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RESEARCH ARTICLE

Colourings of lattices and coincidence site lattices

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The relationship between the coincidence indices of a lattice Γ_1 and a sublattice Γ_2 of Γ_1 is examined via the colouring of Γ_1 that is obtained by assigning a unique colour to each coset of Γ_2 . In addition, the idea of colour symmetry, originally defined for symmetries of lattices, is extended to coincidence isometries of lattices. An example involving the Ammann-Beenker tiling is provided to illustrate the results in the quasicrystal setting.

Keywords: coincidence site lattice, coincidence index, colour coincidence, colour symmetry

1. Introduction

Coincidence site lattices are used in studying and describing the structure of grain boundaries in crystals.[1] The discovery of quasicrystals led to a more mathematical study of coincidence site lattices, including coincidence site lattices in dimensions $d \geq 4$ and their generalizations to modules.[2–9]

Similar to coincidence site lattices, a revived interest on colour symmetries in recent years was brought upon by Schechtman's discovery, see [10–19]. Despite being two different problems, the enumeration and classification of colour symmetries of lattices and the identification of coincidence site lattices come hand in hand.[2, 12, 17, 18, 20, 21].

It is well-known that a lattice and its sublattices have the same set of coincidence isometries. In fact, not only are the coincidence isometries for the three cubic lattices the same, but also are the coincidence indices for a given coincidence isometry. This is due to a particular shell structure of the lattice points: points at the corners of a cube and those in the centre or on the faces of the cube lie on different shells, and thus cannot be mapped onto each other by an isometry.[22] On the other hand, aside from having different point groups, the two four-dimensional hypercubic lattices do not share the same set of coincidence indices (the coincidence indices of the centred type are all odd while those of the primitive type are odd or twice an odd number). [3] This motivates the following questions: Under which conditions does a sublattice have the same coincidence indices as the lattice itself? Is it possible to calculate the coincidence indices of a sublattice once the coincidence indices of its parent lattice have already been determined? In this paper, answers to these questions are discussed by looking at certain colourings of lattices.

A discrete subset Γ of \mathbb{R}^d is a *lattice* (of rank and dimension d) if it is the \mathbb{Z} -span

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of d linearly independent vectors over \mathbb{R} . As a group, Γ is isomorphic to the free abelian group of rank d . A *sublattice* of Γ is a subgroup of Γ of finite index.

An $R \in O(d)$ is a *coincidence isometry* of Γ if $\Gamma(R) := \Gamma \cap R\Gamma$ is a sublattice of Γ . The sublattice $\Gamma(R)$ is called a *coincidence site lattice* (CSL) while the index of $\Gamma(R)$ in Γ , $\Sigma(R) := [\Gamma : \Gamma(R)] = [R\Gamma : \Gamma(R)]$, is called the *coincidence index* of R . The set of coincidence isometries of Γ forms a group, denoted by $OC(\Gamma)$.

If Γ_2 is a sublattice of Γ_1 of index m , then $OC(\Gamma_1) = OC(\Gamma_2)$. [3] However, the coincidence indices and corresponding multiplicities of Γ_1 and Γ_2 are in general different. What is known is if $R \in OC(\Gamma_1)$, then $\Sigma_1(R) \mid m \Sigma_2(R)$ (where $\Sigma_i(R)$ is the coincidence index of R with respect to Γ_i , for $i = 1, 2$). [3] We shall derive a formula for $\Sigma_2(R)$ in terms of $\Sigma_1(R)$ by considering properties of the colouring of Γ_1 that is induced by Γ_2 .

A *colouring* of Γ_1 determined by the sublattice Γ_2 of index m is a bijection between the set of cosets of Γ_2 in Γ_1 and a set C of m colours. For simplicity, we shall say that the coset $c_i + \Gamma_2 \subseteq \Gamma_1$ has *colour* c_i .

