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## Turning big bang into big bounce: II. Quantum dynamics

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### Abstract

We analyze the big bounce transition of the quantum FRW model in the setting of the non-standard loop quantum cosmology (LQC). Elementary observables are used to quantize composite observables. The spectrum of the energy density operator is bounded and continuous. The spectrum of the volume operator is bounded from below and discrete. It has equally distant levels defining a quantum of the volume. The discreteness may imply a foamy structure of spacetime at semiclassical level which may be detected in astro-cosmo observations. The nonstandard LQC method has a free parameter that should be fixed in some way to specify the big bounce transition.

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## I. INTRODUCTION

It is commonly expected that the cosmological singularity (CS) problem [1–4] may be resolved in a theory which unifies gravity and quantum physics. It seems that recent developments concerning quantization of cosmological models by making use of *loop geometry* may bring solution to the problem. It consists in turning the classical *big bang* singularity into *big bounce*, BB, transition. There exist two methods to address the issue: *standard* loop quantum cosmology (LQC) and *nonstandard* LQC. The former has been developed during the last decade (see [5, 6] and references therein) and has been inspired by the loop quantum gravity, LQG (see [7–9] and references therein). The latter has been proposed recently [10–13] and seems to be related to the reduced phase space quantization of LQG [14].

The standard LQC means basically the Dirac method of quantization, which begins with quantization of the *kinematical* phase space followed by imposition of constraints of the gravitational system in the form of operators at the quantum level. Finding kernels of these operators helps to define the *physical* Hilbert space. In the nonstandard LQC (our method) one first solves all the constraints at the classical level to identify the *physical* phase space. Next, one identifies the algebra of *elementary* observables (in the physical phase space) and finds its representation. Then, *composite* observables are expressed in terms of elementary ones and quantized. Final goal is finding *spectra* of composite observables which are used to examine the *nature* of the BB phase in the evolution of the universe.

In what follows we restrict our considerations to the flat Friedmann-Robertson-Walker (FRW) model with massless scalar field. This model of the universe includes the initial cosmological singularity and has been intensively studied recently within the standard LQC.

In our recent paper [10] we have argued that the resolution of the singularity offered by LQC requires specific value of the fundamental *length*. The size of this length has not been determined satisfactory yet. Present paper is an extended version of [11] and together with the classical formalism [12] specifies our nonstandard LQC (when applied to FRW model).

In order to have our paper self-contained, we recall in Sec. II some aspects of the classical formalism of our nonstandard LQC [12]. In Sec. III we present our quantization procedure. It consists in finding representation for composite observables like the energy density and the volume function, and calculating their spectra. We conclude in the last section.

## II. CLASSICAL LEVEL

### A. Hamiltonian

The gravitational part of the classical Hamiltonian,  $H_g$ , of the flat FRW model may be presented in the form [12]

$$H_g = \lim_{\mu \rightarrow 0} H_g^{(\mu)}, \quad (1)$$

where

$$H_g^{(\mu)} = -\frac{\text{sgn}(p)}{2\pi G\gamma^3 \mu^3} \sum_{ijk} N \varepsilon^{ijk} \text{Tr} \left( h_i^{(\mu)} h_j^{(\mu)} (h_i^{(\mu)})^{-1} (h_j^{(\mu)})^{-1} h_k^{(\mu)} \{ (h_k^{(\mu)})^{-1}, V \} \right), \quad (2)$$

and where  $V = |p|^{\frac{3}{2}} = a^3 V_0$  is the volume of the elementary cell  $\mathcal{V} \subset \Sigma$  ( $\Sigma$  is spacelike hypersurface);  $p$  is a canonical variable; the metric of  $k = 0$  FRW model is:  $ds^2 = -N^2(t) dt^2 +$

$a^2(t) (dx^2 + dy^2 + dz^2)$ , where  $a$  is the scale factor and  $N$  denotes the lapse function;  $\varepsilon^{ijk}$  is the alternating tensor;  $\gamma$  is the Barbero-Immirzi parameter, and the holonomy function  $h_k^{(\mu)}$ , along straight line of coordinate length proportional to  $\mu/|a|$ , reads

$$h_k^{(\mu)}(\beta) = \cos(\mu\beta/2) \mathbb{I} + 2 \sin(\mu\beta/2) \tau_k, \quad (3)$$

where  $\tau_k = -i\sigma_k/2$  ( $\sigma_k$  are the Pauli spin matrices).

