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Differential equations as embedding obstructions and vanishing theorems

Damian Brotbek

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Abstract

We generalize a vanishing theorem for the cohomology of symmetric powers of the cotangent bundle of subvarieties of projective space due to Schneider. From this we deduce new vanishing results for Green-Griffiths jet differential bundles, generalizing results of Diverio and Pacienza-Rousseau.

1 Introduction

Given a smooth projective n -dimensional variety X , one can search for the smallest integer N such that there exists an embedding $X \hookrightarrow \mathbb{P}^N$. Obviously, $n \leq N$, and, on the other hand, a simple argument shows that $N \leq 2n + 1$ (see [Har74] §4). In general, these bounds are optimal, and therefore it seems natural to try to understand what geometric properties of X yield obstructions for projective embeddings. Schneider proved in [Sch92] that the existence of symmetric differential forms on X imposes strong restriction on such embeddings. His result, and its generalization for higher order cohomology groups can be stated as follows.

Theorem 1.1 (Schneider [Sch92] Theorems 1.1 and 1.2). *Let $X \subset \mathbb{P}^N$ be a smooth subvariety of dimension n and codimension c . Let $m \in \mathbb{N}$ and $a \in \mathbb{Z}$. Consider an integer $0 \leq j < n - c$. If $a < m - \min\{j, 1\}$, then*

$$H^j(X, S^m \Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

If one regards global sections of $S^k \Omega_X$ as algebraic differential equations of order 1 defined on X , one is naturally led to consider the problem of finding a higher order analogue to this statement. In that direction, Diverio [Div08] established a vanishing theorem for Green-Griffiths jet differential bundles on complete intersection varieties (see section 3 for the definition).

Theorem 1.2 (Diverio [Div08] Theorem 7). *Let $X \subseteq \mathbb{P}^N$ be a smooth complete intersection of dimension n and codimension c . Then*

$$H^0(X, E_{k,m}^{GG} \Omega_X) = 0$$

for all $m \geq 1$ and $1 \leq k < \frac{n}{c}$.

Pacienza and Rousseau [PR08] constructed a generalized version of Green-Griffiths jet differential bundle and they proved a similar result for those more general bundles (see Theorem 8.1).

The aim of our work is to generalize, and unify, Schneider's and Diverio's result. Our main result is the following (see Theorem 6.3).

Theorem A. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider an integer $k \geq 1$, integers $\ell_1, \dots, \ell_k \geq 0$, and $a \in \mathbb{Z}$. If $j < n - k \cdot c$ and $a < \ell_1 + \dots + \ell_k - \min\{j, k\}$, then*

$$H^j(X, S^{\ell_1} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

From this generalization of Schneider's theorem one can deduce a new vanishing result for jet differentials (see Corollary 7.2).

Theorem B. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension c . Let $a \in \mathbb{Z}$. If $j < n - k \cdot c$ and $a < \frac{m}{k} - \min\{j, k\}$ then*

$$H^j(X, E_{k,m}^{GG} \Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Therefore the existence of higher order jet differential equation is an obstruction to projective embeddings, analogues to the ones pointed out by Schneider. We also prove an analogous result for Pacienza-Rousseau's generalized Green-Griffiths jet differential bundles (see Corollary 8.2).

Notation and conventions. We work over the field of complex numbers \mathbb{C} . When X is a complex manifold, we denote the tangent bundle of X by TX , and the cotangent bundle of X by $\Omega_X := TX^*$.

2 The bundle $\tilde{\Omega}$

It will be convenient for us to work with the bundle $\tilde{\Omega}$, studied in particular by Bogomolov and DeOliveira in [BDO08], but also by Debarre in [Deb05]. Therefore we recall some basic fact on this bundle. When $X \subseteq \mathbb{P}^N$ is an n -dimensional variety then $\tilde{\Omega}_X$ is, roughly speaking, the sheaf of differential forms on $\hat{X} \subseteq \mathbb{C}^{N+1}$, the cone over X , which are invariant under the \mathbb{C}^* -action but which do not necessarily satisfy the Euler condition. A more geometrical approach is to consider the Gauss map

$$\begin{aligned} \gamma_X : X &\rightarrow \text{Gr}(n, \mathbb{P}^N) \\ x &\mapsto \mathbb{T}_x X \end{aligned}$$

where $\mathbb{T}_x X \subseteq \mathbb{P}^N$ is the embedded tangent space of X at x , and where $\text{Gr}(n, \mathbb{P}^N)$ denotes the Grassmannian variety of n -dimensional projective subspaces of \mathbb{P}^N . Let \mathcal{S}_n denote the rank- n tautological bundle on $\text{Gr}(n, \mathbb{P}^N)$, then set

