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Approximation of discrete BSDE using least-squares regression

E. Gobet[a,1,*], P. Turkedjiev[b]

[a] *Ecole Polytechnique and CNRS, Centre de Mathématiques Appliquées, Route de Saclay, 91128 Palaiseau, France*
[b] *Humboldt Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany*

Abstract

We consider the dynamic programming equation arising from the time-discretization of backward stochastic differential equations. When the sequence of conditional expectations is computed using empirical least-squares regressions, we show that the Multi step-forward Dynamic Programming (MDP) equation yields better error estimates than the usual One-step forward DP (ODP) equation. We provide full error estimates, depending on the time-grid, the number of simulations and the approximation spaces for regression. The generator is assumed to be locally Lipschitz, which includes some cases of quadratic drivers.

Keywords: Backward stochastic differential equations, dynamic programming equation, empirical regressions, error estimates.
2000 MSC: 49L20, 62Jxx, 65C30, 93E24.

1. Introduction

Framework. Let $T > 0$ be a fixed terminal time and W be a q-dimensional ($q \geq 1$) Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual hypotheses; the filtration may be larger than that generated by W. We are given a deterministic time grid $\pi := \{0 = t_0 < \ldots < t_N = T\}$ for the interval $[0, T]$, whose $(i+1)$-th time-step is denoted $\Delta_i = t_{i+1} - t_i$ and mesh size is defined by $|\pi| := \max_{0 \leq i < N} \Delta_i \leq T$. The $(i+1)$-th Brownian motion increment is defined by $\Delta W_i := W_{t_{i+1}} - W_{t_i}$.

*Corresponding author
Email addresses: emmanuel.gobet@polytechnique.edu (E. Gobet), turkedji@math.hu-berlin.de (P. Turkedjiev)
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In this work, we deal with the numerical resolution of (Y, Z), a *discrete* BSDE with data $(\xi, f_i(y, z))$, which is generated by

$$\begin{cases} Y_N &= \xi, \quad Y_i = \mathbb{E}_i \left(Y_{i+1} + f_i(Y_{i+1}, Z_i)\Delta_i \right), \ 0 \leq i < N, \\ \Delta_i Z_i &= \mathbb{E}_i \left(Y_{i+1} \Delta W_i^\top \right), \quad 0 \leq i < N \end{cases} \tag{1.1}$$

where $\mathbb{E}_i(\cdot) := \mathbb{E}(\cdot|\mathcal{F}_{t_i})$, $\top$ denotes the transpose operator and

- ξ is a given $\mathcal{F}_T$-measurable random variable in $\mathbf{L}_2$,
- $Y := (Y_i)_{0 \leq i \leq N}$ is a scalar process, $Z := (Z_i)_{0 \leq i < N}$ is $\mathbb{R}^q$-valued process (as a row vector),
- for each i, the so-called driver $(\omega, y, z) \mapsto f_i(y, z)$ is $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q)$-measurable.

Equation (1.1) is a backward Dynamic Programming (DP for short) equation, which is solved at $i = N - 1$ by first evaluating Z_{N-1} using $Y_N = \xi$, then Y_{N-1} using Y_N and Z_{N-1}, and then iterating these evaluations until $i = 0$.

Application. Equation (1.1) appears naturally when approximating a continuous-time BSDE by a discrete-time process along the time grid π. The continuous time BSDE may be a generalized BSDE of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s - (L_T - L_t), \tag{1.2}$$

where L is a martingale orthogonal to W. The presence of L occurs, for example, when $\xi = \Phi(X_T)$ where X is a jump-diffusion process driven by W and a Poisson random measure. In this context, and for $f(s, \omega, y, z) = f(s, X_s(\omega), y, z)$, it is shown in [LGW06, Theorem 1] that the discrete time process $(Y_i, Z_i)_{0 \leq i < N}$ generated by (1.1) converges to (Y, Z), in suitable $\mathbf{L}_2$-spaces, as the mesh size $|\pi|$ goes to 0. Note that the general formulation (1.1) using conditional expectations w.r.t. $\mathcal{F}_{t_i}$ allows for path-dependent drivers/terminal conditions: one can take $\xi = \Phi(X_{t_1}, \ldots, X_{t_N})$ (see [GGG11]) or $\xi = \Phi(X_T, \int_0^T X_t dt)$ for a diffusion process X (see [GLW05]).

DP equation (1.1) is written in an explicit form because the driver f_i depends on Y_{i+1}. Discrete BSDEs have traditionally been studied in implicit form, i.e. where f_i depends on Y_i. The explicit and implicit schemes usually give, to the best of our knowledge, the same rate of convergence as $N \to +\infty$ for the discretization error (the error incurred by approximating the continuous BSDE by a discrete process) in an appropriate $\mathbf{L}_2$-space: compare the results of [BT04] (implicit scheme) to [LGW06] (explicit scheme).

