Existence of the harmonic measure for random walks on graphs and in random environments

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EXISTENCE OF THE HARMONIC MEASURE FOR RANDOM WALKS ON GRAPHS AND IN RANDOM ENVIRONMENTS

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Abstract. We give a sufficient condition for the existence of the harmonic measure from infinity of transient random walks on weighted graphs. In particular, this condition is verified by the random conductance model on $\mathbb{Z}^d$, $d \geq 3$, when the conductances are i.i.d. and the bonds with positive conductance percolate. The harmonic measure from infinity also exists for random walks on supercritical clusters of $\mathbb{Z}^2$. This is proved using results of Barlow (2004).

Keywords: Harmonic measure, supercritical percolation clusters, effective conductance, Harnack inequality, random conductance model.

Subject Classification: 60J05, 60K35, 60K37

1. Introduction and results

The harmonic measure from infinity of a closed subset A of $\mathbb{R}^d$, $d \geq 2$, is the hitting distribution of the set A by a d-dimensional Brownian motion started at infinity. A detailed description of this measure is given by Mörters and Peres in [27, section 3.4].

Similarly, given a Markov chain on an infinite graph, the harmonic measure of a finite subset of the graph is defined as the hitting distribution of the set by the Markov chain starting at infinity. The existence of the harmonic measure for the simple symmetric random walk on $\mathbb{Z}^d$ is shown by Lawler in [23, chapter 2] and it is extended to a wider class of random walks on $\mathbb{Z}^d$ by Lawler and Limic in [22, section 6.5].

From these results, one might expect that the existence of the harmonic measure for a Markov chain on $\mathbb{Z}^d$, $d \geq 2$, relies on its Green function asymptotics. The goal of this paper is to show that actually, the existence of the harmonic measure is a fairly robust result in the sense that it exists for a random walk on a weighted graph as soon as there is a weak form of Harnack inequality. In particular, it is verified by a large family of fractal-like graphs and by random conductance models on $\mathbb{Z}^d$, $d \geq 3$, given by a sequence of i.i.d. conductances as soon as there is percolation of the positive conductances. This is done using recent estimates of Andres, Barlow, Deuschel and Hambly [3].

In the recurrent case, although we do not give a general sufficient condition, we show the existence of the harmonic measure for the random walk on the supercritical cluster of $\mathbb{Z}^2$ using estimates of Barlow [7] and Barlow and Hambly [9].

The results of [3] for the random conductance model are part of a long series of works which go back to homogenization of divergence form elliptic operators with random coefficients and to the investigation of the properties of the supercritical percolation cluster.

Some highlights of the properties of the random walk on the supercritical percolation cluster of $\mathbb{Z}^d$ is the proof of the Liouville property for bounded harmonic functions (see Kaimanovich [20] and [11]) and the proof of the transience of the walk when $d \geq 3$ by Grimmett, Kesten and Zhang [19].

In [7], Barlow proved upper and lower gaussian estimates for the probability transitions of a random walk on the supercritical percolation cluster. These are then used to prove a Harnack inequality [7, Theorem 3]. The Liouville property for positive harmonic functions on the percolation cluster follows as well as an estimate of the mean-square displacement of the walk.

Barlow's upper gaussian estimates were also used to prove the invariance principle for the random walk on supercritical percolation clusters by [30], [25], [12]. The invariance principle for the random walk on $\mathbb{Z}^d$ with independent conductances that are bounded below is proved in [8].

Here we show that the existence of the harmonic measure follows from the Green function estimates of [3, Theorem 1.2]. In the case of the two-dimensional percolation cluster, we use both the elliptic and the parabolic Harnack inequalities of [7] and [9].

Whenever the harmonic measure from infinity exists, one can study external diffusion-limited aggregates. Their growth is determined by the harmonic measure which can also be interpreted as the distribution of an electric field on the surface of a grounded conductor with fixed charge of unity. Recent simulations by physicists of the harmonic measure in $\mathbb{Z}^d$ can be found in [1] and of percolation and Ising clusters in [2]. Analytic predictions for the harmonic measure of two dimensional clusters are given by Duplantier in [16] and [17]. See also the survey paper [6].

In contrast, for the internal diffusion-limited aggregates of random walks on percolation clusters, the limiting shape is described in [29] and [15].

The values of the constants $c, C, C' \dots$ may change at each appearance but they are always strictly positive and they do not depend on the environment. The minimum of a and b and the maximum of a and b are respectively denoted by $a \wedge b$ and by $a \vee b$.

1.1. Reversible random walks.

A weighted graph (Γ, a) is given by a countably infinite set Γ and a symmetric function

$$a : \Gamma \times \Gamma \to [0; \infty[$$

which verifies $a(x, y) = a(y, x)$ for all $x, y \in \Gamma$ and

$$\pi(x) := \sum_{y \in \Gamma} a(x, y) > 0 \quad \text{for all} \quad x \in \Gamma.$$

The weight $a(x, y)$ is also called the conductance of the edge connecting x and y since the weighted graph can be interpreted as an electrical or thermic network.

Given a weighted graph (Γ, a), we will write $x \sim y$ if $a(x, y) > 0$. We will always assume that $(\Gamma, \sim)$ is an infinite, locally finite countable graph without multiple edges. A path of length n from x to y is a sequence $x_0, x_1, \dots, x_n$ in Γ such that $x_0 = x$, $x_n = y$ and $x_{i-1} \sim x_i$ for all $1 \leq i \leq n$. The weighted graph (Γ, a) is said to be connected if $(\Gamma, \sim)$ is a connected graph, that is, for all $x, y \in \Gamma$ there is a path of finite length from x to y. The graph distance between two vertices $x, y \in \Gamma$ will be denoted by $D(x, y)$. It is the minimal length of a path from x to y in the graph $(\Gamma, \sim)$. The ball centered at $x \in \Gamma$ of radius R will be denoted by $B(x, R) := \{y \in \Gamma;\ D(x, y) < R\}$.

The random walk on the weighted graph (Γ, a) is the Markov chain on Γ with transition probabilities given by

$$p(x,y) := \frac{a(x,y)}{\pi(x)}, \quad x, y \in \Gamma. \tag{1.1}$$

We denote by P_x the law of the random walk starting at the vertex $x \in \Gamma$. The corresponding expectation is denoted by E_x. The random walk admits reversible measures which are proportional to the measure $\pi(\cdot)$.

For $A \subset \Gamma$, we have the following definitions

$\partial A := \{y \in \Gamma;\ y \notin A \text{ and there is } x \in A \text{ with } x \sim y\}$ and $\overline{A} := \partial A \cup A$,

$\tau_A := \inf\{k \geq 1;\ X_k \in A\}$ and $\overline{\tau}_A := \inf\{k \geq 0;\ X_k \in A\}$

with the convention that $\inf \emptyset = \infty$,

$D(x, A) := \inf\{D(x,y); y \in A\}$,

and for $u : \overline{A} \to \mathbb{R}$ the Laplacian is defined by

$$\mathcal{L}u(x) := \sum_{y \sim x} p(x,y)(u(y) - u(x)), \quad x \in A.$$

A function $u : \overline{A} \to \mathbb{R}$ is *harmonic* in A if for all $x \in A$, $(\mathcal{L}u)(x) = 0$.

The *Green function* of the random walk is defined by

$$G(x,y) := \sum_{k=0}^{\infty} p(x,y,k), \quad x, y \in \Gamma \tag{1.2}$$

where $p(x,y,k) := P_x(X_k = y)$ are the transition probabilities of the walk. Note that $G(\cdot, y)$ is harmonic in $\Gamma \setminus \{y\}$.

For irreducible Markov chains, if $G(x,y) < \infty$ for some $x, y \in \Gamma$ then $G(x,y) < \infty$ for all $x, y \in \Gamma$. The random walk is *recurrent* if $G(x,y) = \infty$ for some $x, y \in \Gamma$ otherwise we say that the walk is *transient.*

1.2. **Results on the existence of the harmonic measure.** Let $X = (X_j)$ be a random walk on a connected weighted graph (Γ, a).

The *hitting distribution* of a set A starting from $x \in \Gamma$ is given by

$$H_A(x,y) := P_x(X_{\tau_A} = y), \quad y \in A.$$

If $P_x(\tau_A < +\infty) > 0$, we also consider

$$\overline{H}_A(x,y) := P_x(X_{\tau_A} = y | \tau_A < +\infty).$$

The *harmonic measure* on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_A(y) := \lim_{D(x,A) \to \infty} \overline{H}_A(x,y), \quad y \in A. \tag{1.3}$$

Our goal is to prove the existence of the harmonic measure for all finite subsets of various weighted graphs. The proof of the existence of the harmonic measure given in [22, section 6.5] for random walks on $\mathbb{Z}^d$, relies on a Harnack inequality and on Green function estimates. Actually, it turns out that only a weak form of Harnack inequality is needed.

In Theorem I, we show that a weaker version of Harnack inequality is a sufficient condition for the existence of the harmonic measure of transient graph. Moreover, weak estimates of the Green function imply the weak Harnack inequality.

As it happens for Brownian motion and for simple random walks (see for instance [27], [23]), the harmonic measure can be expressed in terms of capacities.

The *capacity* of A with respect to B, for $A \subset B \subset \Gamma$, is defined by

$$\mathrm{Cap}_B(A) := \sum_{x \in A} \pi(x) P_x(\overline{\tau}_{B^c} < \tau_A).$$

The *escape probability* of a set A is defined by $\mathrm{Es}_A(x) := P_x(\tau_A = \infty)$ and the capacity of a finite subset $A \subset \Gamma$ is defined by

$$\mathrm{Cap}(A) := \sum_{x \in A} \pi(x) \mathrm{Es}_A(x).$$

The main result for transient graphs is the existence of the harmonic measure for random walks which verify the following weak Harnack inequality.

Definition 1.1. *We say that a weighted graph (Γ, a) satisfies* $\mathbf{wH}(C)$*, the weak Harnack inequality, if there is a constant $C \geq 1$ such that for all $x \in \Gamma$ and for all $R > 0$ there is $R' = R'(x, R)$ such that for any positive harmonic function u on $B(x, R')$,*

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u.$$

Theorem I. *Let (Γ, a) be a weighted graph.*

If (Γ, a) is connected, transient and if it verifies the weak Harnack inequality $\mathbf{wH}(C)$*,*

then for any finite subset $A \subset \Gamma$ the harmonic measure on A exists. That is, for all $y \in A$, the limit (1.3) exists.

