible digraph B'_ω that has also B_ω^4 as associated digraph.

181 Figure 2(a) shows an admissible digraph for $N = 7$ and $C = 2$. Its associated digraph B_ω^5 , which is
 182 depicted in Figure 2(b), induces load 2 but the arc $(1, 4)$ is not routed via a shortest path (although the
 arc (a_1, a_4) was in B_ω).

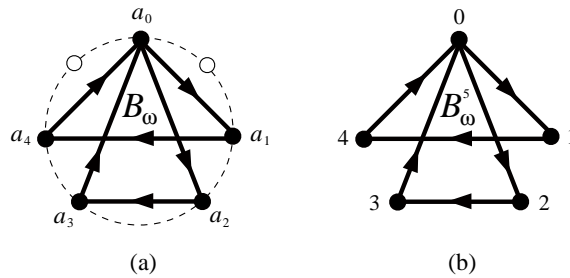


Figure 2: (a) Digraph B_ω admissible for $N = 7$ and $C = 2$; (b) Its associated digraph B_ω^5 .

183

184 In what follows we will compute the load in the associated digraph, but we will have to be careful
 185 that the arcs of B_ω are those of T_N , as pointed out by the above examples.

186 3 Lower Bounds

187 In this section we state general lower bounds on the number of ADMs used by any solution.

188 3.1 Equations of the problem

189 Given a valid solution of the problem, let a_p denote the number of subgraphs of the partition with exactly
 190 p nodes, let A denote the total number of ADMs, let W denote the number of subgraphs of the partition,
 191 and let E_ω be the set of arcs of B_ω . Recall that here $I = T_N$, which has $\frac{N(N-1)}{2}$ arcs. The following
 192 equalities hold:

$$A = \sum_{p=2}^N p a_p \quad (1)$$

$$\sum_{p=2}^N a_p = W \quad (2)$$

$$\sum_{w=1}^W |E_\omega| = \frac{N(N-1)}{2} \quad (3)$$

Proposition 3.1 For $I = T_N$,

$$W \geq \left\lceil \frac{N^2 + \alpha}{8C} \right\rceil, \text{ where } \alpha = \begin{cases} -1, & \text{if } N \text{ is odd} \\ 4, & \text{if } N \equiv 2 \pmod{4} \\ 8, & \text{if } N \equiv 0 \pmod{4} \end{cases}$$

193 **Proof:** The set of arcs of T_N of the form $(i, i + q)$, $0 \leq q < \frac{N}{2}$, load each arc of the ring exactly q times.
194 So if N is odd the load of any arc of the ring is $1 + 2 + \dots + \frac{N-1}{2} = \frac{N^2-1}{8}$.

195 If N is even the load due to these arcs is $1 + 2 + \dots + \frac{N-2}{2} = \frac{N^2-2N}{8}$. We have to add the load due to
196 arcs of T_N of the form $(i, i + \frac{N}{2})$. As there are $\frac{N}{2}$ such arcs, the total load is $\frac{N^2}{4}$ and so one arc of the ring
197 has load at least $\frac{N}{4}$.

198 If $N \equiv 2 \pmod{4}$ that gives a load at least $\lceil \frac{N}{4} \rceil = \frac{N+2}{4}$, so one arc has load at least $\frac{N^2-2N}{8} + \frac{N+2}{4} = \frac{N^2+4}{8}$.

199 If $N \equiv 0 \pmod{4}$ the maximum load due to the arcs $(i, i + \frac{N}{2})$ is at least $\frac{N}{4}$, but in this case we can
200 give a better bound. Indeed, suppose w.l.o.g. that we have the arc $(0, \frac{N}{2})$, and let j be the number of arcs
201 starting in the interval $[1, \frac{N}{2} - 1]$ of the form $(i, i + \frac{N}{2})$ with $0 < i < \frac{N}{2}$. The load of the arc $(\frac{N}{2} - 1, \frac{N}{2})$ of
202 the ring is then $j + 1$. As there are $\frac{N}{2} - 1 - j$ arcs ending in the interval $[1, \frac{N}{2} - 1]$, the load of the arc
203 $(0, 1)$ is $1 + \frac{N}{2} - 1 - j$. Therefore the sum of the loads of the arcs $(0, 1)$ and $(\frac{N}{2} - 1, \frac{N}{2})$ is $\frac{N}{2} + 1$, and so
204 one of these 2 arcs has load $\lceil \frac{N}{4} + \frac{1}{2} \rceil = \frac{N}{4} + 1$. The total load of this arc is $\frac{N^2-2N}{8} + \frac{N}{4} + 1 = \frac{N^2+8}{8}$.

205 As each subgraph can load one arc at most C times, we obtain the lemma. \square

206 3.2 The parameter $\gamma(C, p)$

207 To obtain accurate lower bounds we need to bound the value of $|E_\omega|$ for a digraph with $|V_\omega| = p$ ver-
208 tices, satisfying the load constraint (admissible digraph). As we discussed in the preceding section, we
209 need only to consider the associated digraph embedded in \vec{C}_p . To this end, we introduce the following
210 definitions.

211 **Definition 3.1** Let $\gamma(C, p)$ be the maximum number of arcs of a digraph H with p vertices such that
212 $L(H, e) \leq C$, for every arc e of \vec{C}_p .

Definition 3.2

$$\rho(C) = \max_{p \geq 2} \left\{ \frac{\gamma(C, p)}{p} \right\}.$$

213 In [33] the authors define two parameters which coincide with the parameters $\gamma(C, p)$ and $\rho(C)$ intro-
214 duced above. In [33] the parameter $\rho(C)$ is called *maximal ADM efficiency*, and its value is determined,
215 but no closed formula for $\gamma(C, p)$ is given in [33]. Here we give again the value of $\rho(C)$, using different
216 tools, and give the exact value of $\gamma(C, p)$.

217 The next proposition shows that, in fact, the maximum number of requests we can groom is attained
218 by taking those of minimum length. It is worth mentioning that this property is not true if the physical
219 graph is a path, as shown with a counterexample in [3].

Proposition 3.2 Let $C = \frac{k(k+1)}{2} + r$, with $0 \leq r \leq k$. Then

$$\gamma(C, p) = \begin{cases} \frac{p(p-1)}{2} & , \text{ if } p \leq 2k + 1, \text{ or } p = 2k + 2 \text{ and } r \geq \frac{k+2}{2} \\ kp + 2r - 1 & , \text{ if } p = 2k + 2 \text{ and } 1 \leq r < \frac{k+2}{2} \\ kp + \left\lfloor \frac{rp}{k+1} \right\rfloor & , \text{ otherwise} \end{cases}$$

220 The graphs achieving $\gamma(C, p)$ are either the tournament T_p if p is small (namely, if $p \leq 2k+1$ or $p = 2k+2$
221 and $r \geq \frac{k+2}{2}$), or subgraphs of a circulant digraph containing all the arcs of length $1, 2, \dots, k$, plus some
222 arcs of length $k + 1$ if $r > 0$.

223 **Proof:** We distinguish three cases according to the value of p .

224 **Case 1.** If p is small, that is such that the tournament T_p loads each arc at most C times, then
225 $\gamma(C, p) = \frac{p(p-1)}{2}$. Let us now see for which values of p this fact holds.

226 If p is odd, the load of T_p is $\frac{p^2-1}{8} \leq C$. The inequality $p^2 - 1 \leq 8C$ implies $p^2 - 1 \leq 4k(k+1) + 8r$,
227 and is satisfied if $p \leq 2k + 1$, as $p^2 - 1 \leq 4k(k+1)$.

228 If p is even, the load of T_p is $\frac{p^2}{8} + \frac{1+\delta}{2}$, where $\delta = 1$ if $p \equiv 0 \pmod{4}$ (see proof of Proposition 3.1).

229 If $p \leq 2k$, then $\frac{p^2+8}{8} \leq \frac{4k^2+8}{8} \leq \frac{k(k+1)}{2} \leq C$.

230 For $p = 2k + 2$, then $\frac{p^2}{8} + \frac{1+\delta}{2} = \frac{k^2}{2} + k + 1 + \frac{\delta}{2} \leq \frac{k^2+k}{2} + r = C$ if and only if $r \geq \frac{k+2+\delta}{2}$, with $\delta = 1$ if
231 $p \equiv 0 \pmod{4}$, that is, if k is odd. Therefore, the condition is satisfied if $r \geq \frac{k+2}{2}$.

232 In the next two cases, we provide first a lower bound on $\gamma(C, p)$, and then we prove a matching upper
233 bound.

234 **Case 2.** If $p = 2k + 2$ and $1 \leq r < \frac{k+2}{2}$, a solution is obtained by taking all the arcs of length
235 $1, 2, \dots, k$ ($= \frac{p-2}{2}$) – giving a load of $\frac{k(k+1)}{2}$ – plus $2r - 1$ arcs of length $\frac{p}{2}$. For example, we can take the
236 arcs $(i, i + \frac{p}{2})$ for $i = 0, 2, \dots, 2r - 2$ ($< \frac{p}{2}$) and the arcs $(i, i - \frac{p}{2})$ for $i = 1, 3, \dots, 2r - 3$. The load due to
237 these arcs is at most r . Therefore, in this case $\gamma(C, p) \geq kp + 2r - 1$.