A lot of attention - [Zha04][BT04][GM10] among others - has been paid to the analysis of the discretization error. This is not the focus of the current

article; rather, we focus on the numerical resolution of the DP equations (1.1), allowing $N \to +\infty$.

Numerical approximation. One has to approximate the conditional expectations in (1.1) in order to have a fully implementable scheme. This is because one cannot, in general, calculate the conditional expectation explicitly. Over the last ten years, various different approaches have been developed to do this - see the introduction of [GL10] for an overview - but very few papers [BT04][LGW06][Mos10] have tackled the global error analysis. In the current article, we follow the *empirical regression approach* presented in [LGW06] and estimate the global error that this method incurs in the approximation of (Y, Z). Suppose that

$$(Y_i, Z_i) := (y_i(X_i), z_i(X_i)) \tag{1.3}$$

for some (unknown but deterministic) measurable functions $y_i(\cdot), z_i(\cdot)$ and a d-dimensional explanatory process $X := (X_i)_{0 \leq i \leq N}$ (in the jump-diffusion example above, X would be the Euler approximation at times $(t_i)_i$). Since each conditional expectation $\mathbb{E}_i(\cdot)$ can be viewed as solution of a least-squares problem in $\mathbf{L}_2(\mathbb{P})$, the functions $y_i(\cdot)$ and $z_i(\cdot)$ are then approximated using a finite-dimensional approximation. The coefficients of this approximation are computed using empirical least-squares regression [GKKW02, Chapters 10-11-12] using M simulations of the paths of the explanatory variable X. The use of such an empirical regression scheme is supported by two important features: first, it requires as an input only independent paths of the explanatory process X and of the Brownian motion W; second, using distribution-free tools [GKKW02], one may achieve model-independent error estimates related to the statistical error. These robust estimates (Theorem 4.2) are presumably too conservative; on the other hand, the estimates allow the error analysis to be applied to very general probability spaces, because we make very few assumptions on the explanatory process X: see Section 2.

The resulting global error is known to be very difficult to analyze, because all regression problems are stochastically dependent through the DP equation; moreover, the numerical parameters (the time-grid π, functions basis used for the finite-dimensional approximations of $y_i(\cdot), z_i(\cdot)$ and the number of simulations M) play multiple, often contradictory, roles in the convergence, and it is crucial to find the right trade-off between them. We achieve the global error analysis in Theorem 4.1, which is one of our main results. We then apply these results to optimize the numerical parameters needed for a given accuracy in the asymptotics $N \to +\infty$, see Section 4.3. For the reader interested in empirical analysis, we refer to [Mos10][Ric10][Tur12].

1.1. Our contributions

In this paper, we revisit the error analysis derived in [LGW06], achieving several important improvements and extensions.

1. We derive stability results for discrete BSDEs; see Section 3.1. These stability results are used to prove additional boundedness (Section 3.3) and smoothness (Section 3.5) properties for the discrete BSDEs. These properties are proved under quite weak assumptions.

2. We prove that the One step-forward Dynamic Programming (ODP for short) equation (1.1) provides worse estimates than the Multi step-forward Dynamic Programming (MDP for short) equation given by

$$\begin{cases} Y_i & = \mathbb{E}_i \left(\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k \right), \\ \Delta_i Z_i & = \mathbb{E}_i \left([\xi + \sum_{k=i+1}^{N-1} f_k(Y_{k+1}, Z_k)\Delta_k]\Delta W_i^\top \right). \end{cases} \tag{1.4}$$

   Because of the tower property of conditional expectations, definitions (1.1) and (1.4) coincide. When approximations of the conditional expectations are incorporated, the definitions differ and the MDP-based scheme gives raise to a smaller propagation of errors; see Theorem 3.1. This is in agreement with some features of the Bender-Denk algorithm [BD07] and the Longstaff-Schwartz algorithm [LS01] for optimal stopping problems. The main difference with the [BD07] is that we prove that Picard iterations are unnecessary. This is good news because it greatly simplifies the algorithm without deteriorating the estimates.

3. We derive a full error analysis for the MDP equation, including the effect of statistical errors (finite number of Monte Carlo simulations); see Theorem 4.1. These estimates are obtained by exploiting stability inequalities - Section 3.1 - for discrete BSDEs. We also show that the estimates are an improvement on [LGW06], where ODP equations are analysed; see Section 4.3. Moreover, we do not require Lipschitz continuity assumptions on the Markov functions y_i to obtain our estimates.

4. We allow the driver to satisfy a weaker assumption than in [LGW06, BD07]: only local Lipschitz conditions are considered, see ($\mathbf{A_F}$) below. This allows the results of this paper to be applied to a wider range of approximation problems for BSDEs, including some cases of quadratic BSDEs; see Section 2.

