



**HAL**  
open science

## The lightcone of Gödel-like spacetimes

G Dautcourt

► **To cite this version:**

G Dautcourt. The lightcone of Gödel-like spacetimes. *Classical and Quantum Gravity*, 2010, 27 (22), pp.225024. 10.1088/0264-9381/27/22/225024 . hal-00643030

**HAL Id: hal-00643030**

**<https://hal.science/hal-00643030>**

Submitted on 21 Nov 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The lightcone of Gödel-like spacetimes

G. Dautcourt<sup>‡</sup>

**Abstract.** A study of the lightcone of the Gödel universe is extended to the so-called Gödel-like spacetimes. This family of highly symmetric 4-D Lorentzian spaces is defined by metrics of the form  $ds^2 = -(dt + H(x)dy)^2 + D^2(x)dy^2 + dx^2 + dz^2$ , together with the requirement of spacetime homogeneity, and includes the Gödel metric. The quasi-periodic refocussing of cone generators with startling lens properties, discovered by Ozsváth and Schücking for the lightcone of a plane gravitational wave and also found in the Gödel universe, is a feature of the whole Gödel family. We discuss geometrical properties of caustics and show that (a) the focal surfaces are two-dimensional null surfaces generated by non-geodesic null curves and (b) intrinsic differential invariants of the cone attain finite values at caustic subsets.

PACS numbers: 02.40.Xx, 04.20.-q, 02.40.Hw, 04.20.Jb, 04.20.Gz

## 1. Introduction

The study of null congruences, null hypersurfaces and in particular lightcones in general relativity is complicated by the existence of caustics, i.e. points of intersection of the generating null geodesics. Since light rays become focussed in the presence of matter as well as shear, caustics occur inevitably in realistic situations as frequently encountered in astrophysics. The corresponding strong lens effect is an important astrophysical tool [48]. In numerical relativity caustics are less welcome, they act as barrier for current characteristic codes [19, 13, 52]. The differential geometry of caustics in a spacetime setting is still not well developed, contrary to their mathematical classification using methods of singularity theory [3]. This may be due to the fairly complicated structure of these objects, involving crossings and singularities. Important steps have been taken, among others, by Friedrich and Stewart [19] and by the Newman school [20]-[22], [27].

A way towards a better understanding is the study of curved spacetimes with analytically known focal surfaces. Ozsváth and Schücking presented already in 1962 an exact and detailed analytical picture of the lightcone of a plane gravitational wave [32]. They found a cyclic structure of the focal set, produced by a semi-periodic re-focussing of light rays. An often reproduced illustration of a similar light cone drawn by Penrose [37] served as starting point for investigations in global Lorentzian geometry [18], [6]. A very similar focal structure is present on the lightcone of a quite different spacetime, the rotating Gödel universe [28, 2, 16]. In view of the T-duality of higher-dimensional

<sup>‡</sup> Email: daut@aei.mpg.de, MPI für Gravitationsphysik, 14476 Golm, Germany

supersymmetric versions of the Gödel metric and pp waves [8, 25] it is perhaps not surprising that the same type of caustic is present.

For further analytical studies of caustics it makes sense to discuss spaces of high symmetry first, since here the geodesic equations can be integrated completely. Furthermore, if spacetime homogeneity applies, all lightcones have the same intrinsic geometry, independent of the vertex location. To this class of spacetimes belong metrics of the Ozsváth class III [33], which include the Gödel metric. They have been studied by Rosquist et al [29]. Other examples are the spacetime-homogeneous Gödel-like or Gödel type metrics [49],[39],[11],[40]-[45],[51], [9], also generalizations of the Gödel metric and admitting at least a  $G_5$  Killing symmetry. Their lightcone, and in particular the focal subset, is the subject of the present paper.

Section 2 shortly reviews the two-parameter family of Gödel-like metrics. Basic geometrical properties of these metrics depend on a dimensionless parameter  $k$ , measuring the influence of rotation on the spacetime geometry.  $k^2$  may range from  $-\infty$  to  $\infty$ , but in this article we confine the discussion to a range of positive  $k^2$ . The  $k$ -sequence coincides with the family of  $(2 + 1)$ -dimensional geometries investigated by Rooman and Spindel [46], if one flat space dimension is added to Rooman-Spindel. Their parameter  $\mu$  is our  $k$ .

The lightcone geometry of the Gödel family with  $1 < k^2 < \infty$  ( $k^2 = 2$  corresponds to the Gödel cosmos [23], [24], [26], [5], [34], [35]) is studied in section 3, based on a paper by Calvão, Soares and Tiomno [10]. Further sections consider briefly some limiting cases. In section 4 we treat the special case  $k^2 = 1$ , known as Rebouças-Tiomno metric [40, 44]. Its subspace  $z = const$  is the three-dimensional anti-deSitter space  $AdS_3$ . The causal family  $0 < k^2 < 1$  without closed timelike curves is omitted here, only the static degeneration  $k^2 \rightarrow 0$  with vanishing rotation is shortly considered in section 5. The concluding section notes that a cyclic behaviour of caustics on many lightcones (and on null hypersurfaces in general) may be expected as a consequence of the Sachs equations [47] for divergence and shear of the generator congruence.

## 2. Gödel-like metrics

Raychaudhuri and Guha Thakurta [39] have introduced as "homogeneous spacetimes of the Gödel type" the metrics

$$ds^2 = -(dt + H(x)dy)^2 + D^2(x)dy^2 + dx^2 + dz^2, \quad (1)$$

together with the additional condition of spacetime homogeneity. Spacetime homogeneity requires at least one further Killing field, apart from the three translational Killing vectors along the axes, which evidently exist. This leads to the *necessary* conditions

$$H'/D = const = 2\Omega, \quad D''/D = const = l^2, \quad (2)$$

with two parameter  $\Omega, l$ . We replace  $\Omega$  by the parameter  $k = 2\Omega/l$  and consider the sequence of metrics labeled by  $k$ . Only real  $k$  are taken here and  $l$  is assumed

non-negative. Rebouças and Tiomno found that the conditions (2) are also *sufficient* for spacetime homogeneity [40]. Moreover, it was shown in [40] that for the metrics satisfying (1) and (2) a further Killing vector exists, leading to a  $G_5$  group of motions. A re-examination of the symmetries by Teixeira, Rebouças and Aman [51], who dropped the more or less implicit assumption of time-independent Killing fields made so far, has shown that in the special case  $k^2 = 1$  the group is  $G_7$ , the maximal symmetry group within the Gödel-like class of spacetimes.

We write the metric (1) in cylindrical coordinates  $(t, r, \phi, z)$  as

$$ds^2 = -(dt + \frac{2k}{l} \sinh^2(lr/2)d\phi)^2 + \frac{\sinh^2(lr)}{l^2}d\phi^2 + dr^2 + dz^2. \quad (3)$$

The numbers  $(k^2, l)$  in the two-parameter family (3) specify a metric uniquely: Members with different pairs  $(k^2, l)$  represent different spacetimes. In the limit  $k \rightarrow \infty$  and  $l \rightarrow 0$  such that  $kl = 2\Omega$  remains finite, the function  $\sinh(lr)/l$  can be replaced by  $r$ , and (3) becomes the Som-Raychaudhuri metric [49]. For  $k^2 \rightarrow 2$  one recovers the Gödel metric and for  $k^2 = 1$  the already mentioned  $G_7$  metric is obtained, studied in detail by Rebouças and Tiomno [40].