Case 3. If $p > 2k + 2$ or $p = 2k + 2$ and $r = 0$, a solution is obtained by taking all the arcs of
length $1, 2, \dots, k$ plus $\left\lfloor \frac{rp}{k+1} \right\rfloor$ arcs of length $k + 1$, in such a way that the load due to these arcs is at
most C , which is always possible (for example, if p and $k + 1$ are relatively prime, we take the requests
 $((k+1)i, (k+1)(i+1))$ for $0 \leq i \leq \left\lfloor \frac{rp}{k+1} \right\rfloor - 1$, the indices being taken modulo p). Therefore, in this case

$$\gamma(C, p) \geq kp + \left\lfloor \frac{rp}{k+1} \right\rfloor. \quad (4)$$

238 Let us now turn to upper bounds. Suppose we have a solution with γ arcs, γ_i being of length i on \vec{C}_p .

239 As each arc of length i loads i arcs, and the total load of the arcs of \vec{C}_p is at most Cp , we have

$$\begin{aligned} Cp &\geq \sum_{i=1}^{\infty} i\gamma_i \geq \sum_{i=1}^k i\gamma_i + (k+1)\left(\gamma - \sum_{i=1}^k \gamma_i\right) \\ &= \sum_{i=1}^k ip + (k+1)(\gamma - kp) + \sum_{i=1}^k \underbrace{(k+1-i)(p-\gamma_i)}_{\geq 0} \\ &\geq \frac{k(k+1)}{2} \cdot p + (k+1)(\gamma - kp). \end{aligned}$$

Since $Cp = \frac{k(k+1)}{2} \cdot p + rp$, we obtain $rp \geq (k+1)(\gamma - kp)$, and therefore

$$\gamma(C, p) \leq kp + \frac{rp}{k+1}. \quad (5)$$

240 Combining Relations (4) and (5), we get the result for Case 3. For Case 2, i.e., when $p = 2k + 2$ and
 241 $1 \leq r < \frac{k+2}{2}$, Relation (5) yields $\gamma(C, p) \leq kp + 2r$. If we have equality, then necessarily $\gamma_i = p$ for
 242 $i = 1, \dots, k$, so we have all arcs of length at most k . However, the $2r$ arcs of length at least $k + 1$ induce
 243 a load at least $r + 1$ on some arc of \vec{C}_p , so the total load would be strictly greater than C . Therefore, we
 244 have at most $\gamma(C, p) \leq kp + 2r - 1$, which gives the result. \square

Proposition 3.3 *Let $C = k(k+1)/2 + r$, with $0 \leq r \leq k$. Then*

$$\rho(C) = k + \frac{r}{k+1}. \quad (6)$$

Proof: In Case 1 of the proof of Proposition 3.2, $\rho(C) \leq \frac{p-1}{2}$. If $p \leq 2k + 1$, $\rho(C) \leq k$. If $p = 2k + 2$ and
 $r \geq \frac{k+2}{2}$, $\rho(C) = k + \frac{1}{2} < k + \frac{r}{k+1}$. Otherwise, by Relation (5),

$$\rho(C) \leq \frac{kp + \frac{rp}{k+1}}{p} = k + \frac{r}{k+1}, \quad (7)$$

245 where $C = \frac{k(k+1)}{2} + r$, with $0 \leq r \leq k$. So, in all cases, $\rho(C) \leq k + \frac{r}{k+1}$. Note that when p is a multiple of
 246 $k + 1$, Relation (4) implies that $\gamma(C, p) \geq kp + \frac{rp}{k+1}$, and therefore $\rho(C) \geq k + \frac{r}{k+1}$. The result follows. \square

Note that in [33] the following formula is given, equivalent to Equation (6):

$$\rho(C) = \frac{C}{k+1} + \frac{k}{2}. \quad (8)$$

247 Table 1 shows the parameter $\gamma(C, p)$ for small values of C and p , as well as the parameter $\rho(C)$.

248 3.3 General lower bounds

249 By Propositions 3.1 and 3.2, Equations (1), (2), and (3) become

p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\rho(C)$
$C = 1$	1	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1
$C = 2$	1	3	5	7	9	10	12	13	15	16	18	19	21	22	24	3/2
$C = 3$	1	3	6	10	12	14	16	18	20	22	24	26	28	30	32	2
$C = 4$	1	3	6	10	13	16	18	21	23	25	28	30	32	35	37	7/3
$C = 5$	1	3	6	10	15	18	21	24	26	29	32	34	37	40	42	8/3
$C = 6$	1	3	6	10	15	21	24	27	30	33	36	39	42	45	48	3
$C = 7$	1	3	6	10	15	21	25	29	32	35	39	42	45	48	52	13/4
$C = 8$	1	3	6	10	15	21	27	31	35	38	42	45	49	52	56	14/4
$C = 9$	1	3	6	10	15	21	28	33	37	41	45	48	52	56	60	15/4
$C = 10$	1	3	6	10	15	21	28	36	40	44	48	52	56	60	64	4

Table 1: The parameter $\gamma(C, p)$ for some values of C and p , as well as $\rho(C)$. The **bold** values achieve $\rho(C)$.

$$A = \sum_{p=2}^N pa_p \tag{9}$$

$$\sum_{p=2}^N a_p \geq \left\lceil \frac{N^2 + \alpha}{8C} \right\rceil, \text{ where } \alpha = \begin{cases} -1 & , \text{ if } N \text{ is odd} \\ 4 & , \text{ if } N \equiv 2 \pmod{4} \\ 8 & , \text{ if } N \equiv 0 \pmod{4} \end{cases} \tag{10}$$

$$\sum_{p=2}^N a_p \gamma(C, p) \geq \frac{N(N-1)}{2} \tag{11}$$

250

We are ready to prove the general lower bound on the number of ADMs used by any solution.

Theorem 3.1 (General lower bound) *Let $C = \frac{k(k+1)}{2} + r$, with $0 \leq r \leq k$. The number of ADMs required in a bidirectional ring with N nodes and grooming factor C satisfies*

$$A(C, N) \geq \left\lceil \frac{N(N-1)}{2 \cdot \rho(C)} \right\rceil = \left\lceil \frac{N(N-1)}{2} \frac{k+1}{k(k+1)+r} \right\rceil. \tag{12}$$

Proof: Using Equation (9) and Relation (11), and the definition of $\rho(C)$, we get that the number A of ADMs used by any solution satisfies

$$\frac{N(N-1)}{2} \leq \sum_{p=2}^N a_p \cdot \gamma(C, p) = \sum_{p=2}^N p \cdot a_p \cdot \rho(C) = \rho(C) \cdot A.$$

From the above relation and using Relation (7), we get

$$A \geq \left\lceil \frac{N(N-1)}{2 \cdot \rho(C)} \right\rceil = \left\lceil \frac{N(N-1)}{2} \frac{k+1}{k(k+1)+r} \right\rceil.$$

251

□

252 To achieve the lower bound of Theorem 3.1, the only possibility is to use graphs on p vertices with
 253 $\gamma(C, p)$ arcs. The **bold** values in Table 1 achieve $\rho(C)$, and therefore the subgraphs corresponding to
 254 those values (which exist by Proposition 3.2) are good candidates to construct an optimal partition of the
 255 request graph.

256 **Comparison with existing lower bounds.** In [14] the RING TRAFFIC GROOMING problem in the bidirec-
 257 tional ring is studied. The authors state a lower bound regardless of routing for a general set of requests.
 258 In the particular case of uniform traffic, they get a lower bound of $\frac{N^2-1}{4\sqrt{2C}}$ (see [14, Theorem 1, page 198]).
 259 They indicate in their article that they can improve this bound by a factor of 2 for all-to-all uniform
 260 unitary traffic. We thank T. Chow and P. Lin for sending us the proof of the following theorem, which is
 261 only announced in [14].

Theorem 3.2 ([13, 14]) *If a traffic instance of ring grooming is uniform and unitary, then, regardless of routing,*

$$A(C, N) \geq \frac{1}{2\sqrt{C}} \sqrt{\frac{N^2(N-1)^2}{2} - N(N-1)}.$$

262

The lower bound we obtained in Theorem 3.1 is greater than the bound of Theorem 3.2, but it should be observed that we restrict ourselves to shortest path symmetric routing. Our bound is $\frac{N(N-1)}{2\rho(C)}$ and the lower bound of Theorem 3.2 is less than $\frac{N(N-1)}{2\sqrt{2C}}$. The fact that our bound is better follows from the fact that $\rho(C) < \sqrt{2C}$. Indeed,

$$\rho^2(C) \leq \left(k + \frac{r}{k+1}\right)^2 = k^2 + \frac{2kr}{k+1} + \frac{r^2}{(k+1)^2} < k^2 + 2r + 1 < k^2 + k + 2r = 2C.$$

263 4 Case $C = 1$

264 For $C = 1$, by Proposition 3.2 $\gamma(1, p) = p$ if $p \geq 2$. Furthermore, all the directed cycles achieve $\rho(1)$ (see
 265 Table 1).

Theorem 4.1

$$A(1, N) = \begin{cases} \frac{N(N-1)}{2} & , \text{ if } N \text{ is odd} \\ \frac{N^2}{2} & , \text{ if } N \text{ is even} \end{cases}$$

266 **Proof:** For $C = 1$, the only possible subgraphs involved in the partition of the edges of T_N are cycles
 267 and paths. If only cycles are used, the total number of ADMs is $\frac{N(N-1)}{2}$, which equals the lower bound of
 268 Theorem 3.1. Each path involved in the partition adds one unit of cost with respect to $\frac{N(N-1)}{2}$.