5. We allow the time grid π to be non uniform; see ($\mathbf{A_F}$-**iii**) . Indeed, to reduce the discretization error for BSDE with irregular terminal conditions $\xi = \Phi(X_T)$, it has been recently proposed in [GM10] to choose nonuniform

grids: the grid points are more concentrated close to the terminal time T in order to compensate the lack of regularity of Φ. Similar results are obtained for for path dependent ξ in [GGG11];

6. Finally, we demonstrate how higher orders of smoothness of the Markov functions y_i and z_i defined in (1.3) lead to improvements in the error-computational work trade-off. We also demonstrate that MDP has a better error-computational time trade off than ODP in the context of these theoretical analysis; see Section 4.3.

Organization of the paper. In the remainder of this section, we define notation used throughout the paper. In Section 2, we state our working assumptions and give several examples to show how these assumptions are useful for approximating a wide variety of continuous-time BSDEs. In Section 3, we establish stability estimates for discrete BSDEs, and apply them to derive tight pointwise and $\mathbf{L}_2$-estimates for (Y, Z). We define the MDP-based scheme and we analyze the $\mathbf{L}_2$-error incurred when conditional expectations are approximated by projections on closed convex subsets of $\mathbf{L}_2$. This allows comparison between ODP and MDP-based schemes. Finally, we determine boundedness and smoothness properties of discrete BSDEs under some additional assumptions. In Section 4, the projections are computed using M independent simulations of the explanatory process X. The global error is stated in Theorems 4.1 and 4.2. The rest of the section is devoted to (long and technical) proofs. A discussion related to algorithm complexity is given in Subsection 4.3. In particular, we show how higher order of smoothness of the Markov functions y_i and z_i leads to an improved error-computational time trade-off, and compare this to the results for the ODP. Some intermediate results are detailed in Appendix.

Further notation.

- $|x|$ stands for the Euclidean norm of the vector x.
- $|U|_{\mathbf{L}_p} = (\mathbb{E}|U|^p)^{\frac{1}{p}}$ stands for the $\mathbf{L}_p(\mathbb{P})$-norm ($p \geq 1$) of a random variable U. To indicate that U is additionally measurable w.r.t. the σ-algebra $\mathcal{Q}$, we may write $U \in \mathbf{L}_p(\mathcal{Q}, \mathbb{P})$.
- We reserve the letter $\gamma := (\gamma_0, \ldots, \gamma_{N-1}) \in \mathbb{R}_+^N$ for the parameter appearing in the weighted $\mathbf{L}_2$-norms below. Moreover for a given γ, we set $\Gamma_i := \prod_{k=0}^{i-1}(1 + \gamma_k \Delta_k)$ for $0 \leq i < N$ (with the usual convention $\prod_{k=0}^{-1} \cdots = 1$).
- For a $q(> 1)$-dimensional process $U = (U_i)_{0 \leq i \leq N}$, its l-th component is denoted by $U_l = (U_{l,i})_{0 \leq i \leq N}$.

2. Standing assumptions and applicability to practical continuous-time problems

In this section, we give the standing assumptions for this paper. These assumptions are more general than in previous numerical schemes for BSDEs, and we outline several examples to demonstrate how these more general assumptions lead extended applicability of the results of this paper to practical continuous-time BSDE problems.

The standing assumptions are separated into two parts: the first set consists of the minimal assumptions necessary for basic results of Section 3, and the second set consists of the *Markovian* assumptions necessary for Section 4. The minimal assumptions used in this paper are that the terminal condition ξ is square integrable and the driver is *locally* Lipschitz continuous in the sense that the Lipschitz constant depends on t_i. To be more precise:

$(\mathbf{A}_\xi)$ ξ is in $\mathbf{L}_2(\mathcal{F}_T, \mathbb{P})$.

$(\mathbf{A_F})$ i) $(\omega, y, z) \mapsto f_i(y, z)$ is $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q)$-measurable for every $i < N$, and there exist deterministic parameters $\theta_L \in (0, 1]$ and $L_f \in [0, +\infty)$ such that

$$|f_i(y, z) - f_i(y', z')| \leq \frac{L_f}{(T - t_i)^{(1-\theta_L)/2}}(|y - y'| + |z - z'|), \quad (2.5)$$

for any $(y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q$.

ii) There exist deterministic parameters $\theta_c \in (0, 1]$ and $C_f \in [0, +\infty)$ such that

$$|f_i(0, 0)| \leq \frac{C_f}{(T - t_i)^{1-\theta_c}}, \quad \forall 0 \leq i < N. \quad (2.6)$$

iii) The time-grids $\pi := \{0 = t_0 < \ldots < t_N = T\}$ are such that

$$C_\pi = \sup_{k<N} \frac{\Delta_k}{(T - t_k)^{1-\theta_L}} \to 0 \quad \text{as } N \to +\infty, \quad (2.7)$$

$$\limsup_{N\to\infty} R_\pi < +\infty, \quad \text{where } R_\pi = \sup_{0\leq k\leq N-2} \frac{\Delta_k}{\Delta_{k+1}}. \quad (2.8)$$

Under $(\mathbf{A}_\xi)$ and $(\mathbf{A_F}$**-i-ii**$)$, it is straightforward to check from (1.1) that $(Y_i)_{0\leq i\leq N}$ and $(Z_i)_{0\leq i<N}$ are well defined and belong to $\mathbf{L}_2$ (see Proposition 3.1 for tight estimates).