For convenience we note some properties of the metric (3). The non-vanishing components of the Ricci tensor are (we follow the conventions of [50])

$$R_0^0 = -k^2l^2/2, \quad R_1^1 = R_2^2 = l^2(k^2 - 2)/2, \quad R_2^0 = kl(k^2 - 1)(1 - \cosh(lr)). \quad (4)$$

The eigenvalues  $\lambda$  determined from  $\det|R_\mu^\nu - \lambda\delta_\mu^\nu| = 0$  follow as

$$\lambda_1 = 0, \quad \lambda_2 = -k^2l^2/2, \quad \lambda_{3,4} = l^2(k^2 - 2)/2. \quad (5)$$

The Weyltensor, also given in coordinate form, has the non-vanishing components

$$\begin{aligned} C_{0101} &= -C_{1313} = (k^2 - 1)l^2/6, & C_{0112} &= -kl(k^2 - 1)\cosh(lr)/6, \\ C_{0202} &= (k^2 - 1)\sinh(lr)/6, & C_{0303} &= -(k^2 - 1)l^2/3, \\ C_{0323} &= -kl(k^2 - 1)(\cosh(lr) - 1)/3, & & \\ C_{1212} &= (k^2 - 1)(\cosh(lr) - 1)(k^2(\cosh(lr) - 1) + 2\cosh(lr) + 2)/6, & & \\ C_{2323} &= (k^2 - 1)(\cosh(lr) - 1)(-2k^2(\cosh(lr) - 1) - 1 - \cosh(lr))/6. & & \end{aligned} \quad (6)$$

There exist several interpretations of the matter tensor as calculated from the Einstein field equations (including a cosmological constant). The  $k \rightarrow \infty$  limit, the Som-Raychaudhuri metric, describes the gravitational field of a homogeneous distribution of charged rotating dust. For the Gödel family  $1 < k^2 < \infty$ , the combination of a perfect fluid, a scalar field and a homogeneous source-free electromagnetic field may serve as matter [40]. A perfect fluid description alone applies only to the Gödel metric  $k^2 = 2$ , as shown by Bampi and Zordan [4]. For more recent discussions of Gödel-like spacetimes in gravity theories derived from Lagrangians which are arbitrary functions of curvature invariants, see [12],[43].

The interpretation of the Gödel family as solutions of the Einstein or other field equations may be considered as dubious in the sense that unusual or unphysical forms of matter are involved. Therefore these metrics (except Gödel) are not treated in

the standard book on exact solutions [50]. But this aspect is not important for the geometrical discussion in this paper. The metrics are mainly interesting for their *high degree of symmetry*. All admit at least a  $G_5$  group of motions.

### 3. The Gödel family $1 < k^2 < \infty$

#### 3.1. Null geodesics

For generic Gödel-like metrics, use is made of the results by Calvão, Soares and Tiomno (CST) [10], also largely keeping their notation for comparison. The authors follow a previous paper by Novello, Soares and Tiomno dealing with the Gödel metric [30]. They give a complete discussion of timelike geodesics and treat also null geodesics. We consider only the lightlike case. The high symmetry of the metric allows to write down a sufficient number of first integrals for the geodesic equations  $Dx^\mu/Ds^2 = 0$ , using the fact that for a Killing field  $k_\mu$ ,  $\frac{dx^\mu}{ds}k_\mu$  is constant along a geodesic. With the Killing translations  $\partial_t$ ,  $\partial_\phi$ ,  $\partial_z$  and the corresponding integration constants  $p_t$ ,  $p_\phi$ ,  $p_z$  or equivalently  $p_t$ ,  $\beta = p_z/p_t$ ,  $\gamma = p_\phi/p_t$ , three first integrals may be written

$$\dot{t}/p_t = 1 + kl/2 - k^2 \sinh^2(lr/2)/\cosh^2(lr/2), \quad (7)$$

$$\dot{\phi}/p_t = kl/(2 \cosh^2(lr/2)) - l^2\gamma/(4 \sinh^2(lr/2) \cosh^2(lr/2)), \quad (8)$$

$$\dot{z}/p_t = -\beta. \quad (9)$$

The dot denotes the derivative with respect to an affine parameter  $s$ . A further relation follows from  $\frac{dx^\mu}{ds} \frac{dx^\nu}{ds} g_{\mu\nu} = 0$ :

$$\dot{r}^2/p_t^2 = 1 - \beta^2 - \left( k \frac{\sinh(lr/2)}{\cosh(lr/2)} - \frac{l\gamma}{2 \sinh(lr/2) \cosh(lr/2)} \right)^2. \quad (10)$$

As shown by CST, it is convenient to introduce instead of  $r$  another radial variable  $\xi$ , which increases monotonically with  $r$ :

$$\xi = \sinh^2(lr/2). \quad (11)$$

Equation (10) then becomes

$$\dot{\xi}^2/p_t^2 = l^2\eta\xi^2 + l^2(1 - \beta^2 + kl\gamma)\xi - l^2\gamma^2/4 \quad (12)$$

with

$$\eta = k^2 + \beta^2 - 1. \quad (13)$$

The equations (7)-(12) refer to the class of *all* null geodesics. We are interested in the subset forming a single cone, e.g., passing through the origin of the coordinate system,  $t = 0$ ,  $r = 0$ ,  $z = 0$ . This subset is obtained by setting  $\gamma = 0$ : Expanding the rhs of (12) around  $r = 0$  or  $\xi \approx l^2r^2/4 = 0$ , one obtains  $(d\xi/ds)^2 \approx -l^4p_t^2\gamma^2/4 < 0$ , hence no geodesics with  $\gamma \neq 0$  can pass the origin. On the other hand, every geodesic with  $\gamma = 0$  passes the origin.

With  $\gamma = 0$  the first integrals simplify considerably. The geodesic equations can be integrated completely, leading to the following parameter representation of the light cone with vertex at  $t = 0$ ,  $r = 0$ ,  $z = 0$ :

$$t = \frac{2k}{l} \arctan\left(\frac{k}{\sqrt{\eta}} \tan w\right) - \frac{2w(k^2 - 1)}{l\sqrt{\eta}}, \quad (14)$$

$$r = \frac{2\epsilon}{l} \operatorname{arsinh}\left(\frac{\sin w \sqrt{1 - \beta^2}}{\sqrt{\eta}}\right), \quad (15)$$

$$\phi = \phi_0 + \arccos\left(\sqrt{1 - \beta^2 - \eta\xi}/(\sqrt{1 - \beta^2}\sqrt{1 + \xi})\right), \quad (16)$$

$$z = -\frac{2w\beta}{l\sqrt{\eta}}, \quad (17)$$

with  $\epsilon = 1(-1)$  for the future (past) cone. We have introduced a new affine parameter  $w$  instead of  $s$  by

$$w = lp_t\sqrt{\eta}(s - s_0)/2 \quad (18)$$

(note  $\eta > 0$ , since  $k^2 > 1$  is assumed, the case  $k^2 = 1$  is treated separately).  $w > 0 (< 0)$  corresponds to the future (past) cone. Equation (16) differs from the corresponding equation (47) - restricted to  $\gamma = 0$  - in [10]. Both equations are correct, but refer to different initial values  $\phi_0$ . We have replaced the CST equation in order to have  $\phi = \phi_0$  at the origin  $r = 0$ .

The cone generators depend on the two parameter  $\beta$  and  $\phi_0$ , which represent a possible pair of transversal coordinates for the light rays. It appears more useful to introduce (primarily for the past cone, but easily extended to the full cone) the two angular coordinates  $\theta, \varphi$  on the sky of a suitable observer at the vertex. The observer is assumed comoving with the cosmic fluid with the timelike velocity vector  $u^\mu = \delta_0^\mu$  (in the case of the Gödel metric with  $k^2 = 2$ ) or defined by the normed timelike eigenvector of the Ricci tensor in general. It then is not difficult to see (e.g. by using the method described in the second Appendix in [16]) that  $\beta$  and  $\phi_0$  are related to the coordinates  $\theta, \varphi$  (polar angle and longitude) on the observer sky by

$$\beta = \cos \theta, \quad \phi_0 = \varphi. \quad (19)$$

We note some well-known or easily accessible results. From (15) follows that null geodesics from the origin re-converge after reaching (for rays labeled  $\theta$ ) a maximal radial extension  $r_\theta$ , so we always have

$$r \leq r_\theta = \frac{2}{l} \operatorname{arsinh}\left(\frac{\sin \theta}{\sqrt{k^2 - \sin^2 \theta}}\right). \quad (20)$$

$r_\theta$  is zero for rays along the polar axis and in the opposite (antipode) direction ( $\theta = 0, \pi$ ) and reaches its largest value  $r_m$  for equatorial rays ( $\theta = \pi/2$ ). The hypersurfaces  $r = \text{const}$  are always timelike, in particular  $r = r_m$  is the so-called "light cylinder" or "optical horizon": Evidently, the spacetime region  $r > r_m$  cannot be reached by null geodesics from the origin.