269 If $N = 2q + 1$ is odd, by [10, Theorem 3.3] we know that the arcs of T_N can be covered with q \vec{C}_3 's
 270 and $\frac{q(q-1)}{2}$ \vec{C}_4 's. The total number of vertices of this construction is $3q + 2q(q-1) = q(2q+1) = \frac{N(N-1)}{2}$.

271 If N is even, each vertex must appear with odd degree in at least one subgraph, so the number
 272 of paths in any construction is at least $N/2$. Therefore, the lower bound becomes $\frac{N(N-1)}{2} + \frac{N}{2} = \frac{N^2}{2}$.

273 By [10, Theorem 3.4] the arcs of T_N can be covered with

- 274 • 4 \vec{C}_3 's and $2q^2 - 3$ \vec{C}_4 's, if $N = 4q$ with $q > 1$;
- 275 • 2 \vec{C}_3 's and $2q^2 + 2q - 1$ \vec{C}_4 's, if $N = 4q + 2$.

276 For $N = 4$, we cover T_4 with a \vec{C}_4 and two arcs. Note that in these constructions, some arcs are covered
 277 more than once. In both cases, the total number of vertices of the construction is $\frac{N^2}{2}$, hence the lower
 278 bound is attained.

279 Finally, one can check that in the constructions of [10], the length of the arcs involved in the covering
 280 of T_N is in all cases bounded above by $\lfloor \frac{N}{2} \rfloor$, and therefore all the cycles induce load 1. \square

281 **Remark 4.1** For the original problem with $G = C_N^*$ and $I = K_N^*$, if we apply Theorem 4.1 we get in the
 282 case N even a value of N^2 ADMS; but if we delete the constraint of symmetric routings we get a value of
 283 $N(N-1)/2$ by using [10, Theorems 4.1 and 4.2] (however these constructions use many K_2 's).

284 5 Case $C = 2$

285 When $C = 2$ the general lower bound of Theorem 3.1 gives $A(2, N) \geq \frac{N(N-1)}{3}$. We first improve this
 286 bound in Section 5.1, and then give solutions with a good approximation ratio in Section 5.2.

287 5.1 Improved lower bounds

288 For $C = 2$, by Proposition 3.2 $\gamma(2, 2) = 1$, $\gamma(2, 3) = 3$, $\gamma(2, 4) = 5$ (note that $\gamma(2, 4) = 6$ if the routing
 289 is not restricted to be symmetric), and $\gamma(2, p) = \lfloor \frac{3p}{2} \rfloor$ for $p \geq 5$. The optimal solutions for $p \geq 4$ even
 290 consist of the p arcs of length 1 ($i, i+1$) for $0 \leq i \leq p-1$, plus the $p/2$ arcs of length 2 ($2i, 2i+2$) for
 291 $0 \leq i \leq p/2 - 1$ (in fact, triangles sharing a vertex; see Figure 3 for $p = 6$). For p odd we have two
 292 classes of optimal graphs (see Figure 3 for $p = 5$).

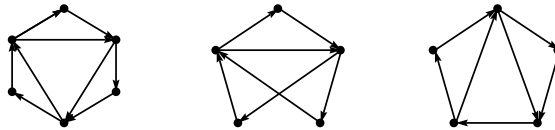


Figure 3: Some admissible digraphs for $C = 2$.

Relation (11) becomes in the case $C = 2$

$$\sum_{p=2}^N a_p \gamma(2, p) = a_2 + 3a_3 + 5a_4 + 7a_5 + 9a_6 + 10a_7 + 12a_8 + \dots \geq \frac{N(N-1)}{2}.$$

293 Therefore,

$$A = \sum_{p=2}^N pa_p \geq \frac{2}{3} \sum_{p=2}^N a_p \gamma(2, p) + \frac{4}{3}a_2 + a_3 + \frac{2}{3}a_4 + \frac{1}{3}(a_5 + a_7 + a_9 + \dots) \quad (13)$$

$$\geq \frac{N(N-1)}{3} + \frac{4}{3}a_2 + a_3 + \frac{2}{3}a_4 + \frac{1}{3}(a_5 + a_7 + a_9 + \dots). \quad (14)$$

294 We can already see that the bound $\frac{N(N-1)}{3}$ cannot be attained. Indeed, to reach it we need to use only
295 graphs with 6, 8, 10, ... vertices. But the number of graphs W satisfies, by Proposition 3.1, $W \geq \frac{N^2-1}{16}$, so

$$296 A \geq 6 \frac{N^2-1}{16} > \frac{N(N-1)}{3}.$$

297 The following proposition gives a lower bound of order $\frac{11}{32}N(N-1)$. Note that $11/32 > 11/33 = 1/3$.

Proposition 5.1 (Tighter lower bound for $C = 2$)

$$A(2, N) \geq \left\lceil \frac{11N^2 - 8N - 3}{32} \right\rceil = \left\lceil \frac{11}{16} \frac{N(N-1)}{2} + \frac{3N-3}{32} \right\rceil. \quad (15)$$

Proof: We can write $A \geq 6(W - a_2 - a_3 - a_4 - a_5) + 2a_2 + 3a_3 + 4a_4 + 5a_5$, that is,

$$A \geq 6W - (4a_2 + 3a_3 + 2a_4 + a_5). \quad (16)$$

From Relations (13) and (14) we get

$$3A \geq N(N-1) + (4a_2 + 3a_3 + 2a_4 + a_5). \quad (17)$$

Summing Relations (16) and (17) gives

$$4A \geq 6W + N(N-1). \quad (18)$$

By Proposition 3.1, we have that

$$W \geq \frac{N(N-1)}{16} + \frac{N+\alpha}{16}. \quad (19)$$

Combining Relations (18) and (19) and using $\alpha \geq -1$ yields

$$A \geq \frac{11N(N-1)}{32} + \frac{3N}{32} + \frac{3\alpha}{32} \geq \frac{11N^2 - 8N - 3}{32}.$$

298

□

299 5.2 Upper bounds

300 In this section we build families of solutions for $C = 2$. We conjecture that there exists a decomposition
 301 using A vertices with ratio $\frac{A}{\frac{N(N-1)}{2}}$ of order $\frac{11}{16}$, which would be optimal by Proposition 5.1. For that, we
 302 should find some (multipartite) graphs achieving this ratio. A candidate is $K_{4,4,4}$, which has 48 edges.
 303 Unfortunately, we have not been able to cover it with 33 vertices (which would achieve the optimal ratio)
 304 but only with 34, giving a 34/33-approximation.

305 For the sake of the presentation, we first present a simple 12/11-approximation inspired from a
 306 construction of [10].

307 5.2.1 A 12/11-approximation

308 This construction is defined recursively. Suppose we have a solution for N vertices using A_N ADMs, with
 309 $N = 2p$ or $N = 2p + 1$. Let the vertex set be labeled $0_A < 1_A < \dots < (p-1)_A < 0_B < 1_B < \dots < (p-1)_B$,
 310 plus ∞ is N is odd. For $N + 2$, we add two vertices x_A and x_B with the order $x_A < 0_A < 1_A < \dots <$
 311 $(p-1)_A < x_B < 0_B < 1_B < \dots < (p-1)_B < \infty$. We use as subdigraphs those of the solution for N
 312 plus the $\lfloor p/2 \rfloor$ digraphs on the 6 vertices $x_A, i_A, (i + \lfloor p/2 \rfloor)_A, x_B, i_B, (i + \lfloor p/2 \rfloor)_B$ and the 8 arcs (x_A, i_A) ,
 313 $(x_A, (i + \lfloor p/2 \rfloor)_A), (i_A, x_B), ((i + \lfloor p/2 \rfloor)_A, x_B), (x_B, i_B), (x_B, (i + \lfloor p/2 \rfloor)_B), (i_B, x_A), ((i + \lfloor p/2 \rfloor)_B, x_A)$, for
 314 $0 \leq i \leq \lfloor p/2 \rfloor - 1$.

315 If $N = 2p$ with p even, there remains uncovered the arc (x_A, x_B) .

316 If $N = 2p + 1$ with p even, there remain the 3 arcs $(x_A, x_B), (x_B, \infty)$, and (∞, x_A) , which we cover
 317 with the circuit (x_A, x_B, ∞) .

318 If $N = 2p$ with p odd, there remain the 5 arcs $(x_A, (p-1)_A), ((p-1)_A, x_B), (x_B, (p-1)_B), ((p-1)_B, x_A)$,
 319 and (x_A, x_B) , which we cover with a digraph on 4 vertices containing all of them.

320 Finally, if $N = 2p + 1$ with p odd, there remain the 7 arcs $(x_A, (p-1)_A), ((p-1)_A, x_B), (x_B, (p-1)_B), ((p-1)_B, x_A), (x_A, x_B), (x_B, \infty)$, and (∞, x_A) , which we cover with a digraph on 5 vertices containing
 321 all of them.
 322

323 One can check that, in all cases, the arcs (u, v) considered satisfy $d_{C_n}^+(u, v) \leq N/2$.

324 To compute the number of ADMs of this construction, we have the recurrence relations $A_{4q+2} =$
 325 $A_{4q} + 6q + 2, A_{4q+4} = A_{4q+2} + 6q + 4, A_{4q+3} = A_{4q+1} + 6q + 3$, and $A_{4q+5} = A_{4q+3} + 6q + 5$. Starting with
 326 $A_2 = 2$ or $A_4 = 6$ (obtained with the partition with the digraph on 4 vertices formed by the C_4 $(0, 1, 2, 3)$
 327 plus the arc $(0, 2)$ and the digraph on 2 vertices $(1, 3)$) and $A_3 = 3$ or $A_5 = 8$ (obtained with the partition
 328 of T_5 using the first digraph on 5 vertices of Figure 3 and the remaining T_3), we get $A_{4q} = 6q^2 = \frac{6N^2}{16}$,
 329 $A_{4q+2} = 6q^2 + 6q + 2 = \frac{6N^2+8}{16}, A_{4q+1} = 6q^2 + 2q = \frac{6N^2-4N-2}{16}$, and $A_{4q+3} = 6q^2 + 8q + 3 = \frac{6N^2-4N+6}{16}$.

