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# EXISTENCE OF THE HARMONIC MEASURE FOR RANDOM WALKS ON GRAPHS AND IN RANDOM ENVIRONMENTS

DANIEL BOIVIN AND CLÉMENT RAU

**ABSTRACT.** We give a sufficient condition for the existence of the harmonic measure from infinity of transient random walks on weighted graphs. In particular, this condition is verified by the random conductance model on  $\mathbb{Z}^d$ ,  $d \geq 3$ , when the conductances are i.i.d. and the bonds with positive conductance percolate. The harmonic measure from infinity also exists for random walks on supercritical clusters of  $\mathbb{Z}^2$ . This is proved using results of Barlow (2004).

*Keywords:* Harmonic measure, supercritical percolation clusters, Harnack inequality, random conductance model.

*Subject Classification:* 60J05, 60K35, 60K37

## 1. INTRODUCTION AND RESULTS

The harmonic measure from infinity of a closed subset  $A$  of  $\mathbb{R}^d$ ,  $d \geq 2$ , is the hitting distribution of the set  $A$  by a  $d$ -dimensional Brownian motion started at infinity. A detailed description of this measure is given by Mörters and Peres in [26, section 3.4].

Similarly, given a Markov chain on an infinite graph, the harmonic measure of a finite subset of the graph is defined as the hitting distribution of the set by the Markov chain starting at infinity. The existence of the harmonic measure for the simple symmetric random walk on  $\mathbb{Z}^d$  is shown by Lawler in [22, chapter 2] and it is extended to a wider class of random walks on  $\mathbb{Z}^d$  by Lawler and Limic in [21, section 6.5].

From these results, one might expect that the existence of the harmonic measure for a Markov chain on  $\mathbb{Z}^d$ ,  $d \geq 2$ , relies on its Green function asymptotics. The goal of this paper is to show that actually, the existence of the harmonic measure is a fairly robust result in the sense that for a random walk on a weighted graph, it holds as soon as there is a weak form of Harnack inequality. In particular, it is verified by a large family of fractal-like graphs and by random conductance models on  $\mathbb{Z}^d$ ,  $d \geq 3$ , given by a sequence of i.i.d. conductances as soon as there is percolation of the positive conductances. This is done using recent estimates of [3].

In  $\mathbb{Z}^2$ , although we do not give a general sufficient condition for recurrent graphs, we show the existence of the harmonic measure for the random walk on the supercritical cluster using some estimates of Barlow [6].

The results of [3] for the random conductance model are part of a long series of works which go back to homogenization of divergence form elliptic operators with random coefficients and to the investigation of the properties of the supercritical percolation cluster.

Some highlights of the properties of the random walk on the supercritical percolation cluster of  $\mathbb{Z}^d$  is the proof of the Liouville property for bounded harmonic functions (see Kaimanovich [19]

and [10]) and the proof of the transience of the walk when  $d \geq 3$  by Grimmett, Kesten and Zhang [18].

In [6], Barlow proved upper and lower gaussian estimates for the probability transitions of a random walk on the supercritical percolation cluster. These are then used to prove a Harnack inequality [6, Theorem 3]. The Liouville property for positive harmonic functions on the percolation cluster follows as well as an estimate of the mean-square displacement of the walk.

Barlow's upper gaussian estimates were also used to prove the invariance principle for the random walk on supercritical percolation clusters by [29], [24], [11]. The invariance principle for the random walk on  $\mathbb{Z}^d$  with independent conductances that are bounded below is proved in [7].

In this paper, we show how to prove the existence of the harmonic measure from the Green function estimates of Andres, Barlow, Deuschel and Hambly [3, Theorem 1.2]. In the case of the two-dimensional percolation cluster, we use the Harnack inequality of [6].

Whenever the harmonic measure from infinity exists, one can study external diffusion-limited aggregates. The shape of the internal diffusion-limited aggregates of random walks on percolation clusters is described in [28] and [14].

The harmonic measure is of interest to physicists as it can be expressed as the normal derivative of an electrical potential on the surface. Recent simulations of the harmonic measure in  $\mathbb{Z}^d$  can be found in [1] and on percolation and Ising clusters in [2]. Analytic predictions for the harmonic measure of two dimensional cluster are given by Duplantier in [15] and [16].

The values of the constants  $c, C, \dots$  may change at each appearance but they are always strictly positive and they do not depend on the environment. The minimum of  $a$  and  $b$  and the maximum of  $a$  and  $b$  are respectively denoted by  $a \wedge b$  and by  $a \vee b$ .

**1.1. Reversible random walks.** A weighted graph  $(\Gamma, a)$  is given by a countably infinite set  $\Gamma$  and a symmetric function

$$a : \Gamma \times \Gamma \rightarrow [0; \infty[$$

which verifies  $a(x, y) = a(y, x)$  for all  $x, y \in \Gamma$  and

$$\pi(x) := \sum_{y \in \Gamma} a(x, y) > 0 \text{ for all } x \in \Gamma.$$

The weight  $a(x, y)$  is also called conductance since it can be interpreted as the electrical or thermic conductance of the edge connecting  $x$  and  $y$ .

Given a weighted graph  $(\Gamma, a)$ , we will write  $x \sim y$  if  $a(x, y) > 0$ . We will always assume that  $(\Gamma, \sim)$  is an infinite, locally finite countable graph without multiple edges. We will say that the weighted graph  $(\Gamma, a)$  is connected if  $(\Gamma, \sim)$  is a connected graph, that is, for all  $x, y \in \Gamma$  there is a sequence  $x_0, x_1, \dots, x_n$  such that  $x_0 = x$ ,  $x_n = y$  and  $x_{i-1} \sim x_i$  for all  $1 \leq i \leq n$ . The graph distance between two vertices  $x, y \in \Gamma$  will be denoted by  $D(x, y)$ . It is the minimal number of edges in a path from  $x$  to  $y$  in the graph  $(\Gamma, \sim)$ . The ball centered at  $x \in \Gamma$  of radius  $R$  will be denoted by  $B(x, R) := \{y \in \Gamma; D(x, y) < R\}$ .

The random walk on the weighted graph  $(\Gamma, a)$  is the Markov chain on  $\Gamma$  with transition probabilities given by

$$p(x, y) := \frac{a(x, y)}{\pi(x)}, \quad x, y \in \Gamma. \tag{1.1}$$

We denote by  $P_x$  the law of the random walk starting at the vertex  $x \in \Gamma$ . The corresponding expectation is denoted by  $E_x$ . The random walk admits reversible measures which are proportional to the measure  $\pi(\cdot)$ .

For  $A \subset \Gamma$ , we have the following definitions

$$\partial A := \{y \in \Gamma; y \notin A \text{ and there is } x \in A \text{ with } x \sim y\} \text{ and } \bar{A} := \partial A \cup A,$$

$$\tau_A := \inf\{k \geq 1; X_k \in A\} \text{ and } \bar{\tau}_A := \inf\{k \geq 0; X_k \in A\}$$

with the convention that  $\inf \emptyset = \infty$ ,

$$D(x, A) = \inf\{D(x, y); y \in A\},$$

and for  $u : \bar{A} \rightarrow \mathbb{R}$  the Laplacian is defined by

$$\mathcal{L}u(x) := \sum_{y \sim x} p(x, y)(u(y) - u(x)), \quad x \in A.$$

A function  $u : \bar{A} \rightarrow \mathbb{R}$  is *harmonic* in  $A$  if for all  $x \in A$ ,  $(\mathcal{L}u)(x) = 0$ .

The *Green function* of the random walk is defined by

$$G(x, y) := \sum_{k=0}^{\infty} p(x, y, k), \quad x, y \in \Gamma \quad (1.2)$$

where  $p(x, y, k) := P_x(X_k = y)$  are the transition probabilities of the walk. Note that  $G(\cdot, y)$  is harmonic in  $\Gamma \setminus \{y\}$ .

For irreducible Markov chains, if  $G(x, y) < \infty$  for some  $x, y \in \Gamma$  then  $G(x, y) < \infty$  for all  $x, y \in \Gamma$ . The random walk is *recurrent* if  $G(x, y) = \infty$  for some  $x, y \in \Gamma$  otherwise we say that the walk is *transient*.

**1.2. Results on the existence of the harmonic measure.** Let  $X = (X_j)$  be a random walk on a connected weighted graph  $(\Gamma, a)$ .

The *hitting probability* of a set  $A$  starting from  $x \in \Gamma$ ,  $y \in A$  by :

$$H_A(x, y) = P_x(X_{\tau_A} = y).$$

If  $P_x(\tau_A < +\infty) > 0$ , we can also define :

$$\bar{H}_A(x, y) = P_x(X_{\tau_A} = y | \tau_A < +\infty).$$

The *harmonic measure* on a finite subset  $A$  of  $\Gamma$  is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_A(y) = \lim_{D(x, A) \rightarrow \infty} \bar{H}_A(x, y), \quad y \in A. \quad (1.3)$$

Our goal is to prove the existence of the harmonic measure for all finite subsets of various weighted graphs. The proof of the existence of the harmonic measure given in [21, section 6.5] for random walks on  $\mathbb{Z}^d$ , relies on a Harnack inequality and on Green function estimates. Actually, it turns out that only a weak form of Harnack inequality is needed.

In Theorem 1.2, we show that a weak Harnack inequality is a sufficient condition for the existence of the harmonic measure of transient graph. Moreover, weak estimates of the Green function imply the weak Harnack inequality.

As it happens for Brownian motion and for the simple random walks (see for instance [26], [22]), the harmonic measure can be expressed in terms of capacities.

The *capacity* of  $A$  with respect to  $B$ , for  $A \subset B \subset \Gamma$ , is defined by

$$\text{Cap}_B(A) = \sum_{x \in A} \pi(x) P_x(\bar{\tau}_{B^c} < \tau_A).$$

The *escape probability* of a set  $A$  is defined by  $\text{Es}_A(x) := P_x(\tau_A = \infty)$  and the capacity of a finite subset  $A \subset \Gamma$  is defined by

$$\text{Cap}(A) = \sum_{x \in A} \pi(x) \text{Es}_A(x).$$

The main result for transient graphs is the existence of the harmonic measure for random walks which verify the following weak form of the Harnack inequality.