$$\tilde{\Omega}_X = \gamma_X^* \mathcal{S}_n^* \otimes \mathcal{O}_X(-1).$$

Observe that $\tilde{\Omega}_{\mathbb{P}(V)} = V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)$. The properties of $\tilde{\Omega}_X$ we will use are summarized in the following commutative diagram.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & N_{X/\mathbb{P}^N}^* & \xlongequal{\quad} & N_{X/\mathbb{P}^N}^* & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{\mathbb{P}^N|_X} & \longrightarrow & \tilde{\Omega}_{\mathbb{P}^N|_X} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Omega_X & \longrightarrow & \tilde{\Omega}_X & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

We refer to [BDO08] and [Deb05] for some details. In this paper the exact sequence

$$0 \rightarrow N_{X/\mathbb{P}^N}^* \rightarrow \tilde{\Omega}_{\mathbb{P}^N|_X} \rightarrow \tilde{\Omega}_X \rightarrow 0$$

will be called the *tilde conormal exact sequence*. And the exact sequence

$$0 \rightarrow \Omega_X \rightarrow \tilde{\Omega}_X \rightarrow \mathcal{O}_X \rightarrow 0$$

will be called the *Euler exact sequence*.

3 The Green-Griffiths jet differential bundles $E_{k,m}^{GG}\Omega_X$

We recall very briefly the definitions of Green-Griffiths jet differential bundles. We refer to [Dem97] for more information, and also to [Mer10] where many details are carried out explicitly. Let X be a projective variety of dimension n . For all $k \geq 1$, we denote by $J_k X \xrightarrow{p_k} X$ the holomorphic bundle of k -jets of germs of holomorphic curves $f : (\mathbb{C}, 0) \rightarrow X$, that is $J_k X := \{f : (\mathbb{C}, 0) \rightarrow X\} / \sim$, where two germs $f, g : (\mathbb{C}, 0) \rightarrow X$ are equivalent ($f \sim g$) if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $0 \leq j \leq k$. The projection is then simply defined by

$$\begin{array}{ccc} J_k X & \xrightarrow{p_k} & X \\ f & \mapsto & f(0). \end{array}$$

Those spaces naturally have the structure of holomorphic fiber bundles over X , but unless $k = 1$ they are not vector bundles. Moreover, for any $x \in X$, $J_k X_x \cong (\mathbb{C}^n)^k$. To see this, take local coordinates (z_1, \dots, z_n) around $x \in X$; we can then write $f = (f_1, \dots, f_n)$ and the k -jet will then be entirely determined, by Taylor's formula, by

$$(f_1'(0), \dots, f_n'(0), f_1''(0), \dots, f_n''(0), \dots, f_1^{(k)}(0), \dots, f_n^{(k)}(0)).$$

With the above notation, for each $k \geq 1$, there is a natural \mathbb{C}^* -action on $J_k X$. Namely if $\lambda \in \mathbb{C}^*$ and $f : (\mathbb{C}, 0) \rightarrow (X, x)$ then

$$\begin{array}{ccc} \lambda \cdot f : (\mathbb{C}, 0) & \rightarrow & (X, x) \\ t & \mapsto & f(\lambda t). \end{array}$$

This action is easily expressed fiberwise. Namely, if one considers local coordinates around $x = f(0) = \lambda \cdot f(0)$ then f is represented by $(f'(0), f''(0), \dots, f^{(k)}(0))$ and

$$\lambda \cdot (f'(0), f''(0), \dots, f^{(k)}(0)) = (\lambda f'(0), \lambda^2 f''(0), \dots, \lambda^k f^{(k)}(0)).$$

Then the Green-Griffiths jet differentials are defined as follows. Fiberwise we consider

$$E_{k,m}^{GG}\Omega_{X,x} := \left\{ Q \in \mathbb{C}[f', \dots, f^{(k)}] / Q(\lambda \cdot (f', \dots, f^{(k)})) = \lambda^m Q(f', \dots, f^{(k)}) \right\}.$$

One can make this even more explicit: when $x \in X$ is fixed, one can consider coordinates

$$(f_1', \dots, f_n', \dots, f_1^{(k)}, \dots, f_n^{(k)})$$

on $J_k X_x$. Then an element $Q \in E_{k,m}^{GG}\Omega_{X,x}$ is exactly a polynomial in the variables $(f_i^{(j)})$ of the form

$$Q = \sum_{\substack{I_1, \dots, I_k \in \mathbb{N}^n \\ |I_1| + 2|I_2| + \dots + k|I_k| = m}} a_{I_1, \dots, I_k} (f')^{I_1} \dots (f^{(k)})^{I_k}$$

where we use the standard multi-index notation for $I = (i_1, \dots, i_n)$ and $1 \leq j \leq k$ we set $(f^{(j)})^I := (f_1^{(j)})^{i_1} \dots (f_n^{(j)})^{i_n}$. It turns out that these fibers can be arranged into a vector bundle over X .

