Faithful polygonal representation of the convex and concave parts of a digital curve

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Reversible Polygon that Faithfully Represents the Convex and Concave Parts of a Digital Curve

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Abstract

From results about digital convexity, we define a reversible polygon that faithfully represents the maximal convex and concave parts of a digital curve. Such a polygon always exists and is unique in the general case. It is computed from a given digital curve in linear-time using well-known routines: adding a point at the front of a digital straight segment and removing a point from the back of a digital straight segment. It may helps to extract perceptually meaningful parts of shape outlines or lines.

Keywords: digital curve; polygonal representation; convex and concave parts

1. Introduction

In [7, 11, 8, 9], Eckhardt and Dorksen-Reiter looked for a reversible polygon that faithfully represents the convex and concave parts of the boundary of a connected digital set. We call such a polygonal representation *faithful polygon* (FP for short) in this paper. Because of the chosen digitization scheme (Gauss digitization), they failed to properly define the FP of any open digital curve and show some digital curves for which it does not exist [7]. Therefore, the problem of defining and computing the FP of a digital curve remained open. Though,

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this problem is closely related to digital convexity and a lot of progress has been made on this topic in the last decade [5, 8, 9, 1, 29, 25]. Two recent algorithms [29, 25] retrieve some kinds of locally extremal points, but the authors did not notice that their algorithm is also able to compute a polygonal representation close to the FP.

The FP of a digital curve may lead to many applications. One of them consists in decomposing digital objects into perceptually meaningful parts, which is of great interest in shape recognition. The concept of convexity plays an important role because a visual part, which is not necessarily convex, is assumed to be convex at a given scale [23]. Finding a good scale with respect to a given purpose or studying the convexity at various scales may solve the problem.

In addition, the convex and concave parts of an object straightforwardly give the sign of the curvature of its boundary, which is known to be a relevant piece of information about its shape. For instance, curvature zero-crossings at various scales lead to an interesting shape representation, known as the curvature scale space representation [24]. Moreover, at a given scale, the endpoints of maximal convex (resp. concave) parts correspond to points of minimal negative (resp. maximal positive) curvature [23].

In Euclidean geometry, a given region $\mathcal{R}$ is said to be convex if and only if for any pair of points $p, q \in \mathcal{R}$ the line segment $[pq]$ is included in $\mathcal{R}$. However, in digital image processing, when each pixel is viewed as a point of $\mathbb{Z}^2$, the only convex regions (in the Euclidean sense) are isolated points, which is not satisfactory at all. Many authors defined the convexity of digital sets, i.e. sets of points of $\mathbb{Z}^2$ (see for instance, Sklansky [30], Kim [16, 17], Kim and Rosenfeld [15, 18], Kim and Sklansky [19], Chassery [2] and Ronse [27]). Most of these definitions may be proved to be equivalent for simply connected sets [17, 16, 27, 10].

However, they failed to properly define the convex and concave parts of open digital curves (because a convex open digital curve may also be considered as a digital set that is not convex). In many applications such as line drawings processing, dealing with open digital curves is important. That is why we define in this paper convex and concave parts by means of the slope of the maximal digital straight segments, i.e. digital straight segments of a digital curve that cannot be extended neither at the front nor at the back (see [20] for a review about digital straightness). This definition was first proposed in [8] and is also used in [12, 25]. In a practical point of view, this definition enables us to deal with any digital curve, which may correspond to the boundary of a given digital set or not, like digital spirals (Fig. 2). In a theoretical point of view, this definition appears to be quite natural since convexity is closely related to straightness: convex sets are defined by means of line segments and line segments are convex sets.

In section 2, we recall some definitions related to digital straightness and digital convexity. In section 3, we extend some important previous results about digital convexity and present new ones in order to precisely define the FP of a digital curve in section 4. We prove its main properties (existence and uniqueness, reversibility, faithful representation of the convex and concave parts) before concluding in section 5.

2. Definitions and Preliminaries

Two points $P, Q \in \mathbb{Z}^2$ are 4-neighbors (resp. 8 neighbors) if and only if $\|\overrightarrow{PQ}\|_1 = 1$ (resp. $\|\overrightarrow{PQ}\|_\infty = 1$).

For all $k \in \{4, 8\}$, a k-connected digital curve (also called k-arc), denoted by C, is a sequence of points $C_1, C_2, \ldots, C_n \in \mathbb{Z}^2$ such that for all $i \in 1, \ldots, n-1$, C_i has exactly two k-neighbors, which are C_{i-1} and C_{i+1}. Any sequence of points $C_i, \ldots, C_j$ is conveniently denoted by $C_{i,j}$.

Moreover, C is open if C_1 and C_n have one k-neighbor only, and closed if C_1 and C_n also have exactly two k-neighbors, which are respectively C_n and C_2, C_{n-1} and C_1.

In the sequel, we define convex and concave digital curves by means of the slope of the maximal digital straight segments. That's why we need to define digital straightness first.

2.1. Digital Straightness

The digital straight line (DSL for short) $D(a, b, \mu)$ of slope $\frac{a}{b}$ and intercept μ is the set of points $(x, y) \in \mathbb{Z}^2$ verifying $\mu \leq ax - by < \mu + \omega$ with a, b, μ, ω integer and $gcd(a, b) = 1$[26]. The DSL is a 4-connected sequence of points if $\omega = |a| + |b|$ and a 8-connected sequence of points if $\omega = \max(|a|, |b|)$.

We choose to work in this paper with the concept of 4-neighborhood, so that $\omega = |a| + |b|$ and $k = 4$. The results derived in the rest of the paper are applicable for 8-neighborhood but with few changes.

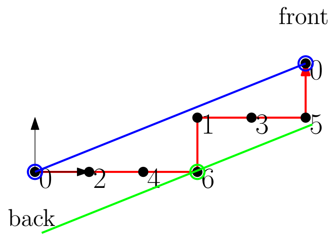
The points lying on the *upper* (resp. *lower*) *leaning line* verifying $ax - by = \mu$ (resp. $ax - by = \mu + \omega - 1$) are called the *upper* (resp. *lower*) *leaning points*. In the sequence, upper leaning points alternate with lower ones.