**Definition 1.1.** *We say that a weighted graph  $(\Gamma, a)$  satisfies  $\mathbf{wH}(C)$ , the weak Harnack inequality, if there is a constant  $C \geq 1$  such that for all  $x \in \Gamma$  and for all  $R > 0$  there is  $R' = R'(x, R)$  such that for any positive harmonic function  $u$  on  $B(x, R')$ ,*

$$\max_{B(x, R)} u \leq C \min_{B(x, R)} u.$$

**Theorem 1.2.** *Let  $(\Gamma, a)$  be a weighted graph.*

*If  $(\Gamma, a)$  is connected, transient and if it verifies the weak Harnack inequality  $\mathbf{wH}(C)$*

*then for any finite subset  $A \subset \Gamma$  the harmonic measure on  $A$  exists. That is, for all  $y \in A$ , the limit (1.3) exists.*

*Moreover, we have:*

$$\lim_{D(x, A) \rightarrow \infty} \bar{H}_A(x, y) = \lim_{m \rightarrow +\infty} H_A^m(y),$$

*where, for  $m$  large enough,*

$$H_A^m(y) = \frac{\pi(y) P_y(\tau_A > \tau_{\partial B(x_0, m)})}{\text{Cap}_m(A)}$$

*where  $\text{Cap}_m(A)$  is the capacity of  $A$  with respect to  $B(x_0, m)$  for some  $x_0 \in \Gamma$ . The limit does not depend on the choice of  $x_0$ .*

The following Green function estimates imply the weak Harnack inequality.

**Definition 1.3.** *We say that a weighted graph  $(\Gamma, a)$  satisfies the Green function estimate  $\mathbf{GE}_\gamma$  for  $\gamma > 0$  if there are constants  $0 < C_i \leq C_s < \infty$  and if for all  $z \in \Gamma$ , there exists  $R_z < \infty$  such that for all  $x, y \in \Gamma$  with  $D(x, y) \geq R_x \wedge R_y$  we have:*

$$\frac{C_i}{D(x, y)^\gamma} \leq G(x, y) \leq \frac{C_s}{D(x, y)^\gamma}. \quad (\mathbf{GE}_\gamma)$$

This condition is a weak version of [30, Definition 1] where  $\gamma$  is called a Greenian index. It is used by Telcs [30] to give an upper bound for the probability transitions of a Markov chain in terms of the growth rate of the volume and of the Greenian index.

**Proposition 1.4.** *Let  $(\Gamma, a)$  be a weighted graph which verifies  $(\mathbf{GE}_\gamma)$  for some  $\gamma > 0$ . Then the graph is connected, transient and  $\mathbf{wH}(C)$  holds with  $C = 2^\gamma \frac{C_s}{C_i}$ .*

In the following corollaries, we describe some weighted graphs where the harmonic measure from infinity exists.

A weighted graph  $(\Gamma, a)$  is said to be *uniformly elliptic* if there is a constant  $c \geq 1$  such that for all edges  $e$ ,

$$c^{-1} \leq a(e) \leq c. \quad (1.4)$$

**Corollary 1.5.** *Let  $(\mathbb{Z}^d, a)$ ,  $d \geq 3$ , be a uniformly elliptic graph.*

*Then for all finite subsets  $A$  of  $\mathbb{Z}^d$  and for all  $y \in A$ , the limit (1.3) exists.*

*Moreover, we have:*

$$\lim_{|x| \rightarrow +\infty} \overline{H}_A(x, y) = \lim_{m \rightarrow +\infty} H_A^m(y),$$

$$\text{where } H_A^m(y) = \frac{\pi(y) P_y(\tau_A > \tau_{\partial B(0, m)})}{\text{Cap}_m(A)}.$$

Indeed, by [13, Proposition 4.2] the Green function of a uniformly elliptic graph  $(\mathbb{Z}^d, a)$ ,  $d \geq 3$ , verifies the estimates  $(\mathbf{GE}_\gamma)$  with  $\gamma = d - 2$ . The existence of the harmonic measure then follows from proposition 1.4 and Theorem 1.2.

The harmonic measure also exists for a large class of fractal like graphs with some regularity properties. Some examples are given in [9]. See also [31, section 1.1] and the references therein.

The volume of a ball  $B(x, R)$  is defined by  $V(x, R) := \sum_{x \in B(x, R)} \pi(x)$  and the mean exit time from the ball is  $E(x, R) := E_x(\sigma_R)$  where  $\sigma_R := \inf\{k \geq 0; X_k \notin B(x, R)\}$ .

A weighted graph  $(\Gamma, a)$  has *polynomial volume growth with exponent  $\alpha > 0$*  if there is a constant  $c \geq 1$  such that for all  $x \in \Gamma$  and for all  $R \geq 1$ ,

$$c^{-1} R^\alpha \leq V(x, R) \leq c R^\alpha. \quad (V_\alpha)$$

A weighted graph  $(\Gamma, a)$  has *polynomial mean exit time with exponent  $\beta > 0$*  if there is a constant  $c \geq 1$  such that for all  $x \in \Gamma$  and for all  $R \geq 1$ ,

$$c^{-1} R^\beta \leq E(x, R) \leq c R^\beta. \quad (E_\beta)$$

A weighted graph  $(\Gamma, a)$  verifies the condition  $(p_0)$  if there is a constant  $p_0 > 0$  such that for all  $x \sim y$ ,

$$p(x, y) \geq p_0. \quad (p_0)$$

Note that under  $(p_0)$ , if the graph verifies the elliptic Harnack inequality with a shrinking parameter  $M > 1$  (see definition 2.1) then it verifies the elliptic Harnack inequality with any shrinking parameter  $M > 1$ . See [31, proposition 3.5] for instance.

Barlow [9, Theorem 2] proved that for  $\alpha \geq 1$  and for  $2 \leq \beta \leq 1 + \alpha$ , there is a weighted graph with polynomial growth with exponent  $\alpha$ , with polynomial mean exit time with exponent  $\beta$  and which satisfies the elliptic Harnack inequality with shrinking parameter  $M = 2$ . Moreover if  $\beta \geq 2$  and if the graph verifies  $(p_0)$  then from [17, Theorem 3.1], the graph verifies the so-called  $\beta$ -Gaussian estimates and consequently, for  $\beta < \alpha$ ,  $(\mathbf{GE}_\gamma)$  holds with  $\gamma = \alpha - \beta$ . These results are summarized in the corollary below.

**Corollary 1.6.** *Let  $(\Gamma, a)$  be a weighted graph. If  $(\Gamma, a)$  verifies  $(p_0)$ ,  $(V_\alpha)$ ,  $(E_\beta)$  for  $\alpha > \beta \geq 2$  and the elliptic Harnack inequality  $\mathbf{H}(C)$  then for all finite subsets  $A \subset \Gamma$  and  $y \in A$  the limit (1.3) exists.*

The harmonic measure from infinity also exists for random walks in random environment and in particular for the random walk on the supercritical percolation cluster. Before stating this result, we give a brief description of the percolation model. See [20] for more details.

Consider the lattice  $\mathbb{Z}^d$ ,  $d \geq 2$ , where  $x \sim y$  if  $|x - y|_1 = 1$  where  $|\cdot|_1$  is the  $\ell_1$ -distance. Denote the set of edges by  $\mathbb{E}^d$ .

Assume that  $(a(e); e \in \mathbb{E}^d)$  are i.i.d. non-negative random variables on a probability space  $(\Omega, \mathbb{P})$ . Call a bond  $e$  open if  $a(e) > 0$  and closed if  $a(e) = 0$ . Let  $p = \mathbb{P}(a(e) > 0)$ . By percolation theory, there exists a critical value  $p_c = p_c(\mathbb{Z}^d) \in ]0; 1[$  such that for  $p < p_c$ ,  $\mathbb{P}$  almost surely, all open clusters of  $\omega$  are finite and for  $p > p_c$ ,  $\mathbb{P}$  almost surely, there is a unique infinite cluster of open edges which is called the supercritical cluster. It will be denoted by  $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$ . The edges of this graph are the open edges of the cluster and the end points of these edges are the vertices of the graph.

For  $x, y \in \mathcal{C}_\infty(\omega)$ , we will write  $x \sim y$  if the edge with endpoints  $x$  and  $y$  is open. The transition probabilities of the random walk on  $\mathcal{C}_\infty(\omega)$  are given by (1.1). The law of the paths starting at  $x \in \mathcal{C}_\infty(\omega)$  will be denoted by  $P_x^\omega$ . The random walk on the supercritical percolation cluster corresponds to the case of Bernoulli random variables. In this case, we will write  $\mathbb{P}_p$  instead of  $\mathbb{P}$ . For  $x, y \in \mathbb{Z}^d$ , we will write  $x \leftrightarrow y$  if there is an open path joining  $x$  and  $y$ .

$D_\omega(x, y)$  will denote the graph distance between  $x$  and  $y$  in the graph  $\mathcal{C}_\infty(\omega)$  and the ball centered at  $x \in \mathcal{C}_\infty(\omega)$  of radius  $R$  will be denoted by  $B_\omega(x, R) = \{y \in \mathcal{C}_\infty(\omega); D_\omega(x, y) < R\}$ .

The existence of the harmonic measure for i.i.d. conductances on  $\mathbb{Z}^d$ ,  $d \geq 3$ , is given in corollary 1.7 below. It follows from the Green function estimates of [3, Theorem 1.2a]. A weaker condition which might hold even if the conductances are not i.i.d. is given in [7, Theorem 6.1].

**Corollary 1.7.** *Let  $(\mathbb{Z}^d, a)$ ,  $d \geq 3$ , be a weighted graph where the weights  $(a(e); e \in \mathbb{E}^d)$  are i.i.d. non-negative random variables on a probability space  $(\Omega, \mathbb{P})$  which verify*

$$\mathbb{P}(a(e) > 0) > p_c(\mathbb{Z}^d).$$

*Then there exist constants  $C_i, C_s$ , which depend on  $\mathbb{P}$  and  $d$ , and  $\Omega_1 \subset \Omega$  with  $\mathbb{P}(\Omega_1) = 1$  such that for each  $\omega \in \Omega_1$ ,  $(\mathbf{GE}_\gamma)$  holds in  $\mathcal{C}_\infty(\omega)$  with the constants  $C_i$  and  $C_s$  and with  $\gamma = d - 2$ .*

*For any finite subset  $A$  of  $\mathcal{C}_\infty$  and for all  $y \in A$ , the limit (1.3) exists.*

*Moreover, we have:*

$$\lim_{|x| \rightarrow +\infty, x \in \mathcal{C}_\infty} \overline{H}_A(x, y) = \lim_{m \rightarrow +\infty} H_A^m(y),$$

*where  $H_A^m(y) = \frac{\pi(y) P_y^\omega(\tau_A > \tau_{\partial B_\omega(x_0, m)})}{\text{Cap}_m(A)}$  for some  $x_0 \in \mathcal{C}_\infty$  and for  $m$  large enough.*

In [3], both the constant speed random walk and the variable speed random walk are considered. From the expression of their generators one immediately sees that they have the same harmonic functions as the discrete time random walk considered here. Moreover, since they are a time change of each other, the Green function is the same. Hence, by [3, Theorem 1.2 a] the Green function of a uniformly elliptic graph  $(\mathbb{Z}^d, a)$ ,  $d \geq 3$ , verifies the estimates  $(\mathbf{GE}_\gamma)$  with  $\gamma = d - 2$ . The existence of the harmonic measure then follows from proposition 1.4 and Theorem 1.2.