At the horizon  $r = r_m$  the coefficient of  $d\phi^2$  in (3) is zero, thus the  $\phi$ -coordinate lines become closed lightlike (for  $r > r_{max}$ , timelike) lines. They are not geodesics,

however. A theory of *non-geodesic null curves* in a Minkowski spacetime was developed by Bonnor [7]. His approach translates immediately to curved spacetimes. A calculation shows that the closed null curves on the optical horizon are *null helices* with constant Bonnor curvatures  $k_1 = 1, k_2 = l(1 + k^2)/(4k), k_3 = 0$ .

### 3.2. Focal subsets and inner metric

The equations (14)-(17) supplemented by (19) map the intrinsic coordinates  $(w, \theta, \varphi)$  of the lightcone to the spacetime coordinates  $(t, r, \phi, z)$ . The critical points of this map are those where the Jacobian matrix has not the maximal rank 3. This happens if close cone generators intersect. For the critical or focal points all four subdeterminants of the Jacobian must vanish simultaneously:

$$\frac{\partial(r, \phi, z)}{\partial(w, \theta, \varphi)} = 0, \quad \frac{\partial(\phi, z, t)}{\partial(w, \theta, \varphi)} = 0, \quad \frac{\partial(z, t, r)}{\partial(w, \theta, \varphi)} = 0, \quad \frac{\partial(t, r, \phi)}{\partial(w, \theta, \varphi)} = 0. \quad (21)$$

A straightforward calculation of (21) leads to the condition  $f(w, \theta) = 0$  for the focal set, where

$$f(w, \theta) \equiv k^2 \sin w \cos^2 \theta + (k^2 - 1)w \cos w \sin^2 \theta. \quad (22)$$

Another way to find singularities is to look for higher degeneration of the induced lightcone metric. Numbering the inner coordinates as  $y^1 = w, y^2 = \theta, y^3 = \varphi$ , the intrinsic three-dimensional cone metric is determined by  $(i, k = 1 \dots 3)$

$$\gamma_{ik} = \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^k} g_{\mu\nu}. \quad (23)$$

$\gamma_{ik}$  is already degenerate of rank 2. A direct calculation shows that the only nonvanishing independent components are

$$\gamma_{22} = \frac{4(f^2 - 2fk^2 \cos^2 \theta \sin^3 w + k^2 q \cos^2 \theta \sin^4 w)}{l^2 \eta^3 \cos^2 w}, \quad (24)$$

$$\gamma_{23} = \frac{4k \sin \theta \cos \theta \sin^2 w (f - q \sin w)}{l^2 \eta^{5/2} \cos w}, \quad (25)$$

$$\gamma_{33} = \frac{4q \sin^2 \theta \sin^2 w}{l^2 \eta^2}. \quad (26)$$

To obtain these compact expressions we have introduced - besides the focal function  $f(w, \theta)$  - a non-negative function  $q(w, \theta)$ :

$$q(w, \theta) = (k^2 + \cos^2 \theta - 1) \cos^2 w + k^2 \cos^2 \theta \sin^2 w. \quad (27)$$

For later use we note that  $q$  is zero on some closed  $\varphi$ -coordinate lines, defined by  $\theta = \pi/2, w = (2n - 1)\pi/2, n = 0, \pm 1, \pm 2, \pm 3 \dots$  and arbitrary  $\varphi$  in the range  $(0, 2\pi)$ .

The determinant of the two-dimensional metric (24)-(26) can be written as square of a function  $h(w, \theta)$ :

$$\gamma_{22}\gamma_{33} - \gamma_{23}^2 = h^2, \quad (28)$$

$$h(w, \theta) = 4f(w, \theta) \sin \theta \sin w / (l^2 \eta^2). \quad (29)$$

Higher order degeneration of the metric requires  $h = 0$  and is therefore given by

- (i) the set of focal points  $f(w, \theta) = 0$ , where neighbouring light rays intersect,
- (ii) the set of points with  $w = n\pi$ ,  $n$  integer, called "keel" singularities in [16], where all rays with equal  $\theta$  and different  $\varphi$  meet in a point on the  $n$ th keel, a spacelike line of finite length, and
- (iii) the pole rays  $\theta = 0, \pi$ , resulting from the  $\sin \theta$ -factor, i.e. from the singularity of the polar coordinate system.

### 3.3. Newman-Penrose coefficients on the cone

Additionally to the intrinsic metric, the geometry of a null hypersurface may be described by some of the Newman-Penrose spin coefficients, mainly by divergence and shear and their change along a ray.

To illustrate this we first shortly consider a fairly known example, a generic null hypersurface in a Minkowski spacetime. Here the real divergence and complex shear evolve along a given ray according to the Penrose equations [36]

$$\rho = (\rho_0 + w[\sigma_0\bar{\sigma}_0 - \rho_0^2])/f, \quad \sigma = \sigma_0/f \quad (30)$$

with the focal function

$$f = 1 - 2w\rho_0 + w^2(\rho_0^2 - \sigma_0\bar{\sigma}_0). \quad (31)$$

$\rho_0$  and  $\sigma_0$  depend on the two transversal parameter fixing a ray. From (30) we have

$$\rho^2 - |\sigma|^2 = (\rho_0^2 - \sigma_0\bar{\sigma}_0)/f. \quad (32)$$

This equation shows that a parabolic point (a point with  $\rho^2 = |\sigma|^2$ ) on a Minkowskian ray implies that the whole ray consists of parabolic points - provided the denominator  $f$  in (32) does not vanish. The denominator vanishes and thus both  $\rho$  and  $|\sigma|$  diverge, if the affine parameter  $w$  takes one of the two values

$$w_f = 1/(\rho_0 \pm |\sigma_0|). \quad (33)$$

At each  $w_f$  a focal surface is passed, and the sign of  $\rho^2 - |\sigma|^2$  changes. The quotient  $j = \rho/|\sigma|$  remains finite, more exactly,  $j \rightarrow \pm 1$  at a focal point. One also notes that a focal point can be considered as degenerate parabolic point.

A similar behaviour of the first-order invariant  $j$  at caustics holds for the Ozsváth-Schücking plane wave lightcone [1] and was found in [16] for the lightcone of the Gödel metric. The difference is only that in both cases one meets an *unlimited* number of focal points if one moves along a ray. It is easy to see that this holds for Gödel-like metrics in general: If one starts from a cone metric  $\gamma_{AB}$  ( $A, B$  always run 2, 3) with  $h = \sqrt{\det|\gamma_{AB}|}$  and  $w$  as running (not necessarily affine) parameter on the generating rays, divergence and shear amount can be calculated from

$$\rho = -\frac{1}{2h} \frac{\partial h}{\partial w}, \quad |\sigma|^2 = \rho^2 - \det\left(\frac{\partial \gamma_{AB}}{\partial w}\right)/(4h^2). \quad (34)$$

Explicitly we find for the metric (24)-(26)

$$\rho = -\cot 2w - \frac{q}{2f \cos w}, \quad (35)$$



$$|\sigma|^2 = \rho^2 + \frac{k^2 \cos^2 \theta}{\eta} + \frac{\eta - 2q}{f \sin w}. \quad (36)$$

It is seen that both  $\rho$  and  $|\sigma|$  diverge at focal points  $f = 0$ . The quantity  $1/j^2 = |\sigma|^2/\rho^2$  measures the anisotropic part of distance change to neighbouring null geodesics along a given ray. It can be written as

$$j^{-2} = 1 + \frac{4f(\eta - 2q) \sin w \cos^2 w}{(f \cos 2w + q \sin w)^2} + \frac{4f^2 k^2 \cos^2 \theta \sin^2 w \cos^2 w}{\eta(f \cos 2w + q \sin w)^2}. \quad (37)$$

$1/j^2$  evidently goes to 1 for  $f \rightarrow 0$ , the same limit is reached at keel singularities  $w = n\pi$ .