A *digital straight segment* (DSS for short) $S(a, b, \mu, \omega)$ of slope $\frac{a}{b}$ and intercept μ is a connected part of the DSL $D(a, b, \mu, \omega)$ containing at least three consecutive leaning points in a clockwise orientation when upper leaning points (in sequence order) are taken before the lower leaning points (in sequence order too). As shown in Fig. 1, the same set of points does not have the same parameters when scanning forward or backward. Actually, the upper (resp. lower) leaning line always lies on the left (resp. right) side of a little man walking along the DSS. As we will see in the next sections, this invariant is key to compute the FP of a digital curve from the leaning points of DSSs.

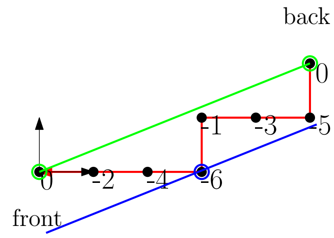
A upper (resp. lower) *digital edge* (also called pattern and reversed pattern in [3]) is a DSS whose first and last points are upper (resp. lower) leaning points. The DSS of Fig. 1 is actually a upper digital edge when scanning forward (a), but a lower digital edge when scanning backward (b).

Due to its definition, any part of a DSS is obviously a DSS. As a corollary, a part that is not a DSS cannot be contained in longer parts that are DSSs. Therefore, any DSS S of C can be defined as maximal iff all the parts C' of C containing S, i.e. such that $S \subset C'$, are not DSSs.

The set of all maximal DSSs (MSs for short, also called fundamental segments in [8]) that lie on a given digital curve is unique. The first point of any



(a) $S(2,5,0,7)$



(b) $S(-2,-5,-6,7)$

Figure 1: The same set of points may be viewed as two DSSs of different parameters when scanning forward or backward. The upper leaning line is depicted with blue, whereas the lower one is depicted with green. The three leaning points are always in a clockwise orientation when upper leaning points (in sequence order) are taken before the lower leaning points (in sequence order too).

two distinct MSs cannot be identical because if it is, the shortest MS is necessarily contained in the longest one and is thus not maximal. Consequently, the MSs can be ordered without any ambiguity according to the position of their first point on the contour. Let us then denote by $(S_i)_{i\in 1,\dots,m}$ the sequence of the MSs lying on C. For the sake of clarity the indices are taken modulo m for closed digital curves.

There exists an elegant algorithm that computes the set of MSs of a digital curve in linear time [13, 22]. An illustration of the output of this algorithm is depicted in Fig. 2.

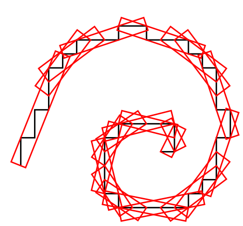


Figure 2: MSs of a digital curve. Each MS is depicted with a red bounding box.

The mechanism can be coarsely described as follows: given a MS, the next one is computed first by removing points from the back of the segment until it can be extended at the front and then by adding points at the front of the segment until it is maximal.

The key tasks are adding [6, 21] and removing [22] a point at one extremity of a DSS in constant time and space.

2.2. Digital Convexity

Let us focus now on the definition of digital convexity for open digital curves. The usual definition is based on the convex hull of a set of points and as a consequence is only satisfactory for closed digital curves or some classes of open digital curves like the monotonic ones in [8, Definition 8.4]:

Definition 1 (Digital convexity from convex hull). *Let the points C_1, C_2, ..., $C_n \in \mathbb{Z}^2$ of a digital curve C have increasing x-coordinates. The digital curve C is upper (resp. lower) convex if and only if there is no point $P \in \mathbb{Z}^2$ between the polygonal line linking the points of C and the upper (resp. lower) part of the convex hull of C.*

In order to extend this definition to arbitrary digital curves, the usual approach consists in decomposing the digital curve into monotonic pieces (as done for instance in [25]). However, this approach does not address special cases occuring in junctions. That is why we propose below a local definition based on the slope of the MSs. It extends the definition of [8, Definition 8.6] only stated for digital curves having points of increasing x-coordinates.

Definition 2 (Digital convexity from MSs). *Two consecutive MSs $S_i(a, b, \mu, \omega)$ and $S_{i+1}(a', b', \mu', \omega')$ make a* convex turn *iff $ab' - a'b > 0$, but make a* concave turn *iff $ab' - a'b < 0$.*

A digital curve C is locally convex (resp. concave) everywhere if and only if any two consecutive MSs make a convex (resp. concave) turn.

This last definition is valid for any open digital curves. For instance, the spiral-shaped digital curve of Fig. 2 is locally convex everywhere because the slope of any two consecutive MSs make a convex turn.

For sake of clarity, we merely say that a digital curve is convex (resp. concave) when it is locally convex (resp. concave) everywhere.

Note that a convex digital curve C is concave if the points are scanned in the reverse way and conversely.

Definition 1 and Definition 2 are perfectly equivalent for digital curves having points of increasing x-coordinates as proven in [8, Theorem 8.1]. Due to the importance of this equivalence in the next sections, we recall this result in the following lemma:

Lemma 1 ([8, Theorem 8.1]). *Let the points $C_1, C_2, \ldots, C_n \in \mathbb{Z}^2$ of a digital curve C have increasing x-coordinates. Any two consecutive MSs of C make a convex (resp. concave) turn if and only if C is upper (resp. lower) convex.*

In the rest of the paper, we refer to this lemma several times, because we always consider small parts of digital curves having no more than two MSs. Two consecutive MSs intersect in at least two distinct points $P, Q \in \mathbb{Z}^2$ so that $|x_P - x_Q| = 1$ or $|y_P - y_Q| = 1$. Since the coordinates of the points of any DSS monotonically increase or decrease, the x-coordinates (resp. y-coordinates) of the points of two consecutive MSs increase or decrease if $|x_P - x_Q| = 1$ (resp. $|y_P - y_Q| = 1$). Due to symmetries, we can assume that the points have increasing x-coordinates. As a result, we can arbitrarily use Definition 1 or Definition 2 in such cases due to Lemma 1.