The total Hamiltonian for FRW universe with a massless scalar field,  $\phi$ , is found to be

$$H = H_g + H_\phi, \quad (4)$$

where  $H_\phi = N p_\phi^2 |p|^{-3/2}/2$ , and where  $\phi$  and  $p_\phi$  are canonical variables. Equation (4) satisfies the condition  $H \approx 0$  as it corresponds to the *scalar constraint* of general relativity; the Gauss and diffeomorphism constraints are equal zero (in the strong sense) due to the choice of gauges.

Making use of (3) we calculate (2) and get the modified total Hamiltonian  $H_g^{(\lambda)}$  corresponding to (4) in the form [12]

$$H^{(\lambda)}/N = -\frac{3}{8\pi G\gamma^2} \frac{\sin^2(\lambda\beta)}{\lambda^2} v + \frac{p_\phi^2}{2v} \approx 0, \quad (5)$$

where

$$\beta := \frac{c}{|p|^{1/2}}, \quad v := |p|^{3/2} \quad (6)$$

are the canonical variables of the so called improved scheme, and where  $c$  and  $p$  are canonical variables. The variable  $\beta = \gamma\dot{a}/a$  corresponds to the Hubble parameter  $\dot{a}/a$ , and  $v^{1/3} = aV_0^{1/3}$  is proportional to the scale factor  $a$ . The relationship between the coordinate length  $\mu$  (which depends on  $p$ ) and the physical length  $\lambda$  (which is a constant) reads:  $\lambda = \mu |p|^{1/2} = \mu a V_0^{1/3}$ .

At this stage, it should be emphasized that (5) presents a purely classical Hamiltonian *modified* by the holonomy (3), i.e. Eq. (5) includes no quantum physics. Contrary, in the standard LQC (5) is called a semi-classical or effective Hamiltonian and is interpreted to include some ‘imprints’ of quantization. In our nonstandard LQC quantum physics enters the formalism only when quantizing the algebra of observables (presented in the subsequent section).

The complete Poisson bracket for the canonical variables  $(\beta, v, \phi, p_\phi)$  is defined to be

$$\{\cdot, \cdot\} := 4\pi G\gamma \left[ \frac{\partial \cdot}{\partial \beta} \frac{\partial \cdot}{\partial v} - \frac{\partial \cdot}{\partial v} \frac{\partial \cdot}{\partial \beta} \right] + \frac{\partial \cdot}{\partial \phi} \frac{\partial \cdot}{\partial p_\phi} - \frac{\partial \cdot}{\partial p_\phi} \frac{\partial \cdot}{\partial \phi}. \quad (7)$$

The dynamics of a canonical variable  $\xi$  is defined by

$$\dot{\xi} := \{\xi, H^{(\lambda)}\}, \quad \xi \in \{\beta, v, \phi, p_\phi\}, \quad (8)$$

where  $\dot{\xi} := d\xi/d\tau$ , and where  $\tau$  is an evolution parameter.

The dynamics in the *physical* phase space,  $\mathcal{F}_{phys}^{(\lambda)}$ , is defined by solutions to (8) satisfying the constraint  $H^{(\lambda)} \approx 0$ . The solutions of (8) ignoring the constraint are in the *kinematical* phase space.

## B. Elementary observables

A function,  $\mathcal{O}$ , defined on phase space is a Dirac *observable* if

$$\{\mathcal{O}, H^{(\lambda)}\} \approx 0. \quad (9)$$

Equation (9) leads to [12]

$$\frac{\sin(\lambda\beta)}{\lambda} \frac{\partial \mathcal{O}}{\partial \beta} - v \cos(\lambda\beta) \frac{\partial \mathcal{O}}{\partial v} - \frac{\kappa \operatorname{sgn}(p_\phi)}{4\pi G} \frac{\partial \mathcal{O}}{\partial \phi} = 0, \quad (10)$$

where  $\kappa^2 := 4\pi G/3$ . Solution to (10) in  $\mathcal{F}_{phys}^{(\lambda)}$  is found to be [12]

$$\mathcal{O}_1 := p_\phi, \quad \mathcal{O}_2 := \phi - \frac{\operatorname{sgn}(p_\phi)}{3\kappa} \operatorname{arth}(\cos(\lambda\beta)). \quad (11)$$