330 In all cases, the number of ADMs is of order $\frac{6}{8} \frac{N(N-1)}{2}$, so asymptotically the ratio between the number
 331 of ADMs of this construction and the lower bound of Proposition 5.1 tends to $\frac{6}{8} \frac{16}{11} = \frac{12}{11}$.

332 5.2.2 A 34/33-approximation

333 It will be useful to use the notation G_5 and G_6 to refer to the digraphs depicted in Figure 4. The key idea of
 334 this construction is that an oriented tripartite graph $K_{4,4,4}$ can be partitioned into admissible subdigraphs
 335 for $C = 2$ using 34 vertices overall, as follows.

336 Let the tripartition classes of the $K_{4,4,4}$ be $\{1_A, 1_B, 1_C, 1_D\}$, $\{2_A, 2_B, 2_C, 2_D\}$, $\{3_A, 3_B, 3_C, 3_D\}$, and let
 337 the vertices be ordered in the ring $1_A < 2_A < 3_A < 1_B < 2_B < 3_B < 1_C < 2_C < 3_C < 1_D <$
 338 $2_D < 3_D$. The arcs of an oriented $K_{4,4,4}$ can be partitioned into 4 G_6 's with $\{x_1, x_2, x_3, x_4, x_5, x_6\} =$
 339 $\{1_A, 2_A, 3_B, 1_C, 2_C, 3_D\}$, $\{1_B, 2_B, 3_B, 1_D, 2_D, 3_D\}$, $\{1_B, 2_C, 3_C, 1_D, 2_A, 3_A\}$, and $\{1_A, 3_A, 2_B, 1_C, 3_C, 2_D\}$, plus
 340 2 G_5 's with $\{x_1, x_2, x_3, x_4, x_5\} = \{3_A, 1_C, 2_C, 1_D, 2_D\}$ and $\{3_D, 2_A, 2_B, 1_D, 1_C\}$ (see Figure 4). The total
 341 number of vertices of this partition is 34.

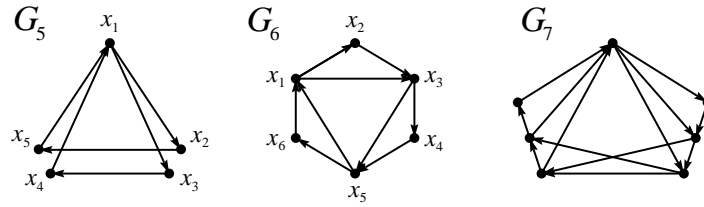


Figure 4: Digraphs G_5 and G_6 used in the 34/33-approximation for $C = 2$, and digraph G_7 suitable for $C = 3$ referred to in the proof of Proposition 6.2.

342 We are now ready to explain the construction. We take an integer $p \equiv 1$ or $3 \pmod{6}$, hence K_p can
 343 be partitioned into triangles. We replace each vertex i of K_p with 4 vertices i_A, i_B, i_C, i_D , and order the
 344 vertices $1_A < \dots < p_A < 1_B < 2_B < \dots < p_B < 1_C < \dots < p_C < 1_D < \dots < p_D$. To a triple $\{i, j, k\}$
 345 corresponding to a triangle of K_p , with $i < j < k$, we associate the decomposition described above of
 346 the $K_{4,4,4}$ on vertices $\{\ell_A, \ell_B, \ell_C, \ell_D : \ell = i, j, k\}$. In this way, $K_{p \times 4}$ can be partitioned into $\frac{p(p-1)}{6} K_{4,4,4}$'s,
 347 or equivalently into $\frac{p(p-1)}{6} \cdot 4 G_6$'s and $\frac{p(p-1)}{6} \cdot 2 G_5$'s. Overall, we use $\frac{34p(p-1)}{6}$ vertices. Each of the
 348 subdigraphs of this partition is admissible, as the distance in the ring between the endpoints of an arc is
 349 strictly smaller than $2p$.

350 To partition an oriented K_{4p} , there remain only the K_4 's induced inside each class of the $K_{p \times 4}$. As
 351 $A(2, 4) = 6$, we use $6p$ vertices to cover all the K_4 's.

352 Therefore, if $p \equiv 1$ or $3 \pmod{6}$, an oriented K_{4p} can be partitioned using $6p + \frac{34p(p-1)}{6} = \frac{34p^2+2p}{6} =$
 353 $\frac{34N^2+8N}{96}$ vertices. To decompose K_{4p+1} , we add a vertex ∞ , and we partition the $p K_5$'s using 8 vertices
 354 for each one of them. Overall, we use $8p + \frac{34p(p-1)}{6} = \frac{34p^2+14p}{6} = \frac{34N^2-12N-24}{96}$ vertices.

355 If $p \not\equiv 1$ or $3 \pmod{6}$, we introduce dummy vertices to get $p' \equiv 1$ or $3 \pmod{6}$, we do the construc-
 356 tion described above, and then we remove the dummy edges and vertices. It is clear that these dummy
 357 vertices add $O(N)$ vertices to the construction, hence the coefficient of the term N^2 remains the same.

358 Since $\frac{33N^2-24N-9}{96}$ is a lower bound by Proposition 5.1, we get the following result.

359 **Proposition 5.2** *The above construction approximates $A(2, N)$ within a factor 34/33.*

360 6 Case $C = 3$

361 We first provide improved lower bounds for some congruence classes in Section 6.1 and then we provide
362 constructions in Section 6.2, which are either optimal or asymptotically optimal.

363 6.1 Improved lower bounds

364 In this case (see Table 1) we have $\gamma(3, 2) = 1$, $\gamma(3, 3) = 3$, $\gamma(3, 4) = 6$, and $\gamma(3, p) = 2p$ for $p \geq 5$, so
365 $\rho(3) = 2$. Therefore, by Theorem 3.1, we get

366 **Proposition 6.1** $A(3, N) \geq \frac{N(N-1)}{4}$.

367 By Relations (9) and (11) we have

$$2A = \sum_{p=2}^N 2pa_p = 4a_2 + 6a_3 + 8a_4 + \sum_{p=5}^N 2pa_p$$

$$\frac{N(N-1)}{2} \leq \sum_{p=2}^N a_p \gamma(3, p) = a_2 + 3a_3 + 6a_4 + \sum_{p=5}^N 2pa_p$$

So

$$A \geq \frac{N(N-1)}{4} + \frac{3}{2}a_2 + \frac{3}{2}a_3 + a_4.$$

368 Therefore, if the lower bound is attained, then necessarily $a_2 = a_3 = a_4 = 0$. We will see in Section 6.2
369 that this is the case for $N \equiv 1$ or $5 \pmod{12}$, using optimal digraphs on 5 vertices (namely T_5) and on 6
370 vertices (namely $\vec{K}_{2,2,2}$, see Figure 5). Optimal graphs are obtained by using arcs of length 1 and 2, so the
371 degree of any vertex in an optimal subdigraph is 4. That is possible only if the total degree of a vertex,
372 namely $N - 1$, is a multiple of 4. Otherwise, the following proposition shows that the lower bound of
373 Proposition 6.1 cannot be attained.

374 Proposition 6.2

375 If $N \equiv 3 \pmod{4}$, $A(3, N) \geq \frac{N(N-1)}{4} + \frac{N}{6} = \frac{3N^2 - N}{12}$.

376 If $N \equiv 0 \pmod{2}$, $A(3, N) \geq \frac{N(N-1)}{4} + \frac{N}{4} = \frac{N^2}{4}$.

377 **Proof:** We use the following observation: If a vertex x has out-degree 3 (resp. in-degree 3) in a digraph
378 B_ω , then its nearest out-neighbor A_x^+ (resp. in-neighbor A_x^-) has in-degree 1 and out-degree at most 1
379 (resp. out-degree 1 and in-degree at most 1). Indeed, suppose x has out-degree 3, and let A_x^+, B_x^+, C_x^+ be
380 the out-neighbors of x . Then the load of the arc entering A_x^+ is already 3, so A_x^+ has no other in-neighbor
381 than x . The load of the arc leaving A_x^+ is already 2, so A_x^+ has at most 1 out-neighbor y . If y has 2 or more
382 in-neighbors, then A_x^+ is not its nearest one. Hence, to each vertex x of out-degree 3 (resp. in-degree 3)
383 is associated a distinct vertex A_x^+ (resp. A_x^-) of degree at most 2.

