Madelung, Gross-Pitaevskii and Korteweg

Rémi Carles, Raphaël Danchin, Jean-Claude Saut

MADELUNG, GROSS–PITAEVSKII AND KORTEWEG

RÉMI CARLES, RAPHAËL DANCHIN, AND JEAN-CLAUDE SAUT

Abstract. This paper surveys various aspects of the hydrodynamic formulation of the nonlinear Schrödinger equation obtained via the Madelung transform in connexion to models of quantum hydrodynamics and to compressible fluids of the Korteweg type.

1. Introduction

In his seminal work [37] (see also [33]), E. Madelung introduced the so-called *Madelung transform* in order to relate the linear Schrödinger equation to a *hydrodynamic type system.* This system takes (slightly) different forms according to the context: linear or nonlinear equation with various nonlinearities. For the semi-classical nonlinear Schrödinger equation (shortened in NLS in what follows):

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = f\left(|\psi^\varepsilon|^2\right)\psi^\varepsilon,$$

the Madelung transform amounts to setting

$$\psi^\varepsilon(t,x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)/\varepsilon},$$

so as to get the following system for ρ and $v = \nabla\phi$:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla f(\rho) = \dfrac{\varepsilon^2}{2}\nabla\left(\dfrac{\Delta\left(\sqrt{\rho}\right)}{\sqrt{\rho}}\right), \\ \partial_t\rho + \operatorname{div}(\rho v) = 0. \end{cases} \tag{1.1}$$

This system is referred to as the *hydrodynamic form* of NLS because of its similarity with the compressible Euler equation (which corresponds to ε=0). The additional term on the right-hand side is the so-called *quantum pressure.*

Madelung transform is crucial to investigate qualitative properties of the nonlinear Schrödinger equation with nonzero boundary conditions at infinity whenever the solution is not expected to vanish "too often". Of particular interest is the so-called *Gross–Pitaevskii equation*

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = (|\psi^\varepsilon|^2-1)\psi^\varepsilon,$$

which corresponds to $f(r) = r-1$ or, more generally, the case where f is a smooth function vanishing at some r_0 and such that $f'(r_0) < 0$. This covers in particular the "cubic-quintic" NLS ($f(r) = -\alpha_1 + \alpha_3 r - \alpha_5 r^2$ with $\alpha_1, \alpha_3, \alpha_5 > 0$).

Using Madelung transform is also rather popular to study the semi-classical limit of NLS. This requires ρ to be nonvanishing, though. We shall see below to what

This work was supported by the French ANR projects Equa-disp (ANR-07-BLAN-0250-01) and R.A.S. (ANR-08-JCJC-0124-01).

extent the presence of vacuum (that is of points where ρ vanishes) is merely a technical problem due to the approach related to Madelung transform.

The hydrodynamic form of NLS may be seen as a particular case of the system of *quantum fluids* (QHD) with a suitable choice of the pressure law, which, in turn, enters in the class of *Korteweg (or capillary) fluids*. We aim here at further investigating the link between these three *a priori* disjoint domains, through various uses of the Madelung transform. In passing, we will also review some more or less known facts pertaining to the theories of quantum fluids and Korteweg fluids. The most original part of this work is a simple proof of a recent result of Antonelli and Marcati [3, 4], on the global existence of weak solutions to a quantum fluids system.

1.1. **Organization of the paper.** The paper is organized as follows. The second section is a review of the connexions of the Madelung transform with the semi-classical limit of the nonlinear Schrödinger equation. In passing we briefly present the state-of-the-art for the Cauchy problem for the Gross-Pitaevskii equation. In Section 3, we solve the quantum hydrodynamical system (1.1) by a direct method based on the use of an extended formulation, and explain how it may be adapted to tackle general Korteweg fluids. In Section 4, we review the use of the Madelung transform and of the hydrodynamical form of the Gross–Pitaevskii equation to study the existence and properties of its traveling wave solutions and of its transonic limit, both in the steady and unsteady cases. We give in Section 5 a simple proof of the aforementioned result of Antonelli and Marcati ([3, 4]). Lastly we list in Appendix the basic conservation laws for the Schrödinger, the QHD and the compressible Euler equations, and show that some of these laws naturally carry over to general Korteweg fluids.

1.2. **Notations.**

- We denote by $|\cdot|_p$ $(1 \leqslant p \leqslant \infty)$ the standard norm of the Lebesgue spaces $L^p(\mathbb{R}^d)$.
- The standard $H^s(\mathbb{R}^d)$ Sobolev norm will be denoted $\|\cdot\|_s$.
- We use the Fourier multiplier notation: $f(D)u$ is defined as $\mathcal{F}(f(D)u)(\xi) = f(\xi)\widehat{u}(\xi)$, where $\mathcal{F}$ and $\widehat{\cdot}$ stand for the Fourier transform.
- The operator $\Lambda = (1-\Delta)^{1/2}$ is equivalently defined using the Fourier multiplier notation to be $\Lambda = (1+|D|^2)^{1/2}$.
- The partial derivatives will be denoted with a subscript, *e.g.* $u_t, u_x, \dots$ or $\partial_t u, \partial_x u, \dots$ or even ∂_j (to designate ∂_{x_j}).
- C will denote various nonnegative absolute constants, the meaning of which will be clear from the context.

2. Madelung transform and the semi-classical limit of NLS

As already mentioned in the Introduction, we consider the equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = f\left(|\psi^\varepsilon|^2\right)\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = \sqrt{\rho_0(x)}e^{i\phi_0(x)/\varepsilon}. \tag{2.1}$$

Here we have in mind the limit $\varepsilon \to 0$. The space variable x belongs to $\mathbb{R}^d$ in this paragraph. The periodic case $x \in \mathbb{T}^d$ being of particular interest for numerical simulations however (see e.g. [6, 23] and references therein), we will state some analogous results in that case, which turn out to be a little simpler than in the case $x \in \mathbb{R}^d$. In this section, we shall focus on two types of nonlinearity:

- Cubic, defocusing nonlinearity[1]:
$$f\left(|\psi^\varepsilon|^2\right)\psi^\varepsilon = |\psi^\varepsilon|^2\psi^\varepsilon.$$
The Hamiltonian associated to (2.1) then reads
$$\mathcal{H}^\varepsilon_{\text{NLS}}\left(\psi^\varepsilon\right) = \|\varepsilon\nabla\psi^\varepsilon\|^2_{L^2} + \|\psi^\varepsilon\|^4_{L^4}.$$
It is well defined on the Sobolev space $H^1(\mathbb{R}^d)$ in dimension $d \leqslant 4$.
- Gross–Pitaevskii equation:
$$f\left(|\psi^\varepsilon|^2\right)\psi^\varepsilon = \left(|\psi^\varepsilon|^2 - 1\right)\psi^\varepsilon.$$
In this case, the natural Hamiltonian associated to (2.1) is
$$\mathcal{H}^\varepsilon_{\text{GP}}\left(\psi^\varepsilon\right) = \|\varepsilon\nabla\psi^\varepsilon\|^2_{L^2} + \left\||\psi^\varepsilon|^2 - 1\right\|^2_{L^2}.$$

In both cases, the Hamiltonian defines an energy space, in which existence and uniqueness for the Cauchy problem (2.1) have been established.

The case of a defocusing cubic nonlinearity is now well understood. In dimension $d \leqslant 4$ the corresponding NLS equation is globally well-posed in $H^1(\mathbb{R}^d)$, and the additional $H^s(\mathbb{R}^d)$ regularity ($s \geqslant 1$) is propagated (see the textbooks [18, 36, 44] and the references therein).

The situation is more complicated for the Gross–Pitaevskii equation, where the finite energy solutions ψ^ε cannot be expected to be in $L^2(\mathbb{R}^d)$, since $|\psi^\varepsilon|^2 - 1 \in L^2(\mathbb{R}^d)$. As noticed in [49, 50], and extended in [24], a convenient space to study the Gross–Pitaevskii equation is the Zhidkov space:
$$X^s(\mathbb{R}^d) = \{\psi \in L^\infty(\mathbb{R}^d)\ ;\ \nabla\psi \in H^{s-1}(\mathbb{R}^d)\} \quad \text{with} \quad s > d/2.$$

Remark 2.1. In the case $x \in \mathbb{T}^d$, the spaces $H^s(\mathbb{T}^d)$ and $X^s(\mathbb{T}^d)$ (with obvious definitions) are the same.

In the case $x \in \mathbb{R}^d$, as a consequence of the Hardy–Littlewood–Sobolev inequality (see e.g. [32, Th. 4.5.9] or [28, Lemma 7]), one may show that if $d \geqslant 2$ and $\psi \in \mathcal{D}'(\mathbb{R}^d)$ is such that $\nabla\psi \in L^p(\mathbb{R}^d)$ for some $p \in]1, d[$, then there exists a constant γ such that $\psi - \gamma \in L^q(\mathbb{R}^d)$, with $1/p = 1/q + 1/d$. Morally, γ is the limit of ψ at infinity. If $d \geqslant 3$ then we can take $p = 2$, so every function in $X^s(\mathbb{R}^d)$ satisfies the above property; for $d = 2$, the above assumption requires a little more decay on $\nabla\psi$ than general functions in $X^s(\mathbb{R}^2)$.

The well-posedness issue in the natural energy space
$$E(\mathbb{R}^d) = \{\psi \in H^1_{\text{loc}}(\mathbb{R}^d),\ \nabla\psi \in L^2(\mathbb{R}^d),\ |\psi|^2 - 1 \in L^2(\mathbb{R}^d)\}$$
associated to the Gross-Pitaevskii equation has been investigated only very recently by C. Gallo in [25] and P. Gérard in [28, 29], in dimension $d \leqslant 4$.

Let us emphasize that in the $\mathbb{R}^d$ case, the energy space is no longer a linear space (contrary to the case of zero boundary condition a infinity where it is $H^1(\mathbb{R}^n)$), hence solving the Gross–Pitaevskii equation in $E(\mathbb{R}^d)$ is more complicated. However, if $d = 3, 4$, one may show that $E(\mathbb{R}^d)$ coincides with
$$\{\psi = c(1 + v), c \in \mathbb{S}^1, \psi \in H^1(\mathbb{R}^d), 2\operatorname{Re}\psi + |\psi|^2 \in L^2(\mathbb{R}^d)\},$$
which allows to endow it with a structure of a metric space. The case $d = 2$ is slightly more technical.

[1]We could consider more general defocusing nonlinearities such as $f\left(|\psi^\varepsilon|^2\right)\psi^\varepsilon = |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon$, $\sigma \in \mathbb{N}$. We consider the exactly cubic case for the simplicity of the exposition only.

2.1. **Some issues related to the use of the Madelung transform.** In the semi-classical context, Madelung transform consists in seeking

$$\psi^\varepsilon(t,x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)/\varepsilon} \tag{2.2}$$

for some $\rho \geqslant 0$ and real-valued function ϕ. Plugging (2.2) into (2.1) and separating real and imaginary values yields:

$$\begin{cases} \sqrt{\rho}\left(\partial_t\phi + \dfrac{1}{2}|\nabla\phi|^2 + f(\rho)\right) = \dfrac{\varepsilon^2}{2}\Delta\sqrt{\rho} \quad ; \quad \phi_{|t=0} = \phi_0, \\ \partial_t\sqrt{\rho} + \nabla\phi\cdot\nabla\sqrt{\rho} + \dfrac{1}{2}\sqrt{\rho}\Delta\phi = 0 \quad ; \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{2.3}$$

Two comments are in order at this stage: the first equation shows that ϕ depends on ε and the second equation shows that so does ρ in general. We shall underscore this fact by using the notation $(\phi^\varepsilon, \rho^\varepsilon)$. Second, the equation for ϕ^ε can be simplified, *provided that ρ^ε has no zero.* Introducing the velocity $v^\varepsilon = \nabla\phi^\varepsilon$, (2.3) yields the system of *quantum hydrodynamics* (1.1) presented in the introduction.

To study the limit $\varepsilon \to 0$ (the *Euler limit*), it is natural to consider the following compressible Euler equation:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla f(\rho) = 0 \quad ; \quad v_{|t=0} = \nabla\phi_0, \\ \partial_t\rho + \operatorname{div}(\rho v) = 0 \quad ; \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{2.4}$$

Note that the solution to (2.4) must not be expected to remain smooth for all time, even if the initial data are smooth. In [38] (see also [47]), it is shown that compactly supported initial data lead to the formation of singularities in finite time. In [42], the author constructs a solution developing singularities in finite time, in the absence of vacuum. In [2], it is shown that for rotationally invariant two-dimensional data that are perturbation of size ε of a rest state, blow-up occurs at time $T_\varepsilon \sim \tau/\varepsilon^2$.

Remark 2.2. In the framework of this section, we have $f' = 1 > 0$. As pointed out above, the case $f(r) = r^\sigma$, $\sigma \in \mathbb{N}$, could be considered as well. On the other hand, if $f' < 0$ (corresponding to the semiclassical limit for a focusing nonlinearity), (2.4) becomes an elliptic system which may be solved locally-in-time for analytic data (see [27, 46]). At the same time, in the case $d = 1$ and $f' = -1$, it has been shown in [40] that there are smooth initial data for which the Cauchy problem (2.4) has no solution. In short, working with analytic data in this context is not only convenient, it is mandatory.

Let us emphasize that the hydrodynamical formulations for the cubic NLS and the Gross–Pitaevskii equations are exactly the same as in both cases we have $\nabla f(\rho) = \nabla\rho$. From this point of view, studying either of the equations is mainly a matter of boundary conditions at infinity: $H^s(\mathbb{R}^d)$ is the appropriate space for the cubic NLS equation whereas $X^s(\mathbb{R}^d)$ is adapted to the Gross–Pitaevskii equation.

In the sequel, Z^s denotes either $H^s(\mathbb{R}^d)$ or $X^s(\mathbb{R}^d)$. In addition, we set

$$Z^\infty = \bigcap_{s>d/2} Z^s.$$

Theorem 2.3. *Let $\rho_0, \phi_0 \in C^\infty(\mathbb{R}^d)$ with $\sqrt{\rho_0}, \nabla\phi_0 \in Z^s$ for some $s > d/2+1$. There exists a unique maximal solution $(v,\rho) \in C([0,T_{\max}); Z^s)$ to (2.4). In addition, $T_{\max}$ is independent of $s > d/2+1$ and*

$$T_{\max} < +\infty \Longrightarrow \int_0^{T_{\max}} \|(v,\sqrt{\rho})(t)\|_{W^{1,\infty}}\, dt = +\infty.$$

Finally, if $\nabla\phi_0$ and ρ_0 are smooth, nonzero and compactly supported then $T_{\max}$ is finite.