The bundle $E_{k,m}^{GG}\Omega_X$ admits a natural filtration. We briefly recall its construction; however, we will not go into details, in particular we don't justify why everything is well-defined.

Fix local coordinates around $x \in X$ as above. For each $p \in \mathbb{N}$ and for each $1 \leq s \leq k$ define

$$F_s^p = F_s^p(E_{k,m}^{GG}\Omega_{X,x}) = \left\{ Q \in E_{k,m}^{GG}\Omega_{X,x} \text{ involving only monomials } (f')^{I_1} \dots (f^{(k)})^{I_k} \text{ with } |I_1| + 2|I_2| + \dots + s|I_s| \geq p \right\}.$$

This gives a filtration

$$\{0\} = F_s^{m+1} \subseteq F_s^m \subseteq \dots \subseteq F_s^1 \subseteq F_s^0 = E_{k,m}^{GG} \Omega_X.$$

We consider the associated graded terms

$$\mathrm{Gr}_s^p = \mathrm{Gr}_s^p(E_{k,m}^{GG} \Omega_X) := F_s^p / F_s^{p+1}.$$

Observe that

$$\begin{aligned} \mathrm{Gr}_{k-1}^p &\cong \left\{ \begin{array}{l} Q \text{ involving only monomials } (f')^{I_1} \dots (f^{(k)})^{I_k} \\ \text{with } |I_1| + 2|I_2| + \dots + (k-1)|I_{k-1}| = p \end{array} \right\} \\ &\cong \left\{ \begin{array}{l} Q \text{ involving only monomials } (f')^{I_1} \dots (f^{(k)})^{I_k} \\ \text{with } k|I_k| = m - p \end{array} \right\}. \end{aligned}$$

Therefore $\mathrm{Gr}_{k-1}^p \neq 0$ if and only if there is an integer $\ell_k \in \mathbb{N}$ such that $p = m - k\ell_k$. Whenever this is satisfied, if one looks closely at the coordinate changes, one observe that

$$\mathrm{Gr}_{k-1}^p = \mathrm{Gr}_{k-1}^{m-k\ell_k} \cong E_{k-1, m-k\ell_k}^{GG} \Omega_X \otimes S^{\ell_k} \Omega_X.$$

And therefore,

$$\mathrm{Gr}_{k-1}^\bullet = \bigoplus_{1 \leq p \leq m} \mathrm{Gr}_{k-1}^p \cong \bigoplus_{0 \leq \ell_k \leq \lfloor \frac{m}{k} \rfloor} E_{k-1, m-k\ell_k}^{GG} \Omega_X \otimes S^{\ell_k} \Omega_X.$$

Combining all those filtrations, we find inductively a filtration F^\bullet on $E_{k,m}^{GG} \Omega_X$ such that the associated graded bundle is

$$\mathrm{Gr}^\bullet(E_{k,m}^{GG} \Omega_X) = \bigoplus_{\ell_1 + 2\ell_2 + \dots + k\ell_k = m} S^{\ell_1} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X.$$

4 Pacienza-Rousseau generalized jet differential bundles

The Green-Griffiths jet differential bundles were constructed to study entire maps $f : \mathbb{C} \rightarrow X$. Pacienza and Rousseau [PR08] generalized this construction to study more generally holomorphic maps $f : \mathbb{C}^p \rightarrow X$. For each $p \geq 1$ they constructed bundles $E_{p,k,m}^{GG} \Omega_X$ generalizing $E_{k,m}^{GG} \Omega_X = E_{1,k,m}^{GG} \Omega_X$. Their construction is very similar to the one described above, however there are some unexpected difficulties appearing. We just really briefly recall the definitions. For the details we refer to [PR08].