Consider the digraphs in which a given vertex x appears. Let α_i^x be the number of times x appears with degree i , and let $\alpha_i = \sum_x \alpha_i^x$. Vertex x appears in $\sum_i \alpha_i^x$ digraphs, so

$$A = \sum_x \sum_i \alpha_i^x = \sum_i \alpha_i. \quad (20)$$

As each vertex has degree $N - 1$, $N - 1 = \sum_i i \cdot \alpha_i^x$, and so

$$N(N - 1) = \sum_x \sum_i i \cdot \alpha_i^x = \sum_i i \cdot \alpha_i. \quad (21)$$

Due to the load constraint, a vertex has out-degree (resp. in-degree) at most 3 in all the digraphs in which it appears. Therefore, its degree is at most 6, that is, $\alpha_i = 0$ for $i \geq 7$. Furthermore, by the above observation if a vertex has degree 6 (resp. 5), to this vertex are associated 2 vertices (resp. 1 vertex) of degree at most 2, and all these vertices are distinct, so

$$\alpha_1 + \alpha_2 \geq 2\alpha_6 + \alpha_5. \quad (22)$$

Combining Equations (20) and (21) we get

$$4A = N(N - 1) + 3\alpha_1 + 2\alpha_2 + \alpha_3 - \alpha_5 - 2\alpha_6. \quad (23)$$

384 We distinguish two cases: N even or $N = 4t + 3$.

If N is even, $N - 1$ is odd and each vertex must appear at least in one B_ω with odd degree, so

$$\alpha_1 + \alpha_3 + \alpha_5 \geq N. \quad (24)$$

385 Using Relation (22) multiplied by 2 in Relation (23) we get $4A \geq N(N - 1) + \alpha_1 + \alpha_3 + \alpha_5 + 2\alpha_6$, so by
386 Relation (24), $4A \geq N(N - 1) + N$, as claimed. Note that to obtain equality we need $\alpha_6 = 0$, $\alpha_1 + \alpha_2 = \alpha_5$,
387 and $\alpha_1 + \alpha_3 + \alpha_5 = N$.

If $N = 4t + 3$, the degree of each vertex satisfies $N - 1 \equiv 2 \pmod{4}$, so no vertex can appear with degree 4 in all the digraphs. Each vertex must appear either at least once with degree 6 or 2, or at least twice with odd degree (for example, 5 and 5, 3 and 3, 1 and 1, or 5 and 1), so

$$\alpha_2 + \alpha_6 + \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_5) \geq N. \quad (25)$$

Equation (23) can be rewritten as

$$4A = N(N - 1) + \frac{2}{3} \left(\alpha_2 + \alpha_6 + \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_5) \right) + \frac{4}{3}(\alpha_2 + \alpha_1 - 2\alpha_6 - \alpha_5) + \frac{2}{3}\alpha_3 + \frac{4}{3}\alpha_1. \quad (26)$$

388 Using Relations (22) and (25) in Relation (26) yields $4A \geq N(N - 1) + \frac{2}{3}N + \frac{2}{3}\alpha_3 + \frac{4}{3}\alpha_1$, or $A \geq \frac{N(N-1)}{4} + \frac{N}{6}$,

389 as claimed. Note that to reach the equality, we need to have $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = 2\alpha_6 + \alpha_5$ by Relation (22),
 390 and $2\alpha_6 + 2\alpha_2 + \alpha_5 = 2N$ by Relation (25), so $\alpha_2 = \frac{2N}{3}$, hence an optimal decomposition should use $\frac{N}{3}$
 391 digraphs like the digraph G_7 depicted in Figure 4, having 1 vertex of degree 6 and 2 vertices of degree 2.
 392 \square

393 6.2 Constructions

394 Our constructions rely on the existence of 3-*GDD*'s, that is, decompositions of complete multipartite
 395 graphs into K_3 's. We recall the definition and some basic results below.

396 **Decompositions of complete multipartite graphs into K_3 's.** Let v_1, v_2, \dots, v_q be non-negative inte-
 397 gers; the complete multipartite graph with group sizes v_1, v_2, \dots, v_q is defined to be the graph with vertex
 398 set $V_1 \cup V_2 \cup \dots \cup V_q$ where $|V_i| = v_i$, and two vertices $u \in V_i$ and $v \in V_j$ are adjacent if $i \neq j$. Using
 399 terminology of design theory, the graph of type $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_h^{\alpha_h}$ is the complete multipartite graph with
 400 α_i groups of size p_i . The existence of a partition of this multipartite graph into K_k 's is equivalent to the
 401 existence of a k -*GDD* (*Group Divisible Design*) of type $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_h^{\alpha_h}$ (see [15]). Here we are interested
 402 in the existence of 3-*GDD*'s, that is, partitions into K_3 's. When $|V_i| = p$ for all i , we denote by $K_{p \times q}$ the
 403 multipartite graph of type p^q . Trivial necessary conditions for the existence of a 3-*GDD* are

- 404 (i) the degree of each vertex is even; and
- 405 (ii) the number of edges is a multiple of 3.

406 These conditions are in general sufficient. In particular, the following results will be used later.

407 **Theorem 6.1 ([15])**

408 *A 3-GDD of type 2^q with $q \geq 3$ exists if and only if $q \equiv 0$ or $1 \pmod{3}$.*

409 *A 3-GDD of type $2^{q-1}4$ with $q \geq 4$ exists if and only if $q \equiv 1 \pmod{3}$.*

410 *A 3-GDD of type 3^q with $q \geq 3$ exists if and only if q is odd.*

411 *A 3-GDD of type $3^{q-1}1$ with $q \geq 3$ exists if and only if q is odd.*

412 *A 3-GDD of type $3^{q-1}5$ with $q \geq 5$ exists if and only if q is odd.*

413 *A 3-GDD of type $3^{q-1}11$ with $q \geq 7$ exists if and only if q is odd.*

414 **The basic partition.** In what follows $\vec{K}_{2,2,2}$ will denote the digraph on 6 vertices and 12 arcs depicted
 415 in Figure 5. This digraph can be viewed as being obtained from $K_3(i, j, k)$ with $i < j < k$ by replacing
 416 each vertex i with two vertices i_A and i_B forming an independent set.

417 Note that $\vec{K}_{2,2,2}$ is an optimal digraph for $C = 3$, since it attains the ratio $\rho(3) = 2$ (see Table 1).
 418 The idea of the constructions consists of starting from some graph G (mainly a multipartite graph) which
 419 can be decomposed into K_3 's, replacing each vertex with two non-adjacent vertices, and then using the
 420 following lemma.

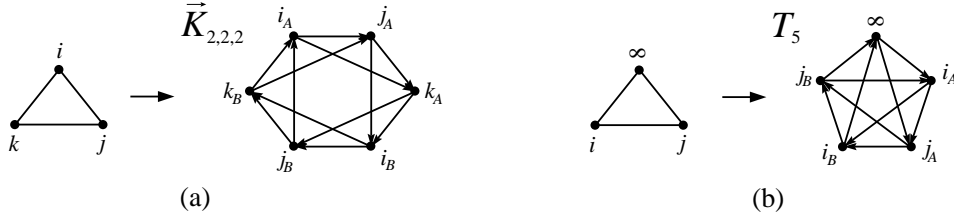


Figure 5: (a) Digraph $\vec{K}_{2,2,2}$ obtained from $K_3(i, j, k)$, with $i < j < k$; (b) digraph T_5 obtained from a K_3 of the form (∞, i, j) .

421 **Lemma 6.1** *If a graph $G = (V, E)$ with vertex set $\{1, 2, \dots, |V|\}$ can be decomposed into h K_3 's, then*
 422 *the digraph H obtained from G by replacing each vertex i with two non-adjacent vertices i_A and i_B , and*
 423 *where the vertices are ordered $1_A, 2_A, \dots, |V|_A, 1_B, 2_B, \dots, |V|_B$, has a valid decomposition into $\vec{K}_{2,2,2}$'s*
 424 *with a total of $6h$ vertices.*

425 **Proof:** To each triangle (i, j, k) with $1 \leq i < j < k \leq |V|$ is associated the $\vec{K}_{2,2,2}$ with vertices $1 \leq i_A <$
 426 $j_A < k_A \leq |V| < i_B < j_B < k_B \leq 2|V|$. To show that the decomposition is valid for $C = 3$, it suffices to
 427 show that the distance between the end-vertices of any arc of any $\vec{K}_{2,2,2}$ is at most $|V|$. That is true for
 428 the arcs (x_A, y_A) or (x_B, y_B) as they satisfy $x < y$, and also for the arcs (x_A, y_B) or (x_B, y_A) as they satisfy
 429 $x > y$ (see Figure 5(a)). \square

430 **Some small cases.** We provide here decompositions of some particular small digraphs that will be used
 431 in the constructions of Propositions 6.4 and 6.5.

432 **Lemma 6.2** $A(3, 5) = 5$, $A(3, 6) \leq 10$, $A(3, 7) \leq 12$, $A(3, 8) \leq 18$, $A(3, 9) \leq 21$, $A(3, 10) \leq 28$,
 433 $A(3, 11) \leq 31$, and $A(3, 23) \leq 132$.

434 **Proof:** Case $N = 5$. The decomposition is given in Figure 5(b), and can be viewed as obtained from the
 435 $K_3(\infty, i, j)$ by replacing each of i, j with two vertices.

436 Case $N = 6, 7$. The complete graph K_4 can be decomposed into one $K_{1,3}(0; \infty, 1, 2)$ and one
 437 $K_3(\infty, 1, 2)$. Replace each of the vertices i, j, k with two vertices. The T_7 on the ordered vertices
 438 $\infty, 0_A, 1_A, 2_A, 0_B, 1_B, 2_B$ can be partitioned into a T_5 on $\infty, 1_A, 2_A, 1_B, 2_B$ ((see Figure 5(b) with $i = 1, j =$
 439 $2))$ and the admissible digraph on 7 vertices and 11 arcs depicted in Figure 6(b) with $i = 0, j = 1, k = 2$.
 440 So we obtain a valid decomposition using 12 vertices. Deleting vertex ∞ yields a decomposition of T_5
 441 with 10 vertices.

