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Quantum fluids, Josephson tunneling and gravitational waves

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Abstract. In this paper we study the Josephson tunneling for a quantum fluid (a Bose-Einstein condensate) in presence of a weak gravitational wave. Starting from a lagrangian formulation in the framework of linearized gravity, we deduce the Gross-Pitaevskii equation for the fluid in a weak gravitational background. We use such equation to investigate the influence of a gravitational wave on Josephson effect. Considering a double-well trap, made of two identical boxes placed orthogonally each other and weakly coupled through a small junction, we show how the trap geometry influences the coupling between the gravitational wave and the quantum field modes in the two sides of the trap. Namely, the *macroscopic* wavefunction of the condensate acquires a different phase shift in the two sides, hence giving rise to a gravitationally-induced ac-Josephson effect through the trap junction. Although small, such effect is theoretically interesting, since it represents the influence of a gravitational wave on a mesoscopic quantum system. Also, the effect exhibits polarimetric properties. Possible experimental detection of such effect is briefly discussed at the end of the paper.

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1. Introduction

Investigation of the interaction of quantum systems with gravity is a quite fascinating and interesting topic, as it deals with two fundamental theories - Quantum Mechanics and General Relativity - which rely on very different theoretical grounds; namely the former describes physical reality through probabilistic laws, while the latter is a deterministic theory on its own.

A consistent application of the quantum framework to gravity - the so-called *quantum gravity* - has still to come. In this respect, theoretical and experimental study of interaction between gravitational and quantum fields seems a promising arena in which a deep understanding of such interaction could be crucial for further development of a full quantum theory of gravitation.

On the other hand, the interplay between the typical non-local character of quantum theory and the local one pertaining to General Relativity could give rise

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to some unexpected and interesting results, perhaps experimentally detectable with present or future technologies.

Because of its intrinsic weakness, gravity is usually expected to be described *classically*, i.e. as an external background field interacting with the considered quantum system. Through the years, the theoretical framework required to analyze quantum fields in a curved gravitational background has been extensively developed [1]; also, a number of interesting effects concerning the influence of gravity on quantum fields have been studied and - in some circumstances - experimentally observed. We only recall the well-known COW experiment performed by Colella *et al* [2], in which neutron interference was observed as a consequence of the phase shift of the neutron wavefunction induced by the gravitational field of the Earth. More recently, quantization of the energy levels of thermal neutrons *falling* in a *gravitational trap* has been detected and measured [3]. Both the cited experiments show a fine agreement with the theory.

Influence of gravity on quantum effects could be also observed on mesoscopic scales taking into account *quantum fluids*, as superconductors, superfluids or atomic Bose-Einstein condensates (BEC), exhibiting macroscopic coherence properties. Interaction between quantum fluids and gravity has been extensively investigated. We recall the works of DeWitt [4] and Papini [5, 6, 7] concerning the behavior of a superconductor in presence of external gravito-inertial fields, as well as the induced quantum phase shift. Anandan developed a theory for superfluid helium in a general relativistic framework, discussing the influence of Lense-Thirring effect on the dc-Josephson effect [8]. Several papers have dealt with the interaction between superfluids and gravitational waves [9, 10, 11, 12]. Coupling of gravitational waves to superconductors has been discussed, e.g., by Anandan [13] and by Peng *et al* [14]; also, interaction between electromagnetic, gravitational and gravitoelectromagnetic waves in superconductors has been investigated by Speliotopoulos [15] and by Tajmar and de Matos [16, 17]. Recently, some claims have been reported in the literature about the possibility of generation and detection of (high-frequency) gravitational waves by means of superfluids (see, e.g., [18]).

In the present paper we propose a weak-field approach to describe the interaction between a quantum fluid and a weak gravitational wave, with special concern for Josephson effect [19]. Working in the linearized gravity we write a suitable Lagrangian from which we derive a Gross-Pitaevskii equation for the quantum fluid. This equation contains a contribution depending on the weak external gravitational field. We apply such equation to the study of a quantum fluid confined in a *double-well trap* with a Josephson junction. We will show that a careful choice of the geometry and profile of the trap can lead to a relative phase shift of the condensate wavefunction in the two sides of the trap in presence of a plane gravitational wave. Such phase shift can be responsible, in turn, for a gravitationally induced ac-Josephson effect.