### 3.4. Geometry of caustics

The affine parameter  $w$  gives rise to a foliation of the cone, but spacelike surfaces  $w = \text{const}$  have no invariant meaning since  $w$  is not uniquely determined. There exist however invariantly defined two-surfaces on the cone, e.g. the spacelike "zero divergence" surfaces. Here  $\rho = 0$ , and from (35) one obtains their equations as

$$-\frac{\tan 2w}{2w} = \frac{(k^2 - 1) \sin^2 \theta}{k^2 - 1 + (k^2 + 1) \cos^2 \theta}. \quad (38)$$

Since the *rhs* is not negative for the metrics considered here, such surfaces can only occur at points where  $\tan 2w/(2w)$  is negative or null, that is in the range  $(2m-1)\pi/2 \leq 2w \leq m\pi$ ,  $m = \pm 1, \pm 2, \pm 3, \dots$

Other invariantly defined subsets of the cone are the focal surfaces  $\mathcal{F}$  (described as "points of the second kind" in [32]). Their equation  $f = 0$  can be written as

$$-\frac{\tan w}{w} = \frac{k^2 - 1}{k^2} \tan^2 \theta, \quad (39)$$

thus focal surfaces occur at points with  $(2n-1)\pi/2 \leq w \leq n\pi$ ,  $n = \pm 1, \pm 2, \pm 3, \dots$ , where  $\tan w/w$  is negative. Contrary to the zero-divergence surfaces, focal surfaces are two-dimensional (finite and, as will be argued, non-geodesic) *null surfaces*: Solving (39) for  $\theta$  and introducing this function  $\theta_f(w)$  in (14)-(17), we obtain a parametric representation of  $\mathcal{F}$ . The intrinsic metric of the focal surface follows from (24)-(26) as

$$ds^2 = f_{ww}dw^2 + 2f_{w\varphi}dw d\varphi + f_{\varphi\varphi}d\varphi^2 \quad (40)$$

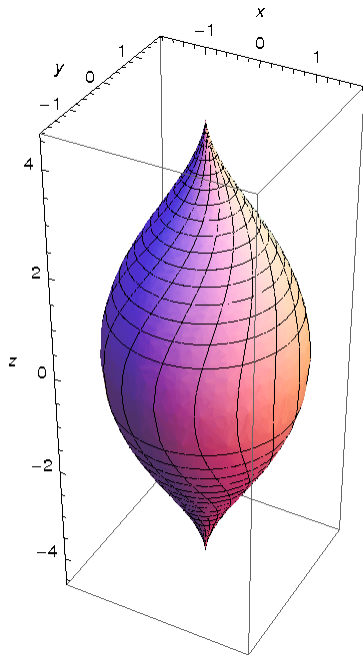
with

$$f_{ww} = \frac{T \sin^2 w (1 + T \cos^2 w)^3}{l^2 (k^2 - 1)^4 \cos^4 w (1 + T)^3 (k^2(1 + T) - 1)}, \quad (41)$$

$$f_{w\varphi} = -\frac{2T \sin^2 w (1 + T \cos^2 w)^2}{l^2 (k^2 - 1)^{5/2} (1 + T)^{5/2} \cos^2 w \sqrt{k^2(1 + T) - 1}}, \quad (42)$$

$$f_{\varphi\varphi} = \frac{4T \sin^2 w (1 + T \cos^2 w)}{l^2 (k^2 - 1) (1 + T)^2} \quad (43)$$

and  $T = -\tan w/w$ . The range of the coordinates  $w, \varphi$  for the  $n$ th focal surface is  $0 \leq \varphi \leq 2\pi$ ,  $(2n-1)\pi/2 \leq w \leq n\pi$ . Since  $f_{ww}f_{\varphi\varphi} - f_{w\varphi}^2 = 0$ , we have the metric of two-dimensional null surfaces, with metric components depending only on  $w$ .



**Figure 1.** The first focal surface  $\mathcal{F}_1$  of the  $l = 1$ ,  $k^2 = 2$  (Gödel) light cone is shown as projection into the 3-space  $t = \text{const}$  using spacetime coordinates  $x = r \cos \phi$ ,  $y = r \sin \phi$ ,  $z$ . The cusps at top and bottom lie on the exceptional rays and are the intersection points with the first keel. The projected keel is the part of the  $z$ -axis between top and bottom.- We add a comment of caution. The 3-space  $t = \text{const}$  with the metric tensor  $g_{ik}$  is a curved Riemannian  $V^3$ , while the plot must use the Euclidean  $R^3$ . Thus distances and angles cannot be represented correctly. In particular, distortions can occur if the map between  $V^3$  and  $R^3$  becomes singular, e.g. at points where  $\det |g_{ik}| = 4(\xi + (1 - k^2)\xi^2)/l^2$  is zero. This happens at  $\xi = 1/(k^2 - 1)$ , corresponding to the "equatorial line" of  $\mathcal{F}_1$ , the intersection of  $\mathcal{F}_1$  with the plane  $z = 0$ . The plot give the wrong impression that this line has nonzero length, while the true  $V^3$ -length is zero, cf. (44).

We consider the first focal surface  $n = 1$  in more detail.  $\mathcal{F}_1$  is shown in Fig. 1 as projection into the 3-space  $t = \text{const}$ , using  $(x = r \cos \phi, y = r \sin \phi, z)$  as spacetime coordinates. The surface is smooth except at  $w = \pi/2$  and  $w = \pi$ . At  $w = \pi/2$ ,  $f_{\varphi\varphi}$  tends to zero,  $f_{ww}$  to infinity, their product  $f_{ww}f_{\varphi\varphi}$  (or  $f_{w\varphi}^2$ ) is finite and equal to the constant  $\pi^4/(4l^4k^2(k^2 - 1))$ .

We may try to explain this geometrically. As noted above, the sign of the metric component  $g_{\phi\phi}$  in (1) decides whether the closed coordinate lines of  $\phi$  are spacelike or timelike. Calculated on the light cone,  $g_{\phi\phi}$  becomes  $\gamma_{33}$ , and calculated on a focal surface on the cone,  $\gamma_{33}$  becomes  $f_{\varphi\varphi}$ , the  $\phi$ -coordinate lines are  $\varphi$ -coordinate lines on  $\mathcal{F}_1$ . Contrary to  $g_{\phi\phi}$ ,  $\gamma_{33}$  and  $f_{\varphi\varphi}$  cannot become negative, they reach zero only at isolated points or lines. Apart from coordinate singularities, the zeros of  $\gamma_{33}$  are found at keels  $w = n\pi$  and as zeros of  $q$ .  $q$  becomes zero only for the  $\varphi$ -coordinate line at  $\theta = \pi/2$ ,  $w = \pi/2$  (or  $w = (2n - 1)\pi/2, n = 0, \pm 1, 2, 3, \dots$  at other focal surfaces  $\mathcal{F}_n$ ). This particular line is a closed null line, with zero length also from the viewpoint of the

cone geometry (it is one of the closed null curves on the optical horizon). The other  $\varphi$ -coordinate lines on  $\mathcal{F}_1$  (the "parallels" in Fig. 1) are spacelike for  $\pi/2 < w < \pi$ . Their total length

$$L(w) = \int_0^{2\pi} d\varphi \sqrt{f_{\varphi\varphi}} = \frac{8\pi \cos w \sin^3 w (\sin w \cos w - w)}{l^2(k^2 - 1)(w \cos w - \sin w)^2} \quad (44)$$

increases for  $w \geq \pi/2$  from zero to a maximum and declines to zero for  $w \rightarrow \pi$ , when the keel is reached. Here  $\mathcal{F}_1$  shrinks to the two cusp points, and all components  $f_{ww}, f_{w\varphi}, f_{\varphi\varphi}$  vanish.

Instead of slicing by  $\varphi$ -lines we can represent  $\mathcal{F}_1$  by the lines  $y^i = (w, \theta_f(w), \varphi = \text{const})$ , the twisted "meridians" in Fig.1. Their tangent vector is  $\frac{dy^i}{dw} = (1, \frac{d\theta_f}{dw}, 0)$ , with a norm given by  $(d\theta_f/dw)^2 \gamma_{22} = f_{ww} \geq 0$ . Thus these lines are spacelike except at the end points  $w = \pi/2, w = \pi$ .