We investigate the following natural questions:

Question 1. *Assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$. Can we say that $\rho^\varepsilon(t,x) > 0$ for all $x \in \mathbb{R}^d$ and $t \in [0,T_{\max})$? If not, what is the maximal interval allowed for t?*

We will also recall that despite the appearance, the presence of vacuum (existence of zeroes of ψ^ε) is merely a technical problem: the Madelung transform ceases to make sense, but a rigorous WKB analysis is available, regardless of the presence of vacuum. See §2.3.

Question 2. *Let $\rho_0, \phi_0 \in C^\infty(\mathbb{R}^d)$ with $\rho_0, \nabla\phi_0 \in H^s(\mathbb{R}^d)$ for some $s > d/2+1$. Suppose that the solution (v,ρ) to (2.4) satisfies $\rho(t,x) > 0$ for $(t,x) \in [0,\tau]\times\mathbb{R}^d$. Can we construct a solution to (1.1) in $C([0,\tau_*]; H^s)$ (possibly with $0 < \tau_* < \tau$)?*

We will see that in general, the answer for this question is no. Even though from the answer to the first question, (1.1) makes sense *formally*, the analytical properties associated to (1.1) are not as favorable as for (2.4). Typically, the right-hand side of the equation for the quantum velocity need not belong to $L^2(\mathbb{R}^d)$.

Question 3. *Let $\rho_0, \phi_0 \in C^\infty(\mathbb{R}^d)$ with $\rho_0, \nabla\phi_0 \in X^s(\mathbb{R}^d)$ for some $s > d/2+1$. Suppose that the solution (v,ρ) to (2.4) satisfies $\rho(t,x) > 0$ for $(t,x) \in [0,\tau]\times\mathbb{R}^d$. Can we construct a solution to (1.1) in $C([0,\tau_*]; X^s)$ (possibly with $0 < \tau_* < \tau$)?*

2.2. **Proof of Theorem 2.3.** We shall simply give the main ideas of the proof of Theorem 2.3. Complete proofs can be found in [38] for the cubic defocusing NLS equation in Sobolev spaces, and in [1] for the Gross-Pitaevskii equation in Zhidkov spaces.

In the framework of this paper, we have

$$\nabla f(\rho) = \nabla\rho.$$

Introduce formally the auxiliary function $a = \sqrt{\rho}$. This nonlinear change of variable makes (2.4) hyperbolic symmetric:

$$\begin{cases} \partial_t v + v\cdot\nabla v + 2a\nabla a = 0 \quad ; \quad v_{|t=0} = \nabla\phi_0, \\ \partial_t a + v\cdot\nabla a + \dfrac{1}{2} a\nabla\cdot v = 0 \quad ; \quad a_{|t=0} = \sqrt{\rho_0}. \end{cases} \tag{2.5}$$

This system is of the form

$$\partial_t u + \sum_{j=1}^d A_j(u)\partial_j u = 0, \text{ where } u = \begin{pmatrix} v_1 \\ \vdots \\ v_d \\ a \end{pmatrix} \in \mathbb{R}^{d+1},$$

and the matrices A_j are symmetrized by the constant multiplier

$$S = \begin{pmatrix} I_d & 0 \\ 0 & 4 \end{pmatrix}.$$

Standard analysis (see e.g. [45]) shows that (2.5) has a unique maximal solution (v, a) in $C([0, T_{\max}); Z^s)$, provided that $s > d/2 + 1$ and that, in addition,

$$T_{\max} < +\infty \Longrightarrow \int_0^{T_{\max}} \|(v,a)(t)\|_{W^{1,\infty}}\, dt = +\infty.$$

We can then define ρ by the linear equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \quad ; \quad \rho_{|t=0} = \rho_0.$$

By uniqueness for this linear equation, $\rho = a^2$ (a is real-valued, so ρ is non-negative), and (v, ρ) solves (2.4).

We now briefly explain why compactly supported initial data lead to the formation of singularities in finite time. The first remark is that in this case, the solution to (2.4) has a finite speed of propagation, which turns out to be zero: so long as (v, ρ) is smooth, it remains supported in the same compact as its initial data. To see this, consider the auxiliary system (2.5): the first equation is a Burgers' equation with source term $2a\nabla a$; the second equation is an ordinary differential equation along the trajectories of the particles. Define the trajectory by

$$\frac{d}{dt}x(t,y) = v\left(t, x(t,y)\right) \quad ; \quad x(0,y) = y. \tag{2.6}$$

For $0 \leqslant t < T_{\max}$, this is a global diffeomorphism of $\mathbb{R}^d$, as shown by the equation

$$\frac{d}{dt}\nabla_y x(t,y) = \nabla v\left(t, x(t,y)\right)\nabla_y x(t,y) \quad ; \quad \nabla_y x(0,y) = \mathrm{Id},$$

and Gronwall lemma. Therefore, for a smooth function f,

$$\left(\partial_t f + v\cdot\nabla f\right)(t, x(t,y)) = \partial_t\left(f\left(t, x(t,y)\right)\right),$$

and (2.5) can be viewed as a system of ordinary differential equations.

Once the non-propagation of the support of smooth solutions is established, the end of the proof relies on a virial computation (like in [48, 30], see also [18]). This computation shows that global in time smooth solutions to (2.5) are dispersive (see also [41]). This is incompatible with the zero propagation speed of smooth compactly supported solutions. Therefore, singularities have to appear in finite time.

2.3. **A review of WKB analysis associated to (2.1).** We consider initial data which are a little more general than in (2.1), namely

$$\psi^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\varphi_0(x)/\varepsilon}, \tag{2.7}$$

where the initial amplitude a_0^ε is assumed to be smooth, *complex-valued*, and possibly depending on ε. Typically, we assume that there exist $a_0, a_1 \in Z^\infty$ independent of ε such that

$$a_0^\varepsilon = a_0 + \varepsilon a_1 + \mathcal{O}(\varepsilon^2) \text{ in } Z^s, \quad \forall s > d/2. \tag{2.8}$$

2.3.1. *First order approximation.* Introduce the solution to the quasilinear system

$$\begin{cases} \partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + f\left(|a|^2\right) = 0 \quad ; \quad \varphi_{|t=0} = \varphi_0, \\ \partial_t a + \nabla\varphi\cdot\nabla a + \frac{1}{2}a\Delta\varphi = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases} \tag{2.9}$$

Theorem 2.3 shows that (2.9) has a unique, smooth solution with $a, \nabla\varphi \in Z^\infty$. The main remark consists in noticing that (2.9) implies that $(\nabla\varphi, |a|^2)$ has to solve (2.4) (a_0 may be complex-valued): Theorem 2.3 yields $v, \rho \in Z^\infty$. We can then define a as the solution to the linear transport equation

$$\partial_t a + v\cdot\nabla a + \frac{1}{2}a\nabla\cdot v = 0 \quad ; \quad a_{|t=0} = a_0.$$

Now $|a|^2$ and ρ solve the same linear transport equation, with the same initial data, hence $\rho = |a|^2$. Using this information in the equation for the velocity, define

$$\varphi(t) = \varphi_0 - \int_0^t \left(\frac{1}{2}|v(\tau)|^2 + f\left(|a(\tau)|^2\right)\right)d\tau.$$

We easily check that $\partial_t\left(\nabla\varphi - v\right) = \nabla\partial_t\varphi - \partial_t v = 0$, and that (φ, a) solves (2.9).

Introduce the solution to the linearization of (2.9), with an extra source term:

$$\begin{cases} \partial_t\varphi^{(1)} + \nabla\varphi\cdot\nabla\varphi^{(1)} + 2\,\mathrm{Re}\left(\overline{a}a^{(1)}\right) f'\left(|a|^2\right) = 0 \quad ; \quad \varphi^{(1)}_{|t=0} = 0, \\ \partial_t a^{(1)} + \nabla\varphi\cdot\nabla a^{(1)} + \nabla\varphi^{(1)}\cdot\nabla a + \frac{1}{2}a^{(1)}\Delta\varphi + \frac{1}{2}a\Delta\varphi^{(1)} = \frac{i}{2}\Delta a \quad ; \quad a^{(1)}_{|t=0} = a_1. \end{cases}$$

We also check that it has a unique smooth solution, with $a^{(1)}, \nabla\varphi^{(1)} \in Z^\infty$. The main result that we will invoke is the following:

Proposition 2.4. *Let $a_0^\varepsilon, a_0, a_1, \varphi_0$ be smooth, with $a_0^\varepsilon, a_0, a_1, \nabla\varphi_0 \in Z^\infty$. Assume that* (2.8) *holds. Then*

$$\psi^\varepsilon = \left(ae^{i\varphi^{(1)}} + \mathcal{O}\left(\varepsilon\right)\right)e^{i\varphi/\varepsilon} \textit{ in } L^\infty([0,\tau]; Z^s), \quad \forall \tau < T_{\max} \textit{ and } \forall s \geqslant 0. \tag{2.10}$$

This result was established in [31] when $Z^s = H^s(\mathbb{R}^d)$, and in [1] when $Z^s = X^s(\mathbb{R}^d)$. Note the shift between the order of the approximation between the initial data (known up to $\mathcal{O}(\varepsilon^2)$) and the approximation (of order $\mathcal{O}(\varepsilon)$ only): this is due to the fact that we consider a regime which is super-critical as far as WKB analysis is concerned (see e.g. [16]). In particular, the phase modulation $\varphi^{(1)}$ is a function of φ_0, a_0 and a_1. It is non-trivial in general, and since we are interested here in real-valued a_0, we shall merely mention two cases (see [16, pp. 69–70]):

- If $a_1 \neq 0$ is real-valued, then $\varphi^{(1)} \neq 0$ in general.
- If $a_1 = 0$ (or more generally if $a_1 \in i\mathbb{R}$), then $\varphi^{(1)} = 0$.

To see the first point, it suffices to notice that the equation for $\varphi^{(1)}$ gives (recall that $f' = 1$):

$$\partial_t\varphi^{(1)}_{|t=0} = -2a_0a_1.$$

This shows that for (2.2)–(1.1) to yield a relevant description of the solution to (2.1), we have to assume $a_0 = \sqrt{\rho_0}$ and $a_1 = 0$. Otherwise, a phase modulation is necessary to describe ψ^ε at leading order, by (2.10), which is incompatible with the form (2.2), unless the Madelung phase ϕ^ε admits a corrector of order ε. But formal asymptotics in (1.1) give $v^\varepsilon = v + \mathcal{O}(\varepsilon^2) = \nabla\varphi + \mathcal{O}(\varepsilon^2)$. Hence the Madelung transform has a chance to give a relevant result only if $a_1 = 0$.

To check the second point of the above assertion, we set $\alpha = \operatorname{Re}\left(\overline{a}a^{(1)}\right)$. Direct computations show that $(\varphi^{(1)},\alpha)$ solves, as soon as $a_0\in\mathbb{R}$ and $a_1\in i\mathbb{R}$:

$$\begin{cases} \partial_t\varphi^{(1)}+\nabla\varphi\cdot\nabla\varphi^{(1)}+2\alpha=0 \quad ; \quad \varphi^{(1)}_{|t=0}=0,\\ \partial_t\alpha+\nabla\varphi\cdot\nabla\alpha=-\dfrac{1}{2}\operatorname{div}\left(|a|^2\nabla\varphi^{(1)}\right)-\alpha\Delta\varphi \quad ; \quad \alpha_{|t=0}=0.\end{cases}$$

This is a linear, homogeneous system, with zero initial data, so its solution is identically zero.

To conclude this paragraph, we briefly outline the proof of Proposition 2.4. The approach in [1] is the same as in [31], with slightly different estimates. For simplicity, and in view of the above discussion, we assume $a_0^\varepsilon=a_0$ independent of ε. We write the solution ψ^ε as $\psi^\varepsilon=a^\varepsilon e^{i\varphi^\varepsilon/\varepsilon}$ (exact formula), where we impose

$$\begin{cases} \partial_t\varphi^\varepsilon+\dfrac{1}{2}|\nabla\varphi^\varepsilon|^2+f\left(|a^\varepsilon|^2\right)=0 \quad ; \quad \varphi^\varepsilon_{|t=0}=\varphi_0,\\ \partial_t a^\varepsilon+\nabla\varphi^\varepsilon\cdot\nabla a^\varepsilon+\dfrac{1}{2}a^\varepsilon\Delta\varphi^\varepsilon=i\dfrac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0}=a_0.\end{cases} \tag{2.11}$$

Note that both φ^ε and a^ε depend on ε, because the right-hand side of the equation for a^ε depends on ε, and because of the coupling between the two equations. Note also that with this approach, one abandons the possibility of considering a real-valued amplitude a^ε.

It is not hard to construct a solution to (2.11) in Z^s, for $s>d/2+2$, and then check that asymptotic expansions are available in Z^s:

$$a^\varepsilon=a+\varepsilon a^{(1)}+\mathcal{O}\left(\varepsilon^2\right) \quad ; \quad \varphi^\varepsilon=\varphi+\varepsilon\varphi^{(1)}+\mathcal{O}\left(\varepsilon^2\right).$$

Back to ψ^ε, this yields Proposition 2.4. We see that the general loss in the precision (from $\mathcal{O}(\varepsilon^2)$ in the initial data to $\mathcal{O}(\varepsilon)$ in the approximation for $t>0$) is due to the division of φ^ε by ε. Note finally that even though $a_1=0$, one has $a^{(1)}\neq 0$: the corrector $a^{(1)}$ solves a linear equation, with a purely imaginary (non trivial) source term, and so $a^{(1)}\in i\mathbb{R}$ is not trivial, while $\varphi^{(1)}=0$ since $\operatorname{Re}\left(\overline{a}a^{(1)}\right)=0$.