Lemma 2 is intuitive and its proof straightforwardly stems from Definition 1 for the class of monotonic digital curves. Writing a proof for any digital curve is however not trivial and the simplest way to achieve this requires using Lemma 1.

Lemma 2. *Any part of a convex (resp. concave) digital curve is convex (resp. concave).*

Proof

Let C be a convex digital curve of n points ($n > 1$). The concave case is symmetric. If we can prove that the sequence of points $C_{1,n-1}$ is a convex digital curve, we can similarly prove that the sequence of points $C_{2,n}$ is a convex digital curve. By induction, we can thus prove that any part of a convex digital curve is convex.

Let us now focus on $C_{1,n-1}$. Its sequence of MSs is hardly the same as the one of $C_{1,n}$. More precisely, all the MSs, except the last one, are equal. As a consequence, we can focus on the part $C_{k,n}$ covered by the second-to-last and the last MS of $C_{1,n}$. If $C_{k,n-1}$ contains only one MS, we are done, otherwise $C_{k,n-1}$ contains only two MSs and it remains to show that they make a convex turn. This step is not trivial because the slope of the last MS of $C_{k,n-1}$ may be smaller of greater than the second-to-last MS of $C_{k,n}$.

Let us assume without loss of generality that the points of $C_{k,n}$ have increasing x-coordinates. The two MSs of $C_{k,n}$ make a convex turn because $C_{1,n}$ is assumed to be convex. Due to Lemma 1, $C_{k,n}$ is thus upper convex. It is then obvious from Definition 1 that $C_{k,n-1}$ is also upper convex, which means that $C_{k,n-1}$ is convex due to Lemma 1. □

Due to the previous lemma, any part of a convex (resp. concave) digital curve is convex (resp. concave). As a corollary, a part that is not convex (resp. concave) cannot be contained in a convex (resp. concave) part. Therefore, as for DSSs, we can define any convex (resp. concave) part P of C as maximal if and only if all the parts C' of C containing P, i.e. such that $P \subset C'$, are not convex (resp. concave). The set of all maximal convex or concave parts that lie on a given digital curve is unique and provides a decomposition into maximal convex and concave parts that is studied in the following section.

3. Local convexity properties

In this section, we recall important local convexity properties that may be found in [5, Theorem 9], [22, lemma 1] and [29, Lemma1]. They are crucial for either showing the existence and uniqueness of the FP or designing a linear-time algorithm that computes such a polygon, which is done in the next section.

3.1. Local criterion for checking convexity

In order to decide if two consecutive MS S_i and S_{i+1} make a convex (resp. concave) turn it is actually enough to consider the point of S_{i+1} following the last point of S_i, without considering the whole MS S_{i+1}:

Theorem 1. *Let $S_i(a, b, \mu, \omega)$ and $S_{i+1}(a', b', \mu', \omega')$ be two consecutive MSs lying along a digital curve C. Let $P(x_P, y_P)$ be the point of S_{i+1} following the last point of S_i. The MSs S_i and S_{i+1} make a convex (resp. concave) turn iff $ax_P - by_P > \mu + \omega$ (resp. $ax_P - by_P < \mu - 1$).*

Proof

Let $C_{k,l}$ be the sequence of points covered by S_i and $C_{k',l'}$ be the sequence of points covered by S_{i+1}. Note that $P = C_{l+1}$. Let us assume without loss of generality that the points of $C_{k,l'}$ have increasing x-coordinates (see Section 2.2).

$\Rightarrow$ If $ax_P - by_P > \mu + \omega$, $C_{k,l+1}$ is not lower convex due to the results of [6]. The whole part $C_{k,l'}$ is thus not lower convex either. Due to Lemma 1, S_i and S_{i+1} does not make a concave turn, but necessarily a convex one.

Similarly, if $ax_P - by_P < \mu - 1$, S_i and S_{i+1} make a concave turn.

$\Leftarrow$ If S_i and S_{i+1} make a convex turn, $C_{k,l'}$ is convex due to Definition 2. The part $C_{k,l+1}$ is thus convex too, due to Lemma 2. Since $C_{k,l+1}$ is not a DSS, either $ax_P - by_P < \mu - 1$ or $ax_P - by_P > \mu + \omega$ [6]. The first case leads to a contradiction due to the forward implication. As a result, only the second one is true, i.e. $ax_P - by_P > \mu + \omega$.

If S_i and S_{i+1} make a concave turn, we can similarly show that $ax_P - by_P < \mu - 1$. $\square$

Close results may be found in [5, Theorem 9] and [22, lemma 1]. A corollary of Theorem 1 is that each maximal convex or concave part of C is exactly covered by a sequence of consecutive MSs of C, i.e. its first and last point are respectively the first and last point of MSs of C.

In Fig. 3, the MSs of the maximal convex parts are depicted in blue and yellow, whereas the MSs of the maximal concave parts are depicted in green and yellow. The yellow MSs belong to both a maximal convex part and a maximal concave part.

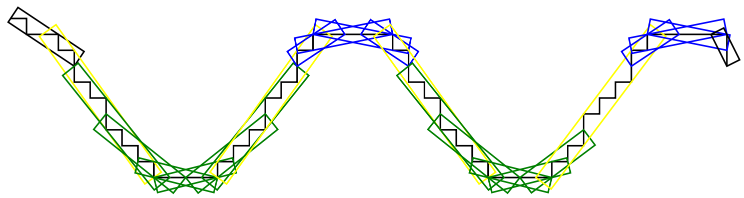


Figure 3: Decomposition of a digital curve into maximal convex and concave parts. A MS is depicted by a bounding box: blue if the MS belongs to a convex part only, green if it belongs to a concave part only, yellow if it belongs to both.