The harmonic measure from infinity also exists for recurrent graphs. The main result here is the existence of the harmonic measure for all finite subsets of two-dimensional supercritical percolation clusters.

**Theorem 1.8.** *Let  $(\mathbb{Z}^2, a)$  be a weighted graph where the weights  $(a(e); e \in \mathbb{E}^2)$  are i.i.d. random variables on a probability space  $(\Omega, \mathbb{P})$  which verify*

$$\mathbb{P}(a(e) = 1) = 1 - \mathbb{P}(a(e) = 0) \quad \text{and} \quad \mathbb{P}(a(e) = 1) > p_c(\mathbb{Z}^2).$$

*Then  $\mathbb{P}$  almost surely, for any finite subset  $A$  of  $\mathcal{C}_\infty(\omega)$  and for all  $y \in A$ , the limit (1.3) exists.*

*Moreover, we have:*

$$\lim_{D(x,A) \rightarrow \infty} H_A(x, y) = -\mathcal{L}u_A(y)$$

*where  $u_A$  is defined in equation (4.5).*

**Theorem 1.9.** *If  $(\mathbb{Z}^2, a)$  is a uniformly elliptic weighted graph then for all finite subset  $A \subset \mathbb{Z}^2$  and for all  $y \in A$ , the limit (1.3) exists.*

**Remark 1.10.** *Note that on a regular tree, the harmonic measure from infinity does not exist for any set  $A$  which contains at least two vertices. It would be interesting to investigate the links between the Poisson boundary of a graph and the existence of the harmonic measures. In particular, the triviality of the Poisson boundary does not imply the existence of the harmonic measure as is shown by the lamplighter group  $\mathbb{Z}^2 \wr \mathbb{Z}/2\mathbb{Z}$ . See [27] and the references therein.*

Various forms of Harnack inequality that will be used both for transient graphs and for recurrent graphs are gathered in section 2. The proof of theorem 1.2 is given in section 3 while theorem 1.8 is proved in section 4. The last section contains the proof of the annulus Harnack inequality that is used in the proof of the existence of the harmonic measure for the random walk on the supercritical cluster of  $\mathbb{Z}^2$ .

## 2. HARNACK INEQUALITIES

We start by recalling a classical form of the Harnack inequality on a graph. Then we give related inequalities and weaker versions.

**Definition 2.1.** *We say that a weighted graph  $(\Gamma, a)$  satisfies  $\mathbf{H}(C)$ , the Harnack inequality with shrinking parameter  $M > 1$ , if there is a constant  $C < \infty$  such that for all  $x \in \Gamma$  and  $R > 0$ , and for any positive harmonic function  $u$  on  $B(x, MR)$ ,*

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u.$$

In our context, we will use the weak form of Harnack inequality given in definition 1.1. We rewrite this definition under a form similar to definition 2.1. The proofs will be given with these notations.

**Definition 2.2.** *We say that a weighted graph  $(\Gamma, a)$  satisfies  $\mathbf{wH}(C)$ , the weak Harnack inequality, if there is a constant  $C > 0$  such that for all  $x \in \Gamma$  and for all  $R > 0$  there is  $M_{x,R} \geq 2$  such that for all  $M > M_{x,R}$  and for any positive harmonic function  $u$  on  $B(x, MR)$ ,*

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u.$$

Barlow [6, Theorem 3] showed that the supercritical percolation cluster verifies another form of Harnack inequality. However, by corollary 1.7 and proposition 1.4 below, the random walk on the supercritical percolation cluster also verifies  $\mathbf{wH}(C)$ . In Theorem 2.3 below, we give Harnack inequality under a form that will be most useful to us. It is an immediate consequence of Theorem 5.11, Proposition 6.11 and of (0.5) of Barlow's work [6].



**Theorem 2.3.** *Let  $d \geq 2$  and let  $p > p_c(\mathbb{Z}^d)$ . There exists  $c_1 = c_1(p, d)$  and  $\Omega_1 \subset \Omega$  with  $\mathbb{P}(\Omega_1) = 1$ , and  $R_0(x, \omega)$  such that  $3 \leq R_0(x, \omega) < \infty$  for each  $\omega \in \Omega_1$ ,  $x \in \mathcal{C}_\infty(\omega)$ .*

*If  $R \geq R_0(x, \omega)$  and if  $D(x, z) \leq \frac{1}{3}R \ln R$  and if  $u : \overline{B(z, R)} \rightarrow \mathbb{R}$  is positive and harmonic in  $B(z, R)$ , then*

$$\max_{B(z, R/2)} u \leq c_1 \min_{B(z, R/2)} u. \quad (2.1)$$

*Moreover, there are positive constants  $c_2, c_3$  and  $\varepsilon$  which depend on  $p$  and  $d$  such that the tail of  $R_0(x, \omega)$  satisfies*

$$\mathbb{P}_p(x \in \mathcal{C}_\infty, R_0(x, \cdot) \geq n) \leq c_2 \exp(-c_3 n^\varepsilon). \quad (2.2)$$

In the proof of Theorem 1.2, we will need the Hölder continuity of harmonic function. It is a consequence of the weak Harnack inequality. Property  $\mathbf{wH}(C)$  leads to the following lemma.

**Lemma 2.4** (weak Hölder continuity). *Let  $(\Gamma, a)$  be a weighted graph which verifies  $\mathbf{wH}(C)$  with shrinking parameters  $(M_{x,R}; x \in \Gamma, R > 0)$  where  $M_{x,R} \geq 2$  for all  $x \in \Gamma$  and  $R > 0$ .*

*Then there exist  $\nu > 0$ ,  $c > 0$  such that for all  $x_0 \in \Gamma$ ,  $R > 0$ ,  $M \geq M_{x_0, R}$  and for any positive harmonic function  $u$  on  $B(x_0, MR)$  and  $x \in B(x_0, R)$ ,*

$$|u(x) - u(x_0)| \leq c \left( \frac{D(x_0, x)}{R} \right)^\nu \max_{B(x_0, MR)} u.$$

*Proof.* Let  $x_0 \in \Gamma$  and  $R > 0$ . Then for all  $M \geq M_{x_0, R}$  and  $R' \leq R$ , if  $u$  is a positive harmonic function on  $B(x_0, MR)$  then

$$\max_{B(x_0, R')} u \leq \max_{B(x_0, R)} u \leq C \min_{B(x_0, R)} u \leq C \min_{B(x_0, R')} u.$$

Let

$$V(i) := \max_{B(x_0, 2^i)} u - \min_{B(x_0, 2^i)} u.$$

Then for  $2^i \leq R$ , the functions  $u - \min_{B(x_0, 2^{i+1})} u$  and  $\max_{B(x_0, 2^{i+1})} u - u$  are harmonic in  $B(x_0, MR)$ . Then by the weak Harnack inequality on  $B(x_0, 2^i)$ ,

$$V(i) + V(i+1) \leq C[V(i+1) - V(i)].$$

And so, we deduce that there exists  $\lambda < 1$  such that

$$V(i) \leq \lambda V(i+1).$$

For any  $x \in B(x_0, R)$ , we can find  $N_1$  such that  $2^{N_1-1} \leq D(x_0, x) \leq 2^{N_1}$ . Then

$$|u(x) - u(x_0)| \leq V(N_1).$$

Let  $N_2$  be such that  $2^{N_2} \leq R < 2^{N_2+1}$ . Then, since  $2^{N_2+1} \leq MR$ ,

$$V(N_1) \leq \lambda^{N_2-N_1+1} V(N_2+1)$$

and in particular,

$$|u(x) - u(x_0)| \leq c \left( \frac{D(x_0, x)}{R} \right)^\nu \max_{B(x_0, MR)} u$$

where  $\nu > 0$  solves  $\lambda^{-1} = 2^\nu$  and  $c > 0$  is a constant. ■

Similarly, from Harnack inequality for the supercritical cluster given in Theorem 2.3, we have the following Hölder continuity property.

**Theorem 2.5.** *Let  $d \geq 2$  and let  $p > p_c(\mathbb{Z}^d)$ . Let  $\Omega_1$  and  $R_0(x, \omega)$  be as in Theorem 2.3. Then there exist positive constants  $\nu$  and  $c$  such that for each  $\omega \in \Omega_1$ ,  $x_0 \in \mathcal{C}_\infty(\omega)$  if  $R \geq R_0(x_0, \omega)$  and  $u$  is a positive harmonic function on  $B_\omega(x_0, R)$  then, for all  $x, y \in B_\omega(x_0, R/2)$ ,*

$$|u(x) - u(y)| \leq c \left( \frac{D(x, y)}{R} \right)^\nu \max_{B(x_0, R)} u.$$

We will also need a Harnack inequality in the annulus of the two-dimensional supercritical percolation cluster. It follows from results of Barlow [6], a percolation result due to Kesten [20] and the following estimates of Antal and Pisztora [5, Theorem 1.1 and Corollary 1.3].

For  $p > p_c(\mathbb{Z}^d)$ ,  $d \geq 2$ , there is a constant  $\mu = \mu(p, d) \geq 1$  such that

$$\limsup_{|y|_1 \rightarrow \infty} \frac{1}{|y|_1} \ln \mathbb{P}_p[0 \leftrightarrow y, D(0, y) > \mu|y|_1] < 0 \quad (2.3)$$

and,  $\mathbb{P}_p$  almost surely, for  $x_0 \in \mathcal{C}_\infty$  and for all  $x \in \mathcal{C}_\infty$  such that  $D(x_0, x)$  is sufficiently large

$$D(x_0, x) \leq \mu|x - x_0|_1. \quad (2.4)$$

**Proposition 2.6.** *Let  $p > p_c(\mathbb{Z}^2)$ . There is a constant  $C > 0$  such that  $\mathbb{Q}$ -a.s., for all  $x_0 \in \mathcal{C}_\infty$  if  $m$  is large enough, then for any positive function  $u$  harmonic in  $B(x_0, 3\mu m) \setminus \{x_0\}$ ,*

$$\max_{x; D(x_0, x) = m} u(x) \leq C \min_{x; D(x_0, x) = m} u(x)$$

where  $\mu$  is the constant that appears in (2.3).