2.3.2. *Higher order approximation and formal link with quantum hydrodynamics.* One can actually consider an asymptotic expansion to arbitrary order,

$$\begin{aligned} a^\varepsilon&=a+\varepsilon a^{(1)}+\ldots+\varepsilon^N a^{(N)}+\mathcal{O}\left(\varepsilon^{N+1}\right),\\ \varphi^\varepsilon&=\varphi+\varepsilon\varphi^{(1)}+\ldots+\varepsilon^N\varphi^{(N)}+\mathcal{O}\left(\varepsilon^{N+1}\right), \quad \forall N\in\mathbb{N}.\end{aligned}$$

For $j\geqslant 1$, the coefficient $a^{(j)}$ is given by a linear system for $(\varphi^{(j)},a^{(j)})$, with source terms involving $(\varphi^{(k)},a^{(k)})_{0\leqslant k\leqslant j-1}$. In the case $a_0\in\mathbb{R}$ and $a_1=0$, we know that $\varphi^{(1)}=0$, and $a^{(2)}$ is given by

$$\partial_t a^{(2)}+\nabla\varphi\cdot\nabla a^{(2)}+\frac{1}{2}a^{(2)}\Delta\varphi+\nabla\varphi^{(2)}\cdot\nabla a+\frac{1}{2}a\Delta\varphi^{(2)}=\frac{i}{2}\Delta a^{(1)} \quad ; \quad a^{(2)}_{|t=0}=0.$$

We check that in the case $a_0^\varepsilon=a_0\in\mathbb{R}$ (which includes the case where Madelung transform is used, $a_0\geqslant 0$), all the profiles a and $a^{(2j)}$, $j\geqslant 1$, are real-valued, while $a^{(2j+1)}$, $j\geqslant 0$, are purely imaginary. Moreover, $\varphi^{(2j+1)}=0$ for all $j\in\mathbb{N}$. This is formally in agreement with (1.1): indeed, (1.1) suggests that ϕ^ε and ρ^ε have asymptotic expansions of the form

$$\phi^\varepsilon\approx\phi+\varepsilon^2\phi^{(2)}+\ldots+\varepsilon^{2j}\phi^{(2j)}+\ldots \quad ; \quad \rho^\varepsilon\approx\rho+\varepsilon^2\rho^{(2)}+\ldots+\varepsilon^{2j}\rho^{(2j)}+\ldots \tag{2.12}$$

On the other hand, we have

$$\rho^\varepsilon = |a^\varepsilon|^2 \approx \left| a + \varepsilon a^{(1)} + \dots \right|^2$$
$$\approx \left(a + \dots + \varepsilon^{2j} a^{(2j)} + \dots \right)^2 - \left(\varepsilon a^{(1)} + \dots + \varepsilon^{2j+1} a^{(2j+1)} + \dots \right)^2 ,$$

since the $a^{(2j+1)}$'s are purely imaginary. This is in agreement with the second formal asymptotics in (2.12). We can check similarly that (2.12) is in agreement with the higher order generalization of (2.10), in view of the special properties of the $\varphi^{(j)}$'s and $a^{(j)}$'s pointed out above.

2.4. **Absence of vacuum before shocks.**

Lemma 2.5. *In Theorem 2.3, assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$ (absence of vacuum). Then $\rho > 0$ on $[0, T_{\max}) \times \mathbb{R}^d$.*

Proof. As in Section 2.2, we use the fact that on $[0, T_{\max})$, the equation for the density is just an ordinary differential equation. Introduce the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t, y),$$

where $x(t, y)$ is given by (2.6). We have seen that $J_t(y) > 0$ for $(t, y) \in [0, T_{\max}) \times \mathbb{R}^d$. Change the unknown ρ to r, with

$$r(t, y) = \rho\left(t, x(t, y)\right) J_t(y).$$

Then for $0 \leqslant t < T_{\max}$, the continuity equation is equivalent to: $\partial_t r = 0$. Therefore,

$$\rho(t, x) = \frac{1}{J_t\left(y(t, x)\right)} \rho_0\left(y(t, x)\right),$$

where $x \mapsto y(t, x)$ denotes the inverse mapping of $y \mapsto x(t, y)$. □

We infer:

Proposition 2.6. *Under the assumptions of Theorem 2.3, assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$ (absence of vacuum). Let $0 < T < T_{\max}$, and K be a compact set in $\mathbb{R}^d$. There exists $\varepsilon(T, K) > 0$ such that for $0 < \varepsilon \leqslant \varepsilon(T, K)$, $|\psi^\varepsilon| > 0$ on $[0, T] \times K$.*

Proof. Proposition 2.4 shows that

$$|\psi^\varepsilon| = |a| + \mathcal{O}(\varepsilon) \text{ in } L^\infty\left([0, T] \times \mathbb{R}^d\right).$$

Note that the constant involved in this $\mathcal{O}(\varepsilon)$ depends on T in general. Recalling that $a = \sqrt{\rho}$, Lemma 2.5 shows that

$$\min_{(t,x) \in [0,T] \times K} a(t, x) = c(T, K) > 0.$$

Now for $0 < \varepsilon \leqslant \varepsilon(T, K) \ll 1$,

$$\left| |\psi^\varepsilon| - |a| \right| \leqslant \frac{1}{2} c(T, K) \text{ in } L^\infty\left([0, T] \times \mathbb{R}^d\right),$$

and the result follows. □

In the case $x \in \mathbb{T}^d$, this shows that before the formation of shocks in the Euler equation, and provided that ε is sufficiently small, the amplitude remains positive: the right-hand side of (1.1) makes sense. This point was remarked initially in [27]. Note that the result of [26] in the one-dimensional case $x \in [0, 1]$ shows that suitable boundary conditions lead to the existence of finite time blow-up for (1.1).

Therefore, the above result is qualitatively sharp (qualitatively only, for it might happen that the solution to (1.1) remains smooth longer than the solution to (2.4)).

Finally, we show that the compactness assumption in Proposition 2.6 can be removed in the case of the Gross–Pitaevskii equation. As regards the nonlinear Schrödinger equation on $\mathbb{R}^d$, this issue seems much more delicate and will not be addressed in this paper. Assume that the Gross–Pitaevskii equation is associated with the boundary condition at infinity

$$|\psi^\varepsilon(t,x)| \underset{|x|\to\infty}{\longrightarrow} 1.$$

Such a condition is used frequently in physics, possibly with a stronger one, of the form (see e.g. [35] and references therein)

$$\psi^\varepsilon(t,x) - e^{iv^\infty\cdot x/\varepsilon} \underset{|x|\to\infty}{\longrightarrow} 0,$$

for some fixed asymptotic "velocity" $v^\infty \in \mathbb{R}^d$.

Now, as regards the Gross-Pitaevskii equation, putting together the continuity of ψ^ε over $[0,T]\times\mathbb{R}^d$, the compactness of the time interval $[0,T]$ and Proposition 2.6, we get

Corollary 2.7. *Under the assumptions of Theorem 2.3, assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$ (absence of vacuum). Assume moreover that the Gross–Pitaevskii equation is associated with the boundary condition at infinity*[2]

$$|\psi^\varepsilon(t,x)| \underset{|x|\to\infty}{\longrightarrow} 1.$$

Let $0 < T < T_{\max}$. There exist $\varepsilon(T), c(T) > 0$ such that

$$|\psi^\varepsilon| \geqslant c(T) \quad \text{on} \quad [0,T]\times\mathbb{R}^d, \quad \text{for all} \quad 0<\varepsilon\leqslant\varepsilon(T).$$

2.5. **Functional spaces associated to the Madelung transform.** It is rather easy to see that the answer to Question 2 is no, in general. Consider for ρ_0 the function in the Schwartz class

$$\rho_0(x) = e^{-|x|^{2k}}, \quad k \geq 1.$$

At time $t=0$, the quantum pressure (right-hand side of (1.1)) grows like $|x|^{2k-1}$ hence the velocity v^ε has no chance to belong to H^s for general initial data in H^∞. Thus, working in Sobolev spaces for general initial data does not make sense for (1.1), while the results in [31] show that it is a fairly reasonable framework to study the semi-classical limit of (2.1).

On the other hand, like for the absence of vacuum, the answer to Question 3 is positive, at least if we consider some special boundary conditions at infinity.

3. Solving the QHD system by a direct approach

In the present section, we describe an efficient method to solve directly the hydrodynamic form of (2.1) given by (1.1), once performed the Madelung transform. This method enables us to study the corresponding initial value problem for (1.1) with data (v_0,ρ_0) such that $(v_0,\nabla\rho_0)$ has a high order Sobolev regularity and ρ_0 is *positive and bounded away from zero.* In addition to local-in-time well-posedness results, we get (see Theorems 3.1 and 3.3 below) nontrivial lower bounds on the

[2] Note that this implies that there exists $c>0$ such that $\rho_0(x)\geqslant c$ for all $x\in\mathbb{R}^d$.

first appearance of a zero for the solution, which are of particular interest for the study of long-wavelength asymptotics if the data are a perturbation of a constant state of modulus one.

We here closely follow the approach that has been initiated in [8]. To help the reader to compare the present results with those of the previous section however, we keep on using the semi-classical scaling given by (2.1) (whereas $\varepsilon = 1$ in [8]).

The use of a suitable *extended formulation* for (1.1) and of weighted Sobolev estimates will be the key to our approach. Let us stress that, recently, similar *extended formulations* have proved to be efficient in other contexts for both numerical (see [21]) and theoretical purposes. As a matter of fact, in the last paragraph of this section, we shall briefly explain how our approach based on such an extended formulation carries over to the more complicated case of Korteweg fluids.

3.1. **Solving the QHD system by means of an extended formulation.** The "improved" WKB method that has been described in the previous section amounts to writing the sought solution ψ^ε as:

$$\psi^\varepsilon = a^\varepsilon e^{i\varphi^\varepsilon/\varepsilon}$$

for some complex valued function a^ε and real valued function φ^ε. In this section, we rather start from the Madelung transform

$$\psi^\varepsilon = \sqrt{\rho^\varepsilon} e^{i\phi^\varepsilon} \quad \text{where} \quad \sqrt{\rho^\varepsilon} = |\psi^\varepsilon|,$$

then write

$$\psi^\varepsilon = e^{i\Phi^\varepsilon/\varepsilon} \quad \text{with} \quad \Phi^\varepsilon = \phi^\varepsilon - i\frac{\varepsilon}{2}\log\rho^\varepsilon,$$

and consider the redundant system that is satisfied by both ρ^ε and $z^\varepsilon = \nabla\Phi^\varepsilon = v^\varepsilon + iw^\varepsilon$ with $v^\varepsilon = \nabla\phi^\varepsilon$ and $w^\varepsilon = -\frac{\varepsilon}{2\rho^\varepsilon}\nabla\rho^\varepsilon$.

In order to obtain the system for z^ε, we first differentiate the density equation in (1.1). This yields

$$\partial_t w^\varepsilon + \nabla(v^\varepsilon \cdot w^\varepsilon) = \frac{\varepsilon}{2}\nabla \operatorname{div} v^\varepsilon.$$

Next, we notice that

$$\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} = -\frac{\varepsilon}{2}\operatorname{div} w^\varepsilon + \frac{1}{2}|w^\varepsilon|^2.$$

In consequence, the equation for v^ε rewrites

$$\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon - \frac{1}{2}\nabla|w^\varepsilon|^2 + \frac{\varepsilon}{2}\nabla\operatorname{div} w^\varepsilon + \nabla f(\rho^\varepsilon) = 0.$$

Of course, z^ε is a potential vector-field, hence $\nabla\operatorname{div} z^\varepsilon = \Delta z^\varepsilon$ so that we eventually get the following "extended" system for $(\rho^\varepsilon, z^\varepsilon)$:

$$\left\{ \begin{array}{l} \partial_t z^\varepsilon + \frac{1}{2}\nabla(z^\varepsilon \cdot z^\varepsilon) + \nabla f(\rho^\varepsilon) = i\frac{\varepsilon}{2}\Delta z^\varepsilon, \\ \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0, \end{array} \right. \tag{3.1}$$

where we agree that $a \cdot b := \sum_{j=1}^{d} a_j b_j$ for a and b in $\mathbb{C}^d$.

Let us now explain how Sobolev estimates may be derived for $(\rho^\varepsilon, z^\varepsilon)$ in the case where $\rho^\varepsilon = 1 + b^\varepsilon$ for some b^ε going to 0 at infinity, and $f(\rho^\varepsilon) = \rho^\varepsilon$ (to simplify). The following computations are borrowed from [8]. For notational simplicity, we omit the superscripts ε.

In order to get the basic energy estimate, we compute[3]:

$$\frac{d}{dt}\int_{\mathbb{R}^d}\big((1+b)|z|^2+b^2\big) = 2\Big(\underbrace{\int_{\mathbb{R}^d}(1+b)\langle z, \partial_t z\rangle}_{I_1} + \underbrace{\int_{\mathbb{R}^d} b\,\partial_t b}_{I_2}\Big) + \underbrace{\int_{\mathbb{R}^d}\partial_t b\,|z|^2}_{I_3}$$

where the notation $\langle a, b\rangle = \sum_{j=1}^d \operatorname{Re} a_j \operatorname{Re} b_j + \operatorname{Im} a_j \operatorname{Im} b_j$ has been used in I_1.

Further computations yield $I_1 = I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5}$ and $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{1,1} = -\int \langle z, \nabla b\rangle, \qquad I_{2,1} = -\int b \operatorname{div} v,$$
$$I_{1,2} = -\int b\langle z, \nabla b\rangle, \qquad I_{2,2} = -\int b \operatorname{div}(bv).$$
$$I_{1,3} = \int \langle z, i\frac{\varepsilon}{2}\Delta z\rangle,$$
$$I_{1,4} = \int b\langle z, i\frac{\varepsilon}{2}\Delta z\rangle,$$
$$I_{1,5} = -\frac{1}{2}\int \rho\langle z, \nabla(z\cdot z)\rangle,$$

Using obvious integrations by parts we readily get

$$I_{1,1} + I_{2,1} = 0, \quad I_{1,2} + I_{2,2} = 0 \ \text{ and } \ I_{1,3} = 0.$$

Therefore, integrating by parts in $I_{1,4}$ also, we get

$$\frac{d}{dt}\int\big((1+b)|z|^2+b^2\big) = -\int \varepsilon\langle z, i\nabla z\cdot\nabla b\rangle - 2\int \rho\langle z, \nabla z\cdot z\rangle + \int \partial_t b\,|z|^2.$$

For "general" functions b and z, the appearance of the terms ∇b and ∇z would preclude any attempt to "close" the estimates. In our case however, as the algebraic relation $-\varepsilon\nabla b = 2\rho w$ holds true, one may avoid this loss of one derivative for one may write

$$-\varepsilon\langle z, i\nabla z\cdot\nabla b\rangle - 2\rho\langle z, \nabla z\cdot z\rangle = -2\rho\langle z, \nabla z\cdot v\rangle.$$

Now, integrating by parts an ultimate time, we conclude that

$$\frac{d}{dt}\int\big(\rho|z|^2+b^2\big) = -\int \rho v\cdot\nabla|z|^2 + \int \partial_t b\,|z|^2 = 0.$$

Hence, $\int\big(\rho|z|^2+b^2\big)$ is a conserved quantity.

The same algebraic cancellations may be used for getting higher order Sobolev (or Besov) estimates. Indeed consider an "abstract" pseudo-differential operator

[3] The method may seem uselessly complicated. However, the algebraic cancellations that are going to be used remain the same when estimating higher order Sobolev norms.