Moreover, we have:

$$\lim_{D(x,A) \to \infty} \overline{H}_A(x, y) = \lim_{m \to +\infty} H_A^m(y),$$

where, for m large enough,

$$H_A^m(y) = \frac{\pi(y) P_y(\tau_A > \tau_{\partial B(x_0, m)})}{\mathrm{Cap}_m(A)}$$

where $\mathrm{Cap}_m(A)$ is the capacity of A with respect to $B(x_0, m)$ for some $x_0 \in \Gamma$. The limit does not depend on the choice of x_0.

The following Green function estimates imply the weak Harnack inequality.

Definition 1.2. *We say that a weighted graph (Γ, a) satisfies the Green function estimate* $\mathbf{GE}_\gamma$ *for $\gamma > 0$ if there are constants $0 < C_i \leq C_s < \infty$ and if for all $z \in \Gamma$, there exists $R_z < \infty$ such that for all $x, y \in \Gamma$ with $D(x, y) \geq R_x \wedge R_y$ we have:*

$$\frac{C_i}{D(x,y)^\gamma} \leq G(x, y) \leq \frac{C_s}{D(x,y)^\gamma}. \tag{$\mathbf{GE}_\gamma$}$$

This condition is a weak version of [31, Definition 1] where γ is called a Greenian index. It is used by Telcs [31] to give an upper bound for the probability transitions of a Markov chain in terms of the growth rate of the volume and of the Greenian index.

Proposition 1.3. *Let (Γ, a) be a weighted graph which verifies* $(\mathbf{GE}_\gamma)$ *for some $\gamma > 0$. Then the graph is connected, transient and* $\mathbf{wH}(C)$ *holds with $C = 2^\gamma \dfrac{C_s}{C_i}$.*

In the following corollaries, we describe some weighted graphs where the harmonic measure from infinity exists.

A weighted graph (Γ, a) is said to be *uniformly elliptic* if there is a constant $c \geq 1$ such that for all edges e,

$$c^{-1} \leq a(e) \leq c. \tag{1.4}$$

Corollary 1.4. *Let $(\mathbb{Z}^d, a)$, $d \geq 3$, be a uniformly elliptic graph.*

Then for all finite subsets A of $\mathbb{Z}^d$ and for all $y \in A$, the limit (1.3) exists.

Moreover, we have:

$$\lim_{|x| \to +\infty} \overline{H}_A(x, y) = \lim_{m \to +\infty} H_A^m(y),$$

where $H_A^m(y) = \dfrac{\pi(y) P_y(\tau_A > \tau_{\partial B(0,m)})}{\mathrm{Cap}_m(A)}$.

Indeed, by [14, Proposition 4.2] the Green function of a uniformly elliptic graph $(\mathbb{Z}^d, a)$, $d \geq 3$, verifies the estimates $(\mathbf{GE}_\gamma)$ with $\gamma = d - 2$. The existence of the harmonic measure then follows from proposition 1.3 and Theorem I.

The harmonic measure also exists for a large class of fractal like graphs with some regularity properties. Various examples are given in [10]. See also [32, section 1.1] and the references therein.

The volume of a ball $B(x, R)$ is defined by $V(x, R) := \sum_{x \in B(x,R)} \pi(x)$ and the mean exit time from the ball is $E(x, R) := E_x(\sigma_R)$ where $\sigma_R := \inf\{k \geq 0; X_k \notin B(x, R)\}$.

A weighted graph (Γ, a) has *polynomial volume growth with exponent* $\alpha > 0$ if there is a constant $c \geq 1$ such that for all $x \in \Gamma$ and for all $R \geq 1$,

$$c^{-1} R^\alpha \leq V(x, R) \leq c R^\alpha. \tag{V_α}$$

A weighted graph (Γ, a) has *polynomial mean exit time with exponent* $\beta > 0$ if there is a constant $c \geq 1$ such that for all $x \in \Gamma$ and for all $R \geq 1$,

$$c^{-1} R^\beta \leq E(x, R) \leq c R^\beta. \tag{E_β}$$

As noticed in [10, Theorem 3.1], by [18, Theorem 5.7 and Theorem 6.1], if a weighted graph verifies (V_α) and (E_β) for $\alpha > \beta \geq 2$ and the elliptic Harnack inequality $\mathbf{H}(C)$ then it is transient. Hence we obtain the following corollary of theorem I.

Corollary 1.5. *Let (Γ, a) be a weighted graph verifying (V_α) and (E_β) for $\alpha > \beta \geq 2$ and the elliptic Harnack inequality* $\mathbf{H}(C)$. *Then for all finite subsets $A \subset \Gamma$ and $y \in A$ the limit (1.3) exists.*

The harmonic measure from infinity also exists for random walks in random environment and in particular for the random walk on the supercritical percolation cluster. Before stating this result, we give a brief description of the percolation model. See [21] for more details.

Consider the lattice $\mathbb{Z}^d$, $d \geq 2$, where $x \sim y$ if $|x - y|_1 = 1$ where $|\cdot|_1$ is the ℓ_1-distance. Denote the set of edges by $\mathbb{E}^d$.

Assume that $(a(e); e \in \mathbb{E}^d)$ are i.i.d. non-negative random variables on a probability space $(\Omega, \mathbb{P})$. Call a bond e open if $a(e) > 0$ and closed if $a(e) = 0$. Let $p = \mathbb{P}(a(e) > 0)$. By percolation theory, there exists a critical value $p_c = p_c(\mathbb{Z}^d) \in]0;1[$ such that for $p < p_c$, $\mathbb{P}$ almost surely, all open clusters of ω are finite and for $p > p_c$, $\mathbb{P}$ almost surely, there is a unique infinite cluster of open edges which is called the supercritical cluster. It will be denoted by $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$. The edges of this graph are the open edges of the cluster and the endpoints of these edges are the vertices of the graph.

For $x, y \in \mathcal{C}_\infty(\omega)$, we will write $x \sim y$ if the edge with endpoints x and y is open. The transition probabilities of the random walk on $\mathcal{C}_\infty(\omega)$ are given by (1.1). The law of the paths starting at $x \in \mathcal{C}_\infty(\omega)$ will be denoted by P_x^ω. The random walk on the supercritical percolation cluster corresponds to the case of Bernoulli random variables. In this case, we will write $\mathbb{P}_p$ instead of $\mathbb{P}$.

$D_\omega(x, y)$ will denote the graph distance between x and y in the graph $\mathcal{C}_\infty(\omega)$ and the ball centered at $x \in \mathcal{C}_\infty(\omega)$ of radius R will be denoted by $B_\omega(x, R) = \{y \in \mathcal{C}_\infty(\omega);\ D_\omega(x, y) < R\}$.

The existence of the harmonic measure for $\mathbb{Z}^d$, $d \geq 3$, with i.i.d. conductances, is given in corollary 1.6 below. It follows from the Green function estimates of [3, Theorem 1.2a]. A weaker condition which might hold even if the conductances are not i.i.d. is given in [8, Theorem 6.1].

Corollary 1.6. *Let $(\mathbb{Z}^d, a)$, $d \geq 3$, be a weighted graph where the weights $(a(e); e \in \mathbb{E}^d)$ are i.i.d. non-negative random variables on a probability space $(\Omega, \mathbb{P})$ which verify*

$$\mathbb{P}(a(e) > 0) > p_c(\mathbb{Z}^d).$$

Then there exist constants C_i, C_s, which depend on $\mathbb{P}$ and d, and $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for each $\omega \in \Omega_1$, $(\mathbf{GE}_\gamma)$ holds in $\mathcal{C}_\infty(\omega)$ with the constants C_i and C_s and with $\gamma = d - 2$.

For any finite subset A of $\mathcal{C}_\infty$ and for all $y \in A$, the limit (1.3) exists.

Moreover, we have:

$$\lim_{|x| \to +\infty, x \in \mathcal{C}_\infty} \overline{H}_A(x, y) = \lim_{m \to +\infty} H_A^m(y),$$

where $H_A^m(y) = \frac{\pi(y) P_y^\omega(\tau_A > \tau_{\partial B_\omega(x_0, m)})}{\mathrm{Cap}_m(A)}$ for some $x_0 \in \mathcal{C}_\infty$ and for m large enough.

In [3], both the constant speed random walk and the variable speed random walk are considered. From the expression of their generators one immediately sees that they have the same harmonic functions as the discrete time random walk considered here. Moreover, since they are a time change of each other, the Green function is the same. Hence, by [3, Theorem 1.2 a] the Green function of a uniformly elliptic graph $(\mathbb{Z}^d, a)$, $d \geq 3$, verifies the estimates $(\mathbf{GE}_\gamma)$ with $\gamma = d - 2$. The existence of the harmonic measure then follows from proposition 1.3 and Theorem I.

The harmonic mesure from infinity also exists for recurrent graphs. The main result here is the existence of the harmonic measure for all finite subsets of two-dimensional supercritical percolation clusters.

Theorem II. *Let $(\mathbb{Z}^2, a)$ be a weighted graph where the weights $(a(e); e \in \mathbb{E}^2)$ are i.i.d. random variables on a probability space $(\Omega, \mathbb{P}_p)$ which verify*

$$p = \mathbb{P}_p(a(e) = 1) = 1 - \mathbb{P}_p(a(e) = 0) > p_c(\mathbb{Z}^2).$$

Then $\mathbb{P}_p$ almost surely, for any finite subset A of $\mathcal{C}_\infty(\omega)$ and for all $y \in A$, the limit (1.3) exists.

An expression for the value of the limit (1.3) is given in equation (4.34).

Theorem 1.7. *If $(\mathbb{Z}^2, a)$ is a uniformly elliptic weighted graph then for all finite subsets $A \subset \mathbb{Z}^2$ and for all $y \in A$, the limit (1.3) exists.*

Remark 1.8. *Note that on a regular tree, the harmonic mesure from infinity does not exist for any set A which contains at least two vertices. It would be interesting to investigate the links between the Poisson boundary of a graph and the existence of the harmonic measures. In particular, the triviality of the Poisson boundary does not imply the existence of the harmonic measure as is shown by the lamplighter group $\mathbb{Z}^2 \wr \mathbb{Z}/2\mathbb{Z}$. See* [28] *and the references therein.*

Various forms of Harnack inequality that will be used both for transient graphs and for recurrent graphs are gathered in section 2. The proof of theorem I is given in section 3 while Theorem II is proved in section 4. The last section contains the proof of the annulus Harnack inequality that is used in the proof of Theorem II.

2. Harnack inequalities

We start by recalling a classical form of the Harnack inequality on a graph. Then we give related inequalities and weaker versions.