One may parameterize  $\mathcal{F}_{phys}^{(\lambda)}$  by the *elementary* observables  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , and define the Poisson structure as follows [12]

$$\{\mathcal{O}_2, \mathcal{O}_1\} = 1, \quad \{\cdot, \cdot\} := \frac{\partial \cdot}{\partial \mathcal{O}_2} \frac{\partial \cdot}{\partial \mathcal{O}_1} - \frac{\partial \cdot}{\partial \mathcal{O}_1} \frac{\partial \cdot}{\partial \mathcal{O}_2}. \quad (12)$$

For simplicity we use the same notation for the Poisson bracket in (12) and (7).

## C. Composite observables

In what follows we consider functions on *physical* phase space like the energy density and the volume, which describe singularity aspects of our cosmological model. Considered functions are expressed in terms of elementary observables and an evolution parameter  $\phi$ . They become observables for each fixed value of  $\phi$ , since in such case they are only functions of observables [12].

An expression for the energy density  $\rho$  of the scalar field  $\phi$  reads [12]

$$\rho(\phi, \lambda) = \frac{1}{2} \frac{1}{(\kappa\gamma\lambda)^2 \cosh^2 3\kappa(\phi - \mathcal{O}_2)}. \quad (13)$$

The volume, a geometrical function, is found to be [12]

$$v(\phi, \lambda) = \kappa\gamma\lambda |\mathcal{O}_1| \cosh 3\kappa(\phi - \mathcal{O}_2). \quad (14)$$

## III. QUANTIZATION

By quantization we mean: (i) finding a self-adjoint representation of composite observables, and (ii) calculating spectra of operators corresponding to the composite observables.

### A. Representation of elementary observables

In what follows we use two representations of the classical algebra (12). Namely,

$$\mathcal{O}_1 \longrightarrow \widehat{\mathcal{O}}_1 f(x) := -i\hbar \partial_x f(x), \quad \mathcal{O}_2 \longrightarrow \widehat{\mathcal{O}}_2 f(x) := \widehat{x}f(x) := xf(x), \quad (15)$$

which leads to  $[\widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2] = -i\hbar \mathbb{I}$ , and

$$\mathcal{O}_1 \longrightarrow \widehat{\mathcal{O}}_1 f(x) := \widehat{x}f(x) := xf(x), \quad \mathcal{O}_2 \longrightarrow \widehat{\mathcal{O}}_2 f(x) := -i\hbar \partial_x f(x), \quad (16)$$

that gives  $[\widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2] = i\hbar \mathbb{I}$ , where  $x \in \mathbb{R}$ . Due to the Stone–von Neumann theorem all self-adjoint representations of the algebra (12) are unitarily equivalent to the representation (15) or (16) defined on a suitable dense subspace of  $L^2(\mathbb{R})$ . In this sense our choice of representation for (12) is unique.

### B. Energy density

The representation (16) is essentially self-adjoint on the dense subspace  $D$  of the Hilbert space  $L^2[-r, r]$ , where  $r \in \mathbb{R}_+$ , defined to be

$$D := \{f \in C^\infty[-r, r] \mid f^{(n)}(-r) = f^{(n)}(r), n \in \{0\} \cup \mathbb{N}\}. \quad (17)$$

The eigenvalue problem:  $\widehat{\mathcal{O}}_2 f_p = p f_p$  has the solution

$$f_p(x) = (2r)^{-1/2} \exp(ixp/\hbar), \quad p(k) := 2\pi\hbar k/r, \quad k \in \mathbb{Z}. \quad (18)$$

The spacing  $\square$  of neighboring eigenvalues

$$\square := p(k+1) - p(k) = 2\pi\hbar/r \quad (19)$$

can be made as small as desired by making  $r$  sufficiently large. Thus, one may say that the spectrum of  $\widehat{\mathcal{O}}_2$  is continuous.

In the representation (16) the energy density operator reads

$$\widehat{\rho} := \frac{1}{2} \frac{1}{(\kappa\gamma\lambda)^2 \cosh^2 3\kappa(\phi + i\hbar \partial_x)}. \quad (20)$$

Since  $\widehat{\mathcal{O}}_2$  is *essentially* self-adjoint on  $\mathcal{F}_r := \{f_{p(k)}\}_{k \in \mathbb{Z}}$ , we may apply the spectral theorem to get

$$\widehat{\rho} f_p = \rho(\phi, \lambda, p) f_p, \quad (21)$$

where

$$\rho(\phi, \lambda, p) := \frac{1}{2} \frac{1}{(\kappa\gamma\lambda)^2 \cosh^2 3\kappa(\phi - p)}, \quad (22)$$

and where  $\rho(\phi, \lambda, p)$  is the eigenvalue corresponding to the eigenvector  $f_p$ .