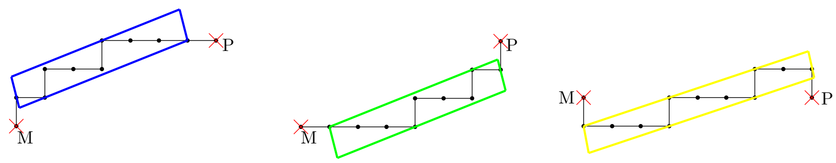
In practice, the maximal convex and concave parts of a digital curve are retrieved in the course of the MSs computation. Let $C_{k,l}$ be the last MS $S(a, b, \mu, \omega)$ and let $C_{l+1} = P$ be the next point on the digital curve. If S belongs to a convex (resp. concave) part and $ax_P - by_P < \mu - 1$ (resp. $ax_P - by_P > \mu + \omega$), then S is the end of a maximal convex (resp. concave) part and the beginning of a concave (resp. convex) part (and would be depicted with a yellow bounding box).

Furthermore, Theorem 1 answers to Eckhardt's question [11]: how far one can decide whether a part of a digital curve is convex or not by a method that is *as local as possible*? The answer is actually that the smallest part required for checking convexity is given by a MS, *plus* at least one of the two points located just before and after this segment. Any smaller part of any digital curve is a DSS, and thus both convex and concave, which is useless.

To sum up, looking at the location of the two points M and P that respectively bound a MS $S(a, b, \mu, \omega)$ from back and front, is a way of classifying the local geometry of the digital curve:

- if $ax_M - by_M > \mu + \omega$ and $ax_P - by_P > \mu + \omega$ (resp. $ax_M - by_M < \mu - 1$ and $ax_P - by_P < \mu - 1$), S belongs to a convex (resp. concave) part as shown in Fig. 4.a (resp. Fig. 4.b).

- if $ax_M - by_M < \mu - 1$ and $ax_P - by_P > \mu + \omega$ (and conversely), S makes the transition between a convex and a concave part (Fig. 4.c).



(a) convex (b) concave (c) inflection

Figure 4: Three different local configurations: MS in a convex part (a), in a concave part (b), in an inflection part (c).

3.2. Local criterion for extracting dominant points

We go further and consider local digital convexity at a thinner scale in order to retrieve some local dominant points, defined as follows:

Definition 3 (dominant point). *Any point C_k that cannot be strictly contained in a upper (resp. lower) digital edge $C_{i,j}$ $(i < k < j)$ is a convex (resp. concave)* dominant point.

Note that the first and last points of an open digital curve are by definition both convex and concave dominant points.

Dominant points are the keystones of the faithful polygon definition, as shown in Section 4. We show below that the study of subsets of MSs is sufficient and necessary to detect local dominant points.

Theorem 2. *Let $C_{i,j}$ be a convex (resp. concave) digital curve and let $C_{k,l}$ be a DSS $S(a, b, \mu, \omega)$ included in $C_{i,j}$ $(i \leq k < l \leq j)$ that is not contained in a longer DSS having the same parameters. Let M and P be the points that respectively bound S from back and front (if they exist), i.e. $M = C_{k-1}$ and $P = C_{l+1}$. The first and last upper (resp. lower) leaning points of S are convex (resp. concave) dominant points iff $ax_M - by_M \geq \mu + \omega$ (resp. $ax_M - by_M \leq \mu - 1$) or $k = i$ and $ax_P - by_P \geq \mu + \omega$ (resp. $ax_P - by_P \leq \mu - 1$) or $l = j$.*

One of the possible configurations involved in Theorem 2 is depicted in Fig. 5.a.

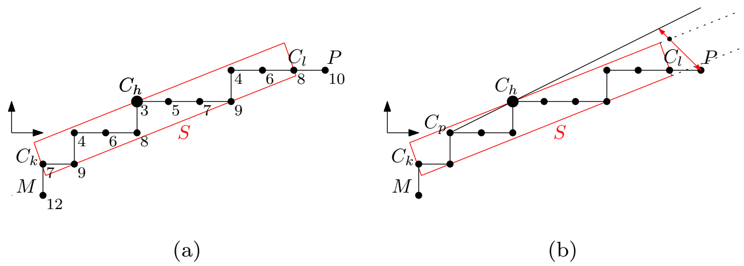


Figure 5: Illustration of Theorem 2. The part $C_{k,l}$ is equal to the DSS $S(2,5,3,7)$. The upper leaning point of $C_{k,l}$, i.e. C_h, is a convex dominant point because $ax_M - by_M > \mu + \omega$ ($12 > 10$) and $ax_P - by_P = \mu + \omega$ ($= 10$). For all $(x,y) \in C_{k-1,l+1}$, the quantity $ax - by$ is given in (a). Case (ii) of the proof is depicted in (b).

Proof

Let us assume that $C_{i,j}$ is convex. A similar reasoning leads to the proof about concave digital curves. Let us assume that $C_{k,l}$ is strictly included in $C_{i,j}$, i.e. $i < k < l < j$. The special cases $i = k$ or $l = j$ shorten some parts of the proof given below. Moreover, let us focus on the last upper leaning point of $C_{k,l}$, denoted by C_h, because the same applies for its first upper leaning point.

Due to Definition 3, the point C_h is a convex dominant point iff there exists no upper digital edge $C_{p,q}$ strictly containing it ($p < h < q$). Since $C_{k,l}$ cannot be contained in a longer DSS having the same parameters, the part $C_{k-1,l+1}$ is not a DSS [6]. Since $C_{p,q}$ is a DSS, only three cases are possible: both C_p and C_q belong to S (i), C_p belongs to S but not C_q (ii), C_q belongs to S but not C_p (iii). Since the last two cases are similar, let us focus on the first two cases.

$\Rightarrow$ We have to prove that there does not exist any upper digital edge $C_{p,q}$ that strictly contains C_h if $ax_P - by_P \geq \mu + \omega$.