Denote by G the symmetry group of Γ_1 . A symmetry in G is called a *colour symmetry* of the colouring of Γ_1 induced by Γ_2 if it permutes the colours in the colouring, that is, all and only those points having the same colour are mapped by the symmetry to a certain colour. The group formed by the colour symmetries of the colouring of Γ_1 is called the *colour group* or *colour symmetry group*, usually denoted by H ([16, 21], see also [23] for an equivalent formulation). Since H acts on C , there exists a homomorphism $f : H \rightarrow P(C)$ where $P(C)$ is the group of permutations of C . The kernel, K , of f is the subgroup of H whose elements fix the colours, in other words, K is the symmetry group of the coloured lattice. We obtain that the group of colour permutations of the lattice, $f(H)$, is isomorphic to H/K . [16]

We shall generalise the notion of a colour symmetry to that of a colour coincidence. In addition, we associate the property that a coincidence isometry R of Γ_1 is a colour coincidence of the colouring of Γ_1 determined by a sublattice Γ_2 with the relationship between the coincidence indices of R with respect to Γ_1 and Γ_2 .

2. Coincidence index with respect to a sublattice

Let Γ_1 be a lattice in \mathbb{R}^d and Γ_2 be a sublattice of Γ_1 of index m . We write $\Gamma_1 = \bigcup_{i=1}^m (c_i + \Gamma_2)$ where $c_1 = 0$, and we consider the colouring of Γ_1 determined by Γ_2 . Fix an $R \in OC(\Gamma_1) = OC(\Gamma_2)$ and denote by $\Sigma_i(R)$ the coincidence index of R with respect to Γ_i , for $i = 1, 2$. Let $I := \{c_i + \Gamma_2 : (c_i + \Gamma_2) \cap \Gamma_1(R^{-1}) \neq \emptyset\}$ and $J := \{c_j + \Gamma_2 : (c_j + \Gamma_2) \cap \Gamma_1(R) \neq \emptyset\}$. Without loss of generality, we assume throughout the paper that $c_i \in \Gamma_1(R^{-1})$ and $c_j \in \Gamma_1(R)$ whenever $c_i + \Gamma_2 \in I$ and $c_j + \Gamma_2 \in J$. We then obtain partitions of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$ that are induced by Γ_2 , given by $\Gamma_1(R^{-1}) = \bigcup_{c_i + \Gamma_2 \in I} (c_i + \Gamma_2) \cap \Gamma_1(R^{-1})$ and $\Gamma_1(R) = \bigcup_{c_j + \Gamma_2 \in J} (c_j + \Gamma_2) \cap \Gamma_1(R)$. These partitions correspond to colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$, respectively, wherein the colours are inherited from the colouring of Γ_1 determined by Γ_2 . Observe that all points of $\Gamma_1(R^{-1})$ are mapped bijectively by R to points of $\Gamma_1(R)$, that is, $R[\Gamma_1(R^{-1})] = \Gamma_1(R)$.

The following lemma tells us that $\Gamma_2(R)$ consists of those points coloured c_1 in the colouring of $\Gamma_1(R)$ whose preimages under R are also points coloured c_1 in the colouring of $\Gamma_1(R^{-1})$.

Lemma 2.1: *Let Γ_2 be a sublattice of Γ_1 and $R \in OC(\Gamma_1)$. Then*

$$[\Gamma_2 \cap \Gamma_1(R)] \cap R[\Gamma_2 \cap \Gamma_1(R^{-1})] = [\Gamma_2 \cap \Gamma_1(R)] \cap [R\Gamma_2 \cap \Gamma_1(R)] = \Gamma_2(R).$$

In particular, if $R\Gamma_2 \cap \Gamma_1(R) = \Gamma_2 \cap \Gamma_1(R)$ then $\Gamma_2(R) = R\Gamma_2 \cap \Gamma_1(R) = \Gamma_2 \cap \Gamma_1(R)$.