Definition 2.1. *We say that a weighted graph (Γ, a) satisfies $\mathbf{H}(C)$, the Harnack inequality with shrinking parameter $M > 1$, if there is a constant $C < \infty$ such that for all $x \in \Gamma$ and $R > 0$, and for any positive harmonic function u on $B(x, MR)$,*

$$\max_{B(x,R)} u \le C \min_{B(x,R)} u.$$

In our context, we will use the weak form of Harnack inequality given in definition 1.1. We rewrite this definition under a form similar to definition 2.1. The proofs will be given with these notations.

Definition 2.2. *We say that a weighted graph (Γ, a) satisfies $\mathbf{wH}(C)$, the weak Harnack inequality, if there is a constant $C > 0$ such that for all $x \in \Gamma$ and for all $R > 0$ there is $M_{x,R} \ge 2$ such that for all $M > M_{x,R}$ and for any positive harmonic function u on $B(x, MR)$,*

$$\max_{B(x,R)} u \le C \min_{B(x,R)} u.$$

Barlow [7, Theorem 3] showed that the supercritical percolation cluster verifies another form of Harnack inequality. However, by corollary 1.6 and proposition 1.3 below, the random walk on the supercritical percolation cluster also verifies $\mathbf{wH}(C)$. Given below is a Harnack inequality under the form that will be most useful to us. It is an immediate consequence of Theorem 5.11, proposition 6.11 and of (0.5) of Barlow's work [7].

Harnack Inequality for the percolation cluster [7]. *Let $d \ge 2$ and let $p > p_c(\mathbb{Z}^d)$. There exists $c_1 = c_1(p, d)$ and $\Omega_1 \subset \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, and $R_0(x, \omega)$ such that $3 \le R_0(x, \omega) < \infty$ for each $\omega \in \Omega_1,\ x \in \mathcal{C}_\infty(\omega)$.*

If $R \ge R_0(x, \omega)$ and if $D(x, z) \le \frac{1}{3} R \ln R$ and if $u : \overline{B(z, R)} \to \mathbb{R}$ is positive and harmonic in $B(z, R)$, then

$$\max_{B(z,R/2)} u \le c_1 \min_{B(z,R/2)} u. \tag{2.1}$$

Moreover, there are positive constants c_2, c_3 and ε which depend on p and d such that the tail of $R_0(x,\omega)$ satisfies

$$\mathbb{P}_p(x \in \mathcal{C}_\infty, R_0(x,\cdot) \geq n) \leq c_2 \exp(-c_3 n^\varepsilon). \tag{2.2}$$

In the proof of Theorem I, we will need the Hölder continuity of harmonic functions. It is a consequence of the weak Harnack inequality. Property $\mathbf{wH}(C)$ leads to the following lemma.

Lemma 2.3 (weak Hölder continuity)**.** *Let (Γ, a) be a weighted graph which verifies $\mathbf{wH}(C)$ with shrinking parameters $(M_{x,R}; x \in \Gamma, R > 0)$ where $M_{x,R} \geq 2$ for all $x \in \Gamma$ and $R > 0$.*

Then there exist $\nu > 0,\ c > 0$ such that for all $x_0 \in \Gamma,\ R > 0,\ M \geq M_{x_0,R}$ and for any positive harmonic function u on $B(x_0, MR)$ and $x \in B(x_0, R)$,

$$|u(x) - u(x_0)| \leq c\ (\frac{D(x_0,x)}{R})^\nu \max_{B(x_0,MR)} u.$$

Proof. Let $x_0 \in \Gamma$ and $R > 0$. Then for all $M \geq M_{x_0,R}$ and $R' \leq R$, if u is a positive harmonic function on $B(x_0, MR)$ then

$$\max_{B(x_0,R')} u \leq \max_{B(x_0,R)} u \leq C \min_{B(x_0,R)} u \leq C \min_{B(x_0,R')} u.$$

Let

$$V(i) := \max_{B(x_0,2^i)} u - \min_{B(x_0,2^i)} u.$$

Then for $2^i \leq R$, the functions $u - \min_{B(x_0,2^{i+1})} u$ and $\max_{B(x_0,2^{i+1})} u - u$ are harmonic in $B(x_0, MR)$. Then by the weak Harnack inequality on $B(x_0, 2^i)$,

$$V(i) + V(i+1) \leq C[V(i+1) - V(i)].$$

And so, we deduce that there exists $\lambda < 1$ such that

$$V(i) \leq \lambda\ V(i+1).$$

For any $x \in B(x_0, R)$, we can find N_1 such that $2^{N_1-1} \leq D(x_0,x) \leq 2^{N_1}$. Then

$$|u(x) - u(x_0)| \leq V(N_1).$$

Let N_2 be such that $2^{N_2} \leq R < 2^{N_2+1}$. Then, since $2^{N_2+1} \leq MR$,

$$V(N_1) \leq \lambda^{N_2-N_1+1}\ V(N_2+1)$$

and in particular,

$$|u(x) - u(x_0)| \leq c\ \left(\frac{D(x_0,x)}{R}\right)^\nu \max_{B(x_0,MR)} u$$

where $\nu > 0$ solves $\lambda^{-1} = 2^\nu$ and $c > 0$ is a constant. ∎

Similarly, from Harnack inequality for the supercritical cluster (2.1), we have the following Hölder continuity property.

Proposition 2.4. *Let $d \geq 2$ and let $p > p_c(\mathbb{Z}^d)$. Let Ω_1 and $R_0(x,\omega)$ be given by the Harnack inequality for the supercritical cluster. Then there exist positive constants ν and c such that for each $\omega \in \Omega_1,\ x_0 \in \mathcal{C}_\infty(\omega)$ if $R \geq R_0(x_0,\omega)$ and u is a positive harmonic function on $B_\omega(x_0,R)$ then, for all $x, y \in B_\omega(x_0, R/2)$,*

$$|u(x) - u(y)| \leq c\ \left(\frac{D(x,y)}{R}\right)^\nu \max_{B(x_0,R)} u.$$

We will also need a Harnack inequality in the annulus of the two-dimensional supercritical percolation cluster. It follows from results of Barlow [7], a percolation result due to Kesten [21] and the following estimates of Antal and Pisztora [5, Theorem 1.1 and Corollary 1.3].

For $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, there is a constant $\mu = \mu(p, d) \geq 1$ such that

$$\limsup_{|x|_1 \to \infty} \frac{1}{|x|_1} \ln \mathbb{P}_p[x_0, x \in \mathcal{C}_\infty, D(x_0, x) > \mu |x|_1] < 0 \tag{2.3}$$

and, $\mathbb{P}_p$ almost surely, for $x_0 \in \mathcal{C}_\infty$ and for all $x \in \mathcal{C}_\infty$ such that $D(x_0, x)$ is sufficiently large

$$D(x_0, x) \leq \mu |x - x_0|_1. \tag{2.4}$$

Proposition 2.5. *Let $p > p_c(\mathbb{Z}^2)$. There is a constant $C > 0$ such that $\mathbb{P}_p$-a.s., for all $x_0 \in \mathcal{C}_\infty$ and $r > 0$, if m is large enough,*
then for any positive function u harmonic in $B(x_0, 3\mu m) \setminus B(x_0, r)$,

$$\max_{x; D(x_0,x)=m} u(x) \leq C \min_{x; D(x_0,x)=m} u(x)$$

where μ is the constant that appears in (2.4).

Since we need a construction that is done in section 4.1, the proof of this Harnack inequality is postponed to section 5.

3. Proofs for transient graphs

In this section, we prove proposition 1.3 and Theorem I.

Proof of proposition 1.3. The key ingredient to prove proposition 1.3 is given by Boukricha's lemma [13]. See also [32, p. 37]. Roughly speaking, this lemma ensures that a Harnack inequality holds for general positive harmonic functions as soon as a Harnack inequality holds for the Green function in an annulus.

For $x \in \Gamma$ and $R > 0$, let

$$M_{x,R} = 3 \vee \frac{1}{R} \max_{w \in B(x,R)} R_w. \tag{3.1}$$

We claim that $\mathbf{wH}(C)$ holds with the shrinking parameters $M_{x,R}$ and the constant $C = 2^\gamma \frac{C_s}{C_i}$.

Fix $x_0 \in \Gamma$, $R > 0$, $M > M_{x_0,R}$ and let u be a positive harmonic function on $B(x_0, MR)$.

First note that under $(\mathbf{GE}_\gamma)$, the graph is transient and we can apply Boukricha's lemma ([13], [32, p. 37]) with $B_0 = B(x_0, R)$, $B_1 = B(x_0, (M+1)R)$, $B_2 = B(x_0, (M+2)R)$ and $B_3 = \Gamma$. So we get that if u is harmonic on B_2, then

$$\max_{B_0} u \leq D \min_{B_0} u,$$

with

$$D = \max_{x,y \in B_0} \max_{z \in \overline{B_2} \setminus B_1} \frac{G(x,z)}{G(y,z)}.$$

So, we have to compare $G(x, z)$ and $G(y, z)$ for $x, y \in B_0$ and $z \in B(x_0, (M+2)R) \setminus B(x_0, (M+1)R)$. For all $w \in B_0$, $D(w, z) > MR > R_w$ by (3.1). Hence, by $(\mathbf{GE}_\gamma)$,

$$\frac{C_i}{D(w,z)^\gamma} \leq G(w, z) \leq \frac{C_s}{D(w,z)^\gamma}.$$

Then, we successively have :

$$\begin{aligned} G(x,z) &\leq \frac{C_s}{D(x,z)^\gamma} \\ &= \frac{C_s}{C_i}\left(\frac{D(y,z)}{D(x,z)}\right)^\gamma \frac{C_i}{D(y,z)^\gamma} \\ &\leq \frac{C_s}{C_i}\left(\frac{R+(M+2)R}{(M+1)R-R}\right)^\gamma G(y,z) \\ &\leq \frac{C_s}{C_i}\left(\frac{M+3}{M}\right)^\gamma G(y,z) \\ &\leq 2^\gamma \frac{C_s}{C_i} G(y,z). \end{aligned}$$

■

We can now state the main lemma to prove Theorem I.

Lemma 3.1. *Let* (Γ, a) *be a weighted graph which verifies* $\mathbf{wH}(C)$. *Fix* $x_0 \in \Gamma$.

Let A *be a finite subset of* Γ. *Let* $r_A > 0$ *be such that* $A \subset B(x_0, r_A)$.