Since we need a construction that is done in section 4.1, the proof of this Harnack inequality is postponed to section 5.

### 3. PROOFS FOR TRANSIENT GRAPHS

In this section, we prove Proposition 1.4 and Theorem 1.2.

*Proof of proposition 1.4.* The key ingredient to prove proposition 1.4 is given by Boukricha's lemma [12]. See also [31, p. 37]. Roughly speaking, this lemma ensures that a Harnack inequality holds for general positive harmonic functions as soon as a Harnack inequality holds for the Green function in an annulus.

For  $x \in \Gamma$  and  $R > 0$ , let

$$M_{x, R} = 3 \vee \frac{1}{R} \max_{w \in B(x, R)} R_w. \quad (3.1)$$

We claim that  $\mathbf{wH}(C)$  holds with the shrinking parameters  $M_{x, R}$  and the constant  $C = 2^\gamma \frac{C_s}{C_i}$ .

Fix  $x_0 \in \Gamma$ ,  $R > 0$ ,  $M > M_{x_0, R}$  and let  $u$  be a positive harmonic function on  $B(x_0, MR)$ .

First note that under  $(\mathbf{GE}_\gamma)$ , the graph is transient and we can apply Boukricha's lemma ([12], [31, p. 37]) with  $B_0 = B(x_0, R)$ ,  $B_1 = B(x_0, (M+1)R)$ ,  $B_2 = B(x_0, (M+2)R)$  and  $B_3 = \Gamma$ . So we get that if  $u$  is harmonic on  $B_2$ , then

$$\max_{B_0} u \leq D \min_{B_0} u,$$

with

$$D = \max_{x, y \in B_0} \max_{z \in B_2 \setminus B_1} \frac{G(x, z)}{G(y, z)}.$$

So, we have to compare  $G(x, z)$  and  $G(y, z)$  for  $x, y \in B_0$  and  $z \in B(x_0, (M+2)R) \setminus B(x_0, (M+1)R)$ . For all  $w \in B_0$ ,  $D(w, z) > MR > R_w$  by (3.1). Hence, by  $(\mathbf{GE}_\gamma)$ ,

$$\frac{C_i}{D(w, z)^\gamma} \leq G(w, z) \leq \frac{C_s}{D(w, z)^\gamma}.$$

Then, we successively have :

$$\begin{aligned} G(x, z) &\leq \frac{C_s}{D(x, z)^\gamma} \\ &= \frac{C_s}{C_i} \left( \frac{D(y, z)}{D(x, z)} \right)^\gamma \frac{C_i}{D(y, z)^\gamma} \\ &\leq \frac{C_s}{C_i} \left( \frac{R + (M+2)R}{(M+1)R - R} \right)^\gamma G(y, z) \\ &\leq \frac{C_s}{C_i} \left( \frac{M+3}{M} \right)^\gamma G(y, z) \\ &\leq 2^\gamma \frac{C_s}{C_i} G(y, z). \end{aligned}$$

■

We can now state the main lemma to prove Theorem 1.2.

**Lemma 3.1.** *Let  $(\Gamma, a)$  be a weighted graph which verifies  $\mathbf{wH}(C)$ . Fix  $x_0 \in \Gamma$ .*

*Let  $A$  be a finite subset of  $\Gamma$ . Let  $R > 1$  be such that  $A \subset B(x_0, R)$ .*

*For all  $M > 2$ , there is  $\lambda_M > 1$  such that for all  $\lambda > \lambda_M$  and for all  $y \in A$  and  $z \in \partial B(x_0, \lambda MR)$ ,*

$$P_y(X_{\tau_A \wedge \tau_{\partial B}} = z | \tau_A > \tau_{\partial B}) = H_{\partial B}(x_0, z) \left[ 1 + O\left(\frac{1}{M^\nu}\right) \right], \quad (3.2)$$

*where  $B = B(x_0, \lambda MR)$  and  $\nu > 0$  is the Hölder exponent given by lemma 2.4. The constant in  $O(\cdot)$  depends only on the constants  $C$  and  $c$  that appear in  $\mathbf{wH}(C)$  and in lemma 2.4 respectively.*

*Proof.* For  $M > 2$ , choose  $M_2$  and  $M_3$  such that

$$M_2 > M(x_0, MR) \quad \text{and} \quad M_3 > M(x_0, M_2 MR)$$

where  $M(x_0, \cdot)$  are the shrinking parameters that appear in  $\mathbf{wH}(C)$ .

Let  $B_1 = B(x_0, MR)$ ,  $B_2 = B(x_0, M_2 MR)$  and  $B_3 = B(x_0, M_3 M_2 MR)$ .

For  $z \in \partial B_3$ , we consider the function

$$f(x) = P_x(X_{\tau_{\partial B_3}} = z), \quad x \in \Gamma.$$

Since  $f$  is harmonic on  $B_2$ , by lemma 2.4, for all  $u \in B_1$ ,

$$|f(u) - f(x_0)| \leq c \left( \frac{D(x_0, u)}{MR} \right)^\nu \max_{B_2} f.$$

In particular, for  $u \in \partial B(x_0, R)$ ,

$$|f(u) - f(x_0)| \leq \frac{c}{M^\nu} \max_{B_2} f. \quad (3.3)$$

Now by considering  $f$  harmonic on  $B_3$ , since the graph verifies  $\mathbf{wH}(\mathbf{C})$ , we have that

$$\max_{B_2} f \leq Cf(x_0). \quad (3.4)$$

Therefore, by (3.3) and (3.4), for all  $u \in \partial B(x_0, R)$ ,

$$P_u(X_{\tau_{\partial B_3}} = z) = H_{\partial B_3}(x_0, z) \left[ 1 + O\left(\frac{1}{M^\nu}\right) \right]. \quad (3.5)$$

Introduce the following notation. For  $U, V$  and  $W$  subsets of  $\Gamma$  with  $U \subset V \subset W$ . We put

$$\partial V[W, U] = \{x \in \partial V; \text{there exist paths in } \Gamma \text{ from } x \text{ to } \partial W \text{ and from } x \text{ to } U\}. \quad (3.6)$$

On the set  $\{\tau_A < \tau_{\partial B_3}\}$ , we let  $\eta = \inf\{j \geq \tau_A; X_j \in \partial B(x_0, R)\}$ .

Then using (3.5), we obtain that for all  $x \in \partial B(x_0, R)[B_3, A]$

$$\begin{aligned} P_x(X_{\tau_{\partial B_3}} = z | \tau_A < \tau_{\partial B}) &= \sum_{u \in \partial B(x_0, R)} P_x(X_\eta = u | \tau_A < \tau_{\partial B_3}) P_u(X_{\tau_{\partial B_3}} = z) \\ &= H_{\partial B_3}(x_0, z) [1 + O\left(\frac{1}{M^\nu}\right)] \end{aligned} \quad (3.7)$$

Let  $x \in \partial B(x_0, R)[B_3, A]$ . By (3.5) and (3.7), we get from the relation

$$\begin{aligned} P_x(X_{\tau_{\partial B_3}} = z) &= P_x(X_{\tau_{\partial B_3}} = z | \tau_A > \tau_{\partial B_3}) P_x(\tau_A > \tau_{\partial B_3}) \\ &\quad + P_x(X_{\tau_{\partial B_3}} = z | \tau_A \leq \tau_{\partial B_3}) (1 - P_x(\tau_A > \tau_{\partial B_3})), \end{aligned}$$

that

$$P_x(X_{\tau_{\partial B_3}} = z | \tau_A > \tau_{\partial B_3}) = H_{\partial B_3}(x_0, z) [1 + O\left(\frac{1}{M^\nu}\right)]$$

This can also be written as,

$$P_x(X_{\tau_{\partial B_3} \wedge \tau_A} = z) = H_{\partial B_3}(x_0, z) P_x(\tau_A > \tau_{\partial B_3}) [1 + O\left(\frac{1}{M^\nu}\right)]. \quad (3.8)$$

Note that every path from  $y$  to  $\partial B_3$  must go through some point of  $\partial B(x_0, R)[B_3, A]$ . So, for all  $y \in A$  and for all  $z \in \partial B_3$ ,

$$\begin{aligned} P_y(X_{\tau_{\partial B_3} \wedge \tau_A} = z) &= \sum_{x \in \partial B(x_0, R)[B_3, A]} P_y(X_{\tau_{\partial B(x_0, R)[B_3, A]} \wedge \tau_A} = x) P_x(X_{\tau_{\partial B_3} \wedge \tau_A} = z) \\ &\stackrel{(3.8)}{=} H_{\partial B_3}(x_0, z) [1 + O\left(\frac{1}{M^\nu}\right)] \\ &\quad \times \sum_{x \in \partial B(x_0, R)[B_3, A]} P_y(X_{\tau_{\partial B(x_0, R)[B, A]} \wedge \tau_A} = x) P_x(\tau_A > \tau_{\partial B_3}) \\ &= H_{\partial B_3}(x_0, z) [1 + O\left(\frac{1}{M^\nu}\right)] P_y(\tau_A > \tau_{\partial B_3}). \end{aligned}$$

This last equation proves that lemma 3.1 holds with  $\lambda_M = M_2 M_3$  where  $M_2 = M(x_0, MR)$  and  $M_3 = M(x_0, M_2 MR)$ .  $\blacksquare$

As in Lawler [22, p. 49], using a last exit decomposition, we obtain the following representation of the hitting distribution in a weighted graph.

The Green function of the random walk in  $B \subset \Gamma$  is defined by

$$G_B(x, y) := \sum_{k=0}^{\infty} p_B(x, y, k), \quad x, y \in \bar{B}$$

where  $p_B(x, y, k) := P_x(X_k = y, k < \bar{\tau}_{B^c})$  are the transition probabilities of the walk with Dirichlet boundary conditions.