When analyzing the influence of M in Section 4, we reinforce the basic assumptions with the following set of *Markovian* assumptions:

$(\mathbf{A_X})$ X is a Markov chain in $\mathbb{R}^d$ $(1 \leq d < +\infty)$ adapted to $(\mathcal{F}_{t_i})_i$.

($\mathbf{A}'_\xi$) i) ξ is a bounded $\mathcal{F}_T$-measurable random variable; we set $C_\xi := \mathbb{P}-\operatorname{ess\,sup}_\omega |\xi(\omega)| < +\infty$.

ii) ξ is of form $\xi = \Phi(X_N)$ for a measurable function Φ.

($\mathbf{A}'_\mathbf{F}$) For every $i < N$, the driver is of the form $f_i(y,z) = f_i(X_i, y, z)$ where $(x,y,z) \mapsto f_i(x,y,z)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^q)$-measurable and ($\mathbf{A}_\mathbf{F}$) is satisfied.

These give us a Markov representation for solutions of the discrete BSDEs: for all $k < N$, there exist measurable, deterministic functions $y_k : \mathbb{R}^d \to \mathbb{R}$ and $z_k : \mathbb{R}^d \to \mathbb{R}^q$ such that $Y_k = y_k(X_k)$ and $Z_k = z_k(X_k)$ holds almost surely. Indeed, taking the Markov chain $(X_i^{k,x})_{i\geq k}$ started at t_k with value $x \in \mathbb{R}^d$ with the same transition probabilities as $(X_k)_k$ yields, by induction, $\Delta_k z_k(x) = \mathbb{E}[\Delta W_k^\top y_{k+1}(X_{k+1}^{k,x})]$ and $y_k(x) = \mathbb{E}[y_{k+1}(X_{k+1}^{k,x}) + f_k(x, y_{k+1}(X_{k+1}^{k,x}), z_k(x))\Delta_k]$; see the proof of Lemma 4.2 for details.

We emphasize that we do not make any further assumptions on X - no non-degeneracy condition, no specific distributions, etc; our error estimates are *model-free* in this sense. This lends flexibility and robustness to the empirical least-squares regression scheme.

The assumptions above derive from particular continuous time settings; see below. This means that it is natural for us to assume that the constants θ_L, L_f, θ_c, and C_f are time-grid independent. This assumption simplifies the complexity analysis in Section 4.3.

At first glance, the boundedness assumption ($\mathbf{A}'_\xi$-**i**) appears to be a serious restriction of our scheme. Indeed, ($\mathbf{A}_\xi$) is the minimal assumption to ensure the existence of a continuous-time BSDE in $\mathbf{L}_2$-spaces [EPQ97]. The *raison d'être* of ($\mathbf{A}'_\xi$-**i**) is to derive robust estimates for the global error (see Theorem 4.1) using the tools of nonparametric regression [GKKW02]. On the other hand, $\xi_n = -n \vee \xi \wedge n$ $(n \geq 0)$ defines a sequence of bounded approximations of ξ and by $\mathbf{L}_2$-stability results on continuous-time BSDEs (see [EPQ97, Proposition 2.1] for instance), the truncation error converges to 0 as $n \to +\infty$. Since in our global error estimates we keep track on the dependence on C_ξ, it would be a priori possible to let this upper bound go appropriately quickly to infinity, while maintaining a converging scheme.

Assumptions ($\mathbf{A}_\mathbf{F}$-**i-ii**) may be surprising because they extend the usual global Lipschitz continuity conditions in an unusual way. Globally Lipschitz drivers are related to the case $\theta_L = 1$, and $\theta_c = 1$ describes the usual situation where drivers are uniformly (in time) bounded at $(y,z) = (0,0)$. The singularity at the terminal time allows us to extend the applicability of our numerical scheme to include a wider class of continuous-time Markovian BSDE related to a $\mathbb{R}^d$-valued Brownian diffusion process $(X_t)_{0\leq t\leq T}$. We outline two canonical examples that motivate ($\mathbf{A}_\mathbf{F}$-**i-ii**) . Take $\xi = \Phi(X_T)$ and $f(t,\omega,y,z) =$

$f(t, X_t(\omega), y, z)$. For simplicity, assume $q = d$ and that the coefficients of X are smooth and bounded and that its diffusion coefficient $\sigma(t, x)$ satisfies a uniform ellipticity condition. We denote by $\mathcal{L}$ the infinitesimal generator of X.