Our results mean that classical (13) and quantum (22) expressions for the energy density *coincide*. It is clear that the maximum density,  $\rho_{max}$ , reads

$$\rho_{max}(\lambda) = \frac{1}{2} \frac{1}{(\kappa\gamma\lambda)^2}. \quad (23)$$

Starting from the representation (15), instead of (16), we would get the quantum model of the energy density presented in [11], which is equivalent to the present one.

### C. Volume operator

To define the quantum operator corresponding to  $v$ , we use

$$v = |w|, \quad w := \kappa\gamma\lambda \mathcal{O}_1 \cosh 3\kappa(\phi - \mathcal{O}_2). \quad (24)$$

Thus, quantization of  $v$  reduces to the quantization problem of  $w$ . Quantization of the latter may be done in a standard way as follows

$$\hat{w} f(x) := \kappa\gamma\lambda \frac{1}{2} \left( \hat{\mathcal{O}}_1 \cosh 3\kappa(\phi - \hat{\mathcal{O}}_2) + \cosh 3\kappa(\phi - \hat{\mathcal{O}}_2) \hat{\mathcal{O}}_1 \right) f(x), \quad (25)$$

where  $f \in L^2(\mathbb{R})$ . For the elementary observables  $\mathcal{O}_1$  and  $\mathcal{O}_2$  we use the representation (15). An explicit form of the operator  $\hat{w}$  is

$$\hat{w} f(x) = i \frac{\kappa\gamma\lambda\hbar}{2} \left( 2 \cosh 3\kappa(\phi - x) \frac{d}{dx} - 3\kappa \sinh 3\kappa(\phi - x) \right) f(x). \quad (26)$$

Considerations simplify if we take  $f$  in the form

$$f(x) := A e^{ih(x)} \cosh^{-1/2} 3\kappa(\phi - x), \quad (27)$$

where  $A \in \mathbb{R}$ , and where  $h$  is a real-valued function.

#### 1. Eigenvalue problem

Let us consider the eigenvalue problem for the operator  $\hat{w}$  in the set of functions of the form (27). We have

$$\hat{w} f(x) = -\kappa\gamma\lambda\hbar \cosh 3\kappa(\phi - x) \frac{dh(x)}{dx} f(x) =: b f(x), \quad (28)$$

where  $b \in \mathbb{R}$  is the eigenvalue of  $\hat{w}$ . One may verify that a general form of  $h$  satisfying (28) is given by

$$h(x) = \frac{2b}{3\kappa^2\gamma\lambda\hbar} \arctan e^{3\kappa(\phi-x)}. \quad (29)$$

Thus, a normalized  $f_b$  satisfying (28) reads

$$f_b(x) := \frac{\sqrt{\frac{3\kappa}{\pi}} \exp\left(i \frac{2b}{3\kappa^2\gamma\lambda\hbar} \arctan e^{3\kappa(\phi-x)}\right)}{\cosh^{\frac{1}{2}} 3\kappa(\phi - x)}. \quad (30)$$

#### 2. Orthogonality

Making use of (30) gives

$$\langle f_b | f_a \rangle = \frac{3\kappa}{\pi} \int_{-\infty}^{\infty} \frac{\exp\left(i \frac{2(a-b)}{3\kappa^2\gamma\lambda\hbar} \arctan e^{3\kappa(\phi-x)}\right)}{\cosh 3\kappa(\phi - x)} dx. \quad (31)$$

We introduce a new variable  $y = e^{3\kappa(\phi-x)}$  and have

$$\langle f_b | f_a \rangle = \frac{2}{\pi} \int_0^\infty \frac{\exp\left(i \frac{2(a-b)}{3\kappa^2\gamma\lambda\hbar} \arctan y\right)}{1+y^2} dy. \quad (32)$$