$A(D)$ (for instance a differential, a fractional derivatives or a spectral localization operator). Then one may write

$$\frac{d}{dt}\int\big((1+b)|A(D)z|^2+(A(D)b)^2\big)$$
$$=2\Big(\underbrace{\int(1+b)\langle A(D)z,\partial_t A(D)z\rangle}_{I_1}+\underbrace{\int A(D)b\,\partial_t A(D)b}_{I_2}\Big)+\underbrace{\int\partial_t b\,|A(D)z|^2}_{I_3}.$$

We notice that $I_1=I_{1,1}+I_{1,2}+I_{1,3}+I_{1,4}+I_{1,5}$ and $I_2=I_{2,1}+I_{2,2}$ with

$$I_{1,1}=-\int\langle A(D)z,\nabla A(D)b\rangle,\qquad I_{2,1}=-\int A(D)b\ \mathrm{div}\,A(D)v,$$
$$I_{1,2}=-\int b\langle A(D)z,\nabla A(D)b\rangle,\qquad I_{2,2}=-\int A(D)b\,A(D)\,\mathrm{div}(bv).$$
$$I_{1,3}=\int\langle A(D)z,i\frac{\varepsilon}{2}\Delta A(D)z\rangle,$$
$$I_{1,4}=\int b\langle A(D)z,i\frac{\varepsilon}{2}\Delta A(D)z\rangle,$$
$$I_{1,5}=-\frac{1}{2}\int\rho\langle A(D)z,A(D)\nabla(z\cdot z)\rangle,$$

As above, obvious integrations by parts ensure that $I_{1,1}+I_{2,1}=0$ and $I_{1,3}=0$. Next, using again integrations by parts, we notice that

$$I_{1,2}+I_{2,2}=\int A(D)b\ \mathrm{div}[b,A(D)]v.$$

Finally, integrating by parts in $I_{1,4}$ and using the fact that $-\varepsilon\nabla b=2\rho w$ yields

$$I_{1,4}=\int\langle A(D)z,i(\nabla A(D)z)\cdot(\rho w)\rangle,$$

and we have

$$I_{1,5}=-\int\langle A(D)z,A(D)\nabla z\cdot(\rho v+i\rho w)\rangle+\int\rho\langle A(D)z,\nabla A(D)z\cdot z-A(D)(\nabla z\cdot z)\rangle.$$

Therefore, using the fact that $\partial_t\rho+\mathrm{div}(\rho v)=0$, we conclude that

$$2(I_{1,4}+I_{1,5})+I_3=2\int\rho\langle A(D)z,\nabla A(D)z\cdot z-A(D)(\nabla z\cdot z)\rangle.$$

Putting all the above equalities together, we thus get

$$\frac{1}{2}\frac{d}{dt}\int\big(\rho|A(D)z|^2+(A(D)b)^2\big)=\int A(D)b\ \mathrm{div}[b,A(D)]v$$
$$+\int\rho\langle A(D)z,\nabla A(D)z\cdot z-A(D)(\nabla z\cdot z)\rangle.$$

If, say, $A(D)$ is a fractional derivatives operator, then one may show by means of classical commutator estimates that the right-hand side may be bounded by

$$C\|\rho\|_{L^\infty}\|(Db,Dz)\|_{L^\infty}\|(A(D)b,A(D)z)\|_{L^2}^2.$$

Therefore,

$$\frac{d}{dt}\int\big(\rho|A(D)z|^2+(A(D)b)^2\big)\leqslant C\|\rho\|_{L^\infty}\|(Db,Dz)\|_{L^\infty}\|(A(D)b,A(D)z)\|_{L^2}^2.$$

Denoting by $E_A^2(t)$ the left-hand side and resorting to Gronwall lemma, we thus get

$$E_A(t) \leqslant E_A(0)\exp\biggl(C\int_0^t \|\rho\|_{L^\infty}\|\rho^{-1}\|_{L^\infty}\|(Db,Dz)\|_{L^\infty}\,d\tau\biggr). \tag{3.2}$$

It is now clear that whenever Db and Dz are bounded in $L^1([0,T];L^\infty)$ and ρ is bounded from below and from above then we get a control of $A(D)b$ and $A(D)z$ in $L^\infty([0,T];L^2)$.

Taking $A(D)=\Lambda^s$ and performing a time integration, (3.2) implies that

$$\|(b,z)(t)\|_{H^s} \leq C\biggl(\|(b_0,z_0)\|_{H^s} + \int_0^T \|(Db,Dz)\|_{L^\infty}\|(b,z)(t)\|_{H^s}\,dt\biggr) \tag{3.3}$$

for some constant $C=C(s,d,\|\rho^{\pm1}\|_{L^\infty})$.

So assuming that $s>1+d/2$ and using Sobolev embedding and Gronwall's inequality, one may conclude by elementary methods to the following statement.

Theorem 3.1. *Let $s>1+d/2$. Assume that $\rho_0=1+b_0$ for some $b_0\in H^{s+1}(\mathbb{R}^d)$ such that $1+b_0>0$, and that $v_0\in H^s(\mathbb{R}^d)$. Then there exists a time*

$$T\geqslant T_0 := \frac{C}{\|b_0\|_{H^s}+\varepsilon\|\nabla b_0\|_{H^s}+\|v_0\|_{H^{s+1}}} \quad \textit{with} \;\; C=C(s,d,\|\rho_0^{\pm1}\|_{L^\infty})$$

such that (1.1) *has a unique solution $(v^\varepsilon,\rho^\varepsilon)$ on $[0,T]\times\mathbb{R}^d$ with $\rho^\varepsilon=1+b^\varepsilon$ bounded away from* 0 *and $(v^\varepsilon,b^\varepsilon)\in C([0,T];H^s\times H^{s+1})\cap C^1([0,T];H^{s-2}\times H^{s-1})$.*

Remark 3.2. Combining basic energy estimates for the wave equation with the above result, one may control the discrepancy between (b,v) and the solution to the acoustic wave equation

$$\left\{\begin{array}{lll} \partial_t\dot v+\nabla\dot b=0 & ; & \dot v_{|t=0}=v_0,\\ \partial_t\dot b+\operatorname{div}\dot v=0 & ; & \dot b_{|t=0}=b_0. \end{array}\right. \tag{3.4}$$

We have, up to time T_0,

$$\|(v^\varepsilon-\dot v,b^\varepsilon-\dot b)(t)\|_{H^{s-2}} \leq C\bigl(t\|(b_0,u_0)\|^2_{H^{s+1}\times H^s}+\varepsilon^2 t\|(b_0,u_0)\|_{H^{s+1}\times H^s}\bigr).$$

Note also that (3.3) provides a blow-up criterion involving the $W^{2,\infty}$ norm of b and the Lipschitz norm of v. In particular, this implies that for given data in H^s ($s>1+d/2$), the lifespan in H^s is the same as the lifespan in $H^{s'}$, for any $1+d/2<s'<s$.

3.2. **Dispersive properties and improved lower bounds for the lifespan.** The system for (b,v) reads

$$\left\{\begin{array}{l} \partial_t v+v\cdot\nabla v+\nabla b=\dfrac{\varepsilon^2}{4}\nabla\Bigl(\dfrac{1}{\rho}\Delta b-\dfrac{1}{2\rho^2}|\nabla b|^2\Bigr),\\ \partial_t b+\operatorname{div} v=-\operatorname{div}(bv). \end{array}\right. \tag{3.5}$$

Therefore the linearized system about $(0,0)$ is not (3.4) but rather

$$\left\{\begin{array}{l} \partial_t\dot v+\nabla\dot b=\dfrac{\varepsilon^2}{4}\nabla\Delta\dot b,\\ \partial_t\dot b+\operatorname{div}\dot v=0. \end{array}\right. \tag{3.6}$$

A straightforward spectral analysis (based on the Fourier transform) shows that the above linear system behaves as the wave equation with speed 1 for frequencies small with respect to $1/\varepsilon$, and as the Schrödinger equation with coefficient $\varepsilon/2$ in the high frequency regime. In fact, in dimension $d \geqslant 2$, it is possible to prove Strichartz inequalities (related to the wave and Schrödinger equations for low and high frequencies, respectively) for (3.6). In the case of small data (b_0, v_0), these inequalities allow to improve the lower bound for the lifespan (see also [22] where a similar idea has been used in the context of the incompressible limit for compressible flows). For the sake of simplicity, let us just state the result in dimension $d \geqslant 4$ (the reader is referred to [8] for the case $d = 2, 3$ and for more details concerning the approximation of the solution by (3.6)):

Theorem 3.3. *Under the assumptions of Theorem 3.1 with $s > 2 + d/2$, then the lifespan T may be bounded from below by*

$$T_1 := \frac{C}{(\|b_0\|_{H^s} + \varepsilon\|\nabla b_0\|_{H^s} + \|v_0\|_{H^{s+1}})^2},$$

and the discrepancy between (b, v) and the solution $(\dot{b}, \dot{v})$ to (3.6) with the same data may be bounded in terms of t and of the data, up to time T_1.

3.3. **Extended formulation for Korteweg fluids.** Compared to the "improved" WKB method, the main drawback of the direct approach based on an extended formulation for solving (1.1) is that vanishing solutions cannot be handled.

On the other hand, the direct method is robust enough so as to be used to solve locally more complicated models such as the following system governing the evolution of inviscid capillary fluids:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla f(\rho) = \nabla\Big(\kappa(\rho)\Delta\rho + \dfrac{1}{2}\kappa'(\rho)|\nabla\rho|^2\Big), \\ \partial_t \rho + \operatorname{div}(\rho v) = 0. \end{cases} \tag{3.7}$$

Physically, the function κ correspond to the capillary coefficient. Obviously, System (1.1) is included in (3.7) (take $\kappa(\rho) = \varepsilon^2/(4\rho)$). In the general case, introducing

$$a(\rho) = \sqrt{\rho\kappa(\rho)}, \quad w = -\sqrt{\frac{\kappa(\rho)}{\rho}}\,\nabla\rho \ \text{ and } \ z = v + iw,$$

we get the following extended formulation for (3.7):

$$\begin{cases} \partial_t z + v \cdot \nabla z + i\nabla z \cdot w + \nabla f(\rho) = i\nabla(a(\rho)\operatorname{div} z), \\ \partial_t \rho + \operatorname{div}(\rho v) = 0. \end{cases} \tag{3.8}$$

Note that in the potential case (namely $\operatorname{curl} z = 0$) then $v \cdot \nabla z + i\nabla z \cdot w = \frac{1}{2}\nabla(z \cdot z)$. Note also that in the general case, the second order term $\nabla(a(\rho)\operatorname{div} z)$ is degenerate.

The case of System (1.1) is particularly simple inasmuch as a is the constant function $\varepsilon/2$ and $\nabla \operatorname{div} z = \Delta z$.

For general capillarity coefficients, one may prove a local well-posedness result, similar to that of Theorem 3.1. This has been done in [7]. The proof relies on the use of weighted Sobolev estimates, with a weight depending both on ρ and on the order of differentiation.

As for the QHD system, we expect the potential part of the solution to System (3.7) to have dispersive properties in dimension $d \geqslant 2$. The general situation is much more complicated however, because those properties are related to those of the quasilinear Schrödinger equation. To our knowledge, this aspect has been investigated only very recently in a work by C. Audiard [5] that concerns potential flows.

4. Asymptotics for the Gross–Pitaevskii equation

This section is concerned with the existence and asymptotics of traveling wave solutions for the Gross-Pitaevskii equation

$$i\psi_t + \Delta\psi + (1 - |\psi|^2)\psi = 0, \tag{4.1}$$

which may be obtained from (2.1) (with $f(r) = r - 1$), up to the factor $\frac{1}{2}$, after performing the change of unknown:

$$\psi^\varepsilon(t, x) = \psi(\varepsilon^{-1}t, \varepsilon^{-1}x).$$

Equation (4.1) is associated to the *Ginzburg-Landau energy* (or Hamiltonian):

$$\mathcal{H}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi|^2 + \frac{1}{4}\int_{\mathbb{R}^d} (1 - |\psi|^2)^2. \tag{4.2}$$

As a consequence, in contrast with the cubic NLS equation, the natural energy space for (4.1) is not $H^1(\mathbb{R}^d)$ but rather

$$E(\mathbb{R}^d) = \left\{\psi \in H^1_{\rm loc}(\mathbb{R}^d),\ \text{s.t. } \mathcal{H}(\psi) < +\infty\right\}\cdot$$

As pointed out before, for $\mathcal{H}(\psi)$ to be finite, $|\psi|$ must, in some sense, tend to 1 at infinity. This "nontrivial" boundary condition provides (4.1) with a richer dynamics than in the case of null condition at infinity which, for a defocusing NLS type equation, is essentially governed by dispersion and scattering. For instance, in nonlinear optics, the "dark solitons" are localized nonlinear waves (or "holes") which exist on a stable continuous wave background. The boundary condition $|\psi(t, x)| \to 1$ at infinity is due to this nonzero background. In the context of superfluids, 1 is the density of the fluid at infinity.

Similarly to the energy, the *momentum*

$$\mathcal{P}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} \langle i\nabla\psi\ , \psi\rangle,$$

is formally conserved. This quantity is well defined for $\psi \in H^1(\mathbb{R}^d)$ but not for solutions with a finite Ginzburg-Landau energy. A major difficulty in the theory of the Gross-Pitaevskii equation is to find an appropriate definition of the momentum which leads to a conserved quantity. A natural definition would be

$$\mathcal{P}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} \langle i\nabla\psi\ , \psi - 1\rangle,$$

but this would require for instance that $\psi - 1 \in L^2(\mathbb{R}^d)$.

In any case we will denote by $\mathfrak{p}$ the (scalar) first component of $\mathcal{P}$ which will play an important role in this section.

Recall that the use of the Madelung transform[4]

$$\psi = \varrho e^{i\phi},$$

leads to the following hydrodynamic form of the equation for ϱ and $v = 2\nabla\phi$:

$$\begin{cases} \partial_t v + v\cdot\nabla v + 2\nabla\varrho^2 = 2\nabla\left(\dfrac{\Delta\varrho}{\varrho}\right), \\ \partial_t\varrho^2 + \operatorname{div}(\varrho^2 v) = 0. \end{cases} \tag{4.3}$$

[4]In this section, we use the normalization of [9, 10], instead of $\psi = \sqrt{\rho}e^{i\phi}$.