Quadratic BSDEs. Consider a quadratic growth driver satisfying

$$|f(t, x, y, z)| \leq c\ (1 + |y| + |z|^2),$$
$$|f(t, x, y, z) - f(t, x, y', z')| \leq c\ (1 + |z| + |z'|)(|y - y'| + |z - z'|)$$

for any $(t, x, y, y', z, z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and for a given constant $c \geq 0$. Assume additionally that the terminal function Φ is Hölder continuous and bounded. Then [DG06, Theorem 2.1] yields that the continuous-time BSDE is given by $Y_t = u(t, X_t)$ and $Z_t = \nabla u(t, X_t)\sigma(t, X_t)$ where u solves the semi-linear PDE $\partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(t, x)) = 0$ with $u(T, x) = \Phi(x)$. Moreover, there exist constants $\theta \in (0, 1]$ and $C_u \in \mathbb{R}^+$ such that

$$(T - t)^{(1-\theta)/2}|\nabla u(t, x)\sigma(t, x)| \leq C_u, \quad \forall (t, x) \in [0, T) \times \mathbb{R}^d.$$

In a personal discussion, François Delarue has brought to our attention that in this BSDE setting, θ is equal to the Hölder exponent of Φ. Now, set $\varphi_t : \zeta \in \mathbb{R} \mapsto \varphi_t(\zeta) = \text{sign}(\zeta) \min\big(|\zeta|, \frac{C_u}{(T-t)^{(1-\theta)/2}}\big)$ and define the new driver $\bar{f}(t, x, y, z) := f\big(t, x, y, \varphi_t(z_1) \ldots, \varphi_t(z_d)\big)$. Observe that $\bar{f}(t, X_t, Y_t, Z_t) = f(t, X_t, Y_t, Z_t)$, thus it is equivalent to solve the BSDE with driver f or $\bar{f}$. Notice also that $\varphi_t(\cdot)$ is 1-Lipschitz continuous and bounded by $\frac{C_u}{(T-t)^{(1-\theta)/2}}$, hence $f_i(y, z) := \bar{f}(t_i, X_{t_i}, y, z)$ satisfies (**$\mathbf{A_F}$-i-ii**) with $C_f = c$, $\theta_c = 1$, $L_f = c(T^{(1-\theta)/2} + 2\sqrt{d}C_u)$, $\theta_L = \theta$. One may argue that the construction of $\bar{f}$ depends on the knowledge of C_u, which is not straightforward; see [Ric10] for some explicit estimates.

Using proxys for numerical stability. Consider a standard Lipschitz driver f. Assume that we know by *expertise* that the solution $(Y_t, Z_t)_t$ is expected to be close to $(v(t, X_t), \nabla v(t, X_t)\sigma(t, X_t))_t$, where v is the explicit solution to a linear parabolic equation $\partial_t v(t, x) + \tilde{\mathcal{L}}v(t, x) + \tilde{f}(t, x) = 0$; the diffusion process associated to $\tilde{\mathcal{L}}$, the terminal condition and the driver may have changed to produce an analytical solution. v is called *proxy* in [BGM09]. It is then natural to numerically compute the residual $(Y_t^0, Z_t^0) := (Y_t - v(t, X_t), Z_t - \nabla v(t, X_t)\sigma(t, X_t))$. It solves a BSDE with terminal function $\Phi(.) - v(T, .)$ and driver

$$f^0(t, x, y, z) := f(t, x, y + v(t, x), z + \nabla v(t, x)\sigma(t, x)) - \tilde{f}(t, x) + (\mathcal{L} - \tilde{\mathcal{L}})v(t, x).$$

The new driver f^0 is uniformly Lipschitz w.r.t. y and z, so (**$\mathbf{A_F}$-i**) is satisfied with $\theta_L = 1$. If $v(T, .)$ is θ-Hölder continuous ($\theta \in (0, 1]$), then usual PDE estimates on the parabolic operator $\tilde{\mathcal{L}}$ give $(T - t)^{(\frac{k-\theta}{2})_+}|D_x^k v(t, x)| \leq C_v$ ($k = 0, 1, 2$), from which (**$\mathbf{A_F}$-ii**) is derived with $\theta_c = \theta/2$. To conclude this example, we mention that in the case $\tilde{\mathcal{L}} = \mathcal{L}$, $v(T, .) = \Phi(.)$ and $\tilde{f} = 0$, it is

proved in [GM10] that the $\mathbf{L}_2$-time-regularity of (Y^0, Z^0) is usually more well-behaved than that of (Y, Z), suggesting that the discretization error from the DP equation for (Y^0, Z^0) would be smaller.