Let us assume without loss of generality that the points of $C_{k,l}$ have increasing x-coordinates. In case (i), the upper leaning point C_h is obviously located above the straight segment $[C_pC_q]$ because C_p and C_q are both located strictly below the upper leaning line of S. As a consequence, $C_{p,q}$ is not an upper digital edge. In case (ii), the upper leaning point C_h is located below the straight segment $[C_pC_q]$ if its slope is greater than the one of S. The point $P = C_{l+1}$, which belongs to $C_{p,q}$, is such that $ax_P - by_P \geq \mu + \omega$, i.e. is located below the lower leaning line of S. As a result, $C_{p,q}$ is clearly not convex due to Lemma 1 and is therefore not an upper digital edge (Fig. 5.b).

$\Leftarrow$ We have to prove now that $ax_P - by_P \geq \mu + \omega$ if there does not exist any upper digital edge $C_{p,q}$ that strictly contains C_h.

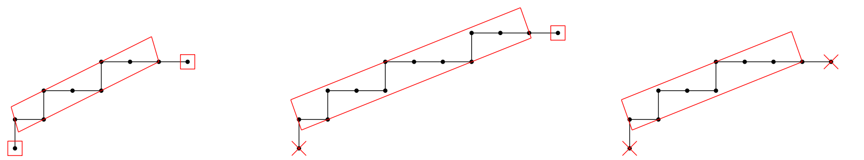
If C_p and C_q both belong to $C_{k,l}$, there does not exist any upper digital edge $C_{p,q}$ that strictly contains C_h, for any value of $ax_P - by_P$. Let us assume now that C_p belongs to $C_{k,l}$ while C_q does not. Since $C_{k,l}$ cannot be contained in a longer DSS of same parameters, $ax_P - by_P \leq \mu - 1$ or $ax_P - by_P \geq \mu + \omega$ [6], but the first inequality cannot be true without raising some contradictions. Indeed, if $ax_P - by_P = \mu - 1$, C_h is contained in a digital edge [6], which raises a contradiction. Finally, if $ax_P - by_P < \mu - 1$, $C_{p,q}$ is not convex due to [6] and Lemma 1, which raises again a contradiction. $\square$

A similar result (but not as general as Theorem 2) may be found in [29, Lemma1]. These results show that MSs are not enough to extract convex or concave dominant points of respectively convex or concave digital curves. A smaller part is often required. In convex (resp. concave) digital curves, it is actually enough to consider parts corresponding to DSSs $S(a, b, \mu, \omega)$ bounded by two points M and P such that $ax_M - by_M \geq \mu + \omega$ (resp. $ax_M - by_M \leq \mu - 1$) and $ax_P - by_P \geq \mu + \omega$ (resp. $ax_P - by_P \leq \mu - 1$):

- common parts of two consecutive MSs that have two bounding points M and P such that $ax_M - by_M = \mu + \omega$ (resp. $ax_M - by_M = \mu - 1$) and $ax_P - by_P = \mu + \omega$ (resp. $ax_P - by_P = \mu - 1$) [3, lemma 2] (Fig. 6.a).

- DSSs maximal at the back only that have two bounding points M and P such that $ax_M - by_M > \mu + \omega$ (resp. $ax_M - by_M < \mu - 1$) and $ax_P - by_P = \mu + \omega$ (resp. $ax_P - by_P = \mu - 1$) and conversely for DSSs maximal at the front only (Fig. 6.b).

- MSs that have two bounding points M and P such that $ax_M - by_M > \mu + \omega$ (resp. $ax_M - by_M < \mu - 1$) and $ax_P - by_P > \mu + \omega$ (resp. $ax_P - by_P < \mu - 1$) (Fig. 6.c).



(a) common part (b) maximal at the back (c) maximal

Figure 6: DSSs $S(a, b, \mu, \omega)$ that necessarily contain at least one convex dominant point: (a) common parts of two consecutive MSs, (b) DSSs maximal at the back or at the front, (c) MSs. The points (x, y) marked by a square are such that $ax - by = \mu + \omega$, whereas those marked by a cross are such that $ax - by > \mu + \omega$

During the computation of the MSs of a convex (resp. concave) digital curve, all the DSSs that are either maximal, maximal at the front, maximal at the back or that are the common part of two consecutive MSs are sequentially scanned. Since all the convex (resp. concave) dominant points of C are contained in such DSSs due to Theorem 2, they all can be retrieved in the course of the MSs computation.

A brief sketch of the algorithm for a convex digital curve is given below. Let $C_{k,l} = S(a, b, \mu, \omega)$ be maximal at the back, *i.e.* the point $M = C_{k-1}$ is such that $ax_M - by_M > \mu + \omega$. If the next point $P = C_{l+1}$ is such that $ax_P - by_P \leq \mu - 1$, the first and last upper leaning points of $C_{k,l}$ are convex dominant points due to Theorem 2. If they are not confounded, only the last upper leaning point of $C_{k,l}$ is stored as a new convex dominant point because the first upper leaning point is assumed to be already stored. In addition, if $ax_P - by_P = \mu - 1$, $C_{k,l+1}$ is a DSS and its first and last upper leaning points are confounded with the last upper leaning point of $C_{k,l}$ [6].

A similar process is performed when points are removed from the back of the current DSS.

It turns out that either during the adding step or the removing step, any two consecutive retrieved points are the first and last point of a digital edge, which guarantees that the points of C lying between two convex dominant points are not retrieved.

Fig. 7 shows what are the convex dominant points of the digital curve of Fig. 2. Note that if the points are scanning in the reverse way, the digital curve would be considered as concave and its concave dominant points would be similarly retrieved from the lower leaning points due to the orientation of the DSSs (see Section 2.1).

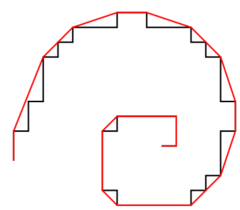


Figure 7: Sequence of convex dominant points linked into a red polygonal line.

4. Faithful Polygon

In [7, 11, 8, 9], Eckhardt and Dorksen-Reiter looked for a reversible polygon Σ that faithfully represents a connected digital set S.

They looked for polygons such that:

- The vertices of Σ belong to S,
- S is the Gauss digitization of Σ,
- Σ respects the convex and concave parts of S.