442 Case $N = 8, 9$. K_5 is the union of two K_3 's $(\infty, 1, 3), (0, 2, 3)$ and a $C_4(\infty, 0, 1, 2)$. Replacing each ver-
 443 tex with two vertices we get a partition of the T_9 on the ordered vertices $\infty, 0_A, 1_A, 2_A, 3_A, 0_B, 1_B, 2_B, 3_B$.
 444 Namely, to the $K_3(\infty, 1, 3)$ we associate a T_5 on $\infty, 1_A, 3_A, 1_B, 3_B$ (see Figure 5(b) with $i = 1, j = 3)$.
 445 To the $K_3(0, 2, 3)$ we associate a $\vec{K}_{2,2,2}$ on $0_A, 2_A, 3_A, 0_B, 2_B, 3_B$. To the $C_4(\infty, 0, 1, 2)$ we associate the
 446 digraph on 7 vertices of Figure 6(a) with $i = 0, j = 1, k = 2$ and the triangle $(1_A, 2_A, 2_B)$. Therefore,
 447 $A(3, 9) \leq 21$. Vertex 1_A appears in 3 digraphs, so $A(3, 8) \leq 21 - 3 = 18$.

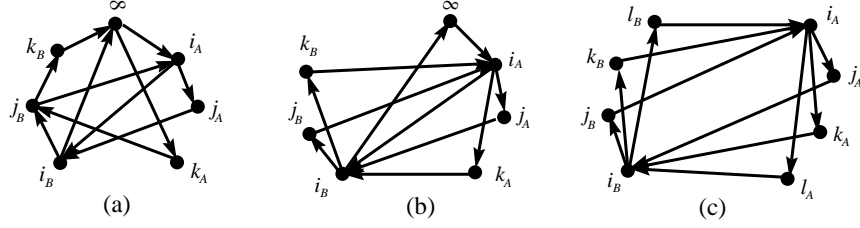


Figure 6: (a) Digraph associated to a $C_4(\infty, i, j, k)$. Digraphs associated to stars ($K_{1,3}$'s), with $\infty < i < j < k < \ell$: (b) star of the form $(i; \infty, j, k)$; (c) star of the form $(i; j, k, \ell)$.

448 Case $N = 10, 11$. K_6 can be partitioned into 3 K_3 's $(\infty, 1, 3), (\infty, 2, 4), (0, 1, 4)$, a star $K_{1,3}(0; \infty, 2, 3)$,
 449 and a $P_4[1, 2, 3, 4]$. Replacing each vertex with two vertices we get a partition of the T_{11} on the ordered
 450 vertices $\infty, 0_A, 1_A, 2_A, 3_A, 4_A, 0_B, 1_B, 2_B, 3_B, 4_B$ into 2 T_5 's on $\infty, 1_A, 3_A, 1_B, 3_B$ and $\infty, 2_A, 4_A, 2_B, 4_B$, a
 451 $\vec{K}_{2,2,2}$ on $0_A, 1_A, 4_A, 0_B, 1_B, 4_B$, a digraph on 7 vertices and 11 arcs depicted in Figure 6(b) with $i = 0, j =$
 452 $2, k = 3$, and an admissible digraph on 8 vertices with arcs $(1_A, 2_A), (2_A, 3_A), (3_A, 4_A), (1_B, 2_B), (2_B, 3_B), (3_B, 4_B),$
 453 $(2_A, 1_B), (2_B, 1_A), (3_A, 2_B), (3_B, 2_A), (4_A, 3_B), (4_B, 3_A)$. Therefore, $A(3, 11) \leq 31$, and as vertex ∞ appears
 454 in 3 subgraphs, we get $A(3, 10) \leq 28$.

455 Case $N = 23$. We decompose K_{12} into 19 K_3 's and 3 $K_{1,3}$'s, where vertex ∞ appears in 5 K_3 's and
 456 in a star $(i; \infty, j, k)$, the two other stars being of the form $(i'; j'k', \ell')$ with $i' < j' < k' < \ell'$. We obtain
 457 a decomposition of T_{23} into 5 T_5 's, 14 $\vec{K}_{2,2,2}$'s, 1 digraph of Figure 6(a), and 2 digraphs of Figure 6(c).
 458 Thus, $A(3, 23) \leq 5 \cdot 5 + 14 \cdot 6 + 7 + 8 + 8 = 132$. \square

459 **Constructions.** We begin with an optimal partition for $N \equiv 0, 1, 4, \text{ or } 5 \pmod{12}$, and then we provide
 460 near-optimal constructions for the remaining values.

461 Proposition 6.3

462 If $N \equiv 0 \text{ or } 4 \pmod{12}$, $A(3, N) = \frac{N^2}{4}$.

463 If $N \equiv 1 \text{ or } 5 \pmod{12}$, $A(3, N) = \frac{N(N-1)}{4}$.

464 **Proof:** The lower bound follows from Propositions 6.1 and 6.2. For the upper bound, we will apply
 465 Lemma 6.1 with $G = K_{2 \times q}$ (type 2^q), which can be decomposed by Theorem 6.1 into $\frac{2q(q-1)}{3} K_3$'s if
 466 $q \equiv 0 \text{ or } 1 \pmod{3}$. As G has $2q$ vertices, the graph H described in Lemma 6.1 has $4q$ vertices and can be
 467 decomposed into admissible $\vec{K}_{2,2,2}$'s. Adding an admissible T_4 on each of the q independent sets of H (of
 468 the form $\{i_A, j_A, i_B, j_B\}$ where $\{i, j\}$ is an independent set of G), we get a valid decomposition of T_{4q} into
 469 $q T_4$'s and $\frac{2q(q-1)}{3}$ admissible $\vec{K}_{2,2,2}$'s. So using $A(3, 4) = 4$, we get $A(3, 4q) \leq qA(3, 4) + 4q(q-1) = 4q^2$
 470 for $q \equiv 0 \text{ or } 1 \pmod{3}$. So $A(3, N) \leq \frac{N^2}{4}$ for $N \equiv 0 \text{ or } 4 \pmod{12}$.

471 For $N = 4q + 1$, we add to the vertex set of H an extra vertex ∞ . Adding to the arcs of H the q
 472 tournaments T_5 built on $\infty, i_A, j_A, i_B, j_B$, where vertices i, j are not adjacent in G , we get a decomposition

473 of T_{4q+1} into q admissible T_5 's plus $\frac{2q(q-1)}{3}$ admissible $\vec{K}_{2,2,2}$'s (the distance being at most $2q-1$ in H
 474 and so $2q$ in T_{4q+1}). Using $A(3, 5) = 5$ (see Lemma 6.2), we get $A(3, 4q+1) \leq qA(3, 5) + 4q(q-1) =$
 475 $4q^2 + q = \frac{(4q+1)4q}{4}$ for $q \equiv 0$ or $1 \pmod{3}$. So $A(3, N) \leq \frac{N(N-1)}{4}$ for $N \equiv 1$ or $5 \pmod{12}$. \square

476 We group the non-optimal constructions in Proposition 6.4 and Proposition 6.5 according to whether
 477 they differ from the lower bound by either a constant or a linear additive term, respectively.

478 **Proposition 6.4**

479 If $N \equiv 8 \pmod{12}$, $A(3, N) \leq \frac{N^2}{4} + 2$.

480 If $N \equiv 9 \pmod{12}$, $A(3, N) = \frac{N(N-1)}{4} + 3$.

481 **Proof:** We start from G of type $2^{q-1}4$ with $q \equiv 1 \pmod{3}$, which can be decomposed by Lemma 6.1
 482 into $\frac{2(q-1)(q+2)}{3} K_3$'s. As in the proof of Proposition 6.3, we get a decomposition of T_{4q+4} into $q-1$ T_4 's,
 483 one T_8 and $\frac{2(q-1)(q+2)}{3} \vec{K}_{2,2,2}$'s (indeed, the independent set V_q of G has 4 vertices, so in H it induces
 484 an independent set of 8 vertices). So using $A(3, 4) = 4$ and $A(3, 8) \leq 18$ (see Lemma 6.2), we get
 485 $A(3, 4q+4) \leq (q-1)A(3, 4) + A(3, 8) + 4(q-1)(q+2) \leq 4q^2 + 8q + 6 = \frac{(4q+4)^2}{4} + 2$ for $q \equiv 1 \pmod{3}$,
 486 so $A(3, N) \leq \frac{N^2}{4} + 2$ for $N \equiv 8 \pmod{12}$.