Assumption ($\mathbf{A_F}$-**iii**) is used to derive stability results for discrete BSDEs (see Proposition 3.1) and for the numerical schemes (see Theorems 3.1 and 4.1) as the number N of grid times becomes large. If $\theta_L = 1$, the condition (2.7) is equivalent to $|\pi| \to 0$. If $\theta_L \in (0,1)$ and π is a time-grid with higher concentration at T as in [GM10] (i.e. $t_k = T - T(1-k/N)^{1/\theta_\pi}$ with $\theta_\pi \in (0,1]$), then one easily checks conditions (2.7) and (2.8) hold (see [GT11]):

$$C_\pi \leq \frac{T^{\theta_L}}{\theta_\pi} \frac{1}{N^{1 \wedge \frac{\theta_L}{\theta_\pi}}}, \qquad R_\pi \leq \frac{1}{\theta_\pi}\left(1 \vee \left(\frac{1}{2\theta_\pi}\right)^{\frac{1}{\theta_\pi}-1}\right).$$

Hence, (2.7) and (2.8) hold true whatever the value θ_π is. This shows that in most usual situations $|\pi| \to 0$ implies ($\mathbf{A_F}$-**iii**) .

3. ODP scheme vs. MDP scheme

The aim of this section is threefold. Firstly, we use the minimal assumptions ($\mathbf{A}_\xi$) and ($\mathbf{A_F}$) to determine a priori stability estimates for discrete BSDEs. Secondly, we use the stability results to show that the MDP scheme combined with $\mathbf{L}_2$-projection yields a smaller error than the ODP scheme. In doing so, we revisit the results of [BD07], but avoid the Picard iterations of their scheme. This also serves as a warm-up to Section 4, where the a priori stability results also play a crucial role. Thirdly, we demonstrate how slightly stronger assumptions yield time uniform almost sure bounds on the solutions of the BSDEs, and smoothness properties in the case of Markovian BSDEs; these properties are extremely useful in Section 4.

3.1. General a priori estimates

Definition 3.1. *The truncation of the i-th Brownian increment at threshold $R = [0, +\infty]$ is defined by*

$$[\Delta W_i]_w = (-R\sqrt{\Delta_i} \vee \Delta W_{1,i} \wedge R\sqrt{\Delta_i}, \cdots, -R\sqrt{\Delta_i} \vee \Delta W_{q,i} \wedge R\sqrt{\Delta_i})^\top.$$

For $R = +\infty$, $[\Delta W_i]_w = \Delta W_i$. Replacing ΔW_i by $[\Delta W_i]_w$ has small impact in the DP equations (1.1), provided that R is large enough (see Proposition 3.3). On the other hand, taking finite R ensures that some quantities are a.s. bounded, which is crucial in our error analysis in Section 4.

In this subsection, we study the difference between two *discrete BSDEs* with truncated Brownian increments, $(Y_i^R, Z_i^R)_i$ and $(\bar{Y}_i^R, \bar{Z}_i^R)_i$, given by

$$\left\{ \begin{array}{rcl} Y_i^R & = & \mathbb{E}_i\left(\xi + \sum_{k=i}^{N-1} f_k(Y_{k+1}^R, Z_k^R)\Delta_k\right), \\ \Delta_i Z_i^R & = & \mathbb{E}_i\left([\xi + \sum_{k=i+1}^{N-1} f_k(Y_{k+1}^R, Z_k^R)\Delta_k][\Delta W_i^\top]_w\right), \end{array} \right. \tag{3.9}$$

and similarly for $(\bar{Y}_i^R, \bar{Z}_i^R)_i$ with data $(\bar{\xi}, \bar{f}_i(y,z))$. The superscript R refers to the fact the Brownian increments are truncated at the threshold $R \in [0, +\infty]$.

We assume that ξ and $\bar{\xi}$ are both in $\mathbf{L}_2$ (assumption $(\mathbf{A}_\xi)$). We allow rather greater generality than afforded by $(\mathbf{A_F})$: firstly, the drivers $f_i(y,z)$ and $\bar{f}_i(y,z)$ are Lipschitz continuous w.r.t. (y,z) and the dependence of their Lipschitz constant w.r.t. i is general; finally, we do not insist that the drivers be adapted. We will require the extension to non-adapted drivers later in Section 4, where we will apply these results to BSDEs with data dependent drivers. However, we assume that each $f_i(Y_{i+1}^R, Z_i^R)$ and $\bar{f}_i(\bar{Y}_{i+1}^R, \bar{Z}_i^R)$ are in $\mathbf{L}_2$, so that Y_i^R, Z_i^R, $\bar{Y}_i^R$, $\bar{Z}_i^R$ are also in $\mathbf{L}_2$ for any i. Using the tower property of conditional expectations, observe that