One can construct a focal surface  $\mathcal{F}$  in a still different way. A two-dimensional null surface always admits a foliation by null lines: The equation  $f_{AB}f^B = 0$  has solutions  $f^A$  different from zero since  $\det|f_{AB}| = 0$ . The tangent lines to these directions can be taken as  $u$ -coordinate lines of a new  $(u, v)$  coordinate system on  $\mathcal{F}$ . In  $(u, v)$  coordinates the inner metric of a two-dimensional null surface is represented by the normal form

$$ds^2 = F(u, v)dv^2. \quad (45)$$

Explicitly, the transformation from  $(w, \varphi)$  to  $(u, v)$  is given for  $\mathcal{F}_1$  by

$$u = w, \quad (46)$$

$$v = \varphi - \int_w^\pi \frac{dw(w - \sin w \cos w)}{\cos w \sqrt{\sin w - w \cos w} \sqrt{k^2(\sin w - w \cos w) + w \cos w}} \quad (47)$$

and the metric function  $F(u, v)$  depends only on  $u = w$ :

$$F = \frac{4T \sin^2 w (1 + T \cos^2 w)}{l^2(k^2 - 1)(1 + T)^2}. \quad (48)$$

The  $u$ -coordinate lines on  $\mathcal{F}_1$  have zero lengths and can therefore be denoted as null lines, but they are different from the null geodesic generators of the light cone. From the four-dimensional viewpoint they are non-geodesic null curves. It should not be too difficult to develop a theory of such curves on null hypersurfaces, analogously to Bonnor's theory in [7].

A way to visualize focal surfaces on the past cone is to locate them on the observer sky. We may think of radiation emitted from different parts of the focal surface. If we walk down the cone into the past with an increasing affine parameter  $|w|$ , after passing the first zero-divergence surface at  $|w| = \pi/4$ , the first focal surface ( $n = 1$ ) starts at  $|w| = \pi/2$ . Radiation from caustic points at this epoch would appear to the observer as luminous ring along the celestial equator  $\theta = \pi/2$ . For larger  $|w|$  the focal surface radiation comes in as two luminous parallels, moving from the equator towards the poles. The pole ( $\theta = 0$ ) and its antipode ( $\theta = \pi$ ) are reached for  $|w| = \pi$ , marking the two singular end points of the focal surface (seen as cusps in Fig.1). The cusps are also intersections of the focal surface with the two exceptional rays on the past cone (these

rays are in the Gödel case related to the rotation direction and its antipode direction [16], and present also for  $k^2 \neq 2$ ).

Keels (denoted as "points of the first kind" in [32]) are another example of invariantly defined subsets on the cone. The keels  $w = n\pi$ , parametrized by  $\theta$ , are pieces of spacelike lines connecting the singular end points of the corresponding focal surface. At each keel point labeled with  $\theta$  all rays with the same  $\theta$  and different  $\varphi$  intersect. The light cone metric further degenerates at keels and becomes a matrix of rank 1, only  $\gamma_{22} = 4f^2/(l^2\eta^3)$  differs in general from zero. At the two end points, where the keel meets the corresponding focal surface, the matrix rank of  $\gamma_{ik}$  is zero, all components  $\gamma_{ik}$  vanish here. The keel appears as second vertex in representations which suppress the  $z$ -coordinate, e.g. in the well-known figure of the Gödel cone in the Hawking-Ellis monograph [26]. But taken all dimensions into account, the keel is an extended spacelike line, which shrinks to a point only in the  $k^2 \rightarrow 1$  limit of the Gödel family. The invariant length of the  $n$ th keel is given by

$$\int_0^\pi d\theta \sqrt{\gamma_{22}} = \frac{2n\pi\sqrt{k^2-1}}{l} \left( E\left(\frac{1}{1-k^2}\right) - K\left(\frac{1}{1-k^2}\right) \right), \quad (49)$$

with  $E$  and  $K$  as complete elliptic integrals.

As known from the Gödel universe or the lightcone of the Ozsváth-Schücking anti-Mach metric [32], focal surfaces, keels and zero-divergence surfaces occur *quasi-periodic*, due to the fact that the focal function  $f(w, \theta)$  is not strictly periodic in  $w$ , while all other functions are *circular* functions of the affine parameter.

### 3.5. Differential invariants

Another quantitative description of null hypersurfaces is provided by their intrinsic differential invariants. The quantity  $j$  defined as quotient of divergence and shear is already an invariant, it is the only invariant depending exclusively on the *first* derivatives of the cone metric. A comment is necessary here. While  $|\sigma|$  is always not negative by definition,  $\rho$  changes the sign when the ray passes a caustic (focal surface or keel), and  $j$  is  $+1$  or  $-1$  before and behind this point. Thus our formal definition produces jumps in  $j$  at these points, as written down without further explanation in [16] for the Gödel cone. This suggests to redefine the first order invariant as  $\tilde{j} = \lambda\rho/|\sigma|$ ,  $\lambda^2 = 1$ , with appropriately chosen  $\lambda = f/|f|$  or  $\lambda = \text{sgn}(\rho^2 - |\sigma|^2)$ , to ensure that  $\tilde{j}$  is a continuous function through caustics. For example, for the Minkowski space null hypersurfaces we have  $\tilde{j} = j_0 + w(1 - j_0^2)|\sigma_0|$  as smooth function of  $w$ , while  $j = \rho/|\sigma|$  shows the unnatural discontinuity at the two focal surfaces. Nevertheless we keep  $j$  as abbreviation for  $\rho/|\sigma|$ .

Besides  $j$  there exist higher-order invariants [14], [15], [31]. For their calculation we use a triad formalism [16]. The degenerate inner metric of the cone can be represented by

$$\gamma_{ik} = t_i \bar{t}_k + \bar{t}_i t_k, \quad (50)$$

where  $t_i$  is a complex covariant vector intrinsic to the cone. The generator direction  $\epsilon^i$

satisfies  $\gamma_{ik}\epsilon^k=0$ . To obtain a complete co- and contravariant triad on the cone we add further vectors  $t^i, \bar{t}^i, \gamma_i$  such that

$$t_i t^i = 0, \quad t_i \bar{t}^i = 1, \quad \gamma_i t^i = 0, \quad \gamma_i \epsilon^i = 1. \quad (51)$$

We use adapted inner cone coordinates  $y^1 = w, y^2 = \theta, y^3 = \varphi$  with  $\epsilon^i = \delta_1^i$ . The degenerate metric  $\gamma_{ik}$  then reduces to the two-dimensional metric  $\gamma_{AB}$ . Comparison with (24-26) gives (together with  $t_1 = 0$ , and up to a rotation)

$$t_2 = i \frac{f}{l\eta\sqrt{q/2}} + \frac{k \cos \theta \sin w}{l\eta^{3/2} \cos w \sqrt{q/2}} (f - q \sin w), \quad (52)$$

$$t_3 = \sin \theta \sqrt{2q} \sin w / (l\eta), \quad (53)$$

The contravariant components  $t^i$  are calculated from  $t_i = \gamma_{ik} t^k$ , the result is (besides  $t^1 = 0$ )

$$t^2 = il\eta\sqrt{2q}/(4f), \quad (54)$$

$$t^3 = \frac{l\eta}{4 \sin \theta \sin w \sqrt{q/2}} + i \frac{kl \cos \theta \sqrt{2\eta}}{4f \sin \theta \cos w \sqrt{q}} (q \sin w - f). \quad (55)$$