**Lemma 3.2.** *Let  $(\Gamma, a)$  be a weighted graph.*

Let  $A \subset B$  be finite subsets of  $\Gamma$ . Then for all  $x \in B^c$  and  $y \in A$ ,

$$H_A(x, y) = \sum_{z \in \partial B} G_{A^c}(x, z) H_{A \cup \partial B}(z, y), \quad (3.9)$$

$$\bar{H}_A(x, y) = \frac{\sum_{z \in \partial B} G_{A^c}(x, z) H_{A \cup \partial B}(z, y)}{\sum_{z \in \partial B} G_{A^c}(x, z) P_z(\tau_A < \tau_{\partial B})} \quad (3.10)$$

and

$$\min_{z \in \partial B} \frac{H_{A \cup \partial B}(z, y)}{P_z(\tau_A < \tau_{\partial B})} \leq \bar{H}_A(x, y) \leq \max_{z \in \partial B} \frac{H_{A \cup \partial B}(z, y)}{P_z(\tau_A < \tau_{\partial B})} \quad (3.11)$$

Then by reversibility,  $\pi(z) H_{A \cup \partial B}(z, y) = \pi(y) H_{A \cup \partial B}(y, z)$  and

$$P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y} \in A} H_{A \cup \partial B}^+(z, \tilde{y}). \text{ Hence,} \quad (3.12)$$

$$\min_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)} \leq \bar{H}_A(x, y) \leq \max_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)}$$

We complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We are given  $x_0 \in \Gamma$  and a finite set  $A \subset \Gamma$ .

Let  $R > 1$  be such that  $A \subset B(x_0, R)$ .

Let  $B = B(x_0, \lambda MR)$  where  $\lambda \geq \lambda_M$  is given by lemma 3.1.

By equation (3.2), for all  $y \in A$  and  $z \in \partial B$ ,

$$\pi(y) H_{A \cup \partial B}(y, z) = H_{\partial B}(x_0, z) [1 + O\left(\frac{1}{M^\nu}\right)] \pi(y) P_y(\tau_A > \tau_{\partial B}). \quad (3.13)$$

By summing over  $y \in A$  the equation (3.13) gives,

$$\sum_{y \in A} \pi(y) P_y(X_{\tau_{\partial B} \wedge \tau_A} = z) = H_{\partial B}(x_0, z) [1 + O\left(\frac{1}{M^\nu}\right)] \sum_{y \in A} \pi(y) P_y(\tau_A > \tau_{\partial B}). \quad (3.14)$$

Since  $(\Gamma, a)$  is connected, both sides of (3.14) are positive. So we can divide (3.13) by (3.14). And a short calculation shows that

$$\frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) P_{\tilde{y}}(X_{\tau_{\partial B} \wedge \tau_A} = z)} = \frac{\pi(y) P_y(\tau_A > \tau_{\partial B})}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) P_{\tilde{y}}(\tau_A > \tau_{\partial B})} [1 + O\left(\frac{1}{N^\nu}\right)]$$

where the constant in  $O(\cdot)$  still depends only on the constants  $C$  and  $c$  that appear in **wH**( $C$ ) and in lemma 2.4 respectively.

By (3.12), we have that for all  $v \notin B$ ,

$$\min_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) P_{\tilde{y}}(X_{\tau_{\partial B} \wedge \tau_A} = z)} \leq \bar{H}_A(v, y) \leq \max_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) P_{\tilde{y}}(X_{\tau_{\partial B} \wedge \tau_A} = z)}$$

So for all  $v \notin B$  we get:

$$\bar{H}_A(v, y) = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B})}{\text{Cap}_B(A)} [1 + O\left(\frac{1}{M^v}\right)] \quad (3.15)$$

As  $v$  goes to  $+\infty$  in an arbitrary way, we will have that  $M \rightarrow \infty$  as well. Hence, by (3.15), we obtain that  $\lim_{v \rightarrow +\infty} \bar{H}_A(v, y)$  exists and

$$\lim_{v \rightarrow +\infty} \bar{H}_A(v, y) = \lim_{m \rightarrow +\infty} \bar{H}_A^m(v, y) = \frac{\pi(y)P_y(\tau_A > +\infty)}{\sum_{\tilde{y} \in A} \pi(\tilde{y})P_{\tilde{y}}(\tau_A > +\infty)}.$$

■

#### 4. RECURRENT GRAPHS

In this section, we prove the existence of the harmonic measure for the random walk on a supercritical percolation cluster of  $\mathbb{Z}^2$ . The proof for the uniformly elliptic random walk on  $\mathbb{Z}^2$  is similar but with many simplifications since we can use the estimates of [13] instead of Barlow's estimates.

##### 4.1. Estimates of the capacity of a box.

**Proposition 4.1.** *Let  $p > p_c(\mathbb{Z}^2)$ . There is a constant  $C \geq 1$  such that  $\mathbb{Q}_p$ -a.s. for  $x_0 \in \mathcal{C}_\infty$ , for all  $n$  sufficiently large,*

$$C^{-1} \leq (\ln n) \text{Cap}_{B_\omega(x_0, n)}(\{x_0\}) \leq C. \quad (4.16)$$

Flows of finite energy on the supercritical percolation cluster with respect to a convex gauge functions are constructed in [4]. To do so, the flow is expressed by a probability on the set of self-avoiding paths. Here, however, the lower estimate of (4.16) is obtained by combining the method used in  $\mathbb{Z}^2$ , see [23, Proposition 2.14], with a percolation lemma of Kesten [20, Theorem 7.11].

*Proof.* The upper bound follows from the variational principle and a comparison with  $\mathbb{Z}^2$  (see for instance [31, section 3.1]).

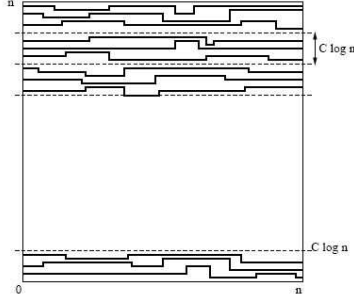
To prove the lower bound, we assume  $0 \in \mathcal{C}_\infty$  and for each  $n$  sufficiently large, we construct a particular flow  $\theta_n$  from 0 to  $\partial B_\omega(0, n)$ . However, it is a difficult task to estimate the energy of a flow from 0 to  $\partial B_\omega(0, n)$  consisting of small flows along simple paths from 0 to  $\partial B_\omega(0, n)$  since the percolation cluster is very irregular. So, as in Mathieu and Remy [25], we construct a "subgrid" of  $B_\omega(0, n)$  by using a Theorem of Kesten ([20]).

Let us introduce some definitions.

**Definition 4.2.** *Let  $B_{m,n} = [0; m] \times [0; n] \cap \mathbb{Z}^2$ .*

*A horizontal [resp. vertical] channel of  $B_{m,n}$  is a path  $(v_0, e_1, e_2, \dots, e_n, v_n)$ , with  $v_i \in \mathbb{Z}^2$  and  $e_i \in \mathbb{E}^2$  for all  $i = 1 \dots n$  such that:*

- $(v_0, e_1, e_2, \dots, e_{n-1}, v_{n-1})$  is contained in the interior of  $B_{m,n}$
- $v_0 \in \{0\} \times [0; n]$  [resp.  $v_0 \in [0; m] \times \{0\}$ ]
- $v_n \in \{m\} \times [0; n]$  [resp.  $v_n \in [0; m] \times \{n\}$ ]

FIGURE 1. The outline of the Kesten 's Grid  $K_G(n)$ .

We say that two channels are disjoint if they have no vertex in common. Let  $N(m, n)$  be the maximal number of disjoint open horizontal channels in  $B_{m,n}$ . Then by [20, Theorem 11.1], for  $p > p_c$ , there is a constant  $c(p)$  and some universal constants  $0 < c_1, c_2, \xi < \infty$ , such that

$$\mathbb{P}_p(N(m, n) > c(p)n) \geq 1 - c_1(m + 1) \exp(-c_2(p - p_c)^\xi n). \quad (4.17)$$

Let us now construct the Kesten grid over  $[-n; n]^2$ . We divide the box  $[-n; n]^2$  in horizontal strips of width  $C_K \ln(n)$  with  $C_K$  large enough so that  $c_2(p - p_c)^\xi C_K > 3$ . Then

$$\sum_n \frac{n}{C_K \ln n} c_1(n + 1) \exp(-c_2(p - p_c)^\xi C_K \ln(n)) < \infty.$$

Hence by Borel-Cantelli lemma, we get that for  $n$  large enough there is at least  $c(p) \ln(n)$  disjoint channels in each horizontal strips of width  $C_K \ln n$ . We do the same construction for vertical strips. Finally, we deduce the existence of a grid  $K_G(n)$  in  $[-n; n]^2$  where each horizontal and each vertical strip of width  $C_K \ln n$  contains at least  $c(p) \ln(n)$  disjoint channels.

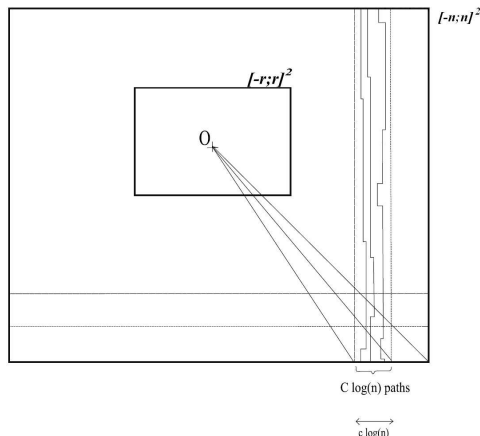
### Construction of the flow

Since  $B_\omega(0, n) \subset \mathcal{C}_\infty \cap [-n; n]^2$ , we have that  $\text{Cap}_{\mathcal{C}_\infty \cap [-n; n]^2}(0) \leq \text{Cap}_{B_\omega(0, n)}(0)$ . Hence, to obtain a lower bound, it suffices to construct a flow  $\theta_n$  from 0 to  $\partial([-n; n]^2) \cap \mathcal{C}_\infty$ .

Then for each a path  $\Pi : (e_1, e_2, \dots, e_L)$  from 0 to  $\partial([-n; n]^2) \cap \mathcal{C}_\infty$  with the induced orientation and consisting only of edges of Kesten's grid  $K_G(n)$ , we associate the unit flow  $\Psi^\Pi = \sum_\ell \mathbf{1}_{\{\vec{e}_\ell\}} - \mathbf{1}_{\{\vec{e}_\ell\}}$ . The flow  $\theta_n$  will be a sum of flows  $\Psi^\Pi$  for a set of well chosen paths.

More precisely, consider a ray from 0 to boundary of the box  $[-n; n]^2$ , ending on a channel of  $K_G(n)$ . There are about  $2 \frac{2n}{C_K \ln n} c(p) \ln n$  such rays.