$$\left\{ \begin{array}{ll} Y_N^R & = \xi, \quad Y_i^R = \mathbb{E}_i \left(Y_{i+1}^R + f_i(Y_{i+1}^R, Z_i^R)\Delta_i \right), \\ \Delta_i Z_i^R & = \mathbb{E}_i \left(Y_{i+1}^R [\Delta W_i^\top]_w \right), \end{array} \right. \tag{3.10}$$

and similarly for $(\bar{Y}^R, \bar{Z}^R)$. We study the differences:

$$\Delta Y_i^R = Y_i^R - \bar{Y}_i^R, \qquad \Delta Z_i^R = Z_i^R - \bar{Z}_i^R$$

and we set

$$\Delta f_i = f_i(Y_{i+1}^R, Z_i^R) - \bar{f}_i(Y_{i+1}^R, Z_i^R), \qquad \Delta\xi = \xi - \bar{\xi}.$$

We shall use the following Lemma repeatedly:

Lemma 3.1 (Local estimates). *For $0 \le i \le N-1$, assume that $\bar{f}_i$ is Lipschitz w.r.t. y and z, with a Lipschitz constant equal to $L_{\bar{f}_i} \in \mathbb{R}^+$. For any $R \in [0, +\infty]$, $\Delta_i \le T$ and $\gamma_i > 0$ satisfying $6q(\Delta_i + \frac{1}{\gamma_i})L_{\bar{f}_i}^2 \le 1$, we have*

$$|\Delta Y_i^R|^2 \le (1 + (\gamma_i + \frac{1}{2})\Delta_i)\mathbb{E}_i(|\Delta Y_{i+1}^R|^2) + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i \mathbb{E}_i(\Delta f_i^2). \tag{3.11}$$

Proof. <u>*Preliminary estimates for ΔZ_i^R.*</u> From (3.10) we have

$$\Delta_i \Delta Z_i^R = \mathbb{E}_i([\Delta Y_{i+1}^R - \mathbb{E}_i(\Delta Y_{i+1}^R)][\Delta W_i^\top]_w).$$

By the Cauchy-Schwarz inequality, note that

$$|\mathbb{E}_i([\Delta Y_{i+1}^R - \mathbb{E}_i(\Delta Y_{i+1}^R)][\Delta W_i^\top]_w)|^2 \le q\ \Delta_i \Big(\mathbb{E}_i[(\Delta Y_{i+1}^R)^2] - (\mathbb{E}_i \Delta Y_{i+1}^R)^2 \Big)$$

uniformly in R, whence

$$\Delta_i |\Delta Z_i^R|^2 \le q \Big(\mathbb{E}_i[(\Delta Y_{i+1}^R)^2] - (\mathbb{E}_i \Delta Y_{i+1}^R)^2 \Big). \tag{3.12}$$

<u>*Estimates for ΔY_i^R.*</u> We have

$$\Delta Y_i^R = \mathbb{E}_i \Delta Y_{i+1}^R + \Delta_i \mathbb{E}_i[\Delta f_i] + \Delta_i \mathbb{E}_i[\bar{f}_i(Y_{i+1}^R, Z_i^R) - \bar{f}_i(\bar{Y}_{i+1}^R, \bar{Z}_i^R)].$$

Combining the Young inequality $(a+b)^2 \leq (1+\gamma_i\Delta_i)a^2 + (1+\frac{1}{\gamma_i\Delta_i})b^2$ and the Lipschitz property of $\bar{f}_i$ and (3.12), we deduce

$$
\begin{aligned}
(\Delta Y_i^R)^2 \leq & (1+\gamma_i\Delta_i)(\mathbb{E}_i\Delta Y_{i+1}^R)^2 \qquad (3.13)\\
& + 3(\Delta_i + \frac{1}{\gamma_i})\Delta_i \left[\mathbb{E}_i[\Delta f_i^2] + L_{\bar{f}_i}^2\mathbb{E}_i[(\Delta Y_{i+1}^R)^2] + L_{\bar{f}_i}^2|\Delta Z_i^R|^2\right]\\
\leq & \left(1+\gamma_i\Delta_i - 3qL_{\bar{f}_i}^2(\Delta_i+\frac{1}{\gamma_i})\right)(\mathbb{E}_i\Delta Y_{i+1}^R)^2 + 3(\Delta_i+\frac{1}{\gamma_i})\Delta_i\mathbb{E}_i[\Delta f_i^2]\\
& + \left[3(\Delta_i+\frac{1}{\gamma_i})\Delta_i L_{\bar{f}_i}^2 + 3qL_{\bar{f}_i}^2(\Delta_i+\frac{1}{\gamma_i})\right]\mathbb{E}_i[(\Delta Y_{i+1}^R)^2]. \qquad (3.14)
\end{aligned}
$$