As pointed out before, if we discard the right-hand side of the first equation and look at ϱ^2 as the density of a fluid with velocity v, then the above system coincides with the Euler equations for a compressible fluid with pressure law $P(\rho) = \rho^2$. In particular, the speed of sound waves near the constant solution $v = 1$ is given by

$$c_s = \sqrt{2}.$$

As we will see below, this sound speed (the value and relevance of which is not so obvious if looking at the original Gross-Pitaevskii equation (4.1)), plays an important role in various aspects of the dynamics of (4.1). The value of c_s may be also found by neglecting the quantum pressure term and linearizing for a perturbation $\psi = (1 + \tilde{\varrho}) \exp(i\tilde{\phi})$. This leads to the wave equation:

$$\partial_t^2 \tilde{\varrho} - 2\Delta\tilde{\varrho} = 0.$$

Note that if the quantum pressure is included (as in (3.6)) then the linearization reads:

$$\partial_t^2 \tilde{\varrho} - 2\Delta\tilde{\varrho} - \Delta^2 \tilde{\varrho} = 0,$$

which, roughly, is the factorization of two linear Schrödinger operators.

The Madelung transform and the hydrodynamic form of (4.1) turn out to be of great interest to study the Gross-Pitaevskii equation with finite Ginzburg–Landau energy since the solution is expected to have very few "vortices" (or cancellations). Even in the "Euler limit" that has been presented in Section 2, one can use it outside the vortices ([14, 15]) to study the traveling waves of sufficiently small velocities. Let us also stress that the hydrodynamic form of the (one dimensional) Gross–Pitaevskii equation is needed in order to define a generalized momentum in the context of the orbital stability of the black solitons (such solitary waves have zeroes. . .); see [11].

In the present section, we shall concentrate on the *transonic limit* of solutions to the Gross-Pitaevskii equation. We shall first present a result pertaining to the asymptotics of traveling waves with speed c tending to the sound speed c_s, in the case $d = 2$, in connexion with the (KP I) equation (see below). Next, for the one-dimensional case, we give an accurate description of the transonic long wave limit of (4.1) in terms of solutions to the KdV equation.

4.1. **The transonic limit of finite energy traveling waves.** Finite energy traveling wave solutions of (4.1) are solutions of the form $\psi(x, t) = \theta(x_1 - ct, x^\perp)$ where $\mathcal{H}(\theta) < +\infty$ and $x^\perp$ denotes the transverse variables $x_2, \cdots, x_d$. The profile θ satisfies the following equation:

$$ic\partial_1\theta + \Delta\theta + \theta(1 - |\theta|^2) = 0. \tag{4.4}$$

A suitable functional setting for the study of such traveling waves is the space:

$$W(\mathbb{R}^d) = \{1\} + V(\mathbb{R}^d),$$

with

$$V(\mathbb{R}^d) = \{\psi : \mathbb{R}^d \mapsto \mathbb{C},\ (\nabla\psi, \operatorname{Re}\psi) \in L^2(\mathbb{R}^d)^2, \operatorname{Im}\psi \in L^4(\mathbb{R}^d),\ \nabla \operatorname{Re}\psi \in L^{4/3}(\mathbb{R}^d)\}\cdot$$

Indeed, given that $W(\mathbb{R}^d)$ is a subset of the energy space $E(\mathbb{R}^d)$, for any data in $W(\mathbb{R}^d)$, Equation (4.1) admits a unique solution. In addition, one may show that this solution stays in $W(\mathbb{R}^d)$ (see [28, 29]). Furthermore, the quantity $\langle i\partial_1\psi, \psi - 1\rangle$ is integrable whenever $\psi \in W(\mathbb{R}^d)$, so that the scalar momentum $\mathfrak{p}(\psi)$ is well-defined. This is a consequence of the identity

$$\langle i\partial_1\psi, \psi - 1\rangle = \partial_1(\operatorname{Re}\psi)\operatorname{Im}\psi - \partial_1(\operatorname{Im}\psi)(\operatorname{Re}\psi - 1), \tag{4.5}$$

and various Hölder's inequalities.

So finally, $\mathcal{H}$ and $\mathfrak{p}$ are continuous on $W(\mathbb{R}^d)$ and all finite energy subsonic solutions to (4.4) have to belong to $W(\mathbb{R}^d)$. Moreover, if $\psi \in W(\mathbb{R}^d)$ may be lifted as $\psi = \varrho\exp(i\phi)$ then

$$\mathfrak{p}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d}\langle i\partial_1\psi\ , \psi - 1\rangle = \frac{1}{2}\int_{\mathbb{R}^d}(1-\varrho^2)\partial_1\phi. \tag{4.6}$$

Notice that for maps which may be lifted, with $\varrho \geqslant \frac{1}{2}$, the last integral makes sense, even if we assume that ψ only belongs to the energy space $E(\mathbb{R}^d)$.

To simplify the presentation, we shall focus on the simpler case $d = 2$. We shall denote $x = x_1$ and $y = x^\perp = x_2$.

It is proven in [9] that, for any $\mathfrak{p} > 0$, the following minimization problem

$$\mathcal{H}_{\min}(\mathfrak{p}) = \inf\{\mathcal{H}(\psi), \psi \in W(\mathbb{R}^2), \mathfrak{p}(\psi) = \mathfrak{p}\}, \tag{4.7}$$

has a solution $u_{\mathfrak{p}}$ which is a nontrivial traveling wave. We call it a *ground state*.

In the rest of this subsection, we focus on the asymptotics $\mathfrak{p}$ going to 0 for $u_{\mathfrak{p}}$, in connexion with the Kadomtsev–Petviashvili I (KP I) equation

$$u_t + uu_x + u_{xxx} - \partial_x^{-1}u_{yy} = 0,$$

where the antiderivative is defined in Fourier variables by $\widehat{\partial_x^{-1}f}(\xi) = \frac{1}{i\xi}\widehat{f}(\xi)$.

The following proposition states that the corresponding speed $c(u_{\mathfrak{p}})$ tends to c_s and gives the first term in the asymptotic expansion for both $\mathcal{H}_{\min}(\mathfrak{p})$ and $c(u_{\mathfrak{p}})$.

Proposition 4.1. *There exist positive constants $\mathfrak{p}_1$, K_0 and K_1 such that we have the asymptotic behaviors*

$$\frac{48\sqrt{2}}{\mathcal{S}_{KP}^2}\mathfrak{p}^3 - K_0\mathfrak{p}^4 \leqslant \sqrt{2}\mathfrak{p} - \mathcal{H}_{\min}(\mathfrak{p}) \leqslant K_1\mathfrak{p}^3, \quad \forall 0 \leqslant \mathfrak{p} \leqslant \mathfrak{p}_1, \tag{4.8}$$

where $\mathcal{S}_{KP}$ stands for the action of the KP I ground state N of velocity 1, that is

$$\mathcal{S}_{KP} = \frac{1}{2}\int_{\mathbb{R}^2}(\partial_x N)^2 + \frac{1}{2}\int_{\mathbb{R}^2}(\partial_x^{-1}(\partial_y N))^2 - \frac{1}{6}\int_{\mathbb{R}^2}N^3 + \frac{1}{2}\int_{\mathbb{R}^2}N^2. \tag{4.9}$$

Moreover, the map $u_{\mathfrak{p}}$ has no zeroes and there exist some positive constants $\mathfrak{p}_2$, K_2 and K_3 such that

$$K_2\mathfrak{p}^2 \leqslant \sqrt{2} - c(u_{\mathfrak{p}}) \leqslant K_3\mathfrak{p}^2, \quad \forall 0 \leqslant \mathfrak{p} < \mathfrak{p}_2. \tag{4.10}$$

Actually, it was established in [9] that if ψ is a finite energy traveling wave of sufficiently small energy, then

$$\frac{1}{2} < |\psi| < 1.$$

This implies that small energy traveling waves have no zeroes and thus can be lifted according to the Madelung transformation. This is in particular the case of minimizers $u_{\mathfrak{p}}$ corresponding to small enough values of $\mathfrak{p}$.

The *transonic limit* to KP I for traveling waves is obtained through the following change of scales:

$$\tilde{x} = \epsilon(\psi)x, \quad \tilde{y} = \frac{\epsilon(\psi)^2}{\sqrt{2}}y.$$

We then set

$$\eta = 1 - |\psi|^2, \quad N_\psi(x, y) = \frac{6}{\epsilon(\psi)^2}\eta\left(\frac{x}{\epsilon(\psi)}, \frac{\sqrt{2}y}{\epsilon(\psi)^2}\right).$$

It turns out that N_ψ converges to a traveling wave solution of the (differentiated) KP I equation [5] as $\epsilon(\psi) \to 0$, that is it approximately solves the equation

$$-w_{xx} - w_{yy} + w_{xxxx} + (ww_x)_x = 0. \tag{4.11}$$

More precisely, denoting $N_\mathfrak{p} = 1 - |u_\mathfrak{p}|^2$ where $u_\mathfrak{p} = |u_\mathfrak{p}|e^{i\phi_\mathfrak{p}}$ is a minimizer of the energy with fixed momentum $\mathfrak{p}$, and

$$\Theta_\mathfrak{p}(x, y) = \frac{6\sqrt{2}}{\epsilon_\mathfrak{p}}\phi_\mathfrak{p}\left(\frac{x}{\epsilon_\mathfrak{p}}, \frac{\sqrt{2}y}{\epsilon_\mathfrak{p}^2}\right),$$

we have (see [10]):

Theorem 4.2. *There exist a subsequence $(\mathfrak{p}_n)_{n\in\mathbb{N}}$ tending to 0, as $n \to +\infty$, and a ground state w of the KP I equation such that both $N_{\mathfrak{p}_n}$ and $\Theta_{\mathfrak{p}_n}$ tend to w in $W^{k,q}(\mathbb{R}^2)$ (for any $k \in \mathbb{N}$ and $q \in (1, +\infty]$) as n goes to $+\infty$.*

Remark 4.3. Those results on the transonic limit of solitary waves have been recently extended in [19] to the three-dimensional case with also general nonlinearities (see also [39] for other existence results). The additional serious difficulty is that the Gross–Pitaevskii ground states solutions are *no longer* global minimizers of the energy with fixed momentum and thus not expected to be stable.

4.2. **The unsteady transonic long wave limit of the Gross–Pitaevskii equation.** The Madelung transform is also crucial to derive and justify the transonic (weak amplitude, long wave) limit of the Gross–Pitaevskii equation.

With the scaling which is used in this section and for data which are perturbations of order ε^2 of a constant state with modulus 1, Theorem 3.1 and the remark that follows ensure that the linear wave equation gives a good approximation of the solution for $t = o(\varepsilon^{-3})$. In the present paragraph, we describe what happens at next order. We shall see in particular that, up to times of order ε^{-3}, the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0,$$

gives an accurate approximation of the one-dimensional Gross-Pitaevskii equation

$$i\partial_t\psi + \partial_{xx}\psi + (1 - |\psi|^2)\psi = 0 \tag{4.12}$$

with data which are small long-wave perturbations of the constant one, namely $\psi = \varrho e^{i\phi}$ with

$$\varrho_0 = \left(1 - \frac{\varepsilon^2}{6}N_\varepsilon^0(\varepsilon x)\right)^{1/2}, \quad \phi_0 = \frac{\varepsilon}{6\sqrt{2}}\Theta_\varepsilon^0(\varepsilon x), \tag{4.13}$$

[5] Actually, to a ground state solution of the KP I equation, that is a minimizer of the Hamiltonian with fixed L^2 norm.

for $0 < \varepsilon \ll 1$, and N_ε^0 and $W_\varepsilon^0 = \partial_x \Theta_\varepsilon^0$ are uniformly bounded in some Sobolev space $H^k(\mathbb{R})$ for sufficiently large k.

We refer to [12, 13] for a detailed analysis and will only summarize the limit to (long) waves propagating in two directions and following a coupled system of KdV equations.

Recall that the one-dimensional Gross–Pitaevskii equation (4.12) is globally well posed in the Zhidkov type spaces Y^k,

$$Y^k(\mathbb{R}) = \left\{\psi \in L^1_{\rm loc}(\mathbb{R};\mathbb{C}),\quad 1-|\psi|^2 \in L^2(\mathbb{R}),\quad \partial_x\psi \in H^{k-1}(\mathbb{R})\right\},$$

for any integer $k \geqslant 1$.

Moreover the Ginzburg-Landau energy $\mathcal{H}(\psi(t))$ is conserved by the flow and, provided $\mathcal{H}(\psi_0) < \frac{2\sqrt{2}}{3}$, the corresponding solution $\psi(t)$ does not vanish so that one may write $\psi = \varrho \exp(i\phi)$, for some continuous function ϕ.

We consider data as in (4.13) with small enough ε and assume in addition that

$$\|N_\varepsilon^0\|_{\mathcal{M}(\mathbb{R})} + \|\partial_x \Theta_\varepsilon^0\|_{\mathcal{M}(\mathbb{R})} < +\infty.$$

Here, $\|\cdot\|_{\mathcal{M}(\mathbb{R})}$ denotes the norm defined on $L^1_{\rm loc}(\mathbb{R})$ by

$$\|f\|_{\mathcal{M}(\mathbb{R})} = \sup_{(a,b)\in\mathbb{R}^2}\left|\int_a^b f(x)dx\right|.$$

We next introduce the *slow coordinates*

$$x^- = \varepsilon(x+\sqrt{2}t),\ x^+ = \varepsilon(x-\sqrt{2}t),\quad \text{and } \tau = \frac{\varepsilon^3}{2\sqrt{2}}t.$$

The definition of the coordinates x^- and x^+ corresponds to reference frames traveling to the left and to the right, respectively, with speed $\sqrt{2}$ in the original coordinates (t,x). We define accordingly the rescaled functions $N_\varepsilon^\pm$ and $\Theta_\varepsilon^\pm$ as follows:

$$\left\{\begin{aligned} &N_\varepsilon^\pm(\tau,x^\pm) = \frac{6}{\varepsilon^2}\eta(t,x) = \frac{6}{\varepsilon^2}\eta\left(\frac{2\sqrt{2}\tau}{\varepsilon^3}, \frac{x^\pm}{\varepsilon} \pm \frac{4\tau}{\varepsilon^3}\right) \qquad \text{with}\quad \eta = 1-\varrho^2,\\ &\Theta_\varepsilon^\pm(\tau,x^\pm) = \frac{6\sqrt{2}}{\varepsilon}\phi(t,x) = \frac{6\sqrt{2}}{\varepsilon}\phi\left(\frac{2\sqrt{2}\tau}{\varepsilon^3}, \frac{x^\pm}{\varepsilon} \pm \frac{4\tau}{\varepsilon^3}\right). \end{aligned}\right. \tag{4.14}$$