The paper is organized as follows. In sections 2 and 3 we derive the Gross-Pitaevskii equation for a quantum fluid in the framework of the linearized gravity. In section 4 we consider - as an example - the quantum phase shift induced by a weak gravitational wave on an *ordinary* scalar quantum field (i.e., neglecting the field self-interaction). We stress that our results are similar to that obtained in [4, 6, 7], although the approach used in those papers (concerning weak fields *and slow motions*) is not suitable to describe gravitational waves. In section 5 - the core of the paper - we discuss Josephson tunneling for a quantum fluid in presence of a weak gravitational wave. We consider a double-well trap, made of two mutually orthogonal boxes coupled

through a small junction. Because of the different coupling between the gravitational wave and the quantum field modes in the two sides of the trap, the quantum fluid undergoes a gravitationally induced ac-Josephson effect through the potential barrier of the double-trap. In section 6 we consider a possible implementation of such effect in the specific case of a superconductor. Section 7 is devoted to some final comments and concluding remarks.

Throughout the paper, unless otherwise specified, use has been made of natural units: $c = 1$, $\hbar = 1$, $G = 1$. Greek indices take values from 0 to 3; latin ones take values from 1 to 3. The metric signature is -2 , with determinant g .

2. Quantum fluids: a Lagrangian approach

Quantum fluids can be considered as mesoscopic system which have undergone some type of gauge symmetry breaking (a phase transition). Such symmetry breaking is usually described by an *order parameter* that, in the framework of quantum field theory, behaves as a *macroscopic* wavefunction $\psi(x, t)$ for the quantum condensate. In the simplest cases (as those which we will consider) that function can be represented as a complex scalar $\psi = \sqrt{N}e^{i\theta}$, where N is the superfluid density and θ is the macroscopic quantum phase. ψ obeys the so-called Gross-Pitaevskii equation, which we will derive from a lagrangian approach (see, e.g., [20]).

Let us consider the lagrangian \mathcal{L}_0 for a self-interacting complex scalar field $\Psi(x)$, $x \equiv (t, \vec{x})$

$$\begin{aligned} \mathcal{L}_0 = & \partial_\mu \Psi^*(x) \partial^\mu \Psi(x) - m^2 \Psi^*(x) \Psi(x) \\ & - \lambda (\Psi^*(x) \Psi(x))^2 - 2mU(x) \Psi^*(x) \Psi(x), \end{aligned} \quad (2.1)$$

where m represents the field mass, λ is a parameter describing the field self-coupling strength and $U(x)$ is a potential due to an external interaction. From the Euler-Lagrange equations we readily obtain the equation of motion for the field

$$[\square + m^2 + 2\lambda|\Psi(x)|^2 + 2mU(x)]\Psi(x) = 0, \quad (2.2)$$

as well as a similar equation for $\Psi^*(x)$. Taking the non-relativistic limit [21] we perform the substitution $\Psi(x) = e^{-imt}\psi(x)$ in (2.2), obtaining

$$2im\partial_t\psi + \partial_t^2\psi - \nabla^2\psi + 2\lambda|\psi|^2\psi + 2mU\psi = 0. \quad (2.3)$$

Since in that limit $\partial_t\Psi \simeq E \ll m$, with E being the kinetic field energy, we can neglect the $\partial_t^2\psi$ term in the above equation. It follows

$$-\frac{1}{2m}\nabla^2\psi + U(x)\psi + \frac{\lambda}{m}|\psi|^2\psi = i\partial_t\psi. \quad (2.4)$$

Restoring, for sake of clarity, SI units we finally have

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) + U(x)\psi(x) + g_0|\psi(x)|^2\psi(x) = i\hbar\partial_t\psi(x), \quad (2.5)$$

where $g_0 = \hbar^2 \frac{\lambda}{m}$ describes the field self-interaction strength. Equation (2.5) is usually referred to as the Gross-Pitaevskii equation (GPE) for the phenomenological order parameter $\psi(x)$ of a Bose-Einstein condensate. In the framework of quantum field theory, GPE resembles a non-linear Schroedinger equation for the fluid macroscopic wavefunction $\psi(x)$.

In the next Section we will deduce a generalization of (2.5) in presence of a weak gravitational wave background.

3. Including a weak gravitational wave background

In linearized gravity a weak gravitational field is usually described through small corrections (gravitational *potentials*) $h_{\mu\nu}$, to the flat minkowskian space-time metric $\eta_{\mu\nu}$, so that the background metric reads

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (3.1)$$

In such framework [22] the interaction between a quantum field and a given background gravitational field can be considered as a field theory in a *flat* spacetime with total lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_G, \quad (3.2)$$

where \mathcal{L}_0 of eq.(2.1) is the field lagrangian in flat space-time, while the gravitational contribution \mathcal{L}_G reads

$$\mathcal{L}_G = -\frac{1}{2}h_{\mu\nu}T^{\mu\nu}, \quad (3.3)$$

with $T^{\mu\nu}$ being the stress-energy tensor of the quantum field Ψ derived from the lagrangian density \mathcal{L}_0 , namely

$$T_\nu^\mu = \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\Psi^*)}\partial_\nu\Psi^* + \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\Psi)}\partial_\nu\Psi - \delta_\nu^\mu\mathcal{L}_0. \quad (3.4)$$