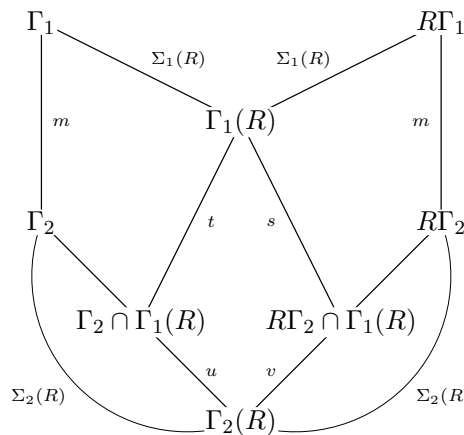


Figure 1. Lattice diagram of the lattices Γ_1 , $R\Gamma_1$, Γ_2 , $R\Gamma_2$, $\Gamma_1(R)$, and $\Gamma_2(R)$ (as groups) and corresponding indices

Proof: We have $[\Gamma_2 \cap \Gamma_1(R)] \cap [R\Gamma_2 \cap \Gamma_1(R)] = (\Gamma_2 \cap R\Gamma_2) \cap \Gamma_1(R) = \Gamma_2(R)$. \square

Let $s := [\Gamma_1(R) : R\Gamma_2 \cap \Gamma_1(R)]$, $t := [\Gamma_1(R) : \Gamma_2 \cap \Gamma_1(R)]$, $u := [\Gamma_2 \cap \Gamma_1(R) : \Gamma_2(R)]$, and $v := [R\Gamma_2 \cap \Gamma_1(R) : \Gamma_2(R)]$ (see Figure 1). By comparing indices in Figure 1, a formula for $\Sigma_2(R)$ in terms of $\Sigma_1(R)$ is readily obtained. In addition, restrictions on the values of s , t , u , and v , as well as interpretations of these values in relation to the colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$ determined by Γ_2 , can be deduced through repeated applications of the second isomorphism theorem (for groups). These results are stated in the following theorem, whose proof will be published elsewhere. [24]

Theorem 2.2: Consider the colouring of a lattice Γ_1 determined by a sublattice Γ_2 of Γ_1 of index m where each coset $c_i + \Gamma_2$ is assigned the colour c_i (for $i = 1, \dots, m$) with $c_1 = 0$. Let $R \in OC(\Gamma_1)$ and $\Sigma_i(R)$ be the coincidence index of R with respect to Γ_i , for $i = 1, 2$. Then

$$\Sigma_2(R) = \frac{t \cdot u \cdot \Sigma_1(R)}{m} = \frac{s \cdot v \cdot \Sigma_1(R)}{m} \tag{1}$$

where s and t are the number of colours in the colouring of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$, respectively, determined by Γ_2 ; u is the number of colours c_i with the property that some points of $\Gamma_1(R^{-1})$ coloured c_i are mapped by R to points coloured c_1 in the colouring of $\Gamma_1(R)$; and v is the number of colours in the colouring of $\Gamma_1(R)$ that are intersected by the images under R of those points of $\Gamma_1(R^{-1})$ coloured c_1 . Moreover, $s|m$, $t|m$, $u|s$, and $v|t$.

An immediate consequence of Theorem 2.2 is the well-known bound on the value of $\Sigma_2(R)$: $\Sigma_1(R) \mid m \Sigma_2(R)$ and $\Sigma_2(R) \mid m \Sigma_1(R)$. [3] Observe that $|I| = s$, $|J| = t$, and $s = t$ if and only if $u = v$.

Example 2.3 Consider the square lattice $\Gamma_1 = \mathbb{Z}^2$ and the sublattice Γ_2 of index 6 in Γ_1 generated by $[6, 0]^T$ and $[2, 1]^T$. Take the coincidence rotation R of Γ_1 corresponding to a rotation about the origin by $\tan^{-1}(\frac{3}{4}) \approx 37^\circ$ in the counter-clockwise direction. It is known that $\Sigma_1(R) = 5$. Choose the coset representatives $c_i = [i - 1, 0]^T$ for $i \in \{1, \dots, 6\}$ so that $\Gamma_1 = \bigcup_{i=1}^6 (c_i + \Gamma_2)$. The colourings of Γ_1 , $\Gamma_1(R^{-1})$, and $\Gamma_1(R)$ determined by Γ_2 are shown in Figures 2, 3(a), and 3(b) which were obtained by designating the colours black, yellow, blue, red, gray, and green to the cosets $c_1 + \Gamma_2$, $c_2 + \Gamma_2$, $c_3 + \Gamma_2$, $c_4 + \Gamma_2$, $c_5 + \Gamma_2$, and $c_6 + \Gamma_2$, respectively.