Setting

$$\left\{\begin{aligned} &U_\varepsilon^-(\tau,x^-) = \frac{1}{2}\left(N_\varepsilon^-(\tau,x^-) + \partial_{x^-}\Theta^-(\tau,x^-)\right),\\ &U_\varepsilon^+(\tau,x^+) = \frac{1}{2}\left(N_\varepsilon^+(\tau,x^+) - \partial_{x^+}\Theta^+(\tau,x^+)\right), \end{aligned}\right. \tag{4.15}$$

the main result is (see [13] for details):

Theorem 4.4. *Let $k \geqslant 0$ and $\varepsilon > 0$ be given. Assume that the initial data ψ_0 belongs to $Y^{k+6}(\mathbb{R})$ and satisfies the assumption*

$$\|N_\varepsilon^0\|_{\mathcal{M}(\mathbb{R})} + \|\partial_x\Theta_\varepsilon^0\|_{\mathcal{M}(\mathbb{R})} + \|N_\varepsilon^0\|_{H^{k+5}(\mathbb{R})} + \varepsilon\|\partial_x^{k+6}N_\varepsilon^0\|_{L^2(\mathbb{R})} + \|\partial_x\Theta_\varepsilon^0\|_{H^{k+5}(\mathbb{R})} \leqslant K_0.$$

Let $\mathcal{U}^-$ and $\mathcal{U}^+$ denote the solutions to the Korteweg–de Vries equations

$$\partial_\tau\mathcal{U}^- + \partial^3_{x^-}\mathcal{U}^- + \mathcal{U}^-\partial_{x^-}\mathcal{U}^- = 0,$$

and

$$\partial_\tau\mathcal{U}^+ - \partial^3_{x^+}\mathcal{U}^+ - \mathcal{U}^+\partial_{x^+}\mathcal{U}^+ = 0,$$

with the same initial value as U_ε^- and U_ε^+, respectively. Then, there exist positive constants ε_1 and K_1, depending only on k and K_0, such that

$$\|U_\varepsilon^-(\tau,\cdot)-\mathcal{U}^-(\cdot,\tau)\|_{H^k(\mathbb{R})}+\|U_\varepsilon^+(\tau,\cdot)-\mathcal{U}^+(\cdot,\tau)\|_{H^k(\mathbb{R})}\leqslant K_1\varepsilon^2\exp\left(K_1|\tau|\right),$$

for any $\tau\in\mathbb{R}$ provided $\varepsilon\leqslant\varepsilon_1$.

We now turn to the two (or higher) dimensional case, which is studied in [20] for a general nonlinear Schrödinger equation of the form similar to (4.1)

$$i\partial_t\psi+\Delta\psi=f(|\psi|^2)\psi,$$

where f is smooth and satisfies $f(1)=0$, $f'(1)>0$.

One also uses a "weakly transverse transonic" scaling, namely

$$T=c\varepsilon^3 t,\quad X_1=\varepsilon(x_1-ct),\quad X_j=\varepsilon^2 x_j,\quad j=2,\cdots,d.$$

After performing the ansatz

$$\psi^\varepsilon(t,X)=\left(1+\varepsilon^2A^\varepsilon(t,X)\right)\exp(i\varepsilon\phi^\varepsilon(t,X)),$$

the hydrodynamic reformulation of the Gross–Pitaevskii equation is used to recast the problem as a singular limit for an hyperbolic system in the spirit of [31]. Then smooth H^s solutions are proven to exist on an interval independent of the small parameter ε. Passing to the limit by a compactness argument yields the convergence of the solutions to that of the KP-I equation. Note however that this method does not provide a convergence rate with respect to ε, contrary to the KdV case considered above.

In comparison, for such data, Theorem 3.3 would ensure that the linear system (3.6) gives a good description of the solution only for times that are $o(\varepsilon^{-3})$ (see the introduction of [8] for more details).

5. Global existence of weak solutions to a quantum fluids system

We aim at providing an elementary proof of the result by P. Antonelli and P. Marcati in [3, 4] concerning global finite energy weak solutions to the QHD system. Here is the statement:

Theorem 5.1. *Let the initial data $(\rho_0,\Lambda_0)\in W^{1,1}\times L^2$ be "well-prepared" in the sense that there exists some wave function $\psi_0\in H^1$ such that*

$$\rho_0=|\psi_0|^2\quad\text{and}\quad J_0:=\sqrt{\rho_0}\Lambda_0=\operatorname{Im}(\bar\psi_0\nabla\psi_0).$$

Assume that $f(r)=r^\sigma$ for some integer σ such that $W^{1,1}(\mathbb{R}^d)\hookrightarrow L^{\sigma+1}(\mathbb{R}^d)$.

There exist some vector-field $\Lambda\in L^\infty(\mathbb{R};L^2)$ and some nonnegative function $\rho\in L^\infty(\mathbb{R};L^1\cap L^{\sigma+1})$ with $\nabla\sqrt{\rho}\in L^\infty(\mathbb{R};L^2)$ such that the following system holds true in the distributional sense[6]:

$$\begin{cases}\partial_t\rho+\operatorname{div}J=0,\\[2mm] \partial_t J+\operatorname{div}(\Lambda\otimes\Lambda)+\nabla(P(\rho))=\dfrac14\Delta\nabla\rho-\operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}),\\[2mm] \partial_j J^k-\partial_k J^j=2\Lambda^k\partial_j\sqrt{\rho}-2\Lambda^j\partial_k\sqrt{\rho}\quad\textit{for all }(j,k)\in\{1,\cdots,d\}^2,\\[2mm] (\rho,J)_{|t=0}=(\rho_0,J_0),\end{cases}\tag{5.1}$$

[6]In the smooth non-vanishing case, the right-hand side of the second equation coincides with that of the velocity equation multiplied by ρ in (1.1) with $\varepsilon=1$, and the third equation just means that there exists some function ϕ such that $J=\rho\nabla\phi$.

with $J := \sqrt{\rho}\Lambda$ *and* $P(\rho) := \rho f(\rho) - F(\rho)$ *with* $F(\rho) = \int_0^\rho f(\rho')\, d\rho'$.

In addition, the energy

$$\int_{\mathbb{R}^d} \Big(\frac{1}{2}|\Lambda|^2 + \frac{1}{2}|\nabla\sqrt{\rho}|^2 + F(\rho) \Big)$$

is conserved for all time.

Proof. Let us first prove the statement in the smooth case, namely we assume that the data ψ_0 is in H^s for some large enough s. It is well known that

$$i\partial_t \psi + \frac{1}{2}\Delta\psi = f\left(|\psi|^2\right)\psi \quad ; \quad \psi_{|t=0} = \psi_0 \tag{5.2}$$

has a unique solution ψ in $C(\mathbb{R}; H^s)$ whenever $s \geqslant 1$ (see e.g. [18, 36, 44]) and that

$$\forall t \in \mathbb{R}, \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\psi(t)|^2 + F\left(|\psi(t)|^2\right) \right) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\psi_0|^2 + F\left(|\psi_0|^2\right) \right). \tag{5.3}$$

Let us set $\rho := |\psi|^2$ and

$$\phi(x) := \begin{cases} |\psi(x)|^{-1}\psi(x) & \text{if} \quad \psi(x) \neq 0, \\ 0 & \text{if} \quad \psi(x) = 0. \end{cases} \tag{5.4}$$

We claim that

$$\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\nabla\psi) \quad \text{a. e.} \tag{5.5}$$

Indeed, for $\varepsilon > 0$, let us set $\phi_\varepsilon := \psi/\sqrt{|\psi|^2 + \varepsilon^2}$. Then (ϕ_ε) converges pointwise to ϕ and an easy computation shows that

$$\bar{\phi}_\varepsilon \psi \underset{\varepsilon\to 0}{\longrightarrow} \sqrt{\rho} \quad \text{uniformly.} \tag{5.6}$$

Next, we compute

$$\nabla(\bar{\phi}_\varepsilon\psi) = \mathrm{Re}(\bar{\phi}_\varepsilon\nabla\psi) + \mathrm{Re}(\psi\nabla\bar{\phi}_\varepsilon). \tag{5.7}$$

The first term in the right-hand side converges pointwise to $\mathrm{Re}(\bar{\phi}\nabla\psi)$ hence in L^1_{loc} owing to Lebesgue's theorem as it is bounded by $|\nabla\psi|$.

As for the last term, we have

$$\mathrm{Re}(\psi\nabla\bar{\phi}_\varepsilon) = \frac{\varepsilon^2}{(\varepsilon^2 + |\psi|^2)^{3/2}}\,\mathrm{Re}(\bar{\psi}\nabla\psi).$$

It is clear that the right-hand side converges pointwise to 0 and is bounded by $|\nabla\psi|$. Hence it also converges to 0 in L^1_{loc}.

So finally, putting these two results together with (5.6) and (5.7), one may conclude to (5.5).

Let $\Lambda := \mathrm{Im}(\bar{\phi}\nabla\psi)$ and $J := \sqrt{\rho}\Lambda$. We claim that (ρ, Λ, J) satisfies (5.1). Indeed, from (5.2), we see that

$$\partial_t(|\psi|^2) = 2\,\mathrm{Re}(\bar{\psi}\partial_t\psi) = -\,\mathrm{Im}(\bar{\psi}\Delta\psi) = -\,\mathrm{Im}\,\mathrm{div}\,(\bar{\psi}\nabla\psi),$$

hence

$$\partial_t\rho + \mathrm{div}\, J = 0. \tag{5.8}$$

Next, we compute

$$\partial_j J^k - \partial_k J^j = \mathrm{Im}\big(\partial_j(\bar{\psi}\partial_k\psi) - \partial_k(\bar{\psi}\partial_j\psi)\big) = 2\,\mathrm{Im}\big(\partial_j\bar{\psi}\partial_k\psi\big).$$

Recall (see e.g. Theorem 6.19 in [34]) that

$$\nabla\psi = 0 \quad \text{a. e. on} \quad \psi^{-1}(\{0\}) \tag{5.9}$$

whenever ψ is locally in $W^{1,1}$.

Therefore, given that $|\phi| = 1$ on $\psi^{-1}(\mathbb{C}\setminus\{0\})$, one may write a. e.

$$\mathrm{Im}\big(\partial_j\bar\psi\partial_k\psi\big) = \mathrm{Im}\big(\phi\partial_j\bar\psi\,\bar\phi\partial_k\psi\big) = \mathrm{Re}(\phi\partial_j\bar\psi)\,\mathrm{Im}(\bar\phi\partial_k\psi) + \mathrm{Re}(\bar\phi\partial_k\psi)\,\mathrm{Im}(\phi\partial_j\bar\psi).$$

So we eventually get

$$\partial_j J^k - \partial_k J^j = 2\partial_j\sqrt{\rho}\,\Lambda^k - 2\partial_k\sqrt{\rho}\,\Lambda^j. \tag{5.10}$$

Next, we compute

$$\partial_t J = \mathrm{Im}\big(\partial_t\bar\psi\,\nabla\psi + \bar\psi\nabla\partial_t\psi\big).$$

Hence, using the equation satisfied by ψ, we get

$$\partial_t J = \frac{1}{2}\underbrace{\mathrm{Re}\big(\bar\psi\nabla\Delta\psi - \Delta\bar\psi\nabla\psi\big)}_{A} + \underbrace{\mathrm{Re}\big(f(|\psi|^2)\bar\psi\nabla\psi - \bar\psi\nabla(f(|\psi|^2)\psi)\big)}_{B}.$$

If ψ is C^1 and the function f has a derivative at every point of $\mathbb{R}^+$, then straightforward computations show that

$$B = -\rho f'(\rho)\nabla\rho = -\nabla(P(\rho)). \tag{5.11}$$

Next, we see that (still in the smooth case)

$$\nabla\Delta|\psi|^2 = 4\,\mathrm{Re}\big(\nabla^2\psi : \nabla\bar\psi\big) + 2\,\mathrm{Re}\big(\nabla\psi\Delta\bar\psi\big) + 2\,\mathrm{Re}\big(\bar\psi\nabla\Delta\psi\big).$$

Hence

$$A = \frac{1}{2}\nabla\Delta|\psi|^2 - 2\,\mathrm{Re}\big(\Delta\bar\psi\nabla\psi\big) - 2\,\mathrm{Re}\big(\nabla^2\psi : \nabla\bar\psi\big).$$

So we get

$$\frac{1}{2}A = \frac{1}{4}\nabla\Delta|\psi|^2 - \mathrm{div}\,\mathrm{Re}(\nabla\bar\psi\otimes\nabla\psi). \tag{5.12}$$

Now, using again (5.9), one may write at almost every point of $\mathbb{R}^d$,

$$\begin{aligned}\mathrm{Re}(\nabla\bar\psi\otimes\nabla\psi) &= \mathrm{Re}(\phi\nabla\bar\psi\otimes\bar\phi\nabla\psi),\\ &= \mathrm{Re}(\bar\phi\nabla\psi)\otimes\mathrm{Re}(\bar\phi\nabla\psi) + \mathrm{Im}(\bar\phi\nabla\psi)\otimes\mathrm{Im}(\bar\phi\nabla\psi).\end{aligned}$$

Therefore, we have

$$\mathrm{Re}(\nabla\bar\psi\otimes\nabla\psi) = \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} + \Lambda\otimes\Lambda. \tag{5.13}$$

Putting this together with (5.11) and (5.12), one may conclude that

$$\partial_t J + \mathrm{div}\,(\Lambda\otimes\Lambda) + \nabla(P(\rho)) = \frac{1}{4}\Delta\nabla\rho - \mathrm{div}\,(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}). \tag{5.14}$$

Of course, as owing to (5.9)

$$|\nabla\psi|^2 = (\mathrm{Re}(\bar\phi\nabla\psi))^2 + (\mathrm{Im}(\bar\phi\nabla\psi))^2 = |\nabla\sqrt{\rho}|^2 + |\Lambda|^2 \quad \text{a. e.}, \tag{5.15}$$

the energy equality for ψ recasts in

$$\int_{\mathbb{R}^d}\Big(\frac{1}{2}|\Lambda(t)|^2 + \frac{1}{2}|\nabla\sqrt{\rho(t)}|^2 + F(\rho(t))\Big) = \int_{\mathbb{R}^d}\Big(\frac{1}{2}|\Lambda_0|^2 + \frac{1}{2}|\nabla\sqrt{\rho_0}|^2 + F(\rho_0)\Big). \tag{5.16}$$

This completes the proof in the smooth case.

Let us now treat the rough case where ψ_0 belongs only to H^1. Then we fix some sequence $(\psi_{0,n})_{n\in\mathbb{N}}$ of functions in H^s (with s large) converging to ψ_0 in H^1. Let us denote by ψ_n the solution of (5.2) in $C(\mathbb{R};H^s)$ corresponding to the data $\psi_{0,n}$, and by $\psi\in C(\mathbb{R};H^1)$ the solution to (5.2) with data ψ_0.