At the present first order of approximation $T^{\mu\nu}$ is independent of $h_{\mu\nu}$, so that it satisfies $\partial_\mu T^{\mu\nu} = 0$. It is then straightforward to check that (3.3) is *gauge-invariant* with respect to gauge transformations of the coordinates:

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x^\nu), \quad |\xi^\mu| \sim |h^{\mu\nu}|, \quad (3.5)$$

causing a change in the gravitational potentials such

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \quad (3.6)$$

From (3.2), (3.3) and (3.4), the total lagrangian for the scalar field in a gravitational background reads

$$\mathcal{L} = \left(1 + \frac{1}{2}h\right)\mathcal{L}_0 - h^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi, \quad (3.7)$$

where $h = h^\mu{}_\mu \equiv \eta^{\mu\nu}h_{\mu\nu}$ is the trace of $h_{\mu\nu}$. Recall that raising and lowering of tensor indices is performed by means of the unperturbed flat minkowskian metric tensor $\eta_{\mu\nu}$. Using the Euler-Lagrange equations we obtain the corresponding equation of motion for the scalar field Ψ

$$\begin{aligned} & \left(1 + \frac{1}{2}h\right) [\square\Psi + 2mU\Psi + m^2\Psi + 2\lambda|\Psi|^2\Psi] \\ & - (\partial_\nu h^{\mu\nu})\partial_\mu\Psi - h^{\mu\nu}\partial_\mu\partial_\nu\Psi = 0. \end{aligned} \quad (3.8)$$

In the linearized theory, gravitational waves are described as small ripples on the background flat space-time texture [23], obeying (in the harmonic gauge $\partial_\mu h^{\mu\nu} - (1/2)\partial^\nu h = 0$) the usual wave equation for a massless spin-2 field, namely $\square h_{\mu\nu} = 0$, satisfying $|h_{\mu\nu}| \ll 1$ as well as $|\partial_\rho h_{\mu\nu}| \ll 1$. It is well-known that the condition $\partial_\mu h^{\mu\nu} - (1/2)\partial^\nu h = 0$ does not fix uniquely the gauge. It is customary to employ the residual gauge freedom to describe a gravitational wave in the transverse-traceless gauge (TT or Fock-de Donder gauge) where we exploit the above recalled gauge-invariance of the theory requiring $h = 0$ and $\partial_\mu h^\mu{}_\nu = 0$. The advantage of TT gauge

is to describe the wave through the two physical degrees related to its polarization states h_+ and h_\times . Thus, in the field of a weak gravitational wave (3.8) reads

$$\square\Psi + m^2\Psi + 2mU\Psi + 2\lambda|\Psi|^2\Psi - h^{\mu\nu}\partial_\mu\partial_\nu\Psi = 0. \quad (3.9)$$

Performing the non-relativistic limit by means of the substitution $\Psi(x) = e^{-imt}\psi(x)$ and proceeding as in the previous section we finally obtain (restoring SI units again)

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi(x) + U(x)\psi(x) + g_0|\psi(x)|^2\psi(x) \\ -\frac{\hbar^2 h^{\mu\nu}}{2m}\partial_\mu\partial_\nu\psi(x) = i\hbar\partial_t\psi(x), \end{aligned} \quad (3.10)$$

which we will assume as the GPE equation in presence of a weak gravitational wave background. One could wonder about the fact we have neglected the $\partial_t^2\psi$ contribution in the above limit. However this contribution becomes relevant only in particle creation effect (as we have proved elsewhere [24]), the latter being exponentially small, hence negligible in the present context, in which we are mainly interested only in phase shift effects due to the interaction between the quantum field and the gravitational wave.

We stress that the above recalled gauge-invariance of \mathcal{L}_G [eq.(3.3)] allows us to employ the TT-frame as the reference frame of a fiducial physical observer.