It turns out that even if some digital sets admit such polygons (Euclidean convex hull of convex digital sets for instance), it is impossible to meet the three requirements at the same time for all sets S [7]. However, we show in this section that if we replace the second requirement by a weaker one, a faithful polygonal representation always exists. Moreover, we provide an online and linear-time algorithm that computes such a polygon.

Before considering arbitrary digital curves, we first focus on strictly convex or strictly concave digital curves.

4.1. Faithful Polygon of Convex or Concave Digital Curves

In a convex (resp. concave) digital curve C, the set of convex (resp. concave) dominant points is unique because the set of configurations involved in Theorem 2 is unique. Moreover, the points can be ordered in a sequence because they all belong to the sequence of points C. Our approach consists in linking these convex (resp. concave) dominant points by straight line segments as shown in Fig. 7. The resulting polygon is interesting because of the following theorem:

Theorem 3. *In a convex (resp. concave) digital curve C, the sequence of convex (resp. concave) dominant points $C_1 = C_{k_0}, C_{k_1}, \ldots, C_{k_m} = C_n$ defines a sequence of upper (resp. lower) digital edges $C_{k_0,k_1}, \ldots, C_{k_{m-1},k_m}$ such that any two consecutive digital edges make a convex (resp. concave) turn.*

Proof

Let us assume that C is a convex digital curve. A similar proof can be derived for concave digital curves.

We have to prove that:

(i) for all $i \in 0, \ldots, m-1$, the part $C_{k_i,k_{i+1}}$ is a digital edge.

(ii) for all $i \in 1, \ldots, m-1$, the parts C_{k_{i-1},k_i} and $C_{k_i,k_{i+1}}$ make a convex turn.

However, (i) straightforwardly comes from the algorithm given at the end of Section 3 for retrieving the convex dominant points of a convex digital curve. It therefore remains to show (ii).

For all $i \in 1, \ldots, m-1$, let us assume without loss of generality that C_{k_{i-1},k_i} lies in the first quadrant, i.e. $x_{C_{k_i}} \geq x_{C_{k_{i-1}}}$ and $y_{C_{k_i}} \geq y_{C_{k_{i-1}}}$. The point $C_{k_{i+1}}$ cannot be located above the straight line passing through $C_{k_{i-1}}$ and C_{k_i} without raising contradictions. Indeed, either $C_{k_{i-1},k_{i+1}}$ is a upper digital edge, which implies that C_{k_i} is not a convex dominant point, or $C_{k_{i-1},k_{i+1}}$ is not convex due to Definition 1 and Lemma 1, which also raises a contradiction. As a consequence, $C_{k_{i+1}}$ is located below the straight line passing through $C_{k_{i-1}}$ and C_{k_i}, i.e. C_{k_{i-1},k_i} and $C_{k_i,k_{i+1}}$ make a convex turn. □

Any convex (resp. concave) digital curve C admits a convex (resp. concave) polygonal line that is obtained by linking the convex (resp. concave) dominant points of C by straight line segments (Fig. 7). Furthermore, this polygonal representation is reversible because any part of C lying between two consecutive convex (resp. concave) dominant points is a upper (resp. lower) digital edge, which can be retrieved in the first octant by the floor (resp. ceil) digitization of the straight line segment joining the two points.

If C is not convex, we can independently compute the convex (resp. concave) dominant points of each maximal convex (resp. concave) part. Tricky issues occur in inflection parts, which belong to both a maximal convex part and a maximal concave part. Though, such parts are MSs and are thus arithmetically well defined. We will see below that a convex and a concave dominant point of two consecutive maximal convex and concave parts can be linked by a straight line segment joining the first upper (resp. lower) and last lower (resp. upper) leaning points of the MS of inflection.

4.2. Faithful Polygon of Arbitrary Digital Curves

We can now precisely define the FP of a digital curve and prove its main properties.

Definition 4 (Faithful polygon). *Let $C^1_{i,j}, C^2_{i,j}, \ldots, C^\Lambda_{i,j}$ be the sequence of the maximal convex or concave parts of a digital curve C. The* faithful polygon *(FP) of C is defined by the concatenation of a subsequence of the convex or concave dominant points of the maximal convex or concave parts of C (see Fig. 8 for an illustration):*

$$\odot_{\lambda \in 1,\ldots,\Lambda}(C^\lambda_{k_\mu}, C^\lambda_{k_{\mu+1}}, \ldots, C^\lambda_{k_\nu})$$

such that:

- *If $C^\lambda_{i,j}$ is convex, $C^\lambda_{k_\mu}, C^\lambda_{k_{\mu+1}}, \ldots, C^\lambda_{k_\nu}$ are the convex dominant points of $C^\lambda_{i,j}$ located between:*

- C_i *if* $\lambda = 1$ *and the last upper leaning point of the first MS of* $C^{\lambda}_{i,j}$ *otherwise.*
- C_j *if* $\lambda = \Lambda$ *and the first upper leaning point of the last MS of* $C^{\lambda}_{i,j}$ *otherwise.*

- *If* $C^{\lambda}_{i,j}$ *is concave,* $C^{\lambda}_{k_\mu}, C^{\lambda}_{k_{\mu+1}}, \ldots, C^{\lambda}_{k_\nu}$ *are the concave dominant points of* $C^{\lambda}_{i,j}$ *located between:*
  - C_i *if* $\lambda = 1$ *and the last lower leaning point of the first MS of* $C^{\lambda}_{i,j}$ *otherwise.*
  - C_j *if* $\lambda = \Lambda$ *and the first lower leaning point of the last MS of* $C^{\lambda}_{i,j}$ *otherwise.*

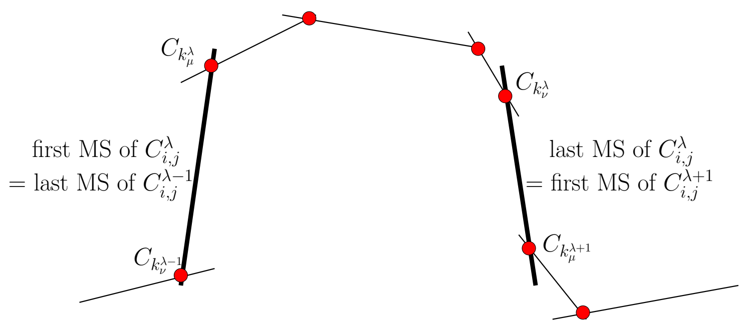


Figure 8: Illustration of the definition of the FP.