For all $M > 2$, *there is* $\lambda_M > 1$ *such that for all* $\lambda > \lambda_M$ *and for all* $y \in A$ *and* $z \in \partial B(x_0, \lambda M r_A)$,

$$P_y(X_{\tau_A \wedge \tau_{\partial B}} = z | \tau_A > \tau_{\partial B}) = H_{\partial B}(x_0, z)[1 + O\Big(M^{-\nu}\Big)], \tag{3.2}$$

where $B = B(x_0, \lambda M r_A)$ *and* $\nu > 0$ *is the Hölder exponent given by lemma 2.3. The constant in* $O(\cdot)$ *depends only on the constants* C *and* c *that appear in* $\mathbf{wH}(C)$ *and in lemma 2.3 respectively.*

Proof. For $M > 2$, choose M_2 and M_3 such that

$$M_2 > M(x_0, M r_A) \quad \text{and} \quad M_3 > M(x_0, M_2 M r_A)$$

where $M(x_0, \cdot)$ are the shrinking parameters that appear in $\mathbf{wH}(C)$.

Let $B_1 = B(x_0, M r_A)$, $B_2 = B(x_0, M_2 M r_A)$ and $B_3 = B(x_0, M_3 M_2 M r_A)$.

For $z \in \partial B_3$, we consider the function

$$f(x) = P_x(X_{\tau_{\partial B_3}} = z), \quad x \in \Gamma.$$

Since f is harmonic on B_2, by lemma 2.3, for all $u \in B_1$,

$$|f(u) - f(x_0)| \leq c\Big(\frac{D(x_0, u)}{M r_A}\Big)^\nu \max_{B_2} f.$$

In particular, for $u \in \partial B(x_0, r_A)$,

$$|f(u) - f(x_0)| \leq \frac{c}{M^\nu} \max_{B_2} f. \tag{3.3}$$

Now by considering f harmonic on B_3, since the graph verifies $\mathbf{wH(C)}$, we have that

$$\max_{B_2} f \leq C f(x_0). \tag{3.4}$$

Therefore, by (3.3) and (3.4), for all $u \in \partial B(x_0, r_A)$,

$$P_u(X_{\tau_{\partial B_3}} = z) = H_{\partial B_3}(x_0, z)\left[1 + O\Big(M^{-\nu}\Big)\right]. \tag{3.5}$$

Introduce the following notation. For U, V and W subsets of Γ with $U \subset V \subset W$. We put

$$\partial V[W,U] = \{x \in \partial V;\ \text{there exist paths in } \Gamma \text{ from } x \text{ to } \partial W \text{ and from } x \text{ to } U\ \}. \tag{3.6}$$

On the set $\{\tau_A < \tau_{\partial B_3}\}$, we let $\eta = \inf\{j \geq \tau_A;\ X_j \in \partial B(x_0, r_A)\}$.

Then using (3.5), we obtain that for all $x \in \partial B(x_0, r_A)[B_3, A]$

$$\begin{aligned} P_x(X_{\tau_{\partial B_3}} = z | \tau_A < \tau_{\partial B}) &= \sum_{u \in \partial B(x_0, r_A)} P_x(X_\eta = u | \tau_A < \tau_{\partial B_3}) P_u(X_{\tau_{\partial B_3}} = z) \\ &= H_{\partial B_3}(x_0, z)[1 + O\Big(M^{-\nu}\Big)] \end{aligned} \tag{3.7}$$

Let $x \in \partial B(x_0, r_A)[B_3, A]$. By (3.5) and (3.7), we get from the relation

$$\begin{aligned} P_x(X_{\tau_{\partial B_3}} = z) &= P_x(X_{\tau_{\partial B_3}} = z | \tau_A > \tau_{\partial B_3}) P_x(\tau_A > \tau_{\partial B_3}) \\ &\quad + P_x(X_{\tau_{\partial B_3}} = z | \tau_A \leq \tau_{\partial B_3})(1 - P_x(\tau_A > \tau_{\partial B_3})), \end{aligned}$$

that

$$P_x(X_{\tau_{\partial B_3}} = z | \tau_A > \tau_{\partial B_3}) = H_{\partial B_3}(x_0, z)[1 + O\Big(M^{-\nu}\Big)]$$

This can also be written as,

$$P_x(X_{\tau_{\partial B_3} \wedge \tau_A} = z) = H_{\partial B_3}(x_0, z) P_x(\tau_A > \tau_{\partial B_3})[1 + O\Big(M^{-\nu}\Big)]. \tag{3.8}$$

Note that every path from y to ∂B_3 must go through some vertex of $\partial B(x_0, r_A)[B_3, A]$. So, for all $y \in A$ and for all $z \in \partial B_3$,

$$\begin{aligned} P_y(X_{\tau_{\partial B_3} \wedge \tau_A} = z) &= \sum_{x \in \partial B(x_0, r_A)[B_3, A]} P_y(X_{\tau_{\partial B(x_0, r_A)[B_3, A]} \wedge \tau_A} = x) P_x(X_{\tau_{\partial B_3} \wedge \tau_A} = z) \\ &\overset{(3.8)}{=} H_{\partial B_3}(x_0, z)[1 + O\Big(M^{-\nu}\Big)] \\ &\qquad \times \sum_{x \in \partial B(x_0, r_A)[B_3, A]} P_y(X_{\tau_{\partial B(x_0, r_A)[B, A]} \wedge \tau_A} = x) P_x(\tau_A > \tau_{\partial B_3}) \\ &= H_{\partial B_3}(x_0, z)[1 + O\Big(M^{-\nu}\Big)] P_y(\tau_A > \tau_{\partial B_3}). \end{aligned}$$

This last equation proves that lemma 3.1 holds with $\lambda_M = M_2 M_3$ where $M_2 = M(x_0, M r_A)$ and $M_3 = M(x_0, M_2 M r_A)$. ■

As in Lawler [23, p. 49], using a last exit decomposition, we obtain the following representation of the hitting distribution in a weighted graph.

Let (Γ, a) be a weighted graph. The Green function of the random walk in $B \subset \Gamma$ is defined by

$$G_B(x, y) := \sum_{k=0}^{\infty} p_B(x, y, k), \quad x, y \in \overline{B}$$

where $p_B(x, y, k) := P_x(X_k = y, k < \overline{\tau}_{B^c})$ are the transition probabilities of the walk with Dirichlet boundary conditions.

Let $A \subset B$ be finite subsets of Γ. Then for all $x \in B^c$ and $y \in A$,

$$H_A(x, y) = \sum_{z \in \partial B} G_{A^c}(x, z) H_{A \cup \partial B}(z, y), \tag{3.9}$$

$$\overline{H}_A(x,y) = \frac{\sum_{z\in\partial B} G_{A^c}(x,z)H_{A\cup\partial B}(z,y)}{\sum_{z\in\partial B} G_{A^c}(x,z)P_z(\tau_A < \tau_{\partial B})}$$

and

$$\min_{z\in\partial B} \frac{H_{A\cup\partial B}(z,y)}{P_z(\tau_A < \tau_{\partial B})} \leq \overline{H}_A(x,y) \leq \max_{z\in\partial B} \frac{H_{A\cup\partial B}(z,y)}{P_z(\tau_A < \tau_{\partial B})}.$$

Then by reversibility, $\pi(z)H_{A\cup\partial B}(z,y) = \pi(y)H_{A\cup\partial B}(y,z)$ and

$P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y}\in A} H_{A\cup\partial B}(z,\tilde{y})$. Hence,

$$\min_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)} \leq \overline{H}_A(x,y) \leq \max_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)} \tag{3.10}$$

We complete the proof of Theorem I with the help of (3.10).

Proof of Theorem I. . Let A be a finite subset of Γ and let $x_0 \in \Gamma$.

Let $r_A > 0$ be such that $A \subset B(x_0, r_A)$.

Let $B = B(x_0, \lambda M r_A)$ where $\lambda \geq \lambda_M$ is given by lemma 3.1.

By equation (3.2), for all $y \in A$ and $z \in \partial B$,

$$\pi(y)H_{A\cup\partial B}(y,z) = H_{\partial B}(x_0,z)[1 + O\Big(M^{-\nu}\Big)]\pi(y)P_y(\tau_A > \tau_{\partial B}). \tag{3.11}$$

By summing over $y \in A$ the equation (3.11) gives,

$$\sum_{y\in A}\pi(y)P_y(X_{\tau_{\partial B}\wedge\tau_A} = z) = H_{\partial B}(x_0,z)[1 + O\Big(M^{-\nu}\Big)]\sum_{y\in A}\pi(y)P_y(\tau_A > \tau_{\partial B}). \tag{3.12}$$

Since (Γ, a) is connected, both sides of (3.12) are positive. So we can divide (3.11) by (3.12). And a short calculation shows that

$$\frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B}\wedge\tau_A} = z)} = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B})}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > \tau_{\partial B})}[1 + O\Big(N^{-\nu}\Big)]$$

where the constant in $O(\cdot)$ still depends only on the constants C and c that appear in $\mathbf{wH(}C\mathbf{)}$ and in lemma 2.3 respectively.

By (3.10), we have that for all $v \notin B$,

$$\min_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B}\wedge\tau_A} = z)} \leq \overline{H}_A(v,y) \leq \max_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B}\wedge\tau_A} = z)}$$

So for all $v \notin B$ we get:

$$\overline{H}_A(v,y) = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B})}{\mathrm{Cap}_B(A)}[1 + O\Big(M^{-\nu}\Big)] \tag{3.13}$$

As v goes to $+\infty$ in an arbitrary way, we will have that $M \to \infty$ as well. Hence, by (3.13), we obtain that $\lim_{v\to+\infty}\overline{H}_A(v,y)$ exists and

$$\lim_{v\to+\infty}\overline{H}_A(v,y) = \lim_{m\to+\infty}\overline{H}^m_A(v,y) = \frac{\pi(y)P_y(\tau_A > +\infty)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > +\infty)}.$$

■

4. Recurrent graphs

In this section, we prove the existence of the harmonic measure for the random walk on a supercritical percolation cluster of $\mathbb{Z}^2$. The proof for the uniformly elliptic random walk on $\mathbb{Z}^2$ is similar but with many simplifications since we can use the estimates of [14] instead of Barlow's estimates.

4.1. **Estimates of the capacity of a box.**

Proposition 4.1. *Let $p > p_c(\mathbb{Z}^2)$. There is a constant $C \geq 1$ such that $\mathbb{P}_p$-a.s. for $x_0 \in \mathcal{C}_\infty$, for all n sufficiently large,*

$$C^{-1} \leq (\ln n)\mathrm{Cap}_{B_\omega(x_0,n)}(\{x_0\}) \leq C. \tag{4.14}$$

Flows of finite energy on the supercritical percolation cluster with respect to convex gauge functions are constructed in [4]. To do so, the flow is expressed by a probability on the set of self-avoiding paths. Here, however, the lower estimate of (4.14) is obtained by combining the method used in $\mathbb{Z}^2$, see [24, Proposition 2.14], with a percolation lemma of Kesten [21, Theorem 7.11].