4. Quantum phase shift in the field of a gravitational wave

Let us briefly review the influence of a gravitational wave on the phase shift of a (non-relativistic) quantum scalar field. In this section we will assume, for simplicity, that the self-interaction is absent, $g_0 = 0$, so that (3.10) simply reduces to its (linear) Schroedinger form plus a gravitational contribution described by the operator \hat{V}

$$\hat{V} = -\frac{\hbar^2 h^{\mu\nu}}{2m}\partial_\mu\partial_\nu \quad (4.1)$$

(throughout the text a caret will be used to label an *operator* quantity). Consider the field confined in a box having dimensions L_x, L_y, L_z . Take the edges of the box along the axes of a reference frame the origin of which is located at one vertex of the box. The confining square-well potential is $U(\vec{x}) = \infty$ outside the cavity and $U(\vec{x}) = 0$ inside the cavity, so requiring the usual Dirichlet boundary conditions for the field at the cavity walls. For a general gravitational plane wave propagating along the z -direction, the corresponding space-time line element (in the TT gauge) is:

$$\begin{aligned} ds^2 = dt^2 - (1 + h_+(t-z))dx^2 - (1 - h_+(t-z))dy^2 \\ - 2h_\times(t-z)dx dy - dz^2. \end{aligned} \quad (4.2)$$

If the cavity size is small with respect to the typical wavelength of the gravitational wave, we can assume $z = 0$ at the cavity location, so that the gravitational perturbation (4.1) in the frame of the fiducial TT-observer reads

$$\hat{V} = -v_+(t)(\partial_x^2 - \partial_y^2) - 2v_\times(t)\partial_x\partial_y, \quad (4.3)$$

$$v_+(t) \equiv \frac{\hbar^2 h_+(t)}{2m}, \quad v_\times(t) \equiv \frac{\hbar^2 h_\times(t)}{2m}. \quad (4.4)$$

Let us consider for simplicity the wave in the h_+ polarization state only. In such a case the only non-null $h_{\mu\nu}$'s will be $h_{xx}(t-z) = -h_{yy}(t-z) \equiv h_+(t-z)$. Then from (3.10), inside the box we have (with $g_0 = 0$)

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}, t) - v_+(t)(\partial_x^2 - \partial_y^2)\psi(\vec{x}, t) = i\hbar\partial_t\psi(\vec{x}, t), \quad (4.5)$$

while $\psi(\vec{x}, t) = 0$ outside the box. The above equation is a time-dependent Schroedinger equation.

Since $|h_+(t)| \ll 1$, we assume that the spatial part of ψ is almost the same as in the stationary case. So we guess

$$\psi(\vec{x}, t) = \alpha(t)\psi^0(\vec{x}), \quad (4.6)$$

with $\psi^0(\vec{x})$ satisfying the time-independent Schroedinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi^0(\vec{x}) = E\psi^0(\vec{x}). \quad (4.7)$$

The well-known solution obeying the required Dirichlet boundary conditions in the above chosen reference frame reads

$$\psi^0(\vec{x}) \sim \sin k_x x \sin k_y y \sin k_z z, \quad (4.8)$$

with eigenvalues

$$E = \frac{\hbar^2 k^2}{2m}, \quad k^2 = \sum_{i=x,y,z} k_i^2, \quad k_i = \frac{n_i \pi}{L_i}, \quad n_i \in N. \quad (4.9)$$

Substituting (4.6) in (4.5) we get

$$\begin{aligned} -\frac{\hbar^2}{2m}\alpha(t)\nabla^2\psi^0(\vec{x}) - \alpha(t)v_+(t)(\partial_x^2 - \partial_y^2)\psi^0(\vec{x}) \\ = i\hbar\alpha(t)\partial_t\psi^0(\vec{x}). \end{aligned} \quad (4.10)$$

In absence of gravitational perturbation [$v_+(t) = 0$], $\alpha(t)$ reduces to the usual factor $e^{-iEt/\hbar}$ and eq.(4.7) is recovered, as expected. When $v_+(t) \neq 0$, we look for $\alpha(t)$ as

$$\alpha(t) = a(t)e^{-iEt/\hbar}. \quad (4.11)$$

From (4.7), (4.8) and (4.10) we get

$$a(t) = a_0 \exp \left[-\frac{i(k_x^2 - k_y^2)}{\hbar} \int_{t_0}^t v_+(t') dt' \right] \quad (4.12)$$

or, in terms of $h_+(t)$

$$a(t) = a_0 \exp \left[-\frac{i\hbar(k_x^2 - k_y^2)}{2m} \int_{t_0}^t h_+(t') dt' \right]. \quad (4.13)$$

If $L_x \neq L_y$ and the field is initially in its ground state, the above equation tells us that the interaction with the gravitational wave gives rise to a *phase shift* in the field wavefunction. As an example, let us model the gravitational wave as a *pulse*. A suitable shape is

$$h_+(t - z) = H \cosh^{-2} \beta(t - z), \quad (4.14)$$

where H and $1/\beta$ represent the amplitude and the typical time duration of the gravitational pulse, respectively. If $1/\beta \gg L_z$ then the spatial variation of the gravitational perturbation inside the cavity is negligible, and we may put $z = 0$. Substituting in (4.13) and performing the integration we get

$$a(t) = a_0 \exp \left[-\frac{i\hbar(k_x^2 - k_y^2)}{2m} \frac{H}{\beta} (\tanh \beta t - \tanh \beta t_0) \right]. \quad (4.15)$$