487 Similarly, adding a vertex ∞ to H we get a decomposition of T_{4q+1} into $q-1$ T_5 's, one T_9 and
 488 $h = \frac{2(q-1)(q+2)}{3} K_3$'s. So using $A(3, 5) = 5$ and $A(3, 9) \leq 21$ we get $A(3, 4q+5) \leq (q-1)A(3, 5) + A(3, 9) +$
 489 $4(q-1)(q+2) \leq 4q^2 + 9q + 8 = \frac{(4q+5)(4q+4)}{4} + 3$ for $q \equiv 1 \pmod{3}$, so $A(3, N) \leq \frac{N(N-1)}{4} + 3$ for $N \equiv 9$
 490 $\pmod{12}$. \square

491 **Proposition 6.5**

492 If $N \equiv 2 \pmod{12}$, $A(3, N) \leq \frac{N^2}{4} + \frac{N+4}{6}$.

493 If $N \equiv 3 \pmod{12}$, $A(3, N) \leq \frac{N^2+3}{4}$.

494 If $N \equiv 6 \pmod{12}$, $A(3, N) \leq \frac{N^2}{4} + \frac{N}{6}$.

495 If $N \equiv 7 \pmod{12}$, $A(3, N) \leq \frac{N^2-1}{4}$.

496 If $N \equiv 10 \pmod{12}$, $A(3, N) \leq \frac{N^2}{4} + \frac{N+8}{6}$.

497 If $N \equiv 11 \pmod{12}$, $A(3, N) \leq \frac{N^2+3}{4} + \varepsilon$, with $\varepsilon = 1$ for $N = 11, 35$.

498 **Proof:** We use as graph G of Lemma 6.1 a multipartite graph of type $3^{q-1}u$ with $3(q-1) + u$ vertices, in
 499 order to get a decomposition of $T_{6(q-1)+2u}$ (resp. $T_{6(q-1)+2u+1}$) into $q-1$ T_6 's (resp. T_7 's), one T_{2u} (resp.
 500 T_{2u+1}) and the digraph H itself decomposed by Lemma 6.1 into $h = \frac{9(q-1)(q-2)}{6} + u(q-1) \vec{K}_{2,2,2}$'s. We
 501 distinguish several cases according to the value of u .

502 **Case 1:** $u = 1, q \geq 3$ odd.

503 Let $N \equiv 2 \pmod{12}$, $N = 6q - 4$. Using $A(3, 2) = 2$ and $A(3, 6) \leq 10$ we get $A(3, 6q - 4) \leq$
 504 $(q-1)A(3, 6) + A(3, 2) + (q-1)(9q-12) \leq 9q^2 - 11q + 4 = \frac{(6q-4)^2}{4} + q = \frac{N^2}{4} + \frac{N+4}{6}$.

505 Let $N \equiv 3 \pmod{12}$, $N = 6q - 3$. Using $A(3, 3) = 3$ and $A(3, 7) \leq 12$ we get $A(3, 6q - 3) \leq$
 506 $(q-1)A(3, 7) + A(3, 3) + (q-1)(9q-12) \leq 9q^2 - 9q + 3 = \frac{(6q-3)^2}{4} + \frac{3}{4} = \frac{N^2+3}{4}$.

507 **Case 2:** $u = 3, q \geq 3$ odd.

508 Let $N \equiv 6 \pmod{12}$, $N = 6q$. Using $A(3, 6) \leq 10$ we get $A(3, 6q) \leq qA(3, 6) + 9q(q-1) \leq 9q^2 + q =$
 509 $\frac{N^2}{4} + \frac{N}{6}$.

510 Let $N \equiv 7 \pmod{12}$, $N = 6q + 1$. Using $A(3, 7) \leq 12$ we get $A(3, 6q + 1) \leq qA(3, 7) + 9q(q-1) \leq$
 511 $9q^2 + 3q = \frac{N^2-1}{4}$.

512 **Case 3:** $u = 5$, $q \geq 5$ odd.

513 Let $N \equiv 10 \pmod{12}$, $N = 6q + 4$. Using $A(3, 6) \leq 10$ and $A(3, 10) \leq 28$ we get $A(3, 6q + 4) \leq$
 514 $(q-1)A(3, 6) + A(3, 10) + (q-1)(9q+12) \leq 9q^2 + 13q + 6 = \frac{(6q+4)^2}{4} + \frac{6q+12}{6} = \frac{N^2}{4} + \frac{N+8}{6}$.

515 Let $N \equiv 11 \pmod{12}$, $N = 6q + 5$. Using $A(3, 7) \leq 12$ and $A(3, 11) \leq 31$ we get $A(3, 6q + 5) \leq$
 516 $(q-1)A(3, 7) + A(3, 11) + (q-1)(9q+12) \leq 9q^2 + 15q + 7 = \frac{N^2+3}{4}$.

517 For $q = 23$ we have $A(3, 23) \leq 132 = \frac{23^2-1}{4}$, one less than the value given by the preceding
 518 construction. Using $u = 11$, $q \geq 7$ odd, $N = 6q + 17$, $A(3, 7) \leq 12$, and $A(3, 23) \leq 132$ we get
 519 $A(3, 6q + 17) \leq (q-1)A(3, 7) + A(3, 23) + (q-1)(9q+48) \leq 9q^2 + 51q + 72 = \frac{(6q+17)^2-1}{4} = \frac{N^2-1}{4}$. It
 520 might be that $A(3, 11) \leq 30$, and then the bound $\frac{N^2-1}{4}$ would be also attained for $N = 11$ and 35. \square

521 7 Case $C > 3$

522 For $C > 3$, we distinguish two cases according to whether C is of the form $\frac{k(k+1)}{2}$ or not. We focus on
 523 those cases in Sections 7.1 and 7.2.

524 7.1 C not of the form $k(k+1)/2$

525 If C is not of the form $\frac{k(k+1)}{2}$, we can improve the lower bound of Theorem 3.1, as we did for $C = 2$
 526 in Proposition 5.1. We provide the details for $C = 4$ and sketch the ideas for $C = 5$, that show how to
 527 improve the lower bound for any value of C not of the form $k(k+1)/2$.

Proposition 7.1

$$A(4, N) \geq \frac{7}{32}N(N-1) = \left(\frac{3}{14} + \frac{1}{224}\right)N(N-1).$$

Proof: The values of $\gamma(4, p)$ are given in Table 1, so Relation (13) becomes in the case $C = 4$

$$A = \sum_{p=2}^N pa_p \geq \frac{3}{7} \sum_{p=2}^N a_p \gamma(4, p) + \frac{11}{7}a_2 + \frac{12}{7}a_3 + \frac{10}{7}a_4 + \frac{5}{7}a_5 + \frac{3}{7}a_6 + \frac{1}{7}(a_7 + 2a_8 + a_{10} + 2a_{11} + a_{13} + 2a_{14} + \dots). \quad (27)$$

Using $\sum_{p=2}^N a_p \gamma(4, p) \geq \frac{N(N-1)}{2}$, Relation (27) becomes

$$14A \geq 3N(N-1) + 22a_2 + 24a_3 + 20a_4 + 10a_5 + 6a_6 + 2a_7 + 4a_8 + \dots \quad (28)$$

On the other hand,

$$A \geq 9 \left(W - \sum_{i=2}^8 a_i \right) + \sum_{i=2}^8 i \cdot a_i = 9W - 7a_2 - 6a_3 - 5a_4 - 4a_5 - 3a_6 - 2a_7 - a_8. \quad (29)$$

Summing Relations (28) and (29) and using $W \geq \frac{N(N-1)}{32} + \frac{N-1}{32}$ by Proposition 3.1 yields

$$15A \geq \frac{105}{32}N(N-1) + \frac{9}{32}(N-1), \text{ and therefore } A \geq \frac{7}{32}N(N-1) + \frac{3}{160}(N-1).$$

528

□

529 For $C = 5$, a similar computation with $\rho(5) = 8/3$ gives

$$8A \geq \frac{3}{2}N(N-1) + 13a_2 + 15a_3 + 14a_4 + 10a_5 + 3a_6 + 2a_7 + a_8. \quad (30)$$

$$A \geq 9W - 7a_2 - 6a_3 - 5a_4 - 4a_5 - 3a_6 - 2a_7 - a_8. \quad (31)$$

So again, summing Relations (30) and (31) and using $W \geq \frac{N(N-1)}{40} + \frac{N-1}{40}$ by Proposition 3.1 yields

$$A \geq \frac{N(N-1)}{6} + \frac{N(N-1)}{40} + \frac{N-1}{40} = \frac{23}{120}N(N-1) + \frac{N-1}{40} = \left(\frac{3}{16} + \frac{1}{240}\right)N(N-1) + \frac{N-1}{40}.$$

530 7.2 C of the form $k(k+1)/2$

531 For $C = \frac{k(k+1)}{2}$ the lower bound of Theorem 3.1 can be attained, according to the existence of a type
532 of k -GDD, called a *Balanced Incomplete Block Design (BIBD)*. A $(v, k, 1)$ -BIBD consists simply of a
533 partition of K_v into K_k 's.

534 **Theorem 7.1** *If there exists a $(k+1)$ -GDD of type k^q (that is, a decomposition of $K_{k \times q}$ into K_{k+1} 's), then*
535 *there exists an optimal admissible partition of T_{2kq+1} for $C = \frac{k(k+1)}{2}$ with $\frac{N(N-1)}{2k}$ ADMs.*

536 **Proof:** The lower bound follows from Theorem 3.1. For the upper bound, as we did in Proposition 6.3
537 (case $k = 2, C = 2$), we replace each vertex i of $K_{k \times q}$ with two vertices i_A and i_B , and add a new vertex
538 ∞ . We label the vertices of the obtained T_{2kq+1} with $\infty, 1_A, \dots, (kq)_A, 1_B, \dots, (kq)_B$. To each K_{k+1} of the
539 decomposition of $K_{k \times q}$ we associate a $T_{2 \times (k+1)}$, which is an optimal digraph for $C = \frac{k(k+1)}{2}$ with $2(k+1)$
540 vertices and $2k(k+1)$ edges, hence attaining $\rho(C) = k$. So adding vertex ∞ to the stable sets of size $2k$
541 we obtain a decomposition of T_{2kq+1} into q T_{2k+1} 's (which are also optimal) and $T_{2 \times (k+1)}$'s.