Rotation coefficients related to this triad and of relevance here can now be obtained from

$$\rho + i\nu = -t^2 \bar{t}_{2,1} - t^3 \bar{t}_{3,1}, \quad (56)$$

$$\sigma = -\bar{t}^2 t_{2,1} - \bar{t}^3 t_{3,1}, \quad (57)$$

$$\tau = (t^2 t^3 - t^2 \bar{t}^3)(\bar{t}_{2,3} - \bar{t}_{3,2}). \quad (58)$$

One may verify that the expressions for  $\rho$  and  $|\sigma|$  obtained from (56,57) agree with (35) and (36). As noted, the components of  $t_i, \bar{t}^i$  are not uniquely determined. This affects some rotation coefficients, but not  $\rho, |\sigma|$  and also not the invariants. The freedom could (but will not here) be used to reach, e.g.,  $\nu = 0$  in (56). For our choice of the triad the real and imaginary part of the complex shear is given by

$$\Re(\sigma) = \frac{2\eta \cos^2 w - q(1 + 2 \cos^2 w)}{q \sin 2w} + \frac{q}{2f \cos w}, \quad (59)$$

$$\Im(\sigma) = \frac{k(k^2 - 1) \cos \theta \sin^2 \theta \sin^2 w}{q\sqrt{\eta}}. \quad (60)$$

A null hypersurface has in general four second-order differential invariants of the inner geometry, written as complex quantities  $I$  and  $J$  and conveniently expressed in terms of rotation coefficients [14]. The quantity  $I$  is linear in the second derivatives of the metric, with derivatives only along the generators and, like  $j$ , dimensionless:

$$I = \frac{i}{|\sigma|} \left( \frac{D\rho}{\rho} - \frac{D\sigma}{\sigma} \right) + 2 \frac{\nu}{|\sigma|}. \quad (61)$$

Explicitly we find for the real part  $I_1$

$$|\sigma|^3 I_1 = \frac{2k(k^2 - 1) \cos \theta \sin^2 \theta (\eta \sin w - f)}{f\eta^{3/2}}. \quad (62)$$

The imaginary part  $I_2$  has a more complicated structure:

$$|\sigma|^3 I_2 = \frac{i_0 + i_1 f + i_2 f^2 + i_3 f^3}{2\eta f^2 \sin^2 w \cos w (f \cos 2w + q \sin w)}. \quad (63)$$

A tedious but straightforward calculation shows that

$$i_0 = \eta q^2 \sin^2 w (2q - \eta), \quad (64)$$

$$i_1 = 2 \sin w \left( -q^3 + \eta q^2 (\sin^2 w - 3) + \eta^2 q (5 - 4 \sin^2 w) - 2\eta^3 \cos^2 w \right), \quad (65)$$

$$i_2 = 4q^2 - 2q\eta \cos 2w - \eta^2, \quad (66)$$

$$i_3 = -2k^2 \cos^2 \theta \sin w. \quad (67)$$

$I_2 = Dj/\rho$  describes the change of the first-order quantity  $j$  along the rays.  $I_1$  is a measure for the rotation of the shear directions (i.e. directions where the distance change to neighbouring rays is a maximum or minimum) relative to the generator congruence. If  $I_1$  is zero (as for null hypersurfaces in a Minkowski or conformally related spacetime), the shear directions always point to the *same* neighbouring null rays if one follows a ray.

The complex invariant  $J$  has the dimension  $(length)^{-1}$ , is nonlinear in the second-order derivatives of the inner metric and involves additionally transversal derivatives [14].  $J$  describes changes of the nullsurface geometry in transversal directions, but is considerably more complicated than  $I$  and will be discussed elsewhere.

The behaviour of invariants at and in the neighbourhood of focal singularities is of interest. While the rotation coefficients  $\rho$  and  $\sigma$  show singularities, the invariants tend to have finite values. We have already noted  $j \rightarrow \pm 1$  at focal points and keels. Expanding  $I$  near  $f = 0$  in powers of  $f$  leads to

$$I_1 = \frac{16k(k^2 - 1) \cos \theta \sin^2 \theta \sin w \cos^3 w}{q^3 \sqrt{\eta}} f^2 + o(f^3), \quad (68)$$

$$I_2 = \frac{4 \cos^2 w (\eta - 2q)}{q^2 \sin w} f + o(f^2). \quad (69)$$

Remarkably, the second-order differential invariants  $I_1, I_2$  vanish at focal points. This also holds at keels  $w = n\pi$ : Writing  $w - n\pi = x$ , one obtains for small  $x$

$$I_1 = 16(-1)^{n+1} k(k^2 - 1) \eta^{-3/2} \cos \theta \sin^2 \theta x^3 + o(x^4), \quad (70)$$

$$I_2 = \frac{4(-1)^n \eta x}{n\pi(k^2 - 1) \sin^2 \theta} + o(x^2). \quad (71)$$

For comparison we note that null hypersurfaces in a Minkowski spacetime satisfy  $I_2 = 1/j - j$ , thus  $I_2$  vanishes at caustics,  $I_1$  is already zero everywhere.

### 3.6. Comments on the Gödel case as treated in [16]

For the Gödel cone ( $k^2 = 2$ ) some differential invariants have been calculated already in [16]. The present paper uses different four-dimensional coordinates as well as different transversal light cone coordinates  $y^A$ . The latter is motivated by the topological fact that

no coordinate system can cover the whole sphere without singularity. The coordinates  $u, v$  in [16] avoid a singularity in the direction of the rotation axes, they become singular in equator directions instead. The polar angles  $\theta, \phi$  here avoid the equator singularities but show the usual pole singularities. The relation between both systems of transversal coordinates is given by  $\cos \phi = (1 - u^2)/(1 + u^2)$ ,  $\sin \theta = \sqrt{2}(v^2 - 1)/(v^2 + 1)$ . For  $k^2 = 2$ , eqn. (39) thus becomes the focal equation  $-\tan w/w = (v^2 - 1)^2/(6v^2 - 1 - v^4)$ , eqn. (81) in [16].- We note a misprint in eqn. (48) of [16]: the denominator should read  $f_2 + 4(1 + f_1)$  instead of  $f_2$ .

#### 4. The Rebouças-Tiomno $G_7$ metric $k^2 = 1$

The case  $k^2 = 1$  was excluded so far, we treat it separately. Rebouças and Tiomno introduced this special case as "the first exact Gödel-type solution of Einstein's equations describing a completely causal spacetime-homogeneous rotating universe" [40]. The lightcone becomes very simple in this model. Since the Weyl tensor vanishes for  $k^2 = 1$  (see (6)), the spacetime metric is conformal to the Minkowski spacetime, thus also the lightcone metric is conformal to the Minkowski cone metric. One obtains in the limit  $k^2 \rightarrow 1, f \rightarrow \sin w \cos^2 \theta, \eta \rightarrow \cos^2 \theta, q \rightarrow \cos^2 \theta$  of preceding fomulae:

$$\gamma_{22} = \frac{4 \sin^2 w}{l^2 \cos^2 \theta}, \quad \gamma_{23} = 0, \quad \gamma_{33} = \frac{4 \sin^2 w}{l^2} \tan^2 \theta. \quad (72)$$

The square  $h$  of the determinant  $|\gamma_{AB}|$ ,

$$h = 4 \sin \theta \sin^2 w / (l^2 \cos^2 \theta), \quad (73)$$

vanishes at the points  $w = n\pi$  (the only other zeros correspond to the coordinate singularity). All light rays from the vertex  $w = 0$  meet again at the points  $w = n\pi$  ( $n$  integer), which are also vertices. Thus every pair of focal surface and keel in the  $k^2 > 1$  family of metrics has collapsed into a single vertex in the limit  $k^2 \rightarrow 1$ . The shear of the cone vanishes, only the divergence differs from zero:

$$\rho = -\cot w. \quad (74)$$

$\rho$  increases from  $-\infty$  at  $w = 0$  to zero at  $w = \pi/2$  and decreases again until  $-\infty$  at the next vertex  $w = \pi$ . The lightcone belongs to a type of null hypersurfaces characterized by  $\rho \neq 0, |\sigma| = 0$  and denoted as "type 5" in the classification of [15]. There exist no second-order inner differential invariants for this class.

The high symmetry of the Rebouças-Tiomno metric is reflected by the existence of a symmetry group  $G_7$  [51], see also [44] for further discussions.