By the notation  $f(n, r) \asymp g(n, r)$  used below, we mean that there is a constant  $c \geq 1$  such that  $\mathbb{P}_p$ -a.s such that for  $n$  large enough and for  $1 \leq r \leq n$ , then  $c^{-1}f(n, r) \leq g(n, r) \leq cf(n, r)$ .

FIGURE 2. Construction of the flow  $\theta_n$ .

Then for  $1 \leq r \leq n$ , the boundary of the box  $[-r; r]^2$  is divided in segments of length

$$\asymp 2r \frac{C_K \ln n}{2n} \frac{1}{c(p) \ln n}.$$

Thus, the number of rays that crosses an edge of the boundary of  $[-r; r]^2$  (of length 1),

$$\asymp \frac{1}{r} \frac{2n}{C_K \ln n} c(p) \ln n. \quad (4.18)$$

To each ray, we associate a simple path, chosen among paths from from 0 to the boundary of  $[-n; n]^2$ . It consists of edges on the channels of  $K_G(n)$  that are close to the ray. To go right from a horizontal channel to up on a vertical channel, the top horizontal channel is attached to the left-most vertical channel. Then similarly from top to bottom for the horizontal channels and from left to right for the vertical ones. We proceed similarly for the other turns.

Let  $\mathcal{P}_n$  be the set of chosen paths. We use these paths to construct the flow  $\theta_n$  of intensity  $cn$  by setting

$$\theta_n = \sum_{\Pi \in \mathcal{P}_n} \Psi^\Pi.$$

The paths of  $\mathcal{P}_n$  might not be disjoint but by (4.18), there is a constant  $C < \infty$  such that an edge on the boundary of  $[-r; r]^2$  belongs to less than  $C \frac{n}{r}$  paths. Hence, there is a constant  $C' < \infty$  such that for an edge  $e$  at  $\ell_1$ -distance  $r$  from the origin, the flow  $\theta_n$  satisfies:

$$\theta_n(e) \leq C' \frac{n}{r}.$$

Hence, by Thomson's principle (see for instance [23, section 2.4]),

$$\frac{1}{\text{Cap}_{B_\omega(0,n)}(0)} \leq \frac{1}{n^2} \mathcal{E}(\theta_n) = \frac{1}{n^2} \sum_e \theta_n(e)^2 \leq \frac{C''}{n^2} \sum_{r=1, \dots, n} \frac{n^2}{r^2} r \leq C'' \ln(n).$$



■

**4.2. The Green kernel and its properties.** In this section, we will use the parabolic Harnack inequality for the random walk on the supercritical percolation cluster proved by Barlow and Hambly in [8].

Besides this, we also use the comparison result for  $D$  and the  $|\cdot|_1$ -distance of Antal and Pisztorá [5], see (2.4).

**Lemma 4.3.**  $\mathbb{P}_p$ -almost surely, for all  $x_0, x \in \mathcal{C}_\infty$ , the series

$$\sum_{k=0}^{\infty} [p(x_0, x_0, k) - p(x, x_0, k)]$$

converges. The limit will be denoted by  $g(x, x_0)$ .

Let  $G_{2n}(x, y)$  and  $p_{2n}(x, y, k)$  be respectively the Green function and the probability transitions of the random walk in the ball  $B(x_0, 2n)$  with Dirichlet boundary conditions. Then

$$g(x, x_0) = \lim_n \sum_{k=0}^{\infty} [p_{2n}(x_0, x_0, k) - p_{2n}(x, x_0, k)] \quad (4.19)$$

$$= \lim_n [G_{2n}(x_0, x_0) - G_{2n}(x, x_0)] \quad (4.20)$$

*Proof.* Let  $R_0$  be as in Theorem 2.3. Then by [8, Proposition 6.1], for  $R \geq R_0(x)$ ,  $B(x, R)$  is very good with  $N_B \leq R^{1/(10(d+2))}$  and it is exceedingly good.

Now let  $R \geq R_0(x) \vee 16$  and let  $R_1 = R \ln R$ . Then, since  $R_1 \geq R_0$ ,  $B = B(x, R_1)$  is very good with  $N_B^{2d+4} \leq R_1^{(2d+4)/(10(d+2))} \leq R_1/(2 \ln R_1)$ . Then by [8, Theorem 3.1], there exists a constant  $C_H$  such that the parabolic Harnack inequality [8, (3.2)] holds in  $Q(x, R, R^2)$ . Therefore [8, Proposition 3.2] holds with  $s(x_0) = R_0(x_0) \vee 16$  and  $\rho(x_0, x) = R_0(x_0) \vee 16 \vee D(x_0, x)$

Fix  $x_0 \in \mathcal{C}_\infty$  then  $v(n, x) := p(x, x_0, n) + p(x, x_0, n+1)$  is a caloric function, that is, it verifies

$$v(n+1, x) - v(n, x) = \mathcal{L}v(n, x), \quad (n, x) \in \mathbb{N} \times \mathcal{C}_\infty.$$

Let  $k > 4D(x_0, x)^2$ . Let  $t_0 = k+1$  and  $r_0 = \sqrt{t_0}$ . Then  $v(n, x)$  is caloric in  $]0, r_0^2] \times B(x_0, r_0)$ ,  $x \in B(x_0, r_0/2)$  since  $D(x_0, x) \leq \sqrt{k} < r_0/2$ , and  $t_0 - \rho(x_0, x)^2 \leq k \leq t_0 - 1$ .

Then by the upper gaussian estimates [6, Theorem 5.7] and [8, (2.18)] and by [8, Proposition 3.2], there is  $\nu > 0$  such that

$$\begin{aligned} |v(k, x) - v(k, x_0)| &\leq C \left( \frac{\rho(x_0, x)}{\sqrt{t_0}} \right)^\nu \sup_{Q_+} v \\ &\leq C \left( \frac{\rho(x_0, x)}{\sqrt{t_0}} \right)^\nu \frac{1}{r_0^2} \\ &\leq C \frac{\rho(x_0, x)^\nu}{k^{1+\nu/2}} \end{aligned}$$

Hence, the series converges. Note that we also have that

$$|p(x, x_0, k) - p(x_0, x_0, k)| \leq C \frac{\rho(x_0, x)^\nu}{k^{1+\nu/2}}.$$

Then (4.19) follows by Lebesgue dominated convergence theorem. ■

**Lemma 4.4.** *There are constants  $0 < c < C < \infty$  such that,  $\mathbb{P}_p$ -a.s., for all  $x_0 \in \mathcal{C}_\infty$  there is  $\rho = \rho(x_0)$  such that if  $D(x_0, x) > \rho$ ,*

$$c \ln D(x_0, x) < g(x, x_0) < C \ln D(x_0, x) \quad (4.21)$$

*Proof.* Let  $x_0 \in \mathcal{C}_\infty$ . For  $m \geq 1$ , write  $\sigma_m$  for  $\bar{\tau}_{B(x_0, m)^c}$ .

Note that for all  $n > 3\mu m$  where  $\mu$  is the constant that appears in (2.3),  $P(\sigma_n < \tau_{x_0})$  is harmonic in  $B(x_0, 3\mu m) \setminus \{x_0\}$ . Then by the annulus Harnack inequality (Proposition 2.6), if  $m$  is sufficient large and if  $m = D(x, x_0)$

$$\begin{aligned} P_{x_0}(\sigma_n < \tau_{x_0}) &= \sum_{x'; D(x_0, x')=m} P_{x_0}(X(\sigma_m) = x', \sigma_m < \tau_{x_0}) P_{x'}(\sigma_n < \tau_{x_0}) \\ &\asymp P_x(\sigma_n < \tau_{x_0}) \sum_{x'; D(x_0, x')=m} P_{x_0}(X(\sigma_m) = x', \sigma_m < \tau_{x_0}) \\ &\asymp P_x(\sigma_n < \tau_{x_0}) \text{Cap}_m(x_0). \end{aligned}$$

By  $f(x) \asymp g(x)$  here, we mean that there are constants  $c$  and  $C$ , which do not depend on  $x, n, m$  or  $\omega$ , such that  $\mathbb{P}_p$ -a.s for  $x \in \mathcal{C}_\infty$ , if  $D(x_0, x)$  is large enough, then

$$0 < cf(x) \leq g(x) \leq Cf(x).$$

Hence by the capacity estimates (4.16)

$$P_x(\sigma_n < \tau_{x_0}) \asymp \frac{\text{Cap}_n(x_0)}{\text{Cap}_m(x_0)} \asymp \frac{\ln m}{\ln n} = \frac{\ln D(x_0, x)}{\ln n}. \quad (4.22)$$

Then by the capacity estimates (4.16) and by (4.22),

$$\begin{aligned} G_n(x_0, x_0) - G_n(x, x_0) &= G_n(x_0, x_0) - P_x(\tau_{x_0} < \sigma_n) G_n(x_0, x_0) \\ &= G_n(x_0, x_0) P_x(\tau_{x_0} > \sigma_n) \\ &\asymp \ln n \frac{\ln D(x, x_0)}{\ln n} \end{aligned}$$

Then (4.21) follows by (4.19). ■

The harmonic measure will be expressed in terms of the function  $u_A$  defined below.

**Definition 4.5.**  $\mathbb{P}_p$ -a.s., for a finite subset  $A$  of  $\mathcal{C}_\infty(\omega)$  and for a fixed  $x_0 \in \mathcal{C}_\infty(\omega)$ , let

$$u_A(x, x_0) := g(x, x_0) - E_{x, \omega} g(X_{\bar{\tau}_A}, x_0).$$

Note that

$$\begin{aligned} u_A(\cdot, x_0) &= 0 \quad \text{on } A, \\ u_A(x, x_0) &\asymp \ln D_\omega(x_0, x) \quad \text{as } x \rightarrow \infty \quad \text{by (4.21),} \\ \text{for all } x \in \mathcal{C}_\infty, \quad P u_A(x, x_0) &= P g(x, x_0) - \sum_{y \sim x} p(x, y) E_y g(X_{\bar{\tau}_A}, x_0) \\ &= g(x, x_0) - \mathbf{1}_{x_0}(x) - E_x g(X_{\bar{\tau}_A}, x_0), \end{aligned}$$

We will need to work in balls defined in terms of  $g(\cdot, x_0)$ . Let

$$\tilde{B}_n := \tilde{B}(x_0, n) := \{x \in \mathcal{C}_\infty; g(x, x_0) \leq \ln n\} \text{ and } \tilde{\sigma}_n := \inf\{k \geq 0; X_k \notin \tilde{B}(x_0, n)\} \quad (4.23)$$

Note that by (4.21), for all  $n$  sufficiently large,

$$B(x_0, n^{1/C}) \subset \tilde{B}(x_0, n) \subset B(x_0, n^{1/c}). \quad (4.24)$$

The next lemma is the analogue of [21, Proposition 6.4.7].