Fix $p \geq 1$. Consider the space $J_{k,p}X$ of k -jets of germs of holomorphic maps $f : (\mathbb{C}^p, 0) \rightarrow X$. As above, this space comes with a natural $(\mathbb{C}^*)^p$ -action. Namely, for $f : (\mathbb{C}^p, 0) \rightarrow X$ and $\lambda = (\lambda_1, \dots, \lambda_p) \in (\mathbb{C}^*)^p$ we define

$$\begin{aligned} \lambda \cdot f : (\mathbb{C}^p, 0) &\rightarrow X \\ (t_1, \dots, t_p) &\mapsto f(\lambda_1 t_1, \dots, \lambda_p t_p). \end{aligned}$$

The bundles $E_{p,k,m}^{GG} \Omega_X$ are defined as follows: for each $x \in X$ we consider

$$E_{p,k,m}^{GG} \Omega_{X,x} := \left\{ \begin{array}{l} Q(f', \dots, f^{(k)}) / Q(\lambda \cdot (f', \dots, f^{(k)})) = \lambda_1^m \dots \lambda_p^m Q(f', \dots, f^{(k)}) \\ \text{for all } \lambda = (\lambda_1, \dots, \lambda_p) \in (\mathbb{C}^*)^p \end{array} \right\}.$$

Those spaces can be arranged into vector bundles. The only thing we will need concerning these bundles is that $E_{p,k,m}^{GG} \Omega_X$ admits a filtration whose graded terms are

$$\bigotimes_{\alpha \in I_1} S^{\alpha} \Omega_X \dots \bigotimes_{\alpha \in I_k} S^{\alpha} \Omega_X,$$

where

$$\sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell = (m, \dots, m),$$

and where for $\ell \in \mathbb{N}$, $I_\ell := \{\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p \mid \alpha_1 + \dots + \alpha_p = \ell\}$. See Remark 2.6 in [PR08].

5 A cohomological lemma

During the proof of our main result, we will need an elementary cohomological lemma.

Lemma 5.1. *Let X be a projective variety. Let G be a vector bundle on X . Let $k \geq 1$. Suppose we have k long exact sequences of vector bundles on X :*

$$\begin{array}{cccccccc} 0 & \rightarrow & E_1^\ell & \rightarrow & E_1^{\ell-1} & \rightarrow & \cdots & \rightarrow & E_1^1 & \rightarrow & E_1^0 & \rightarrow & F_1 & \rightarrow & 0 \\ 0 & \rightarrow & E_2^\ell & \rightarrow & E_2^{\ell-1} & \rightarrow & \cdots & \rightarrow & E_2^1 & \rightarrow & E_2^0 & \rightarrow & F_2 & \rightarrow & 0 \\ & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & \\ 0 & \rightarrow & E_k^\ell & \rightarrow & E_k^{\ell-1} & \rightarrow & \cdots & \rightarrow & E_k^1 & \rightarrow & E_k^0 & \rightarrow & F_k & \rightarrow & 0. \end{array}$$

Fix an integer q . Suppose that

$$H^j(X, E_1^{i_1} \otimes \cdots \otimes E_k^{i_k} \otimes G) = 0 \quad \text{for all } j \leq q + i_1 + \cdots + i_k.$$

Then

$$H^j(X, F_1 \otimes \cdots \otimes F_k \otimes G) = 0 \quad \text{for all } j \leq q.$$

Remark 5.2. The case $k = 1$ appears already in Schneider's article [Sch92].

Proof. The proof is straightforward, but we write it down for the sake of completeness. This is just an induction on k .

Let us start when $k = 1$. We proceed by induction on ℓ .

When $\ell = 0$, this is obvious. When $\ell = 1$, we just have one short exact sequence

$$0 \rightarrow E_1^1 \rightarrow E_1^0 \rightarrow F_1 \rightarrow 0.$$

Tensoring it by G and looking at the associated long exact sequence in cohomology gives the result. Now when $\ell \geq 2$, we cut the long exact sequence into two pieces, and tensor everything by G to obtain

$$\begin{array}{cccccccccccc} 0 & \rightarrow & E_1^\ell \otimes G & \rightarrow & E_1^{\ell-1} \otimes G & \rightarrow & K \otimes G & \rightarrow & 0 \\ & & & & & & & & & & & & & & \\ 0 & \rightarrow & K \otimes G & \rightarrow & E_1^{\ell-2} \otimes G & \rightarrow & \cdots & \rightarrow & E_1^1 \otimes G & \rightarrow & E_1^0 \otimes G & \rightarrow & F_1 \otimes G & \rightarrow & 0. \end{array}$$

Apply the global section functor to the first exact sequence to obtain $H^j(X, K \otimes G) = 0$ for all $j \leq q + \ell - 1$, and then apply the induction hypothesis.