Another substitution  $\tan z = y$  leads to

$$\langle f_b | f_a \rangle = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(i \frac{2(a-b)}{3\kappa^2\gamma\lambda\hbar} z\right) dz = -i \frac{3\kappa^2\gamma\lambda\hbar}{\pi(a-b)} \exp\left(i \frac{2(a-b)}{3\kappa^2\gamma\lambda\hbar} z\right) \Big|_0^{\frac{\pi}{2}}. \quad (33)$$

It is clear that  $\langle f_b | f_a \rangle = 0$  iff

$$a - b = 6\kappa^2\gamma\lambda\hbar m = 8\pi G\gamma\lambda\hbar m, \quad m \in \mathbb{Z}. \quad (34)$$

Thus, the set  $\mathcal{F}_b := \{ f_a \mid a = b + 8\pi G\gamma\lambda\hbar m; m \in \mathbb{Z}; b \in \mathbb{R} \}$  is *orthonormal*. Each subspace  $\mathcal{F}_b \subset L^2(\mathbb{R})$  spans a pre-Hilbert space. The completion of each *span*  $\mathcal{F}_b$ ,  $\forall b \in \mathbb{R}$ , in the norm of  $L^2(\mathbb{R})$  gives  $L^2(\mathbb{R})$ .

### 3. Self-adjointness

Since  $\langle f_b | \hat{w} f_a \rangle - \langle \hat{w} f_b | f_a \rangle = (a - b) \langle f_b | f_a \rangle$ , the operator  $\hat{w}$  is *symmetric* on  $\mathcal{F}_b$  for any  $b \in \mathbb{R}$ , because  $\langle f_b | f_a \rangle = 0$  for  $a \neq b$  due to the orthogonality of the set  $\mathcal{F}_b$ .

To examine the self-adjointness of the unbounded operator  $\hat{w}$ , we first identify the deficiency subspaces,  $\mathcal{K}_\pm$ , of this operator [15, 16]

$$\mathcal{K}_\pm := \{ g_\pm \in D_b(\hat{w}^*) \mid \langle g_\pm | (\hat{w} \pm i\mathbb{I}) f_a \rangle = 0, \forall f_a \in D_b(\hat{w}) \}, \quad (35)$$

where  $D_b(\hat{w}) := \text{span } \mathcal{F}_b$ , and where  $D_b(\hat{w}^*) := \{ f \in L^2(\mathbb{R}) : \exists! f^* \langle f^* | g \rangle = \langle f | \hat{w} g \rangle, \forall g \in D_b(\hat{w}) \}$ .

For each  $f_a \in D_b(\hat{w}) \subset L^2(\mathbb{R})$  we have

$$0 = \langle g_\pm | (\hat{w} \pm i\mathbb{I}) f_a \rangle = (a \pm i) \int_{-\infty}^{\infty} dx \overline{g_\pm(x)} f_a(x) \implies g_+ = 0 = g_-. \quad (36)$$

Thus, the deficiency indices  $n_\pm := \dim[\mathcal{K}_\pm]$  of  $\hat{w}$  satisfy the relation:  $n_+ = 0 = n_-$ , which proves that the operator  $\hat{w}$  is *essentially* self-adjoint on  $D_b(\hat{w})$ .

### 4. Spectrum

Due to the spectral theorem on self-adjoint operators [15, 16], we may carry out quantization of the volume as follows

$$v = |w| \longrightarrow \hat{v} f_a := |a| f_a. \quad (37)$$

A common feature of all  $\mathcal{F}_b$  is the existence of the minimum gap  $\Delta := 8\pi G\gamma\hbar\lambda$  in the spectrum, which defines a *quantum* of the volume.

In the limit  $\lambda \rightarrow 0$ , corresponding to the classical FRW model without the loop geometry modification, there is no quantum of the volume.



It results from (34) that for  $b = 0$  and  $m = 0$  the minimum eigenvalue of  $\hat{v}$  equals zero. It is a special case that corresponds to the classical situation when  $v = 0$ , which due to (5) means that  $p_\phi = 0$  so there is no classical dynamics (for more details see [12]). Thus, we have a direct correspondence between classical and quantum levels corresponding to this very special state. It is clear that all other states describe bouncing *dynamics*.