#### 5. Static degeneration $k^2 \rightarrow 0, l$ finite

The limit  $k \rightarrow 0$ , keeping  $l$  finite, requires  $\Omega \rightarrow 0$ . It represents the static degeneration of the Gödel family and has the simple line element

$$ds^2 = -dt^2 + dr^2 + \frac{\sinh^2(lr)}{l^2} d\theta^2 + dz^2. \quad (75)$$

Teixera, Rebouças and Aman have shown that this metric admits a six-parameter group of motions  $G_6$  [51]. The only nonvanishing components of the Ricci tensor are  $R_{rr} = -l^2$  and  $R_{\theta\theta} = -\sinh^2(lr)$  with a constant Ricci scalar  $R = -l^2$ , and the Riemann tensor has only one independent nonvanishing component  $R_{r\theta r\theta} = -\sinh^2(lr)$ . Thus the metric cannot easily be interpreted as solution of the field equations, it is nevertheless interesting geometrically because of its high symmetry. The null geodesics starting at the origin  $r = 0$ ,  $z = 0$  are given by (here  $w > 0$  corresponds to the past cone)

$$t = -w, \quad r = w \sin \theta, \quad \phi = \phi_0, \quad z = w \cos \theta. \quad (76)$$

The equations show that the cone generators do not re-converge as generally for the Gödel family but extend to null infinity as in the Minkowski spacetime (Minkowski is included for  $l \rightarrow 0$ ). No focal surface or keel exist for finite  $w$ . Nevertheless the cone geometry and in particular the asymptotic behaviour of the cone significantly differ from Minkowski. The cone metric with the intrinsic coordinates  $y^1 = w$ ,  $y^2 = \theta$ ,  $y^3 = \varphi$  is given by

$$\gamma_{22} = w^2, \quad \gamma_{23} = 0, \quad \gamma_{33} = \frac{\sinh^2(lw \sin \theta)}{l^2}, \quad (77)$$

thus the determinant  $|\gamma_{AB}|$  is the square of the function

$$h = \frac{w}{l} \sinh(lw \sin \theta). \quad (78)$$

Hence, apart from the vertex  $w = 0$  and the coordinate singularity on the symmetry axis, there exist no further singularities on the cone. Divergence and shear of the rays follow as

$$\rho = -\frac{1}{2w} - \frac{l \sin \theta}{2} \coth(lw \sin \theta), \quad (79)$$

$$\sigma = \bar{\sigma} = \frac{1}{2w} - \frac{l \sin \theta}{2} \coth(lw \sin \theta). \quad (80)$$

The divergence increases from  $-\infty$  at the vertex  $w = 0$  to  $-l \sin \theta/2$  for  $w \rightarrow \infty$ , if one goes down the past lightcone, thus it always remains negative. The (real) shear starts with zero at the vertex, becomes negative for increasing  $w$  and reaches the same negative limit  $-l \sin \theta/2$  as the divergence for  $w \rightarrow \infty$ . One also has

$$\rho^2 - |\sigma|^2 = \frac{l \sin \theta}{w} \coth(lw \sin \theta), \quad (81)$$

which is always positive, thus the lightcone consists exclusively of elliptic points.

The second-order invariants  $I_1$  and  $J$  are zero, but  $I_2$  is different from zero and given by

$$I_2 = \frac{4lwS \sin \theta (CS - lw \sin \theta)}{(lwC \sin \theta + S)(lwC \sin \theta - S)^2} \quad (82)$$

with  $S \equiv \sinh(lw \sin \theta)$ ,  $C \equiv \cosh(lw \sin \theta)$ . One verifies that asymptotically, for  $w \rightarrow \infty$ ,  $j \rightarrow -1$ ,  $I_2 \rightarrow 0$ , which are standard limiting values at caustics.



## 6. Cyclic structure on general null hypersurfaces

At the end of their pioneering paper [32], Ozsváth and Schücking ask for the origin of the periodicity structure. A certain answer can be given by going back to the Sachs equations [47], the differential equations governing divergence and shear on a null hypersurface  $\mathcal{N}$  in terms of Ricci and Weyl tensor projections into  $\mathcal{N}$ :

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \omega, \quad (83)$$

$$D\sigma = 2(\rho - i\nu)\sigma + \psi, \quad (84)$$

with

$$D = p^\mu \partial_\mu = \frac{\partial}{\partial w}, \quad \rho = -p_{\mu;\nu} \bar{t}^\mu t^\nu, \quad \nu = i \bar{t}_{\mu;\nu} \bar{t}^\mu p^\nu, \quad \sigma = -p_{\mu;\nu} \bar{t}^\mu \bar{t}^\nu, \quad \tau = -\bar{t}_{\mu;\nu} \bar{t}^\mu \bar{t}^\nu, \quad (85)$$

$$\omega = \frac{1}{2} R_{\mu\nu} p^\mu p^\nu, \quad \psi = C_{\mu\nu\rho\sigma} p^\mu \bar{t}^\nu p^\rho \bar{t}^\sigma. \quad (86)$$

The null vector  $p^\mu$  is the direction of the cone generators, the complex null vector  $t^\mu$  spans spacelike directions in  $\mathcal{N}$  orthogonal to  $p^\mu$ . It should be stressed that  $\omega$  and  $\psi$  - in spite of their origin as projections of four-dimensional quantities - depend only on the metric  $\gamma_{ik}$ , they are *objects of the inner geometry of  $\mathcal{N}$* . The rotation coefficients  $\rho, \nu, \sigma, \tau$  defined in (85) agree with those calculated from (56)-(58) for the Gödel family.

The Sachs equations are the first equations to be solved on  $\mathcal{N}$  for a characteristic initial value problem based on the Newman-Penrose formalism and starting from  $\mathcal{N}$ . Thus in a sense they can be considered as the Einstein field equations in a nutshell, being nonlinear and ruling the influence of matter ( $\omega$ ) on the nullsurface geometry. The Penrose equations (30) follow as solution of the Sachs equations in the absence of matter and for a vanishing Weyl tensor ( $\omega = \psi = 0$ ). For the lightcone of the Gödel family, calculation of  $\omega$  and  $\psi$  gives

$$\omega = 2 - \frac{k^2 \cos^2 \theta}{(k^2 - \sin^2 \theta)}, \quad (87)$$

$$\Re(\psi) = \frac{2(k^2 - 1) \sin^2 \theta}{q\eta} (-q + 2\eta \cos^2 w), \quad (88)$$

$$\Im(\psi) = \frac{4k(k^2 - 1) \cos \theta \sin^2 \theta \cos w \sin w}{q\sqrt{\eta}}. \quad (89)$$

The solution of the Sachs equations with these right-hand-sides is given by (35) and (59,60).

From the Sachs equations one can derive a differential equation for an area distance  $r$  (not to be confused with the radial coordinate  $r$ ), introduced by  $Dr = -\rho r$ . One obtains with  $Q = |\sigma|^2 + \omega$  the Jacobi equation

$$DDr + Qr = 0. \quad (90)$$

The caustics are found as zeros of  $r$ . For the Gödel family from (87)  $\omega > 0$ , hence  $Q > 0$ .

The existence of cyclic focal features on many null hypersurfaces (not only cones) may then be considered as property of the Jacobi equation, or, more concretely, as property of the function  $Q$ . The linear second-order differential equation (90) belongs to the most widely studied equations in applied mathematics. Starting with the classical papers by Sturm, Liouville and Kneser in the nineteenth century, there exist many theorems which indeed prove a cyclic or oscillatory behaviour (with arbitrarily large numbers of zeros of  $r$ ) for certain functions  $Q > 0$ . For  $Q < 0$  the solutions are non-oscillatory, but this holds also for some  $Q > 0$ , e.g. for the Minkowski space caustics. The precise dependence of the oscillation feature on properties of  $Q$  is an open mathematical problem, see [53].

## Acknowledgments

This work began as part of a collaborative project with M. Abdel-Megied. I thank him for the reference to the paper by Calvão, Soares and Tiomno [10] and for many useful discussions. I am also grateful to J. Åman and M.A.H. MacCallum for comments on an invariant characterization of Gödel-like metrics.