Now we let $k \geq 2$. Suppose the result holds for any family of k exact sequences. Take one more exact sequence

$$0 \rightarrow E_{k+1}^\ell \rightarrow E_{k+1}^{\ell-1} \rightarrow \cdots \rightarrow E_{k+1}^1 \rightarrow E_{k+1}^0 \rightarrow F_{k+1} \rightarrow 0. \quad (1)$$

Suppose that $H^j(X, E_1^{i_1} \otimes \cdots \otimes E_{k+1}^{i_{k+1}} \otimes G) = 0$ for all $j \leq q + i_1 + \cdots + i_{k+1}$.

Tensoring the exact sequence (1) by $F_1 \otimes \cdots \otimes F_k \otimes G$, we obtain

$$0 \rightarrow \tilde{E}_{k+1}^\ell \rightarrow \tilde{E}_{k+1}^{\ell-1} \rightarrow \cdots \rightarrow \tilde{E}_{k+1}^1 \rightarrow \tilde{E}_{k+1}^0 \rightarrow F_1 \otimes \cdots \otimes F_k \otimes G \rightarrow 0,$$

The case $r = 1$ is the original result of Le Potier [LP75].

Before we continue we recall a well-known fact. Consider a subvariety $X \subseteq \mathbb{P}^N$, and let N_{X/\mathbb{P}^N} denote the normal bundle to X in \mathbb{P}^N . Then $N_{X/\mathbb{P}^N} \otimes \mathcal{O}_X(-1)$ is globally generated. This follows directly from the normal exact sequence and the Euler exact sequence. As a matter of notation, we fix an $(N+1)$ -dimensional complex vector space such that $\mathbb{P}^N = \mathbb{P}(V)$.

Putting all this together, we can prove the following.

Theorem 6.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider an integer $k \geq 1$, integers $\ell_1, \dots, \ell_k \geq 0$, and $a \in \mathbb{Z}$. If $j < n - k \cdot c$ and $a < \ell_1 + \dots + \ell_k$, then*

$$H^j(X, S^{\ell_1} \tilde{\Omega}_X \otimes \dots \otimes S^{\ell_k} \tilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0.$$

Proof. Let $N := N_{X/\mathbb{P}^N}$. Recall the tilde conormal exact sequence

$$0 \rightarrow N^* \rightarrow \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow \tilde{\Omega}_X \rightarrow 0.$$

Taking different symmetric powers we obtain k exact sequences

$$\begin{aligned} 0 \rightarrow \Lambda^{\ell_1} N^* \rightarrow \Lambda^{\ell_1-1} N^* \otimes \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow \dots \rightarrow N^* \otimes S^{\ell_1-1} \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow S^{\ell_1} \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow S^{\ell_1} \tilde{\Omega}_X \rightarrow 0 \\ 0 \rightarrow \Lambda^{\ell_2} N^* \rightarrow \Lambda^{\ell_2-1} N^* \otimes \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow \dots \rightarrow N^* \otimes S^{\ell_2-1} \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow S^{\ell_2} \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow S^{\ell_2} \tilde{\Omega}_X \rightarrow 0 \\ \vdots \\ 0 \rightarrow \Lambda^{\ell_k} N^* \rightarrow \Lambda^{\ell_k-1} N^* \otimes \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow \dots \rightarrow N^* \otimes S^{\ell_k-1} \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow S^{\ell_k} \tilde{\Omega}_{\mathbb{P}^N|X} \rightarrow S^{\ell_k} \tilde{\Omega}_X \rightarrow 0 \end{aligned}$$

For $1 \leq p \leq k$, we let

$$E_p^i = \Lambda^i N^* \otimes S^{\ell_p-i} \tilde{\Omega}_{\mathbb{P}^N|X} = \Lambda^i N^* \otimes S^{\ell_p-i} V \otimes \mathcal{O}_X(i - \ell_p).$$

By Lemma 5.1 to prove that

$$H^j(X, S^{\ell_1} \tilde{\Omega}_X \otimes \dots \otimes S^{\ell_k} \tilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0 \text{ for } j < n - k \cdot c,$$

it suffices to prove that

$$H^j(X, E_1^{i_1} \otimes \dots \otimes E_k^{i_k} \otimes \mathcal{O}_X(a)) = 0 \text{ for } j < n - k \cdot c + i_1 + \dots + i_k.$$