Since all six colours appear in both colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$, $s = t = 6$.

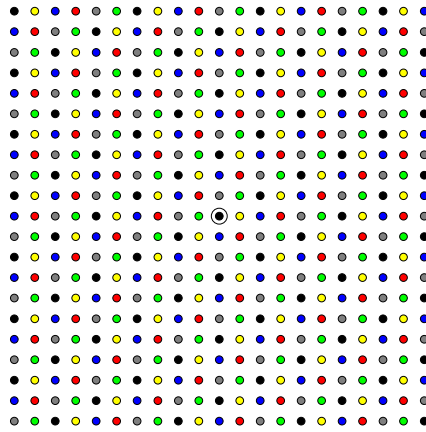


Figure 2. Colouring of $\Gamma_1 = \mathbb{Z}^2$ determined by the sublattice Γ_2 (= black dots) with basis $\{[6, 0]^T, [2, 1]^T\}$. The encircled dot indicates the origin. (Online: coloured version)

Observe that half of the black points in the colouring of $\Gamma_1(R^{-1})$ are sent by R again to black points in the colouring of $\Gamma_1(R)$ while the rest are sent to red points. This implies that $u = 2$. Therefore, by (1), $\Sigma_2(R) = \frac{6 \cdot 2 \cdot 5}{6} = 10$. By Lemma 2.1, $\Gamma_2(R)$ is obtained by taking the intersection of $\Gamma_2 \cap \Gamma_1(R)$ (the black points in the colouring of $\Gamma_1(R)$) and $R[\Gamma_2 \cap \Gamma_1(R^{-1})]$ (the resulting points when the black points in the colouring of $\Gamma_1(R^{-1})$ are rotated by R), and is shown in Figure 3(c).

3. Colour coincidence

We have seen from the previous section how the interaction of the colours in the colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$ determined by Γ_2 affects $\Gamma_2(R)$ and thus the value of $\Sigma_2(R)$. This motivates the following definition.

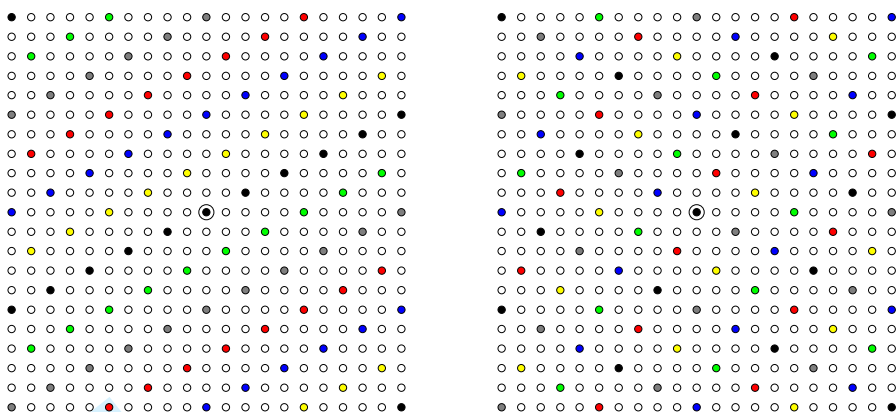
Definition 3.1: A coincidence isometry R of the lattice Γ_1 is called a *colour coincidence* of the colouring of Γ_1 determined by a sublattice Γ_2 of Γ_1 if R defines a bijection between the partitions $\{(c_i + \Gamma_2) \cap \Gamma_1(R^{-1}) : c_i + \Gamma_2 \in I\}$ of $\Gamma_1(R^{-1})$ and $\{(c_j + \Gamma_2) \cap \Gamma_1(R) : c_j + \Gamma_2 \in J\}$ of $\Gamma_1(R)$ given by $R[(c_i + \Gamma_2) \cap \Gamma_1(R^{-1})] = (c_j + \Gamma_2) \cap \Gamma_1(R)$.