Requiring that $\lim_{t \rightarrow -\infty} a(t) = 1$ (lack of gravitational perturbation in the far past) yields

$$a(t) = \exp \left[- \frac{i\hbar(k_x^2 - k_y^2) H}{2m} \frac{H}{\beta} (\tanh \beta t + 1) \right]. \quad (4.16)$$

For $t \rightarrow +\infty$, i.e. a long time after the interaction with the gravitational pulse, we have

$$a(+\infty) = \exp \left[- \frac{i\hbar(k_x^2 - k_y^2) H}{m} \frac{H}{\beta} \right]. \quad (4.17)$$

The overall field wavefunction is then affected by a phase shift after the interaction with the gravitational perturbation. Namely, recalling (4.6) and (4.11) we have

$$\psi(\vec{x}, t) \sim e^{-iEt/\hbar} \exp \left[- \frac{i\hbar(k_x^2 - k_y^2) H}{m} \frac{H}{\beta} \right] \psi^0(\vec{x}). \quad (4.18)$$

5. Josephson tunneling in a double-well trap

Having recalled, on general grounds, the behavior of the phase of a quantum scalar field in presence of a weak gravitational wave, let us now move to explore the possible influence of the latter on Josephson tunneling of a Bose-Einstein fluid. Generally speaking, Josephson effect appears when tunneling is allowed between two weakly linked quantum systems which have undergone some type of gauge symmetry breaking. This is the case of quantum fluids, in which such symmetry breaking - occurring at the superfluid transition phase - leaves an imprint on the overall quantum phase θ of the macroscopic condensate field wavefunction ψ .

Usually, Josephson tunneling is described in a double-well trap enclosing the quantum condensate. In our model we consider two identical boxes joined by a small, short channel (as sketched in fig. 1). The channel is centered at the origin of the fiducial TT reference frame and along the z -axis. The boxes are rotated one with respect to the other through an angle $\pi/2$ about the z -axis. We take the boxes (1) and (2) in the in the $z < 0$ and $z > 0$ half-space, respectively. Also, we assume that the transverse box dimensions, namely ℓ and L , satisfy $\ell = L_{1x} = L_{2y} \ll L_{1y} = L_{2x} = L$. (This model is, in some respect, the rotated three-dimensional one of that studied in ref. [25], although in a quite different context). Such a system acts for the BE condensate as a double-well potential trap. Namely, the trapping potential has an infinite square well profile in the transverse (xy)-plane, meanwhile it exhibits a double-well shape along the z -direction, with the channel acting as a central potential barrier. This external *static* potential can be described as $U(\vec{x}) = U_{\perp}(x, y) + U_{\parallel}(z)$, where $U(z) = U(-z)$ is assumed strongly peaked around $z = 0$ inside the channel, and

$$U_{\perp}(x, y) = \begin{cases} 0 & \text{inside the cavities} \\ +\infty & \text{outside the cavities.} \end{cases} \quad (5.1)$$

Notice also that, because of the peculiar topology of the double-trap, we have the following symmetries

$$U(x, y, z) = U(y, x, -z), \quad (5.2)$$

$$U_{\perp}(x, y)|_1 = U_{\perp}(y, x)|_2. \quad (5.3)$$

As we will see, we will not require for our purposes a detailed knowledge of the analytic form of the trapping potential $U(\vec{x})$. According to [26], we suppose that the

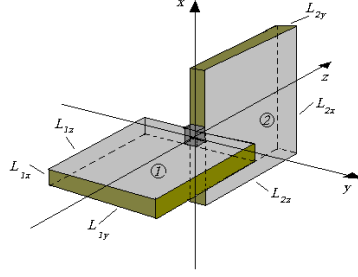


Figure 1. A sketch of the double-trap considered in the text, with the adopted reference frame. Two identical coupled traps (1) and (2) are rotated of $\pi/2$ one respect to the other about the z -axis. The gravitational wave propagates along the z -direction. The grey region between the traps represents the small 'channel' allowing for Josephson tunneling.