From the first part of the proof, we know that there exists some sequence $(\phi_n)_{n\in\mathbb{N}}$ of functions with modulus at most 1 such that if we set $\Lambda_n := \mathrm{Im}(\bar\phi_n\nabla\psi_n)$, $\rho_n := |\psi_n|^2$ and $J_n := \sqrt{\rho_n}\Lambda_n$ then

$$\nabla\sqrt{\rho_n} = \mathrm{Re}(\bar\phi_n\nabla\psi_n) \quad \text{in} \quad L^2,$$

and (ρ_n,Λ_n,J_n) satisfies (5.1) with data $(\rho_{0,n},J_{0,n})$, together with the energy equality

$$\int_{\mathbb{R}^d}\Big(\frac12|\Lambda_n(t)|^2+\frac12|\nabla\sqrt{\rho_n(t)}|^2+F\left(\rho_n(t)\right)\Big) = \int_{\mathbb{R}^d}\Big(\frac12|\Lambda_{0,n}|^2+\frac12|\nabla\sqrt{\rho_{0,n}}|^2+F(\rho_{0,n})\Big). \tag{5.17}$$

We now have to prove the convergence of (ρ_n,Λ_n,J_n) to some solution (ρ,Λ,J) of (5.1) satisfying the energy equality.

On the one hand, standard stability estimates (based on Strichartz inequalities) guarantee that

$$\psi_n \longrightarrow \psi \quad \text{in} \quad L^\infty_{\mathrm{loc}}(\mathbb{R};H^1). \tag{5.18}$$

On the other hand, because $(\phi_n)_{n\in\mathbb{N}}$ is bounded by 1, it converges (up to extraction) in L^∞ weak * to some function ϕ such that $\|\phi\|_{L^\infty}\leqslant 1$. As $\nabla\psi_n$ converges strongly to $\nabla\psi$ in L^2, this implies that

$$\bar\phi_n\nabla\psi_n \rightharpoonup \bar\phi\nabla\psi \quad \text{in} \quad L^2. \tag{5.19}$$

In turn, as obviously $\sqrt{\rho_n}\to\sqrt{\rho}$ in L^2 and as $\nabla\sqrt{\rho_n}=\mathrm{Re}(\bar\phi_n\nabla\psi_n)$, we deduce that

$$\nabla\sqrt{\rho}=\mathrm{Re}(\bar\phi\nabla\psi) \quad \text{and} \quad \nabla\sqrt{\rho_n}\rightharpoonup\nabla\sqrt{\rho} \;\; \text{in} \;\; L^2.$$

Given that $\Lambda_n=\mathrm{Im}(\bar\phi_n\nabla\psi_n)$, (5.19) also ensures that

$$\Lambda_n \rightharpoonup \Lambda := \mathrm{Im}(\bar\phi\nabla\psi) \quad \text{in} \quad L^2.$$

In order to establish that strong convergence in L^2 holds true, it suffices to show that (up to an omitted extraction)

$$\|\nabla\sqrt{\rho_n}\|_{L^2}\to\|\nabla\sqrt{\rho}\|_{L^2} \quad \text{and} \quad \|\Lambda_n\|_{L^2}\to\|\Lambda\|_{L^2}. \tag{5.20}$$

On the one hand, the weak convergence ensures that

$$\|\nabla\sqrt{\rho}\|_{L^2}\leqslant\liminf\|\nabla\sqrt{\rho_n}\|_{L^2} \quad \text{and} \quad \|\Lambda\|_{L^2}\leqslant\liminf\|\Lambda_n\|_{L^2};$$

on the other hand, given that $(\nabla\psi_n)_{n\in\mathbb{N}}$ converges strongly to $\nabla\psi$ in L^2 and that (5.15) holds true for ψ and ψ_n, we may write

$$\begin{aligned}\|\nabla\sqrt{\rho}\|^2_{L^2}+\|\Lambda\|^2_{L^2} &= \|\nabla\psi\|^2_{L^2},\\ &= \lim_{n\to+\infty}\|\nabla\psi_n\|^2_{L^2},\\ &= \lim_{n\to+\infty}\big(\|\nabla\sqrt{\rho_n}\|^2_{L^2}+\|\Lambda_n\|^2_{L^2}\big),\\ &\geqslant \big(\liminf\|\nabla\sqrt{\rho_n}\|_{L^2}\big)^2+\big(\liminf\|\Lambda_n\|_{L^2}\big)^2.\end{aligned}$$

Therefore (5.20) is satisfied. As a conclusion, we thus have established that

$$\sqrt{\rho_n} \to \sqrt{\rho} \ \text{ in } \ H^1 \quad \text{and} \quad \Lambda_n \to \Lambda \ \text{ in } L^2.$$

Of course, this implies that $J_n \to J := \sqrt{\rho}\Lambda$ in L^1 so it is easy to pass to the limit in (5.1) and in the energy equality (5.16). The details are left to the reader. □

Appendix A. Conservation laws

In this Appendix we review conservations laws for the nonlinear Schrödinger, QHD and compressible Euler equations. Even though most of the results are classical (as concerns the Schrödinger equation, they may be found in the textbooks [43, 44] for instance; links between Schrödinger and Euler conservation laws may be found in [16]), we believe the relationships between the aforementioned equations to be of interest. In addition, those conservation laws are still meaningful for the less classical framework of general Korteweg fluids.

A.1. **The case of Schrödinger, QHD and compressible Euler equations.** For the time being, we consider the following system

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla f(\rho) = \dfrac{\varepsilon^2}{2}\nabla\left(\dfrac{\Delta\left(\sqrt{\rho}\right)}{\sqrt{\rho}}\right) \quad ; \quad v_{|t=0} = v_0, \\ \partial_t\rho + \operatorname{div}(\rho v) = 0 \quad ; \quad \rho_{|t=0} = \rho_0, \end{cases} \tag{A.1}$$

which is the QHD system if $\varepsilon > 0$, and the compressible Euler equation if $\varepsilon = 0$, and the nonlinear Schrödinger equation:

$$i\varepsilon\partial_t\psi + \frac{\varepsilon^2}{2}\Delta\psi = f(|\psi|^2)\psi \quad ; \quad \psi_{|t=0} = \psi_0. \tag{A.2}$$

Recall that for $\varepsilon > 0$, one may pass formally from (A.2) to (A.1) by setting

$$\psi = \sqrt{\rho}\, e^{i\phi/\varepsilon} \quad \text{and} \quad v = \nabla\phi.$$

In what follows, the function f is assumed to be continuous on $\mathbb{R}^+$ and, say, C^1 on $(0,+\infty[$, standard cases being $f(r) = r^\sigma$ and $f(r) = r-1$. We denote by F the anti-derivative of f which vanishes at 0, and set $P(\rho) = \rho f(\rho) - F(\rho)$. As pointed out before, from a physical viewpoint, P is the pressure.

The first part of the appendix aims at listing (and deriving formally) the classical conservation laws for (A.2) and (A.1).

Phase invariance. For every $\alpha \in \mathbb{R}$, one has

$$\psi \ \text{ solution of } \ (A.2) \iff e^{i\alpha}\psi \text{ solution of } \ (A.2).$$

The phase invariance is not seen at the level of (A.1) (this amount to changing ϕ into $\phi + \varepsilon\alpha$).

By Noether's theorem or by an easy computation, this leads to the conservation of *mass*:

$$\mathcal{M} := \int |\psi|^2\,dx = \int \rho\,dx. \tag{A.3}$$

Time translation invariance. For every $\tau \in \mathbb{R}$, one has

$$\psi(t,x) \text{ solution of (A.2)} \iff \psi(t+\tau,x) \text{ solution of (A.2)}.$$

By the same time translation, this is expressed at the level of (A.1) and leads to the conservation of the *energy* (or *Hamiltonian*):

$$\mathcal{H} := \int \Big(\frac{\varepsilon^2}{2} |\nabla\psi|^2 + F(|\psi|^2) \Big)\, dx = \int \Big(\frac{1}{2} \rho |v|^2 + \frac{\varepsilon^2}{2} |\nabla \sqrt{\rho}|^2 + F(\rho) \Big)\, dx. \tag{A.4}$$

Space translation invariance. For every $x_0 \in \mathbb{R}^d$, one has

$$\psi(t,x) \text{ solution to (A.2)} \iff \psi(t,x+x_0) \text{ solution to (A.2)}.$$

By the same space translation, this is expressed at the level of (A.1) and leads to the conservation of *momentum*:

$$\mathcal{P} := \operatorname{Im} \int \varepsilon \bar{\psi} \nabla \psi \, dx = \int \rho v \, dx. \tag{A.5}$$

Invariance by spatial rotation. Let R be a spatial rotation. Then

$$\psi(t,x) \text{ solution of (A.2)} \iff \psi(t,Rx) \text{ solution of (A.2)}.$$

By the same spatial rotation on a solution of (A.1) this leads to the conservation of *angular momentum*, which we write in the case of $\mathbb{R}^3$ for the sake of simplicity:

$$\mathcal{A} := \operatorname{Im} \int x \wedge \varepsilon \bar{\psi} \nabla \psi \, dx = \int x \wedge \rho v \, dx. \tag{A.6}$$

Galilean invariance. For every $\xi_0 \in \mathbb{R}^d$, one has

$$\psi(t,x) \text{ solution of (A.2)} \iff e^{-i\xi_0 \cdot x} e^{-i\frac{t\varepsilon}{2}|\xi_0|^2} \psi(t, x+\varepsilon\xi_0 t) \text{ solution of (A.2)}.$$

For (A.1), this implies

$$(v,\rho)(t,x) \text{ solution} \iff \big(v(t,x+\varepsilon\xi_0 t) - \varepsilon\xi_0, \rho(t,x+\varepsilon\xi_0 t) \big) \text{ solution},$$

and leads to

$$\frac{d\mathcal{X}}{dt} = \mathcal{P} \quad \text{with } \mathcal{X} := \int x |\psi|^2 \, dx = \int x \rho \, dx. \tag{A.7}$$

Scale invariance. When $f(r) = r^\sigma$, one has for all $\lambda > 0$,

$$\psi(t,x) \text{ solution of (A.2)} \iff \lambda^{1/\sigma} \psi(\lambda^2 t, \lambda x) \text{ solution of (A.2)},$$

which implies for (A.1)

$$(v,\rho)(t,x) \text{ solution} \iff \big(\lambda v, \lambda^{2/\sigma} \rho \big)(\lambda^2 t, \lambda x) \text{ solution}.$$

The associated conservation law is (recall that d stands for the space dimension):

$$\frac{d\mathcal{F}}{dt} = \int \Big(\varepsilon^2 |\nabla \Psi|^2 + dP(|\psi|^2) \Big)\, dx = 2\mathcal{H} + \int (dP - 2F)(|\psi|^2)\, dx$$

or,

$$\frac{d\mathcal{F}}{dt} = \int \Big(\rho |v|^2 + dP(\rho) \Big)\, dx = 2\mathcal{H} + \int (dP - 2F)(\rho)\, dx$$

with

$$\mathcal{F} := \operatorname{Im} \int \varepsilon \bar{\psi} x \cdot \nabla \psi = \int \rho x \cdot v \, dx \quad \text{and} \quad P(r) := r f(r) - F(r). \tag{A.8}$$

Equality (A.8) remains formally true if the nonlinearity is not a pure power.

Momentum of inertia or virial. A direct computation shows that the momentum of inertia (or virial)

$$\mathcal{I} := \frac{1}{2}\int |x|^2|\psi|^2\,dx = \frac{1}{2}\int |x|^2\rho\,dx \tag{A.9}$$

satisfies $\dfrac{d\mathcal{I}}{dt} = \mathcal{F}$.

Pseudo-conformal invariance. Let φ the pseudo-conformal transform of ψ defined by

$$\varphi(t,x) = \frac{e^{\frac{i|x|^2}{2\varepsilon t}}}{\left(i\frac{t}{\varepsilon}\right)^{d/2}}\bar{\psi}\left(\frac{\varepsilon^2}{t},\frac{x}{t}\right).$$

One notices that

$$\Big(i\varepsilon\partial_t\varphi + \frac{\varepsilon^2}{2}\Delta\varphi\Big)(t,x) = \frac{\varepsilon^2}{t^2}\frac{e^{\frac{i|x|^2}{2\varepsilon t}}}{(i\frac{t}{\varepsilon})^{\frac{n}{2}}}\overline{\Big(i\varepsilon\partial_t\psi + \frac{\varepsilon^2}{2}\Delta\psi\Big)}\left(\frac{\varepsilon^2}{t},\frac{x}{t}\right).$$

Thus

$$i\varepsilon\partial_t\varphi + \frac{\varepsilon^2}{2}\Delta\varphi = \frac{\varepsilon^2}{t^2}f\left(\left(\frac{t}{\varepsilon}\right)^d|\varphi|^2\right)\varphi.$$

For the L^2-critical power nonlinearity $f(r) = r^{2/d}$, one checks that

$$\psi(t,x)\ \text{ solution of }\ (A.2) \iff \varphi(t,x)\text{ solution of }\ (A.2).$$

For (A.1), this yields

$$(v,\rho)(t,x)\ \text{ solution } \iff \left(\frac{x}{t} - \frac{v}{t}\Big(\frac{\varepsilon^2}{t},\frac{x}{t}\Big),\left(\frac{\varepsilon}{t}\right)^d\rho\Big(\frac{\varepsilon^2}{t},\frac{x}{t}\Big)\right)\ \text{ solution.}$$

Using the conservation of energy for φ, one deduces after a lengthy computation that

$$\frac{d\mathcal{Z}}{dt} + t\int (dP - 2F)(|\psi|^2)\,dx = 0 \tag{A.10}$$

with

$$\begin{aligned}\mathcal{Z}(t) \ &:= \ \int\Big(\frac{1}{2}|(x+i\varepsilon t\nabla)\psi|^2 + t^2F(|\psi|^2)\Big)\,dx,\\ &= \ \int\Big(\frac{1}{2}\rho|x-tv|^2 + \frac{1}{2}\varepsilon^2t^2|\nabla\sqrt{\rho}|^2 + t^2F(|\psi|^2)\Big)\,dx.\end{aligned}$$

This equality can be proven more simply by using the fact that the operators $i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta$ and $x + i\varepsilon t\nabla$ commute. The conservation law for $\mathcal{Z}$ is not independent from the preceding ones: by expanding the square of the modulus, one observes that

$$\mathcal{Z}(t) = t^2\mathcal{H} - t\mathcal{F} + \mathcal{I}.$$

Consequently,

$$\frac{d\mathcal{Z}}{dt} = 2t\mathcal{H} - \mathcal{F} - t\frac{d\mathcal{F}}{dt} + \frac{d\mathcal{I}}{dt} = 2t\mathcal{H} - t\frac{d\mathcal{F}}{dt},$$

and one recovers (A.10).