**Proposition 4.6.**  $\mathbb{P}_p$ -a.s., for a finite subset  $A$  of  $\mathcal{C}_\infty(\omega)$  and for a fixed  $x_0 \in \mathcal{C}_\infty(\omega)$ , for all  $x \in A^c$ ,

$$u_A(x, x_0) = \lim_n (\ln n) P_x(\tilde{\sigma}_n < \tau_A).$$

*Proof.* Let  $R_0(x, \omega)$  be as in Barlow's Theorem 2.3. By (2.2), by (2.3) of Antal and Pisztora, and by (4.24)

$$\sum_n \sum_{x \in \partial \tilde{B}(x_0, n)} \mathbb{P}_p(x \in \mathcal{C}_\infty, R_0(x, \cdot) \geq \sqrt{n}) \leq C \sum_n n^{2/c} \exp(-c_3 n^{\varepsilon/2}) < \infty.$$

Therefore, by Borel-Cantelli, there is  $\Omega_1 \subset \Omega$  with  $\mathbb{P}_p(\Omega_1) = 1$ , such that for all  $\omega \in \Omega_1$  there is  $n_0$  such that for all  $n \geq n_0$  and for all  $x \in \partial \tilde{B}(x_0, n)$ ,  $R_0(x) < \sqrt{n}$ .

Let  $x \in \partial \tilde{B}(x_0, n)$  where  $n \geq n_0$ . Then there is  $x' \in \tilde{B}(x_0, n)$  such that  $x' \sim x$  and

$$g(x', x_0) \leq \ln n < g(x, x_0).$$

Moreover, by (4.24),  $D(x, x_0) > n^{1/C}$ . Then by Hölder's continuity property given in Theorem 2.5 and by (4.21),

$$\begin{aligned} g(x, x_0) - \ln n &\leq g(x, x_0) - g(x', x_0) \\ &\leq c \left( \frac{1}{n^{1/C}} \right)^\nu \max_{B(x, n^{1/C})} g(\cdot, x_0) \\ &\leq \frac{c}{C} \left( \frac{1}{n^{1/C}} \right)^\nu \ln n \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

By the optional stopping theorem applied to the martingale  $g(X_k, x_0)$ ,  $k \geq 0$  and for  $n$  large enough,

$$\begin{aligned} g(x, x_0) &= E_x [g(X_{\tau_A \wedge \tilde{\sigma}_n}, x_0)], \quad x \in \tilde{B}(x_0, n) \setminus A \\ &= P_x(\tilde{\sigma}_n < \tau_A) E_x [g(X_{\tilde{\sigma}_n}, x_0) \mid \tilde{\sigma}_n < \tau_A] \\ &\quad + P_x(\tau_A < \tilde{\sigma}_n) E_x [g(X_{\tau_A}, x_0) \mid \tau_A < \tilde{\sigma}_n] \end{aligned}$$

But

$$\begin{aligned} \lim_n P_x(\tau_A < \tilde{\sigma}_n) E_x [g(X_{\tau_A}, x_0) \mid \tau_A < \tilde{\sigma}_n] &= \lim_n E_x [g(X_{\tau_A}, x_0); \tau_A < \tilde{\sigma}_n] \\ &= E_x g(X_{\tau_A}, x_0) \end{aligned}$$

Therefore,  $u_A(x, x_0) = \lim_n (\ln n) P_x(\tilde{\sigma}_n < \tau_A)$ . ■

We can now prove the analogue of lemma 3.1 for the supercritical cluster. Theorem 1.8 will follow from this lemma and from proposition 4.6 above.

**Lemma 4.7.** Let  $p > p_c(\mathbb{Z}^2)$ . Let  $\Omega_1$  and  $R_0(x, \omega)$  be as in Theorem 2.3. There is  $\nu' > 0$  such that the following holds.

Let  $\omega \in \Omega_1$  and let  $A$  be a finite subset of  $\mathcal{C}_\infty(\omega)$ . Fix  $x_0 \in \mathcal{C}_\infty(\omega)$ .

Let  $R > 1$  be such that  $A \subset B(x_0, R)$ .

Then there is  $N_0 = N_0(x_0, \omega)$  such that for all  $n > N_0$ , for all  $y \in A$  and  $z \in \partial\tilde{B}(x_0, n)$ ,

$$P_y(X_{\tilde{\sigma}_n \wedge \tau_A} = z | \tau_A > \tilde{\sigma}_n) = H_{\partial\tilde{B}(x_0, n)}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{n}\right)^{\nu'}\right) \right] \quad (4.25)$$

where  $\tilde{B}_n$  and  $\tilde{\sigma}_n$  are as in (4.23).  $\nu' > 0$  depends on the Hölder exponent given by Theorem 2.5 and the constants given in (4.21). The constants in  $O(\cdot)$  depend only on the constants that appear in theorems 2.3 and 2.5.

*Proof.* For  $R_1 > \max\{R_0(x_0, \omega), R\}$ , let  $B_1 = B(x_0, R_1)$ ,  $B_2 = B(x_0, 2R_1)$ ,  $B_3 = B(x_0, 4R_1)$ . Set  $n = (4R_1)^C$  and let  $\tilde{B}_n = \tilde{B}(x_0, n)$  and  $\tilde{\sigma}_n$  be as in (4.23). Note that by (4.24),  $B_3 \subset \tilde{B}_n$ .

For  $z \in \partial\tilde{B}_n$ , consider the function

$$f(x) = P_x(X_{\tilde{\sigma}_n} = z), \quad x \in \mathcal{C}_\infty(\omega).$$

Since  $f$  is harmonic on  $B_2$ , by Theorem 2.5, for all  $u \in B_1$ ,

$$|f(u) - f(x_0)| \leq c \left( \frac{D(x_0, u)}{R_1} \right)^\nu \max_{B_2} f.$$

In particular, for  $u \in \partial B(x_0, R)$ ,

$$|f(u) - f(x_0)| \leq c \left( \frac{R}{R_1} \right)^\nu \max_{B_2} f. \quad (4.26)$$

Now by considering  $f$  harmonic on  $B_3$ , by Theorem 2.3, we have that

$$\max_{B_2} f \leq c_1 f(x_0). \quad (4.27)$$

Therefore, by (4.26) and (4.27), for all  $u \in \partial B(x_0, R)$ ,

$$P_u(X_{\tilde{\sigma}_n} = z) = H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{R_1}\right)^\nu\right) \right]. \quad (4.28)$$

On the set  $\{\tau_A < \tilde{\sigma}_n\}$ , we let  $\eta = \inf\{j \geq \tau_A; X_j \in \partial B(x_0, R)\}$ .

Then using (4.28), we obtain that for all  $x \in \partial B(x_0, R)[\tilde{B}_n, A]$  (see (3.6) for the notation),

$$\begin{aligned} P_x(X_{\tilde{\sigma}_n} = z | \tau_A < \tilde{\sigma}_n) &= \sum_{u \in \partial B(x_0, R)} P_x(X_\eta = u | \tau_A < \tilde{\sigma}_n) P_u(X_{\tilde{\sigma}_n} = z) \\ &= H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{R_1}\right)^\nu\right) \right]. \end{aligned} \quad (4.29)$$

Let  $x \in \partial B(x_0, R)[\tilde{B}_n, A]$ . By (4.28) and (4.29), we get from the relation

$$\begin{aligned} P_x(X_{\tilde{\sigma}_n} = z) &= P_x(X_{\tilde{\sigma}_n} = z | \tau_A > \tilde{\sigma}_n) P_x(\tau_A > \tilde{\sigma}_n) \\ &\quad + P_x(X_{\tilde{\sigma}_n} = z | \tau_A \leq \tilde{\sigma}_n) (1 - P_x(\tau_A > \tilde{\sigma}_n)), \end{aligned}$$

that

$$P_x(X_{\tilde{\sigma}_n} = z | \tau_A > \tilde{\sigma}_n) = H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{R_1}\right)^\nu\right) \right].$$

This can also be written as,

$$P_x(X_{\tilde{\sigma}_n \wedge \tau_A} = z) = H_{\partial\tilde{B}_n}(x_0, z) P_x(\tau_A > \tilde{\sigma}_n) \left[ 1 + O\left(\left(\frac{R}{R_1}\right)^\nu\right) \right]. \quad (4.30)$$

Note that every path from  $y \in A$  to  $\partial\tilde{B}_n$  must go through some point of  $\partial B(x_0, R)[\tilde{B}_n, A]$ . So, for all  $y \in A$  and for all  $z \in \partial\tilde{B}_n$ ,

$$\begin{aligned}
P_y(X_{\tilde{\sigma}_n \wedge \tau_A} = z) &= \sum_{x \in \partial B(x_0, R)[\tilde{B}_n, A]} P_y(X_{\tau_{\partial B(x_0, R)[\tilde{B}_n, A]} \wedge \tau_A} = x) P_x(X_{\tilde{\sigma}_n \wedge \tau_A} = z) \\
&\stackrel{(4.30)}{=} H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{R_1}\right)^\nu\right) \right] \\
&\quad \times \sum_{x \in \partial B(x_0, R)[\tilde{B}_n, A]} P_y(X_{\tau_{\partial B(x_0, R)[\tilde{B}_n, A]} \wedge \tau_A} = x) P_x(\tau_A > \tilde{\sigma}_n) \\
&= H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{R_1}\right)^\nu\right) \right] P_y(\tau_A > \tilde{\sigma}_n).
\end{aligned}$$

Hence the lemma holds with  $N_0 = (4 \max\{R_0(x_0, \omega), R\})^C$ . ■

**4.3. The existence of the harmonic measure.** We now show how to obtain Theorem 1.8 from lemma 4.7. Let  $R$  be such that  $A \subset B(x_0, R)$ .