The FP of the wave-shaped digital curve of Fig. 3 is shown in Fig. 9.

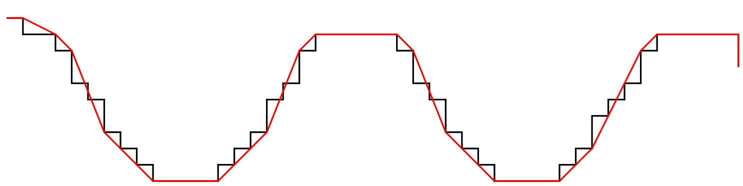


Figure 9: FP of a digital curve in red.

The FP of a digital curve has several interesting properties that are listed and proven below.

Property 1. *Any digital curve C that is not a DSS has a unique FP.*

Proof

The decomposition into maximal convex and concave parts is unique and in each maximal convex (resp. concave) part, the set of convex (resp. concave) dominant points is also unique. □

As shown in Fig. 10, note that a DSS has two different FPs depending on whether the DSS is considered as convex or concave.

(a) convex (b) concave

Figure 10: Two possible representations of a DSS

Property 2. *The vertices of the FP of any digital curve C belong to C.*

Proof

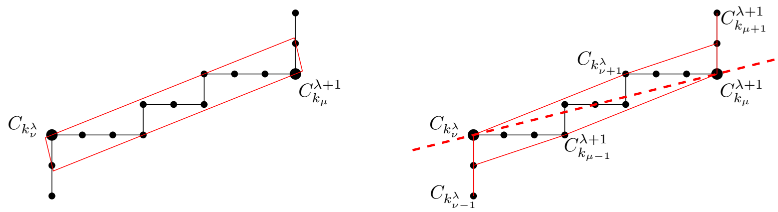
Straightforward from Definition 3 and Definition 4. □

Property 3. *The maximal convex parts, maximal concave parts and inflection MSs of C respectively contain the sequences of edges of decreasing slopes, sequences of edges of increasing slopes and inflection edges of the FP of C.*

Proof

It is enough to show that the inflection MSs of C contain the inflection edges of the FP of C because the maximal convex or concave parts contain the edges of the FP of C having decreasing or increasing slopes due to Definition 4 and Theorem 3.

Let us assume that $C^{\lambda}_{i,j}$ is a maximal convex part and $C^{\lambda+1}_{i,j}$ is the maximal concave part that follows for some $\lambda \in 1, \ldots, \Lambda - 1$. The converse case is similar. Moreover, let $S(a, b, \mu, \omega)$ be the MS of inflection, i.e. the last MS of $C^{\lambda}_{i,j}$ and the first one of $C_{i,j}^{\lambda+1}$. Without loss of generality, we assume that it has points of increasing x-coordinates. The points $C_{k_\nu}^{\lambda}$ and $C_{k_\mu}^{\lambda+1}$ are respectively the first upper and last lower leaning point of S (Fig. 11).



(a) (b)

Figure 11: Illustration of the proof of Property 3. The MS of inflection is depicted with a red bounding box in (a). The points $C_{k_{\nu-1}}^{\lambda}$ and $C_{k_{\mu+1}}^{\lambda+1}$ are located in either side of the dashed line passing through $C_{k_\nu}^{\lambda}$ and $C_{k_\mu}^{\lambda+1}$, the first upper and last lower leaning points of the MS of inflection (b).

Due to Theorem 3, the two upper digital edges $C_{k_{\nu-1},k_\nu}^{\lambda}$ and $C_{k_\nu,k_{\nu+1}}^{\lambda}$ make a convex turn. By definition, the slope of the straight segment joining $C_{k_\nu}^{\lambda}$ and $C_{k_\mu}^{\lambda+1}$ is smaller than the one of the straight segment joining $C_{k_\nu}^{\lambda}$ and $C_{k_{\nu+1}}^{\lambda}$ and thus, the one of the straight segment joining $C_{k_{\nu-1}}^{\lambda}$ and $C_{k_\nu}^{\lambda}$ (Fig. 11). Similarly, the slope of the straight segment joining $C_{k_\nu}^{\lambda}$ and $C_{k_\mu}^{\lambda+1}$ is smaller than the one of the straight segment joining $C_{k_\mu}^{\lambda+1}$ and $C_{k_{\mu+1}}^{\lambda+1}$ (Fig. 11). As a consequence, $[C_{k_\nu}^{\lambda} C_{k_\mu}^{\lambda+1}]$ is an edge of inflection contained in the MS of inflection S. $\square$

Property 4. *Any digital curve C can be retrieved from its FP (stored as a sequence of points).*

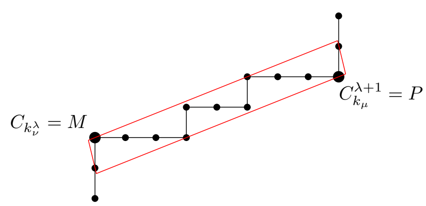
Proof

For all $\lambda \in 1, \dots, \Lambda$, for all $\kappa \in \mu, \dots, \nu - 1$, $C_{k_\kappa,k_{\kappa+1}}^{\lambda}$ is easily retrieved from its two end points because it is a upper digital edge in convex parts and a lower digital edge in concave parts.