542 If $K_{k \times q}$ is decomposable into K_{k+1} 's, the number of K_{k+1} 's (and so the number of $T_{2 \times (k+1)}$'s) is $\frac{kq(q-1)}{k+1}$.
543 Therefore the total number of ADMs is $q(2k+1) + 2kq(q-1) = \frac{(2kq+1)2kq}{2k} = \frac{N(N-1)}{2k}$. □

544 Note that a decomposition of $K_{k \times q}$ into K_{k+1} 's is equivalent to a decomposition of K_{kq+1} into K_{k+1} 's by
545 adding a new vertex ∞ , that is, a $(kq+1, k+1, 1)$ -BIBD. In particular, such designs are known to exist
546 if N is large enough and $(kq+1)kq \equiv 0 \pmod{k(k+1)}$ [15]. For example, for $k = 3$ and $q \equiv 0$ or 1
547 $\pmod{4}$, or $k = 4$ and $q \equiv 0$ or $1 \pmod{5}$.

548 Corollary 7.1

549 *If $C = 6$ and $N \equiv 1$ or $7 \pmod{24}$, $A(6, N) = \frac{N(N-1)}{6}$.*

550 *If $C = 10$ and $N \equiv 1$ or $9 \pmod{40}$, $A(10, N) = \frac{N(N-1)}{8}$.*

551 **Corollary 7.2** For $C \in \{15, 21, 28, 36\}$, there exists a small set of values of N for which the existence of a
 552 BIBD remains undecided (179 values overall, see [15, pages 73-74]). For the values of N different from
 553 these undecided BIBDs, the following results apply.

554 If $C = 15$ and $N \equiv 1$ or $11 \pmod{30}$, $A(15, N) = \frac{N(N-1)}{10}$.

555 If $C = 21$ and $N \equiv 1$ or $13 \pmod{84}$, $A(21, N) = \frac{N(N-1)}{12}$.

556 If $C = 28$ and $N \equiv 1$ or $15 \pmod{112}$, $A(28, N) = \frac{N(N-1)}{14}$.

557 If $C = 36$ and $N \equiv 1$ or $17 \pmod{144}$, $A(36, N) = \frac{N(N-1)}{16}$.

558 Wilson proved [34] that for v large enough, K_v can be decomposed into subgraphs isomorphic to any
 559 given graph G , if the trivial necessary conditions about the degree and the number of edges are satisfied.
 560 Thus, we can assure that optimal constructions exist when $C = \frac{k(k+1)}{2}$ for all $k > 0$.

561 **Corollary 7.3** If $C = \frac{k(k+1)}{2}$, then $A(C, N) = \frac{N(N-1)}{2k}$ for $N \equiv 1$ or $2k + 1 \pmod{4C}$ large enough.

We can also use decompositions of $K_{p \times q}$ into K_{k+1} 's to get constructions asymptotically optimal, but
 not attaining the lower bound like for $C = 3$. For instance, for $C = 6$ the proof of Theorem 7.1 gives
 (without adding the vertex ∞) that for $q \equiv 0$ or $1 \pmod{4}$ and $N \equiv 0$ or $6 \pmod{24}$,

$$A(6, 6q) \leq qA(6, 6) + 6q(q-1) = 6q^2 = \frac{N^2}{6}.$$

562 That might be an optimal value if we could improve the lower bound for $C = 6$ as we did for $C = 3$ in
 563 Proposition 6.2, but the calculations become considerably more complicated.

564 **Corollary 7.4**

565 For $N \equiv 0$ or $6 \pmod{24}$, $\frac{N(N-1)}{6} \leq A(6, N) \leq \frac{N^2}{6}$.

566 For $N \equiv 0$ or $8 \pmod{40}$, $\frac{N(N-1)}{8} \leq A(10, N) \leq \frac{N^2}{8}$.

567 For a general C of the form $C = \frac{k(k+1)}{2}$, the improved lower bound one could expect is $\frac{N^2}{2k}$.

568 Finally, it is worth mentioning here the constructions given in [19] for $C = 8$. Namely, in [19,
 569 Corollary 5] the authors provide a construction that uses asymptotically $\frac{N^2}{2} \frac{5}{16}$ ADMs, using the so-
 570 called *primitive rings*. This construction, according to the lower bound of Theorem 3.1, constitutes a $\frac{35}{32}$ -
 571 approximation for $C = 8$. Note that the construction for $C = 6$ given in Corollary 7.1 uses asymptotically
 572 $\frac{N^2}{2} \frac{1}{3} = \frac{N^2}{2} \frac{5}{15}$ ADMs, which is already very close to the value obtained in [19] for $C = 8$, so it seems
 573 natural to suspect that there is enough room for improvement over the constructions of [19].

574 8 Unidirectional or Bidirectional Rings?

575 This section is devoted to comparing unidirectional and bidirectional rings in terms of minimizing elec-
 576 tronics cost, when these rings are used in a WDM network with traffic grooming and all-to-all requests.

For bidirectional rings, Theorem 3.1 gives the following lower bound by multiplying by 2 the value, in order to take into account requests both clockwise and counterclockwise.

$$\text{LB}_{\text{bi}}(C, N) = \frac{N(N-1)}{2} \cdot \frac{2}{\rho(C)},$$

577 where $\rho(C) = k + \frac{r}{k+1}$ for $C = \frac{k(k+1)}{2} + r$ with $0 \leq r \leq k$.

In [7] the following general lower bound was given for unidirectional rings.

$$\text{LB}_{\text{uni}}(C, N) = \frac{N(N-1)}{2} \cdot \frac{1}{\eta(C)},$$

$$\text{where } \eta(C) = \begin{cases} \frac{k}{2}, & \text{if } C = \frac{k(k+1)}{2} + r \text{ and } 0 \leq r \leq \frac{k}{2} \\ \frac{C}{k+2}, & \text{if } C = \frac{k(k+1)}{2} + r \text{ and } \frac{k}{2} \leq r \leq k \end{cases}$$

Note that for $C = \frac{k(k+1)}{2}$ (that is, for $r = 0$) the bounds are equal. In general, we have

$$1 \leq \frac{\text{LB}_{\text{uni}}(C, N)}{\text{LB}_{\text{bi}}(C, N)} \leq 1 + \frac{1}{2(k+1)}.$$

578 Indeed, either $0 \leq r \leq \frac{k}{2}$ and then

$$\frac{\rho(C)}{2\eta(C)} = 1 + \frac{r}{k(k+1)} \leq 1 + \frac{1}{2(k+1)},$$

or $\frac{k}{2} \leq r \leq k$, and then

$$\frac{\rho(C)}{2\eta(C)} = \frac{(k+2)(k(k+1)+r)}{(k+1)(k(k+1)+2r)} = 1 + \frac{k(k+1)-rk}{(k+1)(k(k+1)+2r)}.$$

Let $r = \frac{k}{2} + r'$, and so $0 \leq r' \leq \frac{k}{2}$. Then

$$\frac{\rho(C)}{2\eta(C)} = 1 + \frac{1}{2(k+1)} \frac{k(k+2)-2r'}{k(k+2)+2r'} \leq 1 = \frac{1}{2(k+1)}.$$

579 Note that there exist constructions for bidirectional rings with cost strictly smaller than $\text{LB}_{\text{uni}}(C, N)$.

580 Indeed, for $C = 2$ we presented in Section 5.2.2 a construction using at most $\frac{17}{48}N(N-1)$ ADMs. Taking
581 into account requests in both directions this construction uses at most $\frac{17}{24}N(N-1)$ ADMs, to be compared
582 with $\text{LB}_{\text{uni}}(2, N) = \frac{3}{4}N(N-1) > \frac{17}{24}N(N-1)$.

583 However, for large C the lower bounds tend to be equal; hence in terms of the number of ADMs
584 there is no real improvement in using bidirectional rings. The real improvement is more in terms of the
585 number of used wavelengths (or, equivalently, the load). Indeed, in unidirectional rings this number is
586 roughly $\frac{N^2}{2C}$ (see for instance [7]), which is twice the number in bidirectional rings (roughly equal to $2 \cdot \frac{N^2}{8C}$
587 by Proposition 3.1).

588 In summary, bidirectional and unidirectional rings are equivalent in terms of the number of ADMs,
589 the trade-off being between better bandwidth utilization in bidirectional rings versus simplicity (and the
590 use of the other ring for fault tolerance) in unidirectional rings.

591 9 Conclusions and Further Research

592 In this article we studied the minimization of ADMs in optical WDM bidirectional ring networks under
593 the assumption of symmetric shortest path routing and all-to-all unitary requests. We precisely formu-
594 lated the problem in terms of graph decompositions, and stated a general lower bound for all the values
595 of C and N . We then studied extensively the cases $C = 2$ and $C = 3$, providing improved lower bounds,
596 optimal constructions for several infinite families, as well as asymptotically optimal constructions and
597 approximations. To the best of our knowledge, these are the first optimal solutions in the literature for
598 traffic grooming in bidirectional rings. We then study the case $C > 3$, focusing specifically on the case
599 $C = k(k + 1)/2$ for some $k \geq 1$. We gave optimal decompositions for several congruence classes of N ,
600 using the existence of some combinatorial designs. We concluded with a comparison of the switching
601 cost in unidirectional and bidirectional WDM rings.

602 Further research is needed to find new families of optimal solutions for other values of C . The first
603 step should be to improve the general lower bound for other values of C , namely, finding a closed for-
604 mula. It would be interesting to consider other kinds of routing in bidirectional rings, not necessarily
605 symmetric or using shortest paths. Stating which kind of routing is the best for each value of N and C
606 would be a nice result. Finally, studying the traffic grooming problem using graph partitioning tools in
607 other topologies, like trees or hypercubes, would be also interesting.

608

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