## References

- [1] Abdel-Megied M 1972 Über die Lichtkegel in speziellen kosmologischen Modellen, PhD thesis, Humboldt University at Berlin
- [2] Abdel-Megied M and Dautcourt G 1972 Zur Struktur des Lichtkegels im Gödel-Kosmos *Mathematische Nachrichten* **54** 33–39
- [3] Arnold V I, Gusein-Zade S M and Varchenko A N 1985 *Singularities of Differentiable Maps* Vol. I (Boston, Basel, Stuttgart: Birkhäuser)
- [4] Bampi F and Zordan C 1978 A note on Gödel's metric *General Relativity and Gravitation* **9** 393
- [5] Barrow J D and Tsagas C G 2004 Dynamics and stability of the Gödel universe *Class Quantum Grav* **21** 1773–90
- [6] Beem J K, Ehrlich P E and Easley K L 1996 *Global Lorentzian Geometry* (New York: Marcel Dekker)
- [7] Bonnor W B 1969 Null Curves in a Minkowski Space-Time, *Tensor* **20** 229–42
- [8] Boyda E K, Ganguli S, Horova P and Varadarajan U 2003 Holographic protection of chronology in universes of the Gödel type *Phys Rev D* **67** 106003
- [9] Calvão M O, Rebouças M J, Teixeira A F F and Silva W M 1988 Notes on a class of homogeneous spate-times *J. Math. Phys.* **29** 1127–29
- [10] Calvão M O, Soares I D and Tiomno J 1990 Geodesics in Gödel-type Space-times *General Relativity and Gravitation* **22** 683–705
- [11] Chakraborty S K and Bandyopadhyay N 1983 Gödel-type universe with a perfect fluid and a scalar field *J. Math. Phys.* **24** 129–32
- [12] Clifton T and Barrow J D 2005 The Existence of Gödel, Einstein and de Sitter Universes *Phys Rev D* **72** (2005) 123003
- [13] Corkill R W and Stewart J M 1983 Numerical Relativity. II. Numerical methods for the characteristic initial value problem and the evolution of the vacuum field equations for space-times with two Killing vectors *Proc. R. Soc. London A* **386** 373–91
- [14] Dautcourt G 1967 Characteristic hypersurfaces in general relativity *J. Math. Phys.* **8** 1492–501

- [15] Dautcourt G 1980 Isotropic hypersurfaces in general relativity admitting groups of motions *Acta Phys. Pol.* **B 11** 791–807
- [16] Dautcourt G and Abdel-Megied M 2006 Revisiting the lightcone of the Gödel universe *Class Quantum Grav* **23** 1269–88
- [17] Ehlers J and Newman E T 2000 The theory of caustics and wave front singularities with physical applications *J. Math. Phys.* **41** 3344–78
- [18] Ehrlich P E 2006 A Personal Perspective on Global Lorentzian Geometry *Lect. Notes Phys.* **692** 3–34
- [19] Friedrich H and Stewart J 1983 Characteristic initial data and wave front singularities in general relativity *Proc. R. Soc. London A* **385** 345–71
- [20] Frittelli S and Newman E T 1999 An Exact Universal Gravitational Lensing Equation *Phys. Rev. D* **59** 124001
- [21] Frittelli S, Newman E T, Silva-Ortigoza G 1999 The Eikonal equation in flat space: Null surfaces and their singularities I *J. Math. Phys.* **40** 383–407
- [22] Frittelli S and Petters A O 2003 Wavefronts, Caustic Sheets, and Caustic Surfing in Gravitational Lensing *J. Math. Phys.* **43** 5578–611
- [23] Gödel K 1949 An example of a new type of cosmological solution of Einstein’s field equations of gravitation *Rev Mod Phys* **21** 447-50
- [24] Gödel K 1952 Rotating universes *Proc Int Cong Math* (Camb, Mass). Ed. L. M. Graves et al. **1**, 175
- [25] Harmark T and Takayanagi T 2003 Supersymmetric Gödel Universes in String Theory *Nucl. Phys.* **B 662** 3–39
- [26] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press)
- [27] Kling T P and Newman E T 1999 Null cones in Schwarzschild geometry *Phys. Rev. D* **59** 124002
- [28] Kundt W 1956 Trägheitsbahnen in einem von Gödel angegebenen kosmologischen Modell *Zs f Ap* **145** 611-20
- [29] Laurent B E, Rosquist K and Sviestins E 1981 The Behaviour of Null Geodesics in a Class of Rotating Space-Time Homogeneous Cosmologies *General Relativity and Gravitation* **13** 1093 **9** 393
- [30] Novello M, Damião Soares and Tiomno J 1983 Geodesic motion and confinement in Gödel’s universe *Phys Rev D* **27** 779–88 Erratum *Phys Rev D* **28** 1561
- [31] Nurowski P, Robinson D C 2000 Intrinsic Geometry of a Null Hypersurface *Class. Quantum Grav.* **17** 4065–84
- [32] Oszváth I and Schücking E 1962 An anti-Mach Metric, in *Recent Developments in General Relativity* (New York: Pergamon Press) 339–50
- [33] Oszváth I 1970 Dust-Filled Universes of Class II and Class III *J. Math. Phys.* **11** 2871-83
- [34] Oszváth I and Schücking E 2001 Approaches to Gödel’s rotating universe *Class. Quantum Grav.* **18** 2243–52
- [35] Oszváth I and Schücking E 2003 Gödel’s trip *Am. J. Phys.* **71** 801-05
- [36] Penrose R 1961 Null Hypersurface Initial Data for Classical Fields of Arbitrary Spin and for General Relativity, published as Golden Oldie *Gen. Rel. Grav.* **12** 225
- [37] Penrose R 1965 A remarkable property of plane waves in general relativity *Rev. Mod. Phys.* **37** 215
- [38] Perlick V 2004 Gravitational Lensing from a Spacetime Perspective *Living Rev. Relativity* **7**, **9**, <http://www.livingreviews.org/lrr-2004-9>
- [39] Raychaudhuri A K and Guha Thakurta S N 1980 Homogeneous space-times of the Gödel type *Phys. Rev.* **22** 802–06
- [40] Rebouças M J and Tiomno J 1983 Homogeneity of Riemannian space-times of Gödel type *Phys. Rev.* **28** 1251–64
- [41] Rebouças M J, Áman J E and Teixeira A F F 1986 A note on Gödel-type space-times *J. Math.*

- Phys.* **27** 1370-72
- [42] Rebouças M J and Áman, J E 1987 Computer-aided study of a class of Riemannian space-times *J. Math. Phys.* **28** 888-92
- [43] Rebouças M J and Santos J 2009 Gödel-type universes in  $f(R)$  gravity *Phys. Rev. D* **28** 063009
- [44] Rebouças M J and Teixeira A F F 1986 Features of a relativistic space-time with seven isometries *Phys. Rev. D* **34** 2985-89
- [45] Rebouças M J and Teixeira A F F 1992 Homogeneous space-times with seven isometries *J. Math. Phys.* **33** 2855-62
- [46] Rooman M and Spindel Ph 1998 Gödel metric as a squashed anti-de Sitter geometry *Class. Quantum Grav.* **15** 3241-49
- [47] Sachs R 1961 Gravitational waves in general relativity VI. The outgoing radiation condition *Proc. R. Soc. London A* **264** 309-38
- [48] Schneider P, Ehlers J and Falco E E 1992 Gravitational Lenses (New York, Berlin, Heidelberg: Springer-Verlag)
- [49] Som M M and Raychaudhuri A K 1968 Cylindrically symmetric charged dust distribution in rigid rotation in general relativity *Proc Roy Soc A* **304** 85-90
- [50] Stephani H, Kramer D, MacCallum M, Hoenselaers C, Herlt E 2003 *Exact Solutions of Einstein's Field Equations* (Second Edition) (Cambridge University Press)
- [51] Teixeira A F F, Rebouças M J and Áman J E 1985 Isometries of homogeneous Gödel type spacetimes *Phys. Rev. D* **32** 3309-10
- [52] Winicour J 2005 Characteristic Evolution and Matching *Living Rev. Relativity* **8**, 10, <http://www.livingreviews.org/lrr-2005-10>
- [53] Zetl A 2005 *Sturm-Liouville Theory* (AMS Mathematical Surveys and Monographs **121**)