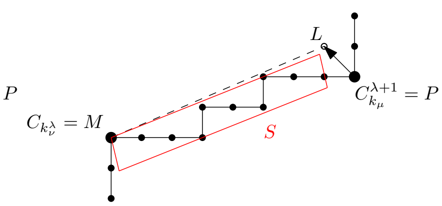
It remains to show that for all $\lambda \in 1, \dots, \Lambda - 1$, the part of C bounded by $C_{k_\nu}^{\lambda}$ and $C_{k_\mu}^{\lambda+1}$ can be retrieved from the end points $C_{k_\nu}^{\lambda}$ and $C_{k_\mu}^{\lambda+1}$

Let us assume that $C_{i,j}^{\lambda}$ is a maximal convex part and $C_{i,j}^{\lambda+1}$ is the maximal concave part that follows for some $\lambda \in 1, \dots, \Lambda-1$. The converse case is similar. For the sake of clarity, let us rename $C_{k_\nu^\lambda}$ and $C_{k_\mu}^{\lambda+1}$ into M and P respectively.

Due to definition 4, M and P are respectively the first upper and the last lower leaning points of the same MS, assumed without loss of generality to have points of increasing x-coordinates (Fig. 12.a).



(a)



(b)

Figure 12: Illustration of the proof of Property 4: M (resp. P) is the first upper (resp. last lower) leaning point of the MS of inflection depicted with a red bounding box in (a). The DSS S, which is depicted with a red bounding box in (b) is the floor digitization of the straight line segment joining M and L (L excluded), where L is derived from P.

Let $S(a, b, \mu, \omega)$ be the DSS lying between $C_{k_\nu^\lambda} = M$ and $C_{k_\mu - 1}^{\lambda+1}$ (Fig. 12.b). If P is the only lower leaning point of the MS of inflection, $ax_P - by_P = \mu + \omega$, otherwise $ax_P - by_P = \mu + \omega - 1$ [6]. Let L be the image of P after a translation by the vector $(-1, 1)$ (Fig. 12.b). The quantity $ax_L - by_L$ is equal to $a(x_P - 1) - b(y_P + 1) = (ax_P - by_P) - (a + b)$ and is thus either equal to μ or $\mu - 1$.

From the algorithm of Debled and Reveilles [6], we can therefore conclude that $S \cup L$ is a DSS and even more precisely an upper digital edge, because M, the first point of S, is also the first upper leaning point of S.

As a consequence, the part of C bounded by $C_{k_\nu}^{\lambda} = M$ and $C_{k_\mu}^{\lambda+1} = P$ can be retrieved from M and P: S is indeed the floor digitization of $[ML]$, but without L and it remains to add P to end the drawing. □

In practice, computing the FP of a digital curve C requires to check whether a given convex (resp. concave) part $C_{i,j}$ is maximal or not and to retrieve its sequence of convex (resp. concave) dominant points. This can be done online in linear-time during the MSs computation as shown in Section 3. A detailed algorithm may be found in [28, Algorithm 3]. It has been implemented in C++ and the code is available on the web site of the LIRIS lab[1]. Fig. 3, Fig. 7 and Fig. 9 are the outputs of the program.

5. Conclusion, discussion and perspectives

In this paper, we have shown that any digital curve C has a reversible faithful polygon (FP) such that:

- Each vertex of the FP of C belongs to C.
- The FP exactly reflects the maximal convex and concave parts of C.
- C may be retrieved from its FP stored as a sequence of points.

An online and linear-time algorithm exists in order to extract the FP of a digital curve. It only uses well-known routines: adding a point to the front of a DSS [6] and removing a point from the back of a DSS [22]. It is thus really easy to implement once these routines are available. Moreover, retrieving a digital curve from its FP can also be achieved in linear-time.

A small translation of each vertex of the FP of a digital curve C is enough to compute the well-known minimum-perimeter polygon (MPP) of the dilatation of C by the closed unit square $\{(x, y) \in \mathbb{R}^2 | \max(|x|, |y|) \leq \frac{1}{2}\}$. The FP is reversible and exactly reflects the convex and concave parts of the digital curve, whereas the MPP minimizes the number of inflection points required to represent the digital curve and is known to provide good estimators of tangent and length. The MPP derived from the FP of Fig. 9 is depicted in Fig. 13. Further details may be found in [28, Section 5.3].

As a consequence, the proposed arithmetical algorithm (dedicated to the FP computation) can also compute the MPP and conversely, the combinatorial algorithm of Provençal and Lachaud [25] (dedicated to the MPP computation)

[1] http://liris.cnrs.fr/m2disco/index_en.html?softwares

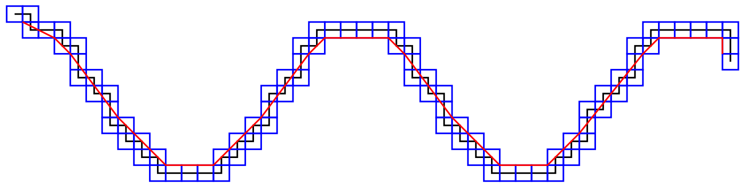


Figure 13: MPP (in red) of a dilated digital curve (in blue).

can also compute the FP. The combinatorial approach of [1] has been shown to be about ten times faster than the arithmetical one [5] in checking the convexity of a digital curve [1, Fig.6]. We can thus expect the same difference in the FP or MPP extraction between the two approaches. However, our approach has the advantage of being purely local, contrary to the one of Provençal and Lachaud [25]. Thus, it may lead to an algorithm solving the dynamic problem: update the polygonal representation during the displacement of a point of the digital curve.

Eventually, note that the parts of the digital curve highlighted in its FP or MPP does not reflect the visual parts of the original shape if the resolution is to high with respect to the scale of its main features or if some stochastic noise is introduced in the digitization. Several ways of copying with this problem can be followed while keeping an arithmetic approach that leads to fast algorithms with integer-only computations: (i) find a deformation process of the digital curve so that it sticks to the expected shape, like in digital deformable models [4], (ii) find a discrete simplification process of the digital curve in the manner of [23] or (iii) work on sub-sampled versions of the initial digital curve as done in [14]. This work and its perspectives lead to think that digital convexity will help to design an efficient and accurate method dedicated to the extraction of perceptually meaningful parts.

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