non-linear interaction of the quantum fluid [controlled by the g_0 parameter in (3.10)] is weak relative to the trapping force due to the transverse potential $U_{\perp}(x, y)$. In such a case the transverse modes of the field ψ in the two cavities will be almost the same as the corresponding transverse ones we found in the previous section for an infinite 3D-square well potential, thus satisfying the following stationary 2D-Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla_{\perp}^2\psi_{\perp}(x, y) + U_{\perp}(x, y)\psi_{\perp}(x, y) = E_{\perp}\psi_{\perp}(x, y) \quad (5.4)$$

with E_{\perp} being the corresponding transverse energy eigenvalues. We assume that the transverse modes $\psi_{\perp}(x, y)$ are in the no-node ground state (see [26]). So, from (5.4) and the above recalled symmetry properties of $U(\vec{x})$, we get, in the adopted reference frame

$$\psi_{\perp}(x, y) = \begin{cases} \psi_{\perp}(x, y)|_1 \sim \cos \frac{\pi x}{\ell} \cos \frac{\pi y}{L}, \\ \psi_{\perp}(x, y)|_2 = \psi_{\perp}(y, x)|_1 \sim \cos \frac{\pi x}{L} \cos \frac{\pi y}{\ell}, \end{cases} \quad (5.5)$$

Notice that, because of the underlying symmetries of the double-trap, the transverse ground states in both the cavities (1) and (2) share the same transverse energy eigenvalue E_{\perp} . Let us look for an overall solution as

$$\psi(\vec{x}, t) = \phi(z, t)\psi_{\perp}(x, y)e^{-iE_{\perp}t/\hbar}. \quad (5.6)$$

Consider a weak, plane-fronted gravitational wave (4.2) propagating along the z direction. Substituting (5.6) in (3.10) and using (5.4) we get

$$\begin{aligned} -\frac{\hbar^2}{2m}\partial_z^2\phi(z, t) + U_{\parallel}(z)\phi(z, t) + g_0|\phi(z, t)\psi_{\perp}(x, y)|^2\phi(z, t)\psi_{\perp}(x, y) \\ + \mathcal{V}(t, z)\phi(z, t)\psi_{\perp}(x, y) = i\hbar\psi_{\perp}(x, y)\partial_t\phi(z, t), \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} \mathcal{V}(t, z) &= \text{sgn}(z)\mathcal{V}_+(t) + \mathcal{V}_{\times}(t), \\ \mathcal{V}_+(t) &= v_+(t)\pi^2[1/\ell^2 - 1/L^2], \\ \mathcal{V}_{\times}(t) &= 2v_{\times}(t)\pi^2[1/(\ell L)], \end{aligned} \quad (5.8)$$

Integrating over the transverse domain D_\perp (notice that $D_\perp \simeq \text{const}$ all through the double-trap, as the two *identical* cavities, although rotated, have the same transverse section) and dividing by D_\perp yields, in a more compact notation

$$\left[\hat{H}_S + \gamma |\phi(t, z)|^2 + \mathcal{V}(t, z) \right] |\phi\rangle = i\hbar \partial_t |\phi\rangle, \quad (5.9)$$

where the normalization $\int_{D_\perp} \psi_\perp dx dy = 1$ has been employed, $\gamma \equiv g_0/D_\perp$ and

$$\hat{H}_S = -\frac{\hbar^2}{2m} \partial_z^2 + U_\parallel(z). \quad (5.10)$$

Eq.(5.9) has the form of a 1D-GP time-dependent equation for the longitudinal profile of the condensate wavefunction $\phi(z, t)$.

To solve the above equation we look for $\phi(z, t)$ according to the following variational ansatz (see, e.g., [27])

$$\phi(z, t) = a_1(t)\chi_1(z) + a_2(t)\chi_2(z), \quad (5.11)$$

where $\chi_{1,2}(z)$ describe the longitudinal profile of the condensate as localized in each trap. If the condensate density in the tunneling region is small, the above ansatz holds true and

$$\langle \chi_i | \chi_j \rangle \equiv \int \chi_i^* \chi_j dz = \delta_{ij} \quad , \quad i, j = 1, 2, \quad (5.12)$$

the longitudinal integration being performed all through the double-trap. Substituting (5.11) in (5.9) and left-multiplying by $|\chi_i\rangle$, ($i = 1, 2$) we get, using also (5.12), the following system of coupled equations

$$\begin{aligned} [E_1 + |a_1(t)|^2 \mathcal{U}_1 - \mathcal{V}_+(t) + \mathcal{V}_\times(t)] a_1(t) + K a_2(t) &= i\hbar \partial_t a_1(t), \\ [E_2 + |a_2(t)|^2 \mathcal{U}_2 + \mathcal{V}_+(t) + \mathcal{V}_\times(t)] a_2(t) + K a_1(t) &= i\hbar \partial_t a_2(t), \end{aligned} \quad (5.13)$$

with the *time-independent* parameters [27]

$$E_i = \langle \chi_i | \hat{H}_S | \chi_i \rangle, \quad i = 1, 2 \quad (5.14)$$

$$\mathcal{U}_i = \gamma \int dz |\chi_i|^4, \quad i = 1, 2 \quad (5.15)$$

$$K = \langle \chi_1 | \hat{H}_S + \gamma |\phi|^2 | \chi_2 \rangle \simeq \langle \chi_1 | \hat{H}_S | \chi_2 \rangle. \quad (5.16)$$