In the above definition, the mapping between $(c_i + \Gamma_2) \cap \Gamma_1(R^{-1})$ and $(c_j + \Gamma_2) \cap \Gamma_1(R)$ means that all points, and only those points, coloured c_i in the colouring of $\Gamma_1(R^{-1})$ are sent by R to points coloured c_j in the colouring of $\Gamma_1(R)$. In such a case, R is said to send or map colour c_i to colour c_j . Furthermore, if R maps a colour c_i onto itself, R is said to *fix the colour* c_i . Hence, a colour coincidence R determines a bijection between the set of colours in the colouring of $\Gamma_1(R^{-1})$ and the set of colours in the colouring of $\Gamma_1(R)$. In particular, if a colour coincidence R is an element of the symmetry group of Γ_1 (that is, $\Gamma_1(R^{-1}) = \Gamma_1(R) = \Gamma_1$), then R is a colour symmetry of the colouring of Γ_1 .

Clearly, the identity in $OC(\Gamma_1)$ is a colour coincidence of any colouring of Γ_1 . In addition, if R is a colour coincidence of a colouring of Γ_1 , then so is R^{-1} . The question of whether the set of colour coincidences of a colouring of a lattice forms a group is yet to be resolved.

The next theorem, which is a generalisation of the first result in Theorem 2 of [16] on colour symmetries of colourings of square and hexagonal lattices, provides a characterisation of colour coincidences of lattice colourings determined by some sublattice. Its proof will be published in [24].

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(a) Colouring of $\Gamma_1(R^{-1})$ (white dots correspond to points of $\Gamma_1 \setminus \Gamma_1(R^{-1})$)

(b) Colouring of $\Gamma_1(R)$

(c) $\Gamma_2(R)$ (= black dots) as a sublattice of Γ_1 .

Figure 3. Colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$ determined by the sublattice Γ_2 of Figure 2, where R is the counterclockwise rotation about the origin by $\tan^{-1}(\frac{3}{4}) \approx 37^\circ$. The CSL $\Gamma_2(R)$ of index 10 in Γ_2 is obtained by taking all black points of $\Gamma_1(R)$ whose preimages under R are also coloured black. (Online: coloured version)

Theorem 3.2: *Let Γ_2 be a sublattice of Γ_1 of index m , with $\Gamma_1 = \bigcup_{i=1}^m (c_i + \Gamma_2)$ where $c_1 = 0$. Then $R \in OC(\Gamma_1)$ is a colour coincidence of the colouring of Γ_1 determined by Γ_2 if and only if R fixes colour c_1 .*

Hence, it is sufficient to verify if $R[\Gamma_2 \cap \Gamma_1(R^{-1})] = \Gamma_2 \cap \Gamma_1(R)$ is satisfied to conclude whether $R \in OC(\Gamma_1)$ is a colour coincidence of the colouring of the lattice Γ_1 induced by the sublattice Γ_2 of Γ_1 . Furthermore, because of Theorem 3.2, a colour coincidence R actually defines a permutation on the set of cosets of $\Gamma_2(R)$ in $\Gamma_1(R)$ with $Rc_i + \Gamma_2(R) = c_j + \Gamma_2(R)$ if R sends colour c_i to c_j .

The next result links the property that a coincidence isometry R of Γ_1 is a colour coincidence of the colouring of Γ_1 determined by Γ_2 with the relationship between $\Sigma_1(R)$ and $\Sigma_2(R)$.

Corollary 3.3: *Let Γ_1 be a lattice having Γ_2 as a sublattice of index m .*

- (1) *If R is a colour coincidence of the colouring of Γ_1 determined by Γ_2 then $\Sigma_2(R) \mid \Sigma_1(R)$.*
- (2) *If $s = t = m$, then R is a colour coincidence of the colouring of Γ_1 determined by Γ_2 if and only if $\Sigma_2(R) = \Sigma_1(R)$.*

Proof:

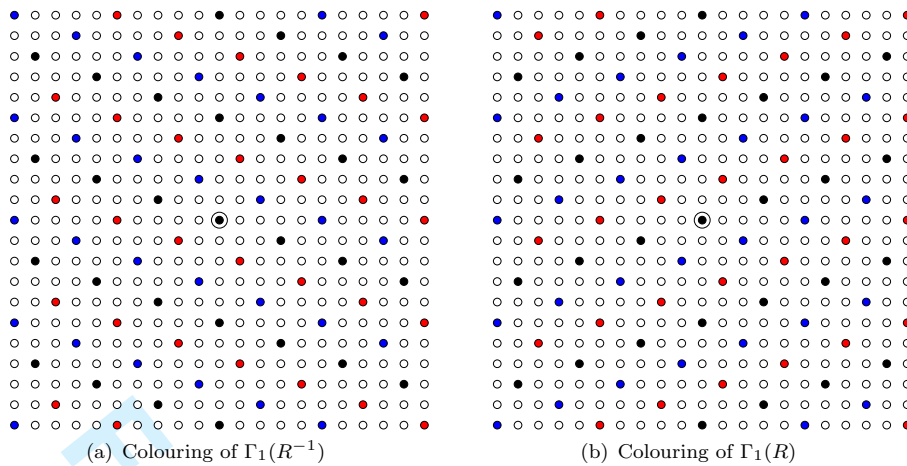


Figure 4. The coincidence rotation R (the same as in Figure 3) is a colour coincidence of the colouring of $\Gamma_1 = \mathbb{Z}^2$ determined by the sublattice Γ_2 generated by $[3,0]^T$ and $[0,1]^T$ because the colour black (corresponding to Γ_2) is fixed by R . The blue and red colours are interchanged by R . The CSL $\Gamma_2(R)$ consists of the black points in the colouring of $\Gamma_1(R)$. (Online: coloured version)

- (1) By Theorem 3.2 and Lemma 2.1, $R[\Gamma_2 \cap \Gamma_1(R^{-1})] = \Gamma_2 \cap \Gamma_1(R) = \Gamma_2(R)$. Thus, $u = v = 1$ and by (1), $\Sigma_1(R) = \frac{m}{t} \cdot \Sigma_2(R)$. Since $t \mid m$, $\Sigma_2(R) \mid \Sigma_1(R)$.
- (2) It follows from above that $\Sigma_2(R) = \Sigma_1(R)$ whenever R is a colour coincidence of the colouring of Γ_1 and $t = m$. Conversely, if $\Sigma_2(R) = \Sigma_1(R)$ then $[\Gamma_1(R) : \Gamma_2(R)] = t \cdot u = s \cdot v = m$ (see Figure 1) and thus, $u = v = 1$. Hence, $R[\Gamma_2 \cap \Gamma_1(R^{-1})] = \Gamma_2 \cap \Gamma_1(R)$ or R fixes colour c_1 . By Theorem 3.2, R is a colour coincidence of the colouring of Γ_1 .

□

The first result is a reformulation of a remark in [3]. The condition $s = t = m$ in the second result means that all m colours of the colouring of Γ_1 appear in the colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$ determined by Γ_2 .

Example 3.4 Take the rectangular sublattice Γ_2 with basis $\{[3,0]^T, [0,1]^T\}$ that is of index 3 in the square lattice $\Gamma_1 = \mathbb{Z}^2$, and the coincidence rotation R of Example 2.3. Both colourings of $\Gamma_1(R^{-1})$ and $\Gamma_1(R)$ include three colours (see Figure 4), and thus $s = t = 3$. Observe that R fixes the colour black (corresponding to the coset $c_1 + \Gamma_2 = \Gamma_2$), which means that R is a colour coincidence of the colouring of Γ_1 by Theorem 3.2. Indeed, R fixes the colour black, and interchanges the colours blue and red. Moreover, by Corollary 3.3, $\Sigma_2(R) = \Sigma_1(R)$. The CSL $\Gamma_2(R)$ is made up of all black points in the colouring of $\Gamma_1(R)$.

4. Example on the Ammann-Beenker tiling

Even though all of the previous results were stated for lattices, they also hold for \mathbb{Z} -modules in \mathbb{R}^d . [24] This suggests that the same techniques are applicable when dealing with quasicrystals. In fact, the coincidence problem for the set of vertex points of a quasicrystalline tiling breaks into two parts: the coincidence problem for the underlying limit translation module, and the computation of the window correction factor. [2, 3] The latter depends on the geometry of the window and equals 1 if the window is circular. In such a case, results on the underlying module are exactly the same as that on the set of vertex points of the tiling.