We now prove this fact. We have:

$$\begin{aligned} H^j(X, E_1^{i_1} \otimes \dots \otimes E_k^{i_k} \otimes \mathcal{O}_X(a)) &= H^j(X, \Lambda^{i_1} N^* \otimes S^{\ell_1-i_1} \tilde{\Omega}_{\mathbb{P}^N|X} \otimes \dots \otimes \Lambda^{i_k} N^* \otimes S^{\ell_k-i_k} \tilde{\Omega}_{\mathbb{P}^N|X} \otimes \mathcal{O}_X(a)) \\ &= H^j(X, \Lambda^{i_1} N^* \otimes \dots \otimes \Lambda^{i_k} N^* \otimes S^{\ell_1-i_1} V \otimes \dots \otimes S^{\ell_k-i_k} V \otimes \mathcal{O}_X(i_1 - \ell_1 + \dots + i_k - \ell_k + a)) \\ &= S^{\ell_1-i_1} V \otimes \dots \otimes S^{\ell_k-i_k} V \otimes H^j(X, \Lambda^{i_1}(N^*(1)) \otimes \dots \otimes \Lambda^{i_k}(N^*(1)) \otimes \mathcal{O}_X(a - \ell_1 - \dots - \ell_k)) \end{aligned}$$

On the other hand using Serre duality, we obtain

$$\begin{aligned} H^j(X, \Lambda^{i_1}(N^*(1)) \otimes \dots \otimes \Lambda^{i_k}(N^*(1)) \otimes \mathcal{O}_X(a - \ell_1 - \dots - \ell_k)) \\ = H^{n-j}(X, \Lambda^{i_1}(N(-1)) \otimes \dots \otimes \Lambda^{i_k}(N(-1)) \otimes \mathcal{O}_X(\ell_1 + \dots + \ell_k - a) \otimes K_X) \end{aligned}$$

Since $\mathcal{O}_X(\ell_1 + \dots + \ell_k - a)$ is ample and $N(-1)$ is a rank- c globally generated vector bundle, we can apply Theorem 6.1 to see that this last cohomology group vanishes if

$$n - j > k \cdot c - i_1 - \dots - i_k$$

or equivalently, if

$$j < n - k \cdot c + i_1 + \dots + i_k.$$

This is exactly what was needed to conclude the proof. \square

We are in a position to deduce the announced generalization of Schneider's theorem.

Theorem 6.3. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider an integer $k \geq 1$ and k integers $\ell_1, \dots, \ell_k \geq 0$, and $a \in \mathbb{Z}$. If $j < n - k \cdot c$ and $a < \ell_1 + \dots + \ell_k - \min\{j, k\}$, then*

$$H^j(X, S^{\ell_1} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Remark 6.4. The particular case $k = 1$ is precisely Schneider's result.

Proof. Consider the k short exact sequences gotten from the Euler exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & S^{\ell_1} \Omega_X & \rightarrow & S^{\ell_1} \tilde{\Omega}_X & \rightarrow & S^{\ell_1-1} \tilde{\Omega}_X \rightarrow 0 \\ 0 & \rightarrow & S^{\ell_2} \Omega_X & \rightarrow & S^{\ell_2} \tilde{\Omega}_X & \rightarrow & S^{\ell_2-1} \tilde{\Omega}_X \rightarrow 0 \\ & & & & \vdots & & \\ 0 & \rightarrow & S^{\ell_k} \Omega_X & \rightarrow & S^{\ell_k} \tilde{\Omega}_X & \rightarrow & S^{\ell_k-1} \tilde{\Omega}_X \rightarrow 0. \end{array}$$

Fix $j < n - k \cdot c$. By Lemma 5.3 it is now sufficient to check that for any $0 \leq i_1, \dots, i_k \leq 1$, for all $0 \leq i \leq j - i_1 - \dots - i_k$ and for all $a < \ell_1 + \dots + \ell_k - \min\{j, k\}$

$$H^i(X, S^{\ell_1-i_1} \tilde{\Omega}_X \otimes \dots \otimes S^{\ell_k-i_k} \tilde{\Omega}_X \otimes \mathcal{O}_X(a)) = 0.$$

This follows from Theorem 6.2 as soon as one observes that under these conditions, $i_1 + \dots + i_k \leq \min\{j, k\}$. \square

7 Application to Green-Griffiths jet differentials

Now we apply those vanishing results to Green-Griffiths jet differential bundle. The idea of converting a vanishing result for symmetric differential form bundles into a vanishing result for Green-Griffiths jet differential bundles is due to Diverio. In [Div08], Diverio uses a vanishing theorem due to Bruckmann and Rackwitz (see [BR90]) concerning symmetric differential forms on complete intersection varieties. We proceed along the lines of [Div08].