The Carles-Nakamura conservation law. Formally, the quantity

$$\mathcal{U}(t) := \mathrm{Re} \int \bar{\psi}\,(x + i\varepsilon t\nabla)\psi\, dx$$

is constant. This can be checked by an indirect way (see [17]) or by a direct computation. The interpretation in terms of (A.1) is

$$\frac{d}{dt}\mathcal{U} = 0 \quad \text{with} \quad \mathcal{U}(t) = \int \rho(x - tv)\, dx. \tag{A.11}$$

This conservation law also results from the conservation of momentum and from the conservation law for $\mathcal{X}$ since in both cases, (A.2) or (A.1), one has

$$\mathcal{U} = \mathcal{X} - t\mathcal{P}.$$

A.2. **Conservation laws for general capillary fluids.** We here study to what extent the conservation laws listed in the previous subsection are relevant for general inviscid capillary fluids. We recall that such fluids are governed by System (3.7). Throughout, we assume the capillarity κ to be a differentiable function on $\mathbb{R}^+$.

For smooth solutions with a non vanishing density, System (3.7) recasts in the following conservative form[7]:

$$\begin{cases} \partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) + \nabla P(\rho) = \mathrm{div}\, K, \\ \partial_t \rho + \mathrm{div}(\rho v) = 0, \end{cases} \tag{A.12}$$

with $K(\rho, \nabla\rho) := \Big(\frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho^2| + \rho\kappa(\rho)\Delta\rho\Big) I_d - \kappa(\rho)\nabla\rho \otimes \nabla\rho.$

Theorem A.1. *If* (ρ, v) *is a sufficiently smooth solution of* (3.7) *which decays at infinity, with* ρ *non vanishing, then the equalities* (A.3), (A.5), (A.6), (A.7) *and* (A.9) *are still valid, and the energy*

$$\mathcal{H} := \int \Big(\frac{1}{2}\rho|v|^2 + \frac{\kappa(\rho)}{2}|\nabla\rho|^2 + F(\rho)\Big)\, dx$$

is conserved.

Furthermore, the quantity $\mathcal{F}$ *defined in* (A.8) *satisfies*

$$\frac{d}{dt}\mathcal{F} = 2\mathcal{H} + \int \Big(dP - 2F + \frac{d}{2}(\rho\kappa)'|\nabla\rho|^2\Big)\, dx.$$

Proof. It is very likely that most of those conservation laws could be derived from Noether theorem. They can also been obtained by pedestrian computations.

For (A.3), (A.7) and (A.9), there is no difference with the QHD since the only equation on ρ is concerned.

For (A.5) the proof is obvious from the conservative form (A.12).

In order to prove the conservation of energy, the simplest method is to take the L^2 scalar product with ρv of the first equation of (3.7). The treatment of the terms on the left-hand side of the equation for v is the same as in the compressible Euler

[7] With the convention $(\mathrm{div}\, K)^j := \sum_i \partial_i K_{ij}$

equation, and one obtains after several integrations by parts and the use of the equation for ρ,

$$\begin{aligned}
\frac{d}{dt}\int\Big(\frac{1}{2}\rho|v|^2+F(\rho)\Big)\,dx &= \int \rho v\cdot\nabla\Big(\kappa\Delta\rho+\frac{1}{2}\kappa'|\nabla\rho|^2\Big)\,dx,\\
&= \int \partial_t\rho\Big(\kappa\Delta\rho+\frac{1}{2}\kappa'|\nabla\rho|^2\Big)\,dx,\\
&= -\int \kappa\nabla\rho\cdot\partial_t\nabla\rho\,dx-\frac{1}{2}\int\kappa'\partial_t\rho|\nabla\rho|^2\,dx,\\
&= -\frac{d}{dt}\int\frac{\kappa}{2}|\nabla\rho|^2\,dx.
\end{aligned}$$

To prove (A.6), one writes (using the Einstein summation convention),

$$A_i=\int \varepsilon_{ijk}x^j\rho v^k$$

with $\varepsilon_{ijk}=0$ if two of the indices are equal, and equal to the signature of $(i\ j\ k)$ otherwise.

Using the conservative form of the equation for ρv and integrating by parts, one thus deduces

$$\begin{aligned}
\frac{d}{dt}A_i &= \int\varepsilon_{ijk}x^j\partial_\ell K_{\ell k}\,dx-\int\varepsilon_{ijk}x^j\partial_k P-\int\varepsilon_{ijk}x^j\partial_\ell(\rho v^k v^\ell)\,dx,\\
&= -\int\varepsilon_{ijk}K_{jk}\,dx+0-\int\varepsilon_{ijk}\rho v^j v^k\,dx.
\end{aligned}$$

The tensors K and $\rho v\otimes v$ being symmetric, the right-hand member of the previous equality vanishes.

The simplest way to prove the conservation law on $\mathcal{F}$, is to use the second equation in (A.12). Integrating by parts the capillary term and treating the other terms as in the compressible Euler equation one obtains

$$\begin{aligned}
\frac{d\mathcal{F}}{dt} &= \int x\cdot\operatorname{div}K\,dx-\int x\cdot\Big(\nabla P+\operatorname{div}(\rho v\otimes v)\Big)\,dx,\\
&= \int x^j\partial_i K_{ij}\,dx+\int(dP+\rho|v|^2)\,dx.
\end{aligned}$$

Using the expression of the tensor K and integrating by parts, one gets

$$\begin{aligned}
\int x^j\partial_i K_{ij}\,dx &= -\int \operatorname{tr}K\,dx,\\
&= -d\int\Big(\frac{1}{2}(\kappa+\rho\kappa')|\nabla\rho|^2+\rho\kappa\Delta\rho\Big)\,dx+\int\kappa|\nabla\rho|^2,\\
&= \frac{d}{2}\int(\rho\kappa)'|\nabla\rho|^2\,dx+\int\kappa|\nabla\rho|^2\,dx,
\end{aligned}$$

from which the claimed equality results. □

Remark A.2. The case of QHD corresponds to $(\rho\kappa)'=0$. For capillary fluids, the presence of extra terms in the conservation law for $\mathcal{F}$ seems relatively harmless if $\rho\mapsto\rho\kappa(\rho)$ is an increasing function. One should be able to prove without difficulty that with a pressure law such that $dP-2F\geqslant 0$, one has

$$I(t)\underset{t\to+\infty}{\sim}\mathcal{H}t^2.$$

In the case where $\rho \mapsto \rho\kappa$ is decreasing, we expect the conservation law for $\mathcal{F}$ to be the key to proving finite time blow-up results similar to those of [42].

References

[1] T. Alazard and R. Carles, *WKB analysis for the Gross–Pitaevskii equation with non-trivial boundary conditions at infinity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 3, 959–977.

[2] S. Alinhac, *Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux*, Invent. Math. **111** (1993), no. 3, 627–670.

[3] P. Antonelli and P. Marcati, *The quantum hydrodynamics system in two dimensions*, Arch. Rational Mech. Anal., to appear. Archived as `arXiv:1011.4545v1`.

[4] ———, *On the finite energy weak solutions to a system in quantum fluid dynamics*, Comm. Math. Phys. **287** (2009), 657–686.

[5] C. Audiard, *Dispersive smoothing for the Euler-Korteweg model*, SIAM J. Math. Anal., in press.

[6] W. Bao, S. Jin, and P. A. Markowich, *Numerical study of time-splitting spectral discretizations of nonlinear Schrödinger equations in the semiclassical regimes*, SIAM J. Sci. Comput. **25** (2003), no. 1, 27–64.

[7] S. Benzoni-Gavage, R. Danchin, and S. Descombes, *On the well-posedness for the Euler-Korteweg model in several space dimensions*, Indiana Univ. Math. J. **56** (2007), no. 4, 1499–1579.

[8] F. Béthuel, R. Danchin, and D. Smets, *On the linear wave regime of the Gross-Pitaevskii equation*, J. Anal. Math. **110** (2010), 297–338.

[9] F. Béthuel, P. Gravejat, and J.-C. Saut, *Existence and properties of travelling waves for the Gross-Pitaevskii equation*, Stationary and time dependent Gross-Pitaevskii equations, Contemp. Math., vol. 473, Amer. Math. Soc., Providence, RI, 2008, pp. 55–103.

[10] ———, *On the KP I transonic limit of two-dimensional Gross-Pitaevskii travelling waves*, Dyn. Partial Differ. Equ. **5** (2008), no. 3, 241–280.

[11] F. Béthuel, P. Gravejat, J.-C. Saut, and D. Smets, *Orbital stability of the black soliton for the Gross-Pitaevskii equation*, Indiana Univ. Math. J. **57** (2008), no. 6, 2611–2642.

[12] ———, *On the Korteweg-de Vries long-wave approximation of the Gross-Pitaevskii equation. I*, Int. Math. Res. Not. IMRN (2009), no. 14, 2700–2748.

[13] ———, *On the Korteweg-de Vries long-wave approximation of the Gross-Pitaevskii equation II*, Comm. Partial Differential Equations **35** (2010), no. 1, 113–164.

[14] F. Béthuel, G. Orlandi, and D. Smets, *Vortex rings for the Gross-Pitaevskii equation*, J. Eur. Math. Soc. (JEMS) **6** (2004), no. 1, 17–94.

[15] F. Béthuel and J.-C. Saut, *Travelling waves for the Gross-Pitaevskii equation. I*, Ann. Inst. H. Poincaré Phys. Théor. **70** (1999), no. 2, 147–238.

[16] R. Carles, *Semi-classical analysis for nonlinear Schrödinger equations*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[17] R. Carles and Y. Nakamura, *Nonlinear Schrödinger equations with Stark potential*, Hokkaido Math. J. **33** (2004), no. 3, 719–729.

[18] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[19] D. Chiron and M. Mariş, *Rarefaction pulses for the nonlinear Schrödinger equation in the transonic limit*, submitted.

[20] D. Chiron and F. Rousset, *The KdV/KP-I limit of the nonlinear Schrödinger equation*, SIAM J. Math. Anal. **42** (2010), no. 1, 64–96.

[21] F. Coquel, G. Dehais, D. Jamet, O. Lebaigue, and N. Seguin, *Extended formulations for Van der Waals models. Application to finite volume methods*, in preparation.

[22] R. Danchin, *Zero Mach number limit in critical spaces for compressible Navier-Stokes equations*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 1, 27–75.

[23] P. Degond, S. Gallego, and F. Méhats, *An asymptotic preserving scheme for the Schrödinger equation in the semiclassical limit*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 9, 531–536.

[24] C. Gallo, *Schrödinger group on Zhidkov spaces*, Adv. Differential Equations **9** (2004), no. 5-6, 509–538.

[25] ———, *The Cauchy problem for defocusing nonlinear Schrödinger equations with non-vanishing initial data at infinity*, Comm. Partial Differential Equations **33** (2008), no. 4-6, 729–771.

[26] I. M. Gamba, M. P. Gualdani, and P. Zhang, *On the blowing up of solutions to quantum hydrodynamic models on bounded domains*, Monatsh. Math. **157** (2009), no. 1, 37–54.

[27] P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.

[28] ———, *The Cauchy problem for the Gross-Pitaevskii equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 5, 765–779.

[29] ———, *The Gross-Pitaevskii equation in the energy space*, Stationary and time dependent Gross-Pitaevskii equations, Contemp. Math., vol. 473, Amer. Math. Soc., Providence, RI, 2008, pp. 129–148.

[30] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), 1794–1797.

[31] E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 523–530.

[32] L. Hörmander, *The analysis of linear partial differential operators. I*, second ed., Springer Study Edition, Springer-Verlag, Berlin, 1990, Distribution theory and Fourier analysis.

[33] L. Landau and E. Lifschitz, *Physique théorique ("Landau-Lifchitz"). Tome III: Mécanique quantique. Théorie non relativiste*, Éditions Mir, Moscow, 1967, Deuxième édition, Traduit du russe par Édouard Gloukhian.

[34] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[35] F. Lin and P. Zhang, *Semiclassical limit of the Gross-Pitaevskii equation in an exterior domain*, Arch. Rational Mech. Anal. **179** (2006), no. 1, 79–107.

[36] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, Universitext, Springer, New York, 2009.

[37] E. Madelung, *Quanten theorie in Hydrodynamischer Form*, Zeit. F. Physik **40** (1927), 322.

[38] T. Makino, S. Ukai, and S. Kawashima, *Sur la solution à support compact de l'équation d'Euler compressible*, Japan J. Appl. Math. **3** (1986), no. 2, 249–257.

[39] M. Mariş, *Travelling waves for nonlinear Schrödinger equations with non zero boundary conditions at infinity*, submitted.

[40] G. Métivier, *Remarks on the well-posedness of the nonlinear Cauchy problem*, Geometric analysis of PDE and several complex variables, Contemp. Math., vol. 368, Amer. Math. Soc., Providence, RI, 2005, pp. 337–356.

[41] D. Serre, *Solutions classiques globales des équations d'Euler pour un fluide parfait compressible*, Ann. Inst. Fourier **47** (1997), 139–153.

[42] T. Sideris, *Formation of Singularities in Three-Dimensional Compressible Fluids*, Comm. Math. Phys. **101** (1985), 475–485.

[43] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation, self-focusing and wave collapse*, Springer-Verlag, New York, 1999.

[44] T. Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006, Local and global analysis.

[45] M. Taylor, *Partial differential equations. III*, Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997, Nonlinear equations.

[46] L. Thomann, *Instabilities for supercritical Schrödinger equations in analytic manifolds*, J. Differential Equations **245** (2008), no. 1, 249–280.

[47] Z. Xin, *Blowup of smooth solutions of the compressible Navier-Stokes equation with compact density*, Comm. Pure Appl. Math. **51** (1998), 229–240.

[48] V. E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP **35** (1972), 908–914.

[49] P. E. Zhidkov, *The Cauchy problem for a nonlinear Schrödinger equation*, JINR Commun., R5-87-373, Dubna (1987), (in Russian).

[50] ———, *Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory*, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001.

CNRS & Univ. Montpellier 2, Mathématiques, UMR 5149, CC 051, 34095 Montpellier, France

E-mail address: Remi.Carles@math.cnrs.fr

LAMA UMR CNRS 8050, Université Paris EST, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France

E-mail address: danchin@univ-paris12.fr

Laboratoire de Mathématiques, UMR 8628, Université Paris-Sud et CNRS, 91405 Orsay, France

E-mail address: jean-claude.saut@math.u-psud.fr