As the universe expands, a discrete spectrum of the volume operator may favour a *foamy* like structure which should finally turn into a continuous spacetime. The quantum of a volume may be used as a measure of a size,  $\lambda_f$ , of a spacetime foam. One may speculate that  $\lambda_f := \Delta^{1/3} = (8\pi G\gamma\hbar\lambda)^{1/3}$ . Thus, an astro-cosmo data that determine a size of spacetime granularity may fix the minimum length parameter  $\lambda$  of LQC. That would enable making an estimate of the critical matter density  $\rho_{\max} = 1/2(\kappa\gamma\lambda)^2$  corresponding to the Big Bounce.

The granularity of volume should lead to the granularity of energy of physical fields. We suggest, making use of the de Broglie relation, that a specific particle representing a quantum of energy may have a momentum  $p_i$  corresponding to its wavelength  $\lambda_i$  such that  $p_i \lambda_i = \hbar$ . The detection of an ultrahigh energy particle with specific  $p_i$  may be used to determine  $\lambda_i$ , and consequently set the upper limit for the fundamental length  $\lambda_f$ . The set of parameters  $\lambda_i$  (for a set of particles) may be treated further as multiplicities of  $\lambda_f$  in which case the greatest common divisor of all  $\lambda_i$  would set the lowest upper limit for  $\lambda_f$ .

## 5. Evolution

The relation between eigenvectors corresponding to the same eigenvalue for different values of the parameter  $\phi$  reads:

$$f_a^{\phi+\psi} = e^{\psi\partial_\phi} f_a^\phi = e^{-i\frac{\psi}{\hbar}\hat{\mathcal{O}}_1} f_a^\phi. \quad (38)$$

One may verify that

$$\hat{w}(\phi + \psi) = \cosh(3\kappa\psi)\hat{w}(\phi) + \frac{\sinh(3\kappa\psi)}{3\kappa}\partial_\phi\hat{w}(\phi), \quad (39)$$

thus

$$\begin{aligned} \langle f_b^\phi | \hat{w}(\phi + \psi) f_a^\phi \rangle &= \langle f_b^\phi | \hat{w}(\phi) f_a^\phi \rangle \cosh(3\kappa\psi) + \frac{\sinh(3\kappa\psi)}{3\kappa} \langle f_b^\phi | \partial_\phi \hat{w}(\phi) f_a^\phi \rangle \\ &= a \cosh(3\kappa\psi) \delta_{ab} + (b - a) \frac{\sinh(3\kappa\psi)}{3\kappa} \langle f_b^\phi | \partial_x f_a^\phi \rangle. \end{aligned} \quad (40)$$

Finally, an evolution of the expectation value of the operator  $\hat{w}$  is found to be

$$\langle f(\phi) | \hat{w}(\phi + \psi) f(\phi) \rangle = A \cosh 3\kappa(\psi + B), \quad (41)$$

where  $f := \sum \alpha_a f_a$ ,  $f_a \in \mathcal{F}_b$ . One may verify that

$$A = \text{sgn}(X) \sqrt{X^2 - Y^2}, \quad B = \frac{1}{6\kappa} \ln \frac{X + Y}{X - Y}, \quad (42)$$

where

$$X := \sum_a |\alpha_a|^2 a, \quad Y := \sum_{a, m} \frac{\bar{\alpha}_b \alpha_a - \bar{\alpha}_a \alpha_b}{i\pi} \frac{m(2a + 6m\kappa^2\gamma\hbar\lambda)}{(2m - 1)(2m + 1)}, \quad (43)$$

and where  $b = a + 6\kappa^2\gamma\lambda\hbar$ ,  $b \in \mathbb{R}$ ,  $m \in \mathbb{Z}$ , and  $|X| > |Y|$ .

We can see that the evolution of the expectation value of the operator  $\hat{w}$  coincides with the classical expression (24).

#### D. Energy density operator in the basis $\mathcal{F}_b$

The operator  $\hat{\rho}$  may be expressed in terms of the basis vectors from  $\mathcal{F}_b$ . One may verify that

$$\langle f_{a+m\Delta} | \hat{\rho} f_a \rangle = \begin{cases} \frac{-1}{8(\kappa\gamma\lambda)^2} & \text{if } m = 1 \text{ or } -1; \\ \frac{1}{4(\kappa\gamma\lambda)^2} & \text{if } m = 0; \\ 0 & \text{otherwise} \end{cases} \quad (44)$$

which leads to

$$\hat{\rho} f_a = \frac{1}{4(\kappa\gamma\lambda)^2} f_a - \frac{1}{8(\kappa\gamma\lambda)^2} f_{a-4\Delta} - \frac{1}{8(\kappa\gamma\lambda)^2} f_{a+4\Delta}, \quad (45)$$

where  $\Delta := 8\pi G\gamma\hbar\lambda$ . Therefore,  $\hat{\rho}$  is bounded and does not commute with  $\hat{v}$ . The latter is consistent with the Poisson bracket relation for these observables, which reads

$$\{w, \rho\} = 3\kappa^2\gamma\lambda \sinh 3\kappa(\phi - \mathcal{O}_2) \rho. \quad (46)$$