To illustrate, we consider the set of vertex points P_1 of the eightfold symmetric

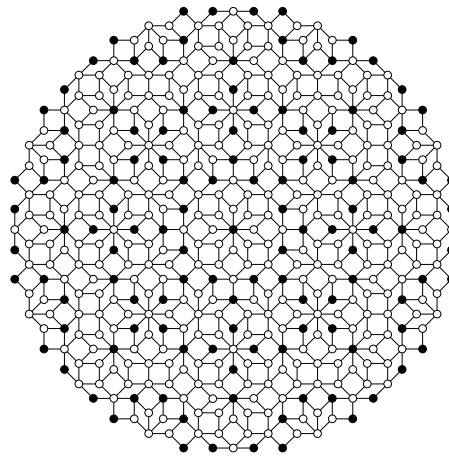


Figure 5. The set of vertices P_1 of the Ammann-Beenker tiling and the subset P_2 (coloured black) of P_1

Ammann-Beenker tiling (see Figure 5). We now have to consider the coincidence problem of the underlying module M_1 which is the standard eight-fold planar module $\mathbb{Z}[\xi]$ of rank 4, where $\xi = e^{\pi i/4}$. That is, we treat P_1 as a discrete subset of M_1 embedded in the complex plane. Due to lack of space, we omit the discussion of the window correction factor here, but notice that the acceptance factor for the Ammann-Beenker tiling is anyway very close to one. [2]

Let P_2 be the set of points of P_1 coloured black in Figure 5. The subset P_2 can be obtained as the intersection of the submodule $M_2 = \langle 1 + \xi^2 \rangle$ (of index 4 in M_1) and P_1 . Choose the coincidence rotation R of M_1 (and P_1) to be the counterclockwise rotation about the origin at an angle of $\tan^{-1}(-2\sqrt{2}) \approx 109.5^\circ$. The coincidence index of R with respect to M_1 is 9.

A colouring of P_1 with four colours determined by M_2 is shown in Figure 1 of [19]. Denote by $P_1(R^{-1})$ and $P_1(R)$ the set of coinciding points of P_1 and $R^{-1}P_1$, and of P_1 and RP_1 , respectively. Figures 6(a) and 6(b) show the colourings of $P_1(R^{-1})$ and $P_1(R)$ induced by the colouring of P_1 determined by M_2 . Observe that all four colours appear in both colourings, and R is a colour coincidence of the colouring (with all colours being fixed). Hence, the coincidence index of R with respect to M_2 is also 9. Also, the set of coinciding points, $P_2(R)$, consists of all points of P_2 in the colouring of $P_1(R)$.

5. Conclusion and outlook

A method of calculating the coincidence index of a coincidence isometry of a lattice with respect to a sublattice is formulated in this paper by considering the colouring of the lattice determined by the sublattice. Examples in the plane were given, for which the actual colourings were shown. However, this does not mean that the results acquired are only applicable when actual colourings of the lattices can be visualised. Examples for lattices-sublattices in higher dimensions are also possible, as well as general results (for instance, sublattices of prime index) in arbitrary dimension. [24] With these results, it is foreseeable that the coincidence indices of other type of lattices may be computed, as long as the lattices can be treated as sublattices of other lattices for which the coincidence problem has already been completely solved. Though more involved, these results are still applicable in the quasicrystal case.