Proof. The upper bound follows from the variational principle and a comparison with $\mathbb{Z}^2$ (see for instance [32, section 3.1]).

To prove the lower bound, we assume $0 \in \mathcal{C}_\infty$ and for each n sufficiently large, we construct a particular flow θ_n from 0 to $\partial B_\omega(0,n)$. However, it is a difficult task to estimate the energy of a flow from 0 to $\partial B_\omega(0,n)$ consisting of small flows along simple paths from 0 to $\partial B_\omega(0,n)$ since the percolation cluster is very irregular. So, as in Mathieu and Remy [26], using a Theorem of Kesten [21], we construct a grid of open paths in $[-n;n]^2$ which we will call Kesten's grid.

Construction of Kesten's grid

Let us introduce some definitions.

Definition 4.2. *Let $B_{m,n} = ([0;m] \times [0;n]) \cap \mathbb{Z}^2$.*

A horizontal [resp. vertical] channel of $B_{m,n}$ is a path in $\mathbb{Z}^2$ $(x_0, x_1, x_2, ..., x_L)$ such that:

- *$\{x_1, x_2, ..., x_{L-1}\}$ is contained in the interior of $B_{m,n}$*
- *$x_0 \in \{0\} \times [0;n]$ [resp. $x_0 \in [0;m] \times \{0\}$]*
- *$x_L \in \{m\} \times [0;n]$ [resp. $x_L \in [0;m] \times \{n\}$]*

We say that two channels are disjoint if they have no vertex in common. Let $N(m,n)$ be the maximal number of disjoint open horizontal channels in $B_{m,n}$. Then by [21, Theorem 11.1], for $p > p_c$, there is a constant $c(p)$ and some universal constants $0 < c_4, c_5, \xi < \infty$, such that

$$\mathbb{P}_p\big(N(m,n) > c(p)n\big) \geq 1 - c_4(m+1)\exp(-c_5(p-p_c)^\xi n). \tag{4.15}$$

We apply this result to the number of disjoint open channels in a horizontal strip of length n and width $C_K \ln n$ contained in $[-n;n]^2$. If C_K is large enough so that $c_5(p-p_c)^\xi C_K > 3$ then

$$\sum_n nc_4(n+1)\exp(-c_5(p-p_c)^\xi C_K \ln n) < \infty.$$

Hence by (4.15) and Borel-Cantelli lemma, we get that for n large enough there is at least $c(p)C_K \ln n$ disjoint open channels in each horizontal strip of length n and width $C_K \ln n$ contained in $[-n;n]^2$. We do the same construction for vertical strips. Finally, we obtain that each horizontal and each vertical strip of length n and width $C_K \ln n$ in $[-n;n]^2$ contains at least $c(p)C_K \ln n$ disjoint open channels.

Fix n large enough so that there is a Kesten's grid and let J be the largest integer such that

$$(J+1/2)C_K \ln n < n.$$

Set $N := (J+1/2)C_K \ln n$. Divide $[-N;N]^2$ into squares $S_{i,j}$ of side $C_K \ln n$ centered at $(iC_K \ln n; jC_K \ln n)$ for $-J \le i,j \le J$, $i,j \in \mathbb{Z}$. Denote this set of $(2J+1)^2$ squares by $\mathcal{S}_n$.

Since $B_\omega(0,N) \subset \mathcal{C}_\infty \cap [-N;N]^2$, we have that $\mathrm{Cap}_{\mathcal{C}_\infty \cap [-N;N]^2}(0) \le \mathrm{Cap}_{B_\omega(0,N)}(0)$. Hence, to obtain a lower bound, it suffices to construct a flow θ_n from 0 to the vertices of $\mathcal{C}_\infty$ on the sides of $[-N;N]^2$.

Construction of the flow

To each open path $\Pi : (x_0, x_1, x_2, ..., x_L)$ from $x_0 = 0$ to a side of $[-N;N]^2$ with the induced orientation, we associate the unit flow with source at 0, $\Psi^\Pi = \sum_{\ell=1}^{L} (\mathbf{1}_{\{\overrightarrow{e}_\ell\}} - \mathbf{1}_{\{\overleftarrow{e}_\ell\}})$ where $\overrightarrow{e}_\ell$ is the edge from $x_{\ell-1}$ to x_ℓ. The flow θ_n will be a sum of flows Ψ^Π for a set $\mathcal{P}_n$ of well chosen open paths.

Definition 4.3. *A sequence $(S_k; -m \le k \le m)$ of squares of $\mathcal{S}_n$ is called a path of squares if for $-m \le k < m$, the squares S_k and S_{k+1} have a common side.*

We now proceed in three steps. First, to each square of $\mathcal{S}_n$ on the left side of $[-N;N]^2$, we construct a path of squares of $\mathcal{S}_n$ from left to right containing the square centered at $(0;0)$. See figure 1. In the second step, for each of these paths of squares, we contruct $c(p)C_K \ln n$ open paths using Kesten's grid. And in the last step, we show how to connect these open paths to $(0;0)$.

We call *diameter*, a line segment from $(-N; jC_K \ln n)$ to $(N; -jC_K \ln n;)$ for $-J \le j \le J$.

To each diameter, we associate a path of squares $(S_k; -m \le k \le m)$ consisting of the squares of $\mathcal{S}_n$ which intersect the diameter. Whenever a diameter goes through a corner of a square of $\mathcal{S}_n$, add an additional square (there are two choices) in order to obtain a path of squares.

Now fix one of these $2J+1$ paths of squares.

Let $S_k, S_{k+1}, \ldots, S_{k+K-1}$, with $2 \le K \le 2J+1$, be a horizontal stretch of squares within a path of squares that is, a horizontal path of squares of maximal length K. Then each of the $c(p)C_K \ln n$ horizontal open channels crossing this horizontal strip contains an open path from the leftmost side of S_k to the rightmost side of S_{k+K-1} which lies entirely inside $S_k \cup S_{k+1} \cup \ldots \cup S_{k+K-1}$. Indeed, running along an open channel from left to right, it consists of the edges of the channel between the last visited vertex of the leftmost side of S_k to the first visited vertex of the rightmost side of S_{k+K-1}. See case (i) of figure 2. We proceed similarly for all horizontal and vertical stretches of the path of squares.

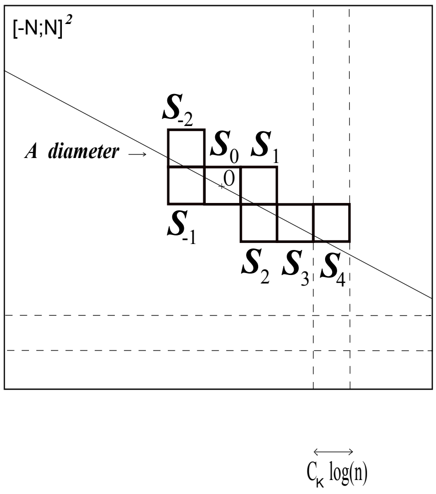

FIGURE 1. A diameter and some squares of the associated path of squares.

Whenever the path of squares turns, the vertical and horizontal open paths of the corresponding stretches must be connected together. For instance, to go right along a horizontal stretch to down along a vertical stretch, the lowest horizontal channel is attached to the leftmost vertical channel, the second lowest horizontal channel is attached to the second leftmost vertical channel and so on. See case (ii) of figure 2. We proceed similarly for the other turns. The open paths obtained by this procedure might not be disjoint but each of their edges belongs to at most two paths. Actually, a slightly more complicated rule would yield disjoint open paths.

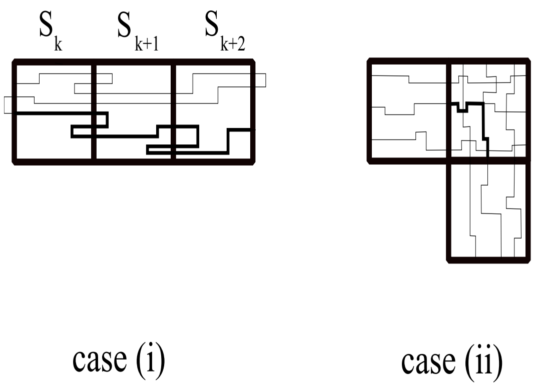

FIGURE 2. In bold, the chosen portion of the open channels.

The second step is completed and we have constructed a set $\mathcal{P}'_n$ of $(2J+1)c(p)C_K \ln n$ open paths from the left to the right side of $[-N;N]^2$.

The last step is to connect these paths to $(0;0)$. Choose two vertices of the infinite cluster $\mathcal{C}_\infty$, x_1 and x_2. x_1 is chosen in the square of $\mathcal{S}_n$ centered at $(0;C_K \ln n)$ and x_2 in the square of $\mathcal{S}_n$ centered at $(0;-C_K \ln n)$. Then for $\ell = 1,2$, let π_ℓ be an open path from x_ℓ to $(0;0)$ for which the chemical distance is attained. By (2.4), for n large enough,

$$\pi_1 \text{ and } \pi_2 \text{ have lengths less than } \ 4\mu C_K \ln n. \tag{4.16}$$

Since each path of $\mathcal{P}'_n$ intersects $\pi_1 \cup \pi_2$, we can associate to each path of $\mathcal{P}'_n$ two open paths from $(0;0)$, one to the left side of $[-N;N]^2$ and the other one to the right side. A specific rule would be to follow a path of $\mathcal{P}'_n$ starting on the left side of $[-N;N]^2$ up to its first intersection with $\pi_1 \cup \pi_2$ which we then follow up to $(0;0)$ and for the second path, start on the right side of $[-N;N]^2$ up to the first intersection with $\pi_1 \cup \pi_2$ which we then follow up to $(0;0)$.

The set $\mathcal{P}_n$ consists of all these open paths from $(0;0)$ to the left or the right side of $[-N;N]^2$ and we set

$$\theta_n = \sum_{\Pi \in \mathcal{P}_n} \Psi^{\Pi}.$$

The intensity of the flow θ_n is

$$(2J+1)c(p)C_K \ln n \geq Cn. \tag{4.17}$$

The energy of the flow

Let us now obtain an upper bound on the energy of the flow θ_n.