Therefore, the operators  $\hat{\rho}$  and  $\hat{v}$  cannot have common eigenfunctions so they provide two alternative ways of describing the system, either in terms of the eigenfunctions of  $\hat{\rho}$  or  $\hat{v}$ .

## IV. SUMMARY AND CONCLUSIONS

Turning the big bang into the big bounce (BB) in the FRW universe is due to the modification of the model at the *classical* level by making use of the loop geometry. The modification is parameterized by a *continuous* parameter  $\lambda$ .

Each value of  $\lambda$  specifies the critical energy density of the scalar field corresponding to the BB. The spectrum of the energy density operator is *bounded* and *continuous*. Classical and quantum expressions for the minimum of the energy density coincide. The spectrum of the volume operator, parameterized by  $\lambda$ , is *bounded* from below and *discrete*. The expectation value of the volume operator coincide with the classical expression. The results concerning the volume operator may be extended to the area and the length operators [12].

An *evolution* parameter,  $\phi$ , does not belong to the *physical* phase space of our nonstandard LQC. Thus, it stays classical during the quantization process [11]. In the standard LQC, contrary to our method,  $\phi$  is a phase space variable so it must be quantized [17]. Being a quantum variable it may *fluctuate* so its use as an evolution parameter at the quantum level has poor interpretation.

The Hamiltonian *constraints* are treated differently in the standard and nonstandard LQC at the classical level. In the former case the Gauss and diffeomorphism constraints are turned into zero by a suitable choice of gauges; the scalar constraint is solved only at the quantum level. In the latter case all the constraints are classically solved leading to the *physical* phase space. Thus, considered observables are not kinematical, but physical. This is why an evolution of geometrical functions like the volume may be used for testing the singularity aspects of a given cosmological model. The standard and our LQC methods

give similar results in the sense that they lead to the Big Bounce transition. However, our method is fully controlled analytically as it does not require any numerical work.

We believe that our nonstandard LQC may be related with the reduced phase space quantization of Loop Quantum Gravity (LQG) (see [14] and references therein). Finding the correspondence may help deriving LQC from LQG.

There exist results concerning the spectrum of the volume operator obtained within LQG (see, e.g. [18, 19]), but cannot be compared easily with our results due to the lack of a direct correspondence between LQG and LQC models.

As there is no specific choice of  $\lambda$ , the BB may occur at any low and high densities [10]. Finding specific value of the parameter  $\lambda$  (and energy scale specific to the BB) is an open problem. It may happen, that the value of the parameter  $\lambda$  cannot be determined, for some reason, theoretically. Fortunately, there is a rapidly growing number of data coming from observational cosmology that may be useful in this context. In particular, the detection of the primordial gravitational waves created at the BB phase may bring valuable information about this phase [20–22]. There exists speculation that the foamy structure of spacetime may lead to the dependence of the velocity of a photon on its energy. Such dependence is weak, but may sum up to give a measurable effect in the case of photons travelling over cosmological distances across the Universe [23]. Presently, available data suggest that such *dispersion* effects do not occur up to the energy scale  $5 \times 10^{17}$  GeV [24] so such effects may be present, but at higher energies.

Various forms of discreteness of spacetime underly many approaches in fundamental physics. Just to name a few: noncommutative geometry [25], causal sets approach [26], gravitational Wilson loops [27], Regge calculus [28], path integral over geometries [29], spin foam model [30], and categories [31]. The discreteness may translate at the semi-classical level into a foamy structure of space. Such expected property of spacetime creates large activity in observational astrophysics and cosmology (see, e.g. Lorentz and CPT violation [32], dispersion of cosmic photons [34], electrons [33] and neutrinos [35], birefringence effects [36]). Our results concerning the physics of geometry at short distances give some support to these approaches and expectations.

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