The generalisation of a colour symmetry to colour coincidences is not only in-

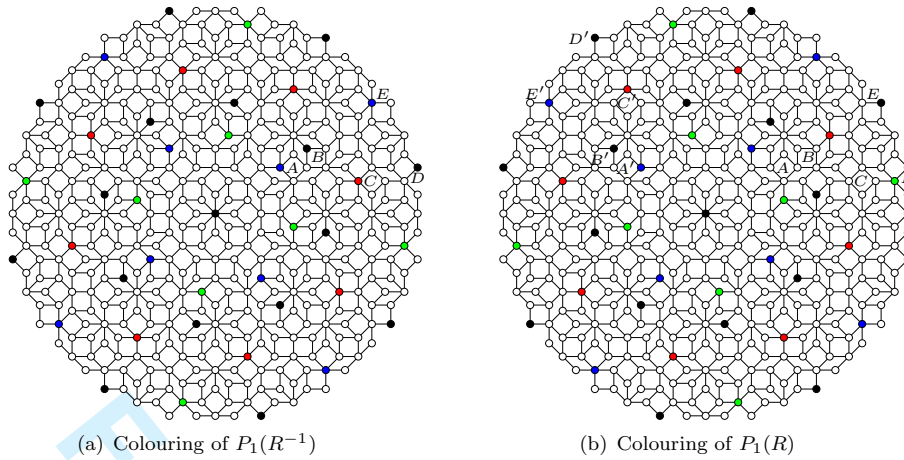


Figure 6. Colourings of $P_1(R^{-1})$ and $P_1(R)$ determined by P_2 (see Figure 5), where the coincidence rotation R corresponds to a rotation about the origin by $\tan^{-1}(-2\sqrt{2}) \approx 109.5^\circ$ in the counterclockwise direction. The points labelled $A, B, C, D,$ and E are mapped by R to the points labelled $A', B', C', D',$ and E' , respectively. R is a colour coincidence of the colouring of Γ_1 that fixes all the colours. The intersection of P_2 and $RP_2, P_2(R)$, is made up of all black points in the colouring of $P_1(R)$. (Online: coloured version)

interesting in its own right, but also provides a further connection between the relationship of the coincidence indices of a lattice and sublattice. Moreover, Theorem 3.2 permits a simpler way of determining the colour symmetry group of colourings of lattices and \mathbb{Z} -modules.

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References

- [1] W. Bollmann, *Crystal Defects and Crystalline Interfaces*, Springer, Berlin, 1970.
- [2] P.A.B. Pleasants, M. Baake and J. Roth, *J. Math. Phys.* 37 (1996) p.1029.
- [3] M. Baake, *Solution of the coincidence problem in dimensions $d \leq 4$* , in *The Mathematics of Long-Range Aperiodic Order*, R.V. Moody, ed., Kluwer Academic, Dordrecht, 1997, p.9-44.
- [4] P. Zeiner, *Z. Krist.* 220 (2005) p.915.
- [5] P. Zeiner, *Z. Krist.* 221 (2006) p.105.
- [6] M. Baake and P. Zeiner, *Phil. Mag.* 88 (2008) p. 2025.
- [7] M. Baake et al., *European J. Combin.* 29 (2008) p.1808.
- [8] Y.M. Zou, *Acta Crystallogr. Sect. A* 62 (2006) p.109.
- [9] C. Huck, *Z. Krist.* 224 (2009) p.341.
- [10] R.V. Moody and J. Patera, *Can. J. Phys.* 72 (1994) p.442.
- [11] R. Lifshitz, *Rev. Mod. Phys.* 69 (1997) p.1181.
- [12] M. Baake, *J. Phys. A* 30 (1997) p.2687.
- [13] M. Baake, U. Grimm and M. Scheffer, *J. Alloys Comp.* 342 (2002) p.195.
- [14] M. Baake, U. Grimm and M. Scheffer, *J. Non-Crystal. Solids* 334-335 (2004) p.83.
- [15] M. Baake and U. Grimm, *Z. Krist.* 219 (2004) p.72.
- [16] M.L.A.N. De Las Peñas and R.P. Felix, *Z. Krist.* 222 (2007) p.505.
- [17] R. Lück, *Phil. Mag.* 88 (2008) p.2049.
- [18] R. Lück and D. Frettlöh, *Z. Krist.* 223 (2008) p.782.
- [19] E.P.C. Bugarin et al., *Z. Krist.* 223 (2008) p.785.
- [20] R.L.E. Schwarzenberger, *N-dimensional Crystallography*, Pitman, London, 1980.
- [21] R.L.E. Schwarzenberger, *Bull. Lond. Math. Soc.* 16 (1984) p.209.
- [22] H. Grimmer, W. Bollmann and D.H. Warrington, *Acta Crystallogr. Sect. A* 30 (1974) p.197.
- [23] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, Freeman, New York, 1987.
- [24] M. Loquias and P. Zeiner, In preparation.