A square $S_{i,j} \in \mathcal{S}_n$, $-J \leq i,j \leq J$, belongs to less than $C\ J/(|i|+1)$ paths of squares for some constant $C > 0$ independent of n. Hence for $-J \leq i,j \leq J$,

$$\text{each edge of } S_{i,j} \ \text{ belongs to less than } \ C\ J/(|i|+1) \text{ paths of } \mathcal{P}'_n. \tag{4.18}$$

Finally, gathering (4.16), (4.17), (4.18), by Thomson's principle (see for instance [24, section 2.4]), we get

$$\begin{aligned}
\frac{1}{\mathrm{Cap}_{B_\omega(0,N)}(0)} &\leq \frac{C}{n^2} \sum_{e \in [-N;N]^2} \theta_n(e)^2 \\
&\leq \frac{C}{n^2} \left[\sum_{-J \leq i,j \leq J} \sum_{e \in S_{i,j}} \theta_n(e)^2 + \sum_{e \in \pi_1 \cup \pi_2} \theta_n(e)^2 \right] \\
&\leq \frac{C}{n^2} \left[\sum_{1 \leq j \leq J} j \left(\frac{J}{j} \right)^2 (2C_K \ln n)^2 + n^2 \ln n \right] \\
&\leq C \ln n
\end{aligned}$$

■

4.2. **The Green kernel and its properties.** In this section, we will use the parabolic Harnack inequality for the random walk on the supercritical percolation cluster proved by Barlow and Hambly in [9].

Besides this, we also use the comparison result for D and the $|\cdot|_1$-distance of Antal and Pisztora [5], see (2.4).

Lemma 4.4. *$\mathbb{P}_p$-almost surely, for all $x_0, x \in \mathcal{C}_\infty$, the series*

$$\sum_{k=0}^{\infty} [p(x_0, x_0, k) - p(x, x_0, k)] \tag{4.19}$$

converges. The limit will be denoted by $g(x, x_0)$.

Let $G_{2n}(x, y)$ and $p_{2n}(x, y, k)$ be respectively the Green function and the probability transitions of the random walk in the ball $B(x_0, 2n)$ with Dirichlet boundary conditions. Then

$$g(x, x_0) = \lim_n \sum_{k=0}^{\infty} [p_{2n}(x_0, x_0, k) - p_{2n}(x, x_0, k)] = \lim_n [G_{2n}(x_0, x_0) - G_{2n}(x, x_0)] \tag{4.20}$$

Proof. Let R_0 be given by the Harnack inequality for the supercritical cluster (2.2). Then as in the proof of [7, Proposition 6.1], we have that for $x \in \mathcal{C}_\infty$ and $R \geq R_0(x)$, $B(x, R)$ is very good (see [7, definition 1.7]) with $N_B \leq R^{1/(10(d+2))}$ and it is exceedingly good (see [7, definition 5.4]) .

Now let $R \geq R_0(x) \vee 16$ and let $R_1 = R \ln R$. Then, since $R_1 \geq R_0$, $B = B(x, R_1)$ is very good with $N_B^{2d+4} \leq R_1^{(2d+4)/(10(d+2))} \leq R_1/(2 \ln R_1)$. Then by [9, Theorem 3.1], there exists a constant C_H such that the parabolic Harnack inequality [9, (3.2)] holds in $Q(x, R, R^2)$. Therefore [9, Proposition 3.2] holds with $s(x_0) = R_0(x_0) \vee 16$ and $\rho(x_0, x) = R_0(x_0) \vee 16 \vee D(x_0, x)$

Fix $x_0 \in \mathcal{C}_\infty$ then $v(n, x) := p(x, x_0, n) + p(x, x_0, n+1)$ is a caloric function, that is, it verifies

$$v(n+1, x) - v(n, x) = \mathcal{L}v(n, x), \quad (n, x) \in \mathbb{N} \times \mathcal{C}_\infty.$$

Let $k > 4D(x_0, x)^2$. Let $t_0 = k + 1$ and $r_0 = \sqrt{t_0}$. Then $v(n, x)$ is caloric in $]0, r_0^2] \times B(x_0, r_0)$, $x \in B(x_0, r_0/2)$ since $D(x_0, x) \leq \sqrt{k} < r_0/2$, and $t_0 - \rho(x_0, x)^2 \leq k \leq t_0 - 1$.

Then by the upper gaussian estimates [7, Theorem 5.7] and [9, (2.18)] and by [9, Proposition 3.2], there is $\nu > 0$ such that

$$\begin{aligned} |v(k, x) - v(k, x_0)| &\leq C \left(\frac{\rho(x_0, x)}{\sqrt{t_0}} \right)^{\nu} \sup_{Q_+} v \\ &\leq C \left(\frac{\rho(x_0, x)}{\sqrt{t_0}} \right)^{\nu} \frac{1}{r_0^2} \\ &\leq C \frac{\rho(x_0, x)^{\nu}}{k^{1+\nu/2}}. \end{aligned}$$

Note that we also have that for all $k > 4D(x_0, x)^2$,

$$|p(x, x_0, k) - p(x_0, x_0, k)| \leq C \frac{\rho(x_0, x)^{\nu}}{k^{1+\nu/2}}.$$

Hence (4.19) converges. Then (4.20) follows by Lebesgue dominated convergence theorem. ∎

Lemma 4.5. *There is a constant $c_6 \geq 1$ such that, $\mathbb{P}_p$-a.s., for all $x_0 \in \mathcal{C}_\infty$ there is $\rho = \rho(x_0)$ such that if $D(x_0, x) > \rho$,*

$$c_6^{-1} \ln D(x_0, x) \leq g(x, x_0) \leq c_6 \ln D(x_0, x) \tag{4.21}$$

Proof. Let $x_0 \in \mathcal{C}_\infty$ and $r > 0$. Write $\tau_r := \inf\{k > 0; X_k \in B(x_0, r)\}$ and for $m \geq 1$, write $\sigma_m := \inf\{k > 0; X_k \notin B(x_0, m)\}$.

Note that for all $n > 3\mu m$ where μ is the constant that appears in (2.3), $P_{\cdot}(\sigma_n < \tau_r)$ is harmonic in $B(x_0, 3\mu m) \setminus B(x_0, r)$. Then by the annulus Harnack inequality (proposition 2.5), if m is sufficiently large and $D(y, x_0) = m$, then $\sum_{x \in B(x_0,r)} \pi(x) P_x(\sigma_n < \tau_r)$

$$\begin{aligned}
&= \sum_{x \in B(x_0,r)} \sum_{x'; D(x_0,x')=m} \pi(x) P_x(X(\sigma_m) = x', \sigma_m < \tau_r) P_{x'}(\sigma_n < \tau_r) \\
&\asymp P_y(\sigma_n < \tau_r) \sum_{x \in B(x_0,r)} \sum_{x'; D(x_0,x')=m} \pi(x) P_x(X(\sigma_m) = x', \sigma_m < \tau_r) \\
&\asymp P_y(\sigma_n < \tau_r) \mathrm{Cap}_m(B(x_0, r)).
\end{aligned}$$

By $f_1(y, m, n) \asymp f_2(y, m, n)$ here, we mean that there is a constant $c \geq 1$ which does not depend on y, m, n nor on ω and r, and such that $\mathbb{P}_p$-a.s for m is sufficiently large and $D(y, x_0) = m$, then

$$0 < c^{-1} f_1(y, m, n) \leq f_2(y, m, n) \leq c f_1(y, m, n).$$

Then by the capacity estimates (4.14), for $m = D(y, x_0)$,

$$P_y(\sigma_n < \tau_r) \asymp \frac{\mathrm{Cap}_n(B(x_0, r))}{\mathrm{Cap}_m(B(x_0, r))} \asymp \frac{\ln m}{\ln n} = \frac{\ln D(y, x_0)}{\ln n}. \tag{4.22}$$

It follows from (4.22) and the capacity estimates (4.14) that for m sufficiently large, $D(x, x_0) = m$ and $n > 3\mu m$,

$$\begin{aligned}
G_n(x_0, x_0) - G_n(x, x_0) &= G_n(x_0, x_0) - P_x(\tau_{x_0} < \sigma_n) G_n(x_0, x_0) \\
&= G_n(x_0, x_0) P_x(\tau_{x_0} > \sigma_n) \\
&\asymp \ln n \frac{\ln D(x, x_0)}{\ln n}
\end{aligned}$$

Then (4.21) follows by (4.20). ∎

We will need to work in sets defined in terms of $g(\cdot, x_0)$. Let

$$\widetilde{B}_n := \widetilde{B}(x_0, n) := \{x \in \mathcal{C}_\infty; g(x, x_0) < \ln n\} \text{ and } \widetilde{\sigma}_n := \inf\{k \geq 0; X_k \notin \widetilde{B}(x_0, n)\}. \tag{4.23}$$

Note that by (4.21), for all n sufficiently large,

$$B(x_0, n^{1/c_6}) \subset \widetilde{B}(x_0, n) \subset B(x_0, n^{c_6}). \tag{4.24}$$

Lemma 4.6. *There is a constant $C > 0$ such that, $\mathbb{P}_p$-a.s., for any non empty finite subset A of $\mathcal{C}_\infty$ and $x_0 \in A$, if m sufficiently large and $n > (3\mu m)^{c_6}$, then*

$$\min_{y; m=D(y,x_0)} P_y(\widetilde{\sigma}_n < \tau_A) \geq C(\ln m / \ln n).$$

Proof. The lemma is a consequence of (4.22). For a finite subset A of $\mathcal{C}_\infty$ such that $x_0 \in A \subset B(x_0, r)$, for m sufficiently large and $n > (3\mu m)^{c_6}$,

$$\begin{aligned}\min_{y;m=D(y,x_0)} P_y(\widetilde{\sigma}_n < \tau_A) &\geq \min_{y;m=D(y,x_0)} P_y(\widetilde{\sigma}_n < \tau_{B(x_0,r)}) \\ &\geq \min_{y;m=D(y,x_0)} P_y(\sigma_{B(x_0,n^{1/c_6})} < \tau_{B(x_0,r)}) \\ &\geq C(\ln m / \ln n^{1/c_6})\end{aligned}$$

■

The harmonic measure will be expressed in terms of the function u_A defined below.