As usual, let us put

$$a_i(t) = \sqrt{N_i(t)} e^{i\theta_i(t)}, \quad (5.17)$$

where $N_i(t) = |a_i(t)|^2$ and $\theta_i(t)$ represent, respectively, the number of particles and the *global* phase of the BE condensate in each trap. Then, in the flat space-time case ($\mathcal{V}(t) \rightarrow 0$) these equations reduce to the well-known ones for a BE condensate in the treatment of Josephson tunneling. The above equations (5.13) are clearly non-linear, due to the presence of the $|a_i(t)|^2 \mathcal{U}_i$ -terms, so usually requiring a numerical solution. They are however quite general, describing the evolution of a *macroscopic* wavefunction of some Bose-Einstein condensate. The latter may be a superfluid or a superconductor. We will focus on this latter case in the next section.

6. The superconductor case

Eqs.(5.13) can be easily implemented for the analysis of Josephson effect in a superconducting environment, assuming that an external circuit provides a compensation for particle unbalance between the two traps due to the tunneling across the junction [27]. Indeed, in such a case we can suppose that the particle numbers N_1 and N_2 are kept *constant* (in spite of the current flow through the channel) in each trap. Then the non-linear terms $|a_i(t)|^2 \mathcal{U}_i \equiv N_i \mathcal{U}_i$ can be considered almost *constant* in the two traps and absorbed into a redefinition of E_i . This allows us to rewrite eqs.(5.13) as

$$\begin{aligned} [E'_1 - \mathcal{V}_+(t) + \mathcal{V}_\times(t)]a_1(t) + Ka_2(t) &= i\hbar\partial_t a_1(t), \\ [E'_2 + \mathcal{V}_+(t) + \mathcal{V}_\times(t)]a_2(t) + Ka_1(t) &= i\hbar\partial_t a_2(t), \end{aligned} \quad (6.1)$$

where $E'_i = E_i + N_i \mathcal{U}_i$.

Substituting (5.17) in (6.1), the latter can be rewritten in terms of $N_i(t)$ and $\theta_i(t)$ (for details see, e.g., [28, 29]), namely

$$\dot{N}_i(t) = \frac{2K}{\hbar} \sqrt{N_i N_j} \sin(\theta_i - \theta_j), \quad i, j = 1, 2, \quad i \neq j, \quad (6.2)$$

and

$$\begin{aligned} \dot{\theta}_1(t) &= \frac{K}{\hbar} \sqrt{\frac{N_2}{N_1}} \cos(\theta_1 - \theta_2) - \frac{E'_1 - \mathcal{V}_+(t) + \mathcal{V}_\times(t)}{\hbar}, \\ \dot{\theta}_2(t) &= \frac{K}{\hbar} \sqrt{\frac{N_2}{N_1}} \cos(\theta_2 - \theta_1) - \frac{E'_2 + \mathcal{V}_+(t) + \mathcal{V}_\times(t)}{\hbar}. \end{aligned} \quad (6.3)$$

We point out that in (6.2) $\dot{N}_i(t)$ represents the *time-dependent* current flow through the junction, while the particle numbers N_i are held almost *constant* in the two traps by means of a suitable external circuit. This means that the time dependence of the flux $\dot{N}_i(t)$ is entirely due to the global phases $\theta_i(t)$.

For equal superconductors we put $N_1 = N_2 \equiv N$ in (6.2) and (6.3). From eqs.(6.3) we immediately get

$$\partial_t(\theta_1 - \theta_2) = -\frac{E'_1 - E'_2 + 2\mathcal{V}_+(t)}{\hbar}. \quad (6.4)$$

We stress that the $\mathcal{V}_\times(t)$ contribution has disappeared. So, because of its peculiar geometry, the double-trap acts - in some respect - as a *polarimeter*, being sensitive to the polarization state of the incoming gravitational wave. Assuming $E'_1 = E'_2$ for identical superconducting traps [this is a quite reasonable assumption for equal superconductors cooled near 0K; see (5.14)], we integrate the above equation, thus obtaining

$$\delta(t) = \delta_0 - \frac{2}{\hbar} \int dt \mathcal{V}(t), \quad (6.5)$$

where $\delta(t) = \theta_1 - \theta_2$ is the phase difference of the macroscopic wavefunction in the two traps. Using also (5.8) and (4.3), we have

$$\delta(t) = \delta_0 - \frac{\hbar\pi^2[1/\ell^2 - 1/L^2]}{m} \int dt h_+(t), \quad (6.6)$$