*Proof.* Let  $\tilde{B}_n$  and  $\tilde{\sigma}_n$  be as in (4.23). Let  $y \in A$ . For  $x \notin \tilde{B}_n$ , by (3.9), by reversibility of the Markov chain and by (4.25), for all  $n > N_0$ ,

$$\begin{aligned}
\pi(x)H_A(x, y) &= \pi(x)P_x(X_{\tau_A} = y) \\
&= \pi(x) \sum_{z \in \partial\tilde{B}_n} G_{A^c}(x, z) H_{A \cup \partial\tilde{B}_n}(z, y) \\
&= \sum_{z \in \partial\tilde{B}_n} G_{A^c}(z, x) \pi(y) H_{A \cup \partial\tilde{B}_n}(y, z) \\
&= \sum_{z \in \partial\tilde{B}_n} G_{A^c}(z, x) \pi(y) P_y(\tilde{\sigma}_n < \tau_A) H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{n}\right)^{\nu'}\right) \right] \\
&= \pi(y) P_y(\tilde{\sigma}_n < \tau_A) \sum_{z \in \partial\tilde{B}_n} G_{A^c}(z, x) H_{\partial\tilde{B}_n}(x_0, z) \left[ 1 + O\left(\left(\frac{R}{n}\right)^{\nu'}\right) \right]
\end{aligned}$$

At this point for the supercritical cluster of  $\mathbb{Z}^d$ ,  $d \geq 3$ , it suffices to sum over  $y \in A$  and divide the equations. However, since the walk is recurrent on the supercritical percolation cluster of  $\mathbb{Z}^2$ ,  $P_y(\tilde{\sigma}_n < \tau_A) \rightarrow 0$  as  $n \rightarrow \infty$ , this would lead to an indeterminate limit. But by proposition 4.6,

$$\begin{aligned}
\pi(x)H_A(x, y) &= \frac{\pi(x)H_A(x, y)}{\pi(x) \sum_{y' \in A} H_A(x, y')} \\
&= \lim_n \frac{\pi(y)P_y(\tilde{\sigma}_n < \tau_A)}{\sum_{y' \in A} \pi(y')P_{y'}(\tilde{\sigma}_n < \tau_A)} \\
&= \lim_n \frac{(\ln n)\pi(y)P_y(\tilde{\sigma}_n < \tau_A)}{(\ln n) \sum_{y' \in A} \pi(y')P_{y'}(\tilde{\sigma}_n < \tau_A)} \\
&= \frac{\pi(y)Pu_A(y)}{\sum_{y' \in A} \pi(y')Pu_A(y')}.
\end{aligned}$$
■

## 5. PROOF OF PROPOSITION 2.6

In this proof, we keep the notations of [6] except for the graph distance which will still be denoted by  $D(x, y)$ .

For a cube  $Q$  of side  $n$ , let  $Q^+ := A_1 \cap \mathbb{Z}^d$  and  $Q^\oplus := A_2 \cap \mathbb{Z}^d$  where  $A_1$  and  $A_2$  are the cubes in  $\mathbb{R}^d$  with the same center as  $Q$  and with side length  $\frac{3}{2}n$  and  $\frac{6}{5}n$  respectively. Note that  $Q \subset Q^\oplus \subset Q^+$ .

$\mathcal{C}(x)$  is the connected open cluster that contains  $x$ .  $\mathcal{C}_Q(x)$ , which will be called the open  $Q$  cluster, is the set of points connected to  $x$  by an open path within  $Q$ . And  $\mathcal{C}^\vee(Q)$  is the largest open  $Q$  cluster (with some rule for breaking ties).

Set  $\alpha_2 = (11(d+2))^{-1}$ .

*Proof.* By [6, lemma 2.24] and by Borel-Cantelli lemma, for all  $x \in \mathbb{Z}^d$ , there is  $N_x$  such that for all  $n > N_x$ ,  $L(Q)$  (see [6, p. 3052]) holds for all cubes  $Q$  of side  $n$  with  $x \in Q$ .

Let  $z \in \mathbb{Z}^d$  and let  $n > N_z = N_z(\omega)$ .

Let  $Q$  be a cube of side  $n$  which contains  $z$ .

Let  $x_0 \in \mathcal{C}^\vee(Q^+) \cap Q^\oplus$  with  $Q(x_0, r+k_0)^+ \subset Q^+$  where  $C_H n^{\alpha_2} \leq r \leq n$  and  $k_0 = k_0(p, d)$  is the integer chosen in [6, p. 3041].

Let  $R$  be such that

$$B_\omega(x_0, (3/2)R \ln R) \subset Q^\oplus \quad \text{and} \quad (5.31)$$

$$(C_H n^{\alpha_2})^{d+2} \leq (C_H n^{\alpha_2})^{4(d+2)} < R < R \ln R < n. \quad (5.32)$$

Then by [6, Theorem 2.18c],  $B_\omega(x_0, R \ln R)$  is  $(C_V, C_P, C_W)$ - very good with

$$N_{B_\omega(x_0, R \ln R)} \leq C_H n^{\alpha_2}$$

with the constants given in [6, section 2].

Then by [6, Theorem 5.11] and (5.32), there is a constant  $C_1$ , which depends only on  $d$  and on the constants  $C_V, C_P, C_W$ , such that if  $D(x_0, x_1) \leq \frac{1}{3}R \ln R$  and if  $h : B(x_1, R) \rightarrow \mathbb{R}$  is positive and harmonic in  $B(x_1, R)$ , then

$$\max_{B(x_1, R/2)} h \leq C_1 \min_{B(x_1, R/2)} h. \quad (5.33)$$

Note that since  $4\alpha_2(d+2) = 4/11 < 1/2$ , the conditions (5.32) are verified for  $R = 2\sqrt{n}$  when  $n$  large enough.

We now apply a standard chaining argument to a well chosen covering by balls (see for instance [31, chapters 3 and 9]). Let  $x_0 \in \mathbb{Z}^2$  and consider environments such that  $x_0 \in \mathcal{C}_\infty(\omega)$ . The main difficulty to carry out the chaining argument is to check that the intersection of ‘‘consecutive’’ balls is not empty. The remainder of the proof is to construct an appropriate covering of  $\{x \in \mathcal{C}_\infty; D(x_0, x) = m\}$ , for  $m$  large enough, with a finite number balls, which does not depend on  $x_0, m$  or  $\omega$ , and such that the Harnack inequality (5.33) holds in each ball.

Let  $\delta_1, \delta_2$  and  $\delta_3$  be three positive real numbers such that

$$2\delta_2 < \delta_1 \quad \text{and} \quad \delta_1 + 2\delta_2 < \delta_3 < \frac{1}{5\mu} \left( \frac{4}{5} - \delta_2 \right). \quad (5.34)$$

For instance, choose  $\delta_3$  so that  $0 < \delta_3 < 4/(50\mu)$ , then choose  $\delta_1$  so that  $0 < 2\delta_1 < \delta_3$  and finally choose  $\delta_2$  so that  $\delta_2 < \min\{\delta_1/2, 4/(50\mu)\}$ .

Let  $n > N_{x_0}$ .

Furthermore, take  $n$  large enough so that there is a Kesten's grid in  $Q$  with constant  $C_K$  and  $R(Q)$  holds (by [6, lemma 2.8]). That is in each vertical and each horizontal strip of width  $C_K \ln n$  contains at least  $c(p) \ln n$  open disjoint channels. Moreover, since  $R(Q)$  holds,  $\mathcal{C}^\vee(Q) \subset \mathcal{C}^\vee(Q^+)$ . In particular,  $x_0 \in \mathcal{C}^\vee(Q^+) \cap Q^\oplus$ .

Furthermore by (2.3) and Borel-Cantelli, if  $m$  is large enough then for all  $x, y \in \mathcal{C}_\infty$  such that  $|x|_1 \leq 3\mu m$ ,  $|y|_1 \leq 3\mu m$  and  $|x - y|_1 \geq m(\delta_1 - 2\delta_2)/\mu$  we have

$$|x - y|_1 \leq D(x, y) \leq \mu|x - y|_1.$$

Set  $\frac{R}{2} = m\delta_3 = \sqrt{n}$ .

Furthermore, take  $m$  large enough so that

$$C_K \ln n \leq m\delta_2/\mu, \quad 3m\mu < \frac{1}{3}R \ln R \quad (5.31) \text{ and } (5.32) \text{ are verified.}$$

Instead of constructing a finite covering of  $\{x \in \mathcal{C}_\infty; D(x_0, x) = m\}$ , it is easier to construct a finite covering of the region  $\{x \in \mathcal{C}_\infty; \frac{4m}{5\mu} \leq |x - x_0|_1 \leq 2m\}$  which is a larger subset of  $\mathbb{Z}^2$ .

Let  $\mathcal{I} := \{(i; j) \in \mathbb{N}^2; 4/(5\delta_1) \leq i + j \leq 2\mu/\delta_1\}$ . Let  $M$  be the cardinal of  $\mathcal{I}$ .

Let  $x_{i,j} = x_0 + (im\delta_1/\mu; jm\delta_1/\mu)$  with  $(i; j) \in \mathcal{I}$ . Then for each  $x_{i,j}$  with  $(i; j) \in \mathcal{I}$ , there is  $\tilde{x}_{i,j} \in \mathcal{C}_\infty$  such that  $|x_{i,j} - \tilde{x}_{i,j}|_1 \leq m\delta_2/\mu$ .

We proceed similarly in the other three quadrants to obtain a set of  $4M$  vertices which we denote by  $\mathcal{D}$ . Note that  $M$  does not depend on  $m$ .

The finite covering of the region  $\frac{4m}{5\mu} \leq |x - x_0|_1 \leq 2m$  is

$$\{B(\tilde{x}, m\delta_3), \quad \tilde{x} \in \mathcal{D}\}.$$

Note that each ball contains the center of the four neighbouring balls except those on the boundary of the region. But these are connected to at least one neighbouring ball. Indeed, if  $\tilde{x}, \tilde{y} \in \mathcal{D}$  are neighbouring centers then by (5.34),

$$D(\tilde{x}, \tilde{y}) < \mu|\tilde{x} - \tilde{y}| < m(\delta_1 + 2\delta_2) < m\delta_3.$$

If  $\tilde{x} \in \mathcal{D}$  then by (5.34),

$$D(x_0, \tilde{x}) > \frac{m}{\mu} \left( \frac{4}{5} - \delta_2 \right) > 5m\delta_3,$$

$D(x_0, \tilde{x}) < \mu|x_0 - \tilde{x}|_1 < 2m\mu$  and  $\mu(2m + m\delta_2/\mu) < 3m\mu$ .

Therefore,  $x_0$  does not belong to a ball of the covering and  $u$  is harmonic in each ball  $B(\tilde{x}, 2m\delta_3)$  with  $\tilde{x} \in \mathcal{D}$ . Then the Harnack inequality holds for  $R = 2m\delta_3$  since for all  $\tilde{x} \in \mathcal{D}$ ,

$$D(x_0, \tilde{x}) < 2m\mu < \frac{1}{3}R \ln R.$$

■

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