Definition 4.7. *$\mathbb{P}_p$-a.s., for a finite subset A of $\mathcal{C}_\infty(\omega)$ and for a fixed $x_0 \in A$, let*

$$u_A(x, x_0) := g(x, x_0) - E_{x,\omega} g(X_{\overline{\tau}_A}, x_0), \quad x \in \mathcal{C}_\infty(\omega).$$

Note that,

$$\begin{aligned}&u_A(\cdot, x_0) = 0 \quad \text{on} \quad A,\\ &u_A(x, x_0) \asymp \ln D_\omega(x_0, x) \quad \text{as} \quad D_\omega(x_0, x) \to \infty \quad \text{by} \quad (4.21),\\ &Pu_A(x, x_0) = Pg(x, x_0) - \sum_{y \sim x} p(x, y) E_y g(X_{\overline{\tau}_A}, x_0)\\ &\qquad = g(x, x_0) - \mathbf{1}_{x_0}(x) - E_x g(X_{\tau_A}, x_0), \quad x \in \mathcal{C}_\infty(\omega).\end{aligned}$$

The next lemma is the analogue of [22, Proposition 6.4.7].

Proposition 4.8. *$\mathbb{P}_p$-a.s., for a finite subset A of $\mathcal{C}_\infty(\omega)$ and for $x_0 \in A$ and $x \in A^c$,*

$$u_A(x, x_0) = \lim_n (\ln n) P_x(\widetilde{\sigma}_n < \tau_A).$$

Proof. Let $R_0(z, \omega)$ be as the Harnack inequality for the supercritical cluster (2.2). By (2.3) of Antal and Pisztora, and by (4.24)

$$\sum_n \sum_{z \in \partial \widetilde{B}(x_0, n)} \mathbb{P}_p(z \in \mathcal{C}_\infty, R_0(z, \cdot) \geq n^{1/c_6}) \leq C \sum_n n^{2c_6} \exp(-c_3 n^{\varepsilon/c_6}) < \infty.$$

Therefore, by Borel-Cantelli, there is $\Omega_1 \subset \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, such that for all $\omega \in \Omega_1$ there is n_0 such that for all $n \geq n_0$ and for all $z \in \partial \widetilde{B}(x_0, n)$, $R_0(z) < n^{1/c_6}$.

Let $z \in \partial \widetilde{B}(x_0, n)$ where $n \geq n_0$. Then there is $z' \in \widetilde{B}(x_0, n)$ such that $z' \sim z$ and

$$g(z', x_0) < \ln n \leq g(z, x_0).$$

Moreover, by (4.24), $D(z, x_0) > n^{1/c_6}$. Then by Hölder's continuity property given in proposition 2.4 and by (4.21),

$$\begin{aligned}0 \leq g(z, x_0) - \ln n &\leq g(z, x_0) - g(z', x_0)\\ &\leq c \left(\frac{1}{n^{1/c_6}}\right)^\nu \max_{B(z, n^{1/c_6})} g(\cdot, x_0)\\ &\leq \frac{c}{C} \left(\frac{1}{n^{1/c_6}}\right)^\nu \ln n. \qquad (4.25)\end{aligned}$$

By the optional stopping theorem applied to the martingale $g(X_k, x_0)$, $k \geq 0$ and for n large enough and $x \in \widetilde{B}(x_0, n) \setminus A$,

$$\begin{aligned} g(x, x_0) &= E_x\left[g(X_{\tau_A \wedge \widetilde{\sigma}_n}, x_0)\right], \\ &= P_x(\widetilde{\sigma}_n < \tau_A) E_x\left[g(X_{\widetilde{\sigma}_n}, x_0) \mid \widetilde{\sigma}_n < \tau_A\right] \\ &\quad + P_x(\tau_A < \widetilde{\sigma}_n) E_x\left[g(X_{\tau_A}, x_0) \mid \tau_A < \widetilde{\sigma}_n\right]. \end{aligned} \tag{4.26}$$

But

$$\begin{aligned} \lim_n P_x(\tau_A < \widetilde{\sigma}_n) E_x\left[g(X_{\tau_A}, x_0) \mid \tau_A < \widetilde{\sigma}_n\right] &= \lim_n E_x\left[g(X_{\tau_A}, x_0); \tau_A < \widetilde{\sigma}_n\right] \\ &= E_x g(X_{\tau_A}, x_0) \end{aligned}$$

Therefore by (4.25) and (4.26), $u_A(x, x_0) = \lim_n (\ln n) P_x(\widetilde{\sigma}_n < \tau_A)$. ∎

We can now prove the analogue of lemma 3.1 for the supercritical cluster. Theorem II will follow from this lemma and from proposition 4.8 above.

Lemma 4.9. *Let $p > p_c(\mathbb{Z}^2)$. Let Ω_1 and $R_0(x, \omega)$ be as in the Harnack inequality for the percolation cluster (2.2). There is $\nu' > 0$ such that the following holds.*

Let $\omega \in \Omega_1$ and let A be a finite subset of $\mathcal{C}_\infty(\omega)$. Fix $x_0 \in A$.

Then there is $N_0 = N_0(x_0, A, \omega)$ such that for all $n > N_0$, for all $y \in A$ and $z \in \partial\widetilde{B}(x_0, n)$,

$$H_{A \cup \partial\widetilde{B}(x_0,n)}(y, z) = P_y(\tau_A > \widetilde{\sigma}_n) H_{\partial\widetilde{B}(x_0,n)}(x_0, z) \left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] \tag{4.27}$$

where $\widetilde{B}_n$ and $\widetilde{\sigma}_n$ are as in (4.23). $\nu' > 0$ depends on the Hölder exponent given by proposition 2.4 and the constants given in (4.21) The constant in $O(\cdot)$ depends on ω and A and on the constants that appear in (2.1), (2.2) and in proposition 2.4.

Proof. Let m be sufficiently large so that $A \subset B(x_0, m)$ and so that (4.25) holds for all $n > (3\mu m)^{c_6}$.

For $R_1 > \max\{R_0(x_0, \omega), (3\mu m)/4, m\}$, let $B_1 = B(x_0, R_1)$, $B_2 = B(x_0, 2R_1)$, $B_3 = B(x_0, 4R_1)$. Set $n = (4R_1)^{c_6}$ and let $\widetilde{B}_n = \widetilde{B}(x_0, n)$ and $\widetilde{\sigma}_n$ be as in (4.23). Note that by (4.24), $B_3 \subset \widetilde{B}_n$ and (4.25) holds.

For $z \in \partial\widetilde{B}_n$, consider the function

$$f(x) = P_x(X_{\widetilde{\sigma}_n} = z), \quad x \in \mathcal{C}_\infty(\omega).$$

Since f is harmonic on B_2, by proposition 2.4, for all $u \in B_1$,

$$|f(u) - f(x_0)| \leq c\Big(\frac{D(x_0, u)}{R_1}\Big)^\nu \max_{B_2} f.$$

In particular, for $u \in \partial B(x_0, m)$,

$$|f(u) - f(x_0)| \leq c\Big(\frac{m}{R_1}\Big)^\nu \max_{B_2} f. \tag{4.28}$$

Now by considering f harmonic on B_3, by (2.1), we have that

$$\max_{B_2} f \leq c_1 f(x_0). \tag{4.29}$$

Therefore, by (4.28) and (4.29), for all $u \in \partial B(x_0, m)$,

$$P_u(X_{\widetilde{\sigma}_n} = z) = H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right]. \tag{4.30}$$

On the set $\{\tau_A < \widetilde{\sigma}_n\}$, we let $\eta = \inf\{j \geq \tau_A;\ X_j \in \partial B(x_0, m)\}$.

Then using (4.30), we obtain that for all $x \in \partial B(x_0, m)[\widetilde{B}_n, A]$ (see (3.6) for the notation),

$$\begin{aligned} P_x(X_{\widetilde{\sigma}_n} = z | \tau_A < \widetilde{\sigma}_n) &= \sum_{u \in \partial B(x_0,m)} P_x(X_\eta = u | \tau_A < \widetilde{\sigma}_n) P_u(X_{\widetilde{\sigma}_n} = z) \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right]. \end{aligned} \tag{4.31}$$

Let $x \in \partial B(x_0, m)[\widetilde{B}_n, A]$. By (4.30), (4.31) and (4.25), we get from the relation

$$\begin{aligned} P_x(X_{\widetilde{\sigma}_n} = z) &= P_x(X_{\widetilde{\sigma}_n} = z | \tau_A > \widetilde{\sigma}_n) P_x(\tau_A > \widetilde{\sigma}_n) \\ &\quad + P_x(X_{\widetilde{\sigma}_n} = z | \tau_A \leq \widetilde{\sigma}_n)(1 - P_x(\tau_A > \widetilde{\sigma}_n)), \end{aligned}$$

that

$$\begin{aligned} P_x(X_{\widetilde{\sigma}_n} = z | \tau_A > \widetilde{\sigma}_n) &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + \frac{1}{P_x(\tau_A > \widetilde{\sigma}_n)} O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big) + O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right] \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{\ln m}\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right] \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] \end{aligned}$$

where $\nu' = \nu / c_6 > 0$ and where the constant in the last $O(\cdot)$ now depends on ω and A. This can also be written as,

$$P_x(X_{\widetilde{\sigma}_n \wedge \tau_A} = z) = H_{\partial \widetilde{B}_n}(x_0, z) P_x(\tau_A > \widetilde{\sigma}_n)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right]. \tag{4.32}$$

Note that every path from $y \in A$ to $\partial \widetilde{B}_n$ must go through some vertex of $\partial B(x_0, m)[\widetilde{B}_n, A]$. So, for all $y \in A$ and for all $z \in \partial \widetilde{B}_n$,

$$\begin{aligned} P_y(X_{\widetilde{\sigma}_n \wedge \tau_A} = z) &= \sum_{x \in \partial B(x_0,m)[\widetilde{B}_n, A]} P_y(X_{\tau_{\partial B(x_0,m)[\widetilde{B}_n,A]} \wedge \tau_A} = x) P_x(X_{\widetilde{\sigma}_n \wedge \tau_A} = z) \\ &\overset{(4.32)}{=} H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] \\ &\qquad \times \sum_{x \in \partial B(x_0,m)[\widetilde{B}_n, A]} P_y(X_{\tau_{\partial B(x_0,m)[\widetilde{B}_n,A]} \wedge \tau_A} = x) P_x(\tau_A > \widetilde{\sigma}_n) \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] P_y(\tau_A > \widetilde{\sigma}_n). \end{aligned}$$

Hence (4.27) holds with $N_0 = (4 \max\{R_0(x_0, \omega), (3\mu m)/4, m\})^{c_6}$. ∎

4.3. **The existence of the harmonic measure.** We now show how to obtain Theorem II from lemma 4.9.