with m being the effective mass of a Cooper pair ($m = 2m_e$). From (6.2) the current flow $J \equiv \dot{N}_1(t) = -\dot{N}_2(t)$ across the junction is $J(t) = J_0 \sin \delta(t)$ [$J_0 = 2KN/\hbar$,

see (5.16)]. Assuming, e.g., a monochromatic gravitational wave of amplitude H and frequency ω_g , we have (at $z = 0$) $h_+(t) = H \sin \omega_g t$, so

$$J(t) = J_0 \sin \delta_0 + \delta J(t), \quad (6.7)$$

where $J_0 \sin \delta_0$ represents the well-known dc-Josephson effect, while the *time-dependent* contribution reads

$$\delta J(t) = -J_0 \frac{\hbar \pi^2 [1/\ell^2 - 1/L^2] H}{m \omega_g} \cos \delta_0 \cos \omega_g t. \quad (6.8)$$

If $\ell \ll L$ (as supposed at the beginning of this section), then the above formula can be rewritten as

$$\delta J(t) \simeq -J_0 \frac{\hbar \pi^2 H}{m \ell^2 \omega_g} \cos \delta_0 \cos \omega_g t. \quad (6.9)$$

Thus, a sinusoidal gravitational wave induces a small oscillating current through the junction between the two orthogonal superconducting traps. Such current oscillates at the same frequency of the gravitational wave. This can be considered as a *gravitationally induced ac-Josephson effect*. From (6.9) we see that the effect is enhanced in presence of a very thin transverse trap section ℓ and of low-frequency gravitational waves.

We can try to give a rough estimate of the ratio between the time-dependent induced current $\delta J(t)$ and the dc-Josephson term J_0 . Assuming a Josephson junction made of two traps the thickness of which is about 2000 Å (as in a typical SQUID junction), we find from (6.9) the maximum value

$$\frac{|\delta J(t)|_{max}}{J_0} \simeq 5.5 \times 10^{10} \frac{H}{\omega_g}. \quad (6.10)$$

So, for a low-frequency $\omega_g \simeq 100 \text{ s}^{-1}$ gravitational wave with amplitude $H \simeq 10^{-21}$ we get

$$\frac{|\delta J(t)|_{max}}{J_0} \simeq 5.5 \times 10^{-13}. \quad (6.11)$$

This is indeed a very small value. However, current technology should allow to integrate a very large number of Josephson junctions (followed by some integrated rectifying circuitry) in a single chip. For example, Cybart *et al.* [30] have recently obtained a Very Large Scale integration by building an array of 15,820 Josephson junctions on a $500 \mu\text{m} \times 3 \text{ mm}$ $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ microstrip. So, it is likely that a suitable (e.g., $\simeq 1,000$) number of such parallel-operating devices can give an output signal which should be large enough ($\frac{|\delta J(t)|_{max}}{J_0} \simeq 10^{-5}$) to be amplified and detected by means of some external electronics.

7. Concluding Remarks

In this paper we have obtained a generalization of the Gross-Pitaevskii equation in the framework of linearized gravity. We have employed such equation to describe the influence of a weak gravitational wave on a quantum fluid confined in a double-well trap. We have chosen a suitable geometry for the trap, namely two mutually orthogonal cavities linked by a Josephson junction. Such a geometry leads to a different coupling between the gravitational wave and the quantum fluid in the two sides of the trap. The resulting different phase shift caused by the wave on the

macroscopic quantum field wavefunction in the two sides gives rise, in turn, to a gravitationally induced ac-Josephson tunneling through the double-trap. We have briefly discussed that effect in the case of an ideal superconducting device.

We would like to point out some relevant aspects. Although the effect is very small, it is nevertheless quite interesting, since it results from the interaction between a gravitational wave and a mesoscopic quantum system. The above results show how topologic and geometric properties of the configuration providing the quantum field confinement can become relevant in defining the coupling with the gravitational wave and the consequent phase shift of the quantum wavefunction. The chosen geometry for the double-trap is also responsible for the sensitivity of the quantum system to the polarization state of the wave. In some respect the device acts as a *polarimeter* for the incoming gravitational wave. Because the above called gravitationally induced ac-Josephson tunneling is proportional to the integrated gravitational signal, the reported effect is enhanced in presence of low-frequency gravitational waves.

Eq.(6.9) suggests that the effect is improved in presence of very thin coupled traps. We are not so aware of the present technology about possible realization of superconducting nanostructures. Nevertheless, we believe that it could be interesting to explore the possibility that a large-scale integration of a very large number of such devices (VLSI), coupled to a suitable external electronics for signal amplification and processing, can compensate for the smallness of the gravitationally induced Josephson current.

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