We start by recalling a cohomological lemma (see [Div08]).

Lemma 7.1. *Let $E \rightarrow X$ be a holomorphic vector bundle with a filtration $\{0\} = E_r \subset \dots \subset E_1 \subset E_0 = E$. If $H^q(X, \text{Gr}^\bullet E) = 0$, then $H^q(X, E) = 0$.*

Combining this lemma with Theorem 6.3 one derives our result.

Corollary 7.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension c . Let $a \in \mathbb{Z}$. If $j < n - k \cdot c$ and $a < \frac{m}{k} - \min\{j, k\}$, then*

$$H^j(X, E_{k,m}^{GG} \Omega_X \otimes \mathcal{O}_X(a)) = 0.$$

Proof. Recall that the Green-Griffiths bundle admits a filtration whose graded bundle is given by

$$\text{Gr}^\bullet E_{k,m}^{GG} \Omega_X = \bigoplus_{\ell_1+2\ell_2+\dots+k\ell_k=m} S^{\ell_1} \Omega_X \otimes S^{\ell_2} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X.$$

By twisting everything by $\mathcal{O}_X(a)$, we obtain a filtration for $E_{k,m}^{GG} \Omega_X \otimes \mathcal{O}_X(a)$ whose graded bundle is given by

$$\text{Gr}^\bullet E_{k,m}^{GG} \Omega_X \otimes \mathcal{O}_X(a) = \bigoplus_{\ell_1+2\ell_2+\dots+k\ell_k=m} S^{\ell_1} \Omega_X \otimes S^{\ell_2} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X \otimes \mathcal{O}_X(a).$$

By Lemma 7.1, we only have to prove that

$$H^j(X, S^{\ell_1} \Omega_X \otimes S^{\ell_2} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X \otimes \mathcal{O}_X(a)) = 0$$

for all $j < n - k \cdot c$ and $\ell_1 + \dots + k\ell_k = m$. The hypothesis then gives us

$$a + \min\{j, k\} < \frac{m}{k} = \frac{\ell_1 + \dots + k\ell_k}{k} \leq \frac{k\ell_1 + \dots + k\ell_k}{k} = \ell_1 + \dots + \ell_k.$$

Thus we can conclude by applying Theorem 6.3. \square

Remark 7.3. When $k = 1$, this is exactly Schneider's theorem. When X is a complete intersection, $a = 0$ and $j = 0$ then this is precisely Diverio's theorem.

Remark 7.4. Assertion 2 in Corollary 7.2 is optimal in k . Let X be a complete intersection variety in \mathbb{P}^N of dimension n and codimension $c = N - n$, denotes its multidegree by (d_1, \dots, d_c) . It is possible to prove that for any $k \geq \frac{n}{c}$, if the d_i are big enough, then $0 \neq H^0(X, E_{k,m}^{GG} \Omega_X \otimes \mathcal{O}_X(a))$ for $m \gg 0$. In fact one obtains a stronger result: under the same hypothesis, it is possible to prove that $0 \neq H^0(X, E_{k,m} \Omega_X \otimes \mathcal{O}_X(a))$, where $E_{k,m} \Omega_X \subseteq E_{k,m}^{GG} \Omega_X$ denotes the Demailly-Semple jet differential bundle. We refer to [Bro11] for a proof.

8 Application to Pacienza-Rousseau's generalized Green-Griffiths jet differentials

In [PR08], Pacienza and Rousseau generalized Green-Griffiths jet differential bundles, $E_{p,k,m}^{GG}$. And, among other things, they generalized Diverio's vanishing theorem to those bundles.

Theorem 8.1 (Pacienza-Rousseau [PR08] Theorem 5.1). *Let $X \subseteq \mathbb{P}^N$ be a smooth complete intersection. Then*

$$H^0(X, E_{p,k,m}^{GG}) = 0,$$

for all $m \geq 1$ and k such that

$$\binom{k+p}{p} - 1 < \frac{\dim(X)}{\text{codim}(X)}.$$

There proof rests on the same idea than the proof of Diverio's Theorem 1.2. Therefore it is not surprising that one can prove a more general statement using Theorem 6.3.