Proof. Let $y \in A$. Let $\widetilde{B}_n$ and $\widetilde{\sigma}_n$ be as in (4.23). For $x \notin \widetilde{B}_n$, by (3.9), by reversibility of the Markov chain and by (4.27), for all $n > N_0$,

$$\begin{aligned}
\pi(x)H_A(x,y) &= \pi(x)P_x(X_{\tau_A}=y) \\
&= \pi(x)\sum_{z\in\partial\widetilde{B}_n} G_{A^c}(x,z)H_{A\cup\partial\widetilde{B}_n}(z,y) \\
&= \sum_{z\in\partial\widetilde{B}_n} G_{A^c}(z,x)\pi(y)H_{A\cup\partial\widetilde{B}_n}(y,z) \\
&= \sum_{z\in\partial\widetilde{B}_n} G_{A^c}(z,x)\pi(y)P_y(\widetilde{\sigma}_n<\tau_A)H_{\partial\widetilde{B}_n}(x_0,z)\left[1+O\Big(n^{-\nu'}\Big)\right] \\
&= \pi(y)P_y(\widetilde{\sigma}_n<\tau_A)\sum_{z\in\partial\widetilde{B}_n} G_{A^c}(z,x)H_{\partial\widetilde{B}_n}(x_0,z)\left[1+O\Big(n^{-\nu'}\Big)\right]. \qquad (4.33)
\end{aligned}$$

At this point for the supercritical cluster of $\mathbb{Z}^d$, $d \geq 3$, it suffices to sum over $y \in A$ and divide the equations. However, since the walk is recurrent on the supercritical percolation cluster of $\mathbb{Z}^2$, $P_y(\widetilde{\sigma}_n < \tau_A) \to 0$ as $n \to \infty$, this would lead to an indeterminate limit. But by (4.33),

$$\begin{aligned}
H_A(x,y) &= \frac{\pi(x)H_A(x,y)}{\pi(x)\sum_{y'\in A}H_A(x,y')} \\
&= \frac{\pi(y)P_y(\widetilde{\sigma}_n<\tau_A)}{\sum_{y'\in A}\pi(y')P_{y'}(\widetilde{\sigma}_n<\tau_A)}\left[1+O\Big(n^{-\nu'}\Big)\right]
\end{aligned}$$

and by proposition 4.8,

$$\begin{aligned}
\lim_{x\to\infty} H_A(x,y) &= \lim_{n\to\infty}\frac{(\ln n)\pi(y)P_y(\widetilde{\sigma}_n<\tau_A)}{(\ln n)\sum_{y'\in A}\pi(y')P_{y'}(\widetilde{\sigma}_n<\tau_A)}\left[1+O\Big(n^{-\nu'}\Big)\right] \\
&= \frac{\pi(y)Pu_A(y,x_0)}{\sum_{y'\in A}\pi(y')Pu_A(y',x_0)}. \qquad (4.34)
\end{aligned}$$

■

5. Proof of proposition 2.5

In this proof, we keep the notations of [7] except for the graph distance which will still be denoted by $D(x,y)$.

For a cube Q of side n, let $Q^+ := A_1 \cap \mathbb{Z}^d$ and $Q^\oplus := A_2 \cap \mathbb{Z}^d$ where A_1 and A_2 are the cubes in $\mathbb{R}^d$ with the same center as Q and with side length $\frac{3}{2}n$ and $\frac{6}{5}n$ respectively. Note that $Q \subset Q^\oplus \subset Q^+$.

$\mathcal{C}(x)$ is the connected open cluster that contains x. $\mathcal{C}_Q(x)$, which will be called the open Q cluster, is the set of vertices connected to x by an open path within Q. And $\mathcal{C}^\vee(Q)$ is the largest open Q cluster (with some rule for breaking ties).

Set $\alpha_2 = (11(d+2))^{-1}$.

Proof. By [7, lemma 2.24] and by Borel-Cantelli lemma, for all $x \in \mathbb{Z}^d$, there is N_x such that for all $n > N_x$, $L(Q)$ (see [7, p. 3052]) holds for all cubes Q of side n with $x \in Q$.

Let $z \in \mathbb{Z}^d$ and let $n > N_z = N_z(\omega)$.

Let Q be a cube of side n which contains z.

Let $x_0 \in \mathcal{C}^\vee(Q^+) \cap Q^\oplus$ with $Q(x_0, r+k_0)^+ \subset Q^+$ where $C_H n^{\alpha_2} \le r \le n$ and $k_0 = k_0(p,d)$ is the integer chosen in [7, p. 3041].

Let R be such that

$$B_\omega(x_0, (3/2)R\ln R) \subset Q^\oplus \qquad \text{and} \tag{5.35}$$

$$(C_H n^{\alpha_2})^{d+2} \le (C_H n^{\alpha_2})^{4(d+2)} < R < R\ln R < n. \tag{5.36}$$

Then by [7, Theorem 2.18c], $B_\omega(x_0, R\ln R)$ is (C_V, C_P, C_W)- very good with

$$N_{B_\omega(x_0, R\ln R)} \le C_H n^{\alpha_2}$$

with the constants given in [7, section 2].

Then by [7, Theorem 5.11] and (5.36), there is a constant C_1, which depends only on d and on the constants C_V, C_P, C_W, such that if $D(x_0, x_1) \le \frac{1}{3} R\ln R$ and if $h : \overline{B(x_1, R)} \to \mathbb{R}$ is positive and harmonic in $B(x_1, R)$, then

$$\max_{B(x_1, R/2)} h \le C_1 \min_{B(x_1, R/2)} h. \tag{5.37}$$

Note that since $4\alpha_2(d+2) = 4/11 < 1/2$, the conditions (5.36) are verified for $R = 2\sqrt{n}$ when n large enough.

We now apply a standard chaining argument to a well chosen covering by balls (see for instance [32, chapters 3 and 9]). Let $x_0 \in \mathbb{Z}^2$ and consider environments such that $x_0 \in \mathcal{C}_\infty(\omega)$. The main difficulty to carry out the chaining argument is to check that the intersection of "consecutive" balls is not empty. The remainder of the proof is to construct an appropriate covering of $\{x \in \mathcal{C}_\infty; D(x_0, x) = m\}$, for m large enough, with a finite number balls, which does not depend on x_0, m or ω, and such that the Harnack inequality (5.37) holds in each ball.

Let δ_1, δ_2 and δ_3 be three positive real numbers such that

$$2\delta_2 < \delta_1 \quad \text{and} \quad \delta_1 + 2\delta_2 < \delta_3 < \frac{1}{5\mu}\left(\frac{4}{5} - \delta_2\right). \tag{5.38}$$

For instance, choose δ_3 so that $0 < \delta_3 < 4/(50\mu)$, then choose δ_1 so that $0 < 2\delta_1 < \delta_3$ and finally choose δ_2 so that $\delta_2 < \min\{\delta_1/2, 4/(50\mu)\}$.

Let $n > N_{x_0}$.

Furthermore, take n large enough so that there is a Kesten's grid in Q with constant C_K and $R(Q)$ holds (by [7, lemma 2.8]). That is in each vertical and each horizontal strip of width $C_K \ln n$ contains at least $c(p) C_K \ln n$ open disjoint channels. Moreover, since $R(Q)$ holds, $\mathcal{C}^\vee(Q) \subset \mathcal{C}^\vee(Q^+)$. In particular, $x_0 \in \mathcal{C}^\vee(Q^+) \cap Q^\oplus$.

Furthermore by (2.3) and Borel-Cantelli, if m is large enough then for all $x, y \in \mathcal{C}_\infty$ such that $|x|_1 \le 3\mu m$, $|y|_1 \le 3\mu m$ and $|x-y|_1 \ge m(\delta_1 - 2\delta_2)/\mu$ we have

$$|x-y|_1 \le D(x,y) \le \mu |x-y|_1.$$

Set $\dfrac{R}{2} = m\delta_3 = \sqrt{n}$.

Furthermore, take m large enough so that (5.35) and (5.36) are verified as well as

$$C_K \ln n < m\delta_2/\mu, \quad 3m\mu < \frac{1}{3}R\ln R \quad \text{and} \quad r < 4m\delta_3.$$

Instead of constructing a finite covering of $\{x \in \mathcal{C}_\infty; D(x_0,x) = m\}$, it is easier to construct a finite covering of the region $\{x \in \mathcal{C}_\infty; \frac{4m}{5\mu} \le |x-x_0|_1 \le 2m\}$ which is a larger subset of $\mathbb{Z}^2$.

Let $\mathcal{I} := \{(i;j) \in \mathbb{N}^2; 4/(5\delta_1) \le i+j \le 2\mu/\delta_1.\}$ Let M be the cardinal of $\mathcal{I}$.

Let $x_{i,j} = x_0 + (im\delta_1/\mu; jm\delta_1/\mu)$ with $(i;j) \in \mathcal{I}$. Then for each $x_{i,j}$ with $(i;j) \in \mathcal{I}$, there is $\widetilde{x}_{i,j} \in \mathcal{C}_\infty$ such that $|x_{i,j} - \widetilde{x}_{i,j}|_1 \le m\delta_2/\mu$.

We proceed similarly in the other three quadrants to obtain a set of $4M$ vertices which we denote by $\mathcal{D}$. Note that M does not depend on m.

The finite covering of the region $\frac{4m}{5\mu} \le |x-x_0|_1 \le 2m$ is

$$\{B(\widetilde{x}, m\delta_3), \quad \widetilde{x} \in \mathcal{D}\}.$$

Note that each ball contains the center of the four neighbouring balls except those on the boundary of the region. But these are connected to at least one neighbouring ball. Indeed, if $\widetilde{x}, \widetilde{y} \in \mathcal{D}$ are neighbouring centers then by (5.38),

$$D(\widetilde{x},\widetilde{y}) < \mu|\widetilde{x}-\widetilde{y}| < m(\delta_1 + 2\delta_2) < m\delta_3.$$

If $\widetilde{x} \in \mathcal{D}$ then by (5.38),

$$D(x_0,\widetilde{x}) > \frac{m}{\mu}\left(\frac{4}{5} - \delta_2\right) > 5m\delta_3,$$

$D(x_0,\widetilde{x}) < \mu|x_0 - \widetilde{x}|_1 < 2m\mu$ and $\mu\,(2m + m\delta_2/\mu) < 3m\mu$.

Therefore, $B(x_0, r)$ does not belong to a ball of the covering and u is harmonic in each ball $B(\widetilde{x}, 2m\delta_3)$ with $\widetilde{x} \in \mathcal{D}$. Then the Harnack inequality holds for $R = 2m\delta_3$ since for all $\widetilde{x} \in \mathcal{D}$,

$$D(x_0,\widetilde{x}) < 2m\mu < \frac{1}{3}R\ln R.$$

■

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