Corollary 8.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension c . Let $a \in \mathbb{Z}$. If $0 \leq j < N - \binom{k+p}{p} \cdot c$ and $a < \frac{m}{k} - \min\left\{j, \binom{k+p}{p} - 1\right\}$, then*

$$H^j(X, E_{p,k,m}^{GG} \otimes \mathcal{O}_X(a)) = 0.$$

Proof. Recall ([PR08] Remark 2.6) that there is a filtration for $E_{p,k,m}^{GG} \Omega_X \otimes \mathcal{O}_X(a)$ such that the graded terms are of the form

$$\bigotimes_{\alpha \in I_1} S^{q_\alpha^1} \Omega_X \cdots \bigotimes_{\alpha \in I_k} S^{q_\alpha^k} \Omega_X \otimes \mathcal{O}_X(a), \quad (2)$$

where $I_\ell := \{\alpha = (\alpha_1, \dots, \alpha_p) / \alpha_1 + \dots + \alpha_p = \ell\}$ and where $\sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell = (m, \dots, m)$.

First observe that, $|I_1| + \dots + |I_k| = \binom{k+p}{p} - 1$. Therefore there are only $\binom{k+p}{p} - 1$ symmetric powers in expression (2). We apply Theorem 6.3, with the above observation, we just have to prove that under our hypothesis

$$a < \sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell - \min\left\{j, \binom{k+p}{p} - 1\right\}.$$

But this follows at once from

$$\frac{(m, \dots, m)}{k} = \sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell \frac{\alpha}{k} \leq \sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell \frac{(k, \dots, k)}{k} = \left(\sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell, \dots, \sum_{\ell=1}^k \sum_{\alpha \in I_\ell} q_\alpha^\ell \right).$$

□

Remark 8.3. Recall Hartshorne's conjecture for complete intersection variety (see [Har74]): if X is a non-singular subvariety of dimension n of \mathbb{P}^N , and if $n > \frac{2}{3}N$, then X is a complete intersection. In view of this conjecture, our vanishing results are not surprising, at least in low codimension. It just shows that one can not use naively Diverio's vanishing theorem nor Pacienza-Rousseau's vanishing theorem to distinguish a complete intersection variety from a variety which is not complete intersection.

9 A further generalization

One can generalize Theorem 6.2 and 6.3 even further if one uses the whole strength of Manivel's results.

Theorem 9.1 (Manivel [Man97] Theorem A). *Let E be a holomorphic vector bundle of rank e , and L a line bundle on a smooth projective complex variety X of dimension n . Suppose that E is ample and L is nef, or that E is nef and L ample. Then, for any sequences of integers k_1, \dots, k_ℓ and j_1, \dots, j_m ,*

$$H^{p,q}(X, S^{k_1} E \otimes \dots \otimes S^{k_\ell} E \otimes \Lambda^{j_1} E \otimes \dots \otimes \Lambda^{j_m} E \otimes (\det E)^{\ell+n-p} \otimes L) = 0$$

as soon as $p + q > n + \sum_{s=1}^m (e - j_s)$.

Redoing the proof of Theorem 6.2 and 6.3, and considering moreover exterior powers of $\tilde{\Omega}_X$ and Ω_X one obtains the following.

Theorem 9.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and codimension $c = N - n$. Consider integers $j, k, p, r \geq 0$, integers $\ell_1, \dots, \ell_k, m_1, \dots, m_r \geq 0$, and $a, q \in \mathbb{Z}$. Let $\ell = \sum_{i=1}^k \ell_i$ and $m = \sum_{i=1}^r m_i$.*

1. *If $j + p < n - kc$ and $K_X^{-q-p-r} \otimes \mathcal{O}_X(\ell + m - (n+1)(p+r) - a)$ is ample, then*

$$H^{p,j}(X, S^{\ell_1} \tilde{\Omega}_X \otimes \dots \otimes S^{\ell_k} \tilde{\Omega}_X \otimes \Lambda^{m_1} \tilde{\Omega}_X \otimes \dots \otimes \Lambda^{m_r} \tilde{\Omega}_X \otimes K_X^q \otimes \mathcal{O}_X(a)) = 0.$$

2. *If $j + p < n - kc$ and $K_X^{-q-p-r} \otimes \mathcal{O}_X(\ell + m - (n+1)(p+r) - a - \min\{j, k+m\})$ is ample, then*

$$H^{p,j}(X, S^{\ell_1} \Omega_X \otimes \dots \otimes S^{\ell_k} \Omega_X \otimes \Lambda^{m_1} \Omega_X \otimes \dots \otimes \Lambda^{m_r} \Omega_X \otimes K_X^q \otimes \mathcal{O}_X(a)) = 0.$$

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