Under our assumptions on γ_i, we have $\gamma_i \geq 3qL_{\bar{f}_i}^2$, which ensures $1+\gamma_i\Delta_i - 3qL_{\bar{f}_i}^2(\Delta_i+\frac{1}{\gamma_i}) \geq 0$ for any Δ_i. This allows us to combine terms of $(\mathbb{E}_i\Delta Y_{i+1}^R)^2$ and $\mathbb{E}_i[(\Delta Y_{i+1}^R)^2]$ using Jensen's inequality in (3.14):

$$
\begin{aligned}
(\Delta Y_i^R)^2 \leq & \left(1+\gamma_i\Delta_i - 3qL_{\bar{f}_i}^2(\Delta_i+\frac{1}{\gamma_i})\right)\mathbb{E}_i[(\Delta Y_{i+1}^R)^2] + 3(\Delta_i+\frac{1}{\gamma_i})\Delta_i\mathbb{E}_i(\Delta f_i^2)\\
& + \left[3(\Delta_i+\frac{1}{\gamma_i})\Delta_i L_{\bar{f}_i}^2 + 3qL_{\bar{f}_i}^2(\Delta_i+\frac{1}{\gamma_i})\right]\mathbb{E}_i[(\Delta Y_{i+1}^R)^2]\\
= & \left(1+\gamma_i\Delta_i + 3(\Delta_i+\frac{1}{\gamma_i})\Delta_i L_{\bar{f}_i}^2\right)\mathbb{E}_i[(\Delta Y_{i+1}^R)^2] + 3(\Delta_i+\frac{1}{\gamma_i})\Delta_i\mathbb{E}_i(\Delta f_i^2),
\end{aligned}
$$

which proves (3.11) since $3(\Delta_i+\frac{1}{\gamma_i})\Delta_i L_{\bar{f}_i}^2 \leq \frac{\Delta_i}{2}$. □

The following Proposition will be used extensively in the statistical analysis:

Proposition 3.1 (Global pointwise estimates). *Assume that, for each i, $\bar{f}_i$ is Lipschitz w.r.t. y and z with Lipschitz constant $L_{\bar{f}_i} \in \mathbb{R}^+$. Then, for any $R \in [0,+\infty]$, and any time grid π and $\gamma \in (0,+\infty)^N$ satisfying $6q(\Delta_k + \frac{1}{\gamma_k})L_{\bar{f}_k}^2 \leq 1$ for all $k \leq N-1$, we have for $0 \leq i \leq N$*

$$
\begin{aligned}
& |\Delta Y_i^R|^2\Gamma_i + \sum_{k=i}^{N-1}\Delta_k\mathbb{E}_i(|\Delta Z_k^R|^2)\Gamma_k\\
& \qquad \leq C_{3.15}\Big(\Gamma_N\mathbb{E}_i(\Delta\xi^2) + 3\sum_{k=i}^{N-1}\big(\frac{1}{\gamma_k}+\Delta_k\big)\Delta_k\mathbb{E}_i(\Delta f_k^2)\Gamma_k\Big), \qquad (3.15)
\end{aligned}
$$

where $\Gamma_i := \prod_{k=0}^{i-1}(1+\gamma_k\Delta_k)$ and $C_{3.15} := 2q + (1+T)e^{T/2}$.

Note that, whenever necessary, the above pointwise estimates can be easily turned into uniform $\mathbf{L}_2$-estimates:

$$
\sup_{i\leq k\leq N}\mathbb{E}(|\Delta Y_k^R|^2)\Gamma_k + \sum_{k=i}^{N-1}\Delta_k\mathbb{E}(|\Delta Z_k^R|^2)\Gamma_k
$$

$$\leq C_{3.15}\Big(\Gamma_N\mathbb{E}(\Delta\xi^2) + 3\sum_{k=i}^{N-1}(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}(\Delta f_k^2)\Gamma_k\Big).$$

Proof. Starting at (3.11), multiply both sides by

$$\lambda_i := (1+(\gamma_{i-1}+\frac{1}{2})\Delta_{i-1})\lambda_{i-1}, \quad \lambda_0 := 1,$$

sum between $k=i$ to $k=N-1$, and take conditional expectations $\mathbb{E}_i$ to deduce:

$$(\Delta Y_i^R)^2\lambda_i \leq \lambda_N\mathbb{E}_i(\Delta\xi^2) + 3\sum_{k=i}^{N-1}(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}_i(\Delta f_k^2)\lambda_k. \tag{3.16}$$

From the simple inequality $\Gamma_i \leq \lambda_i = e^{\sum_{k=0}^{i}\ln(1+(\gamma_k+\frac{1}{2})\Delta_k)} \leq e^{T/2}\Gamma_i$, we get for $0\leq i\leq N$

$$(\Delta Y_i^R)^2\Gamma_i \leq e^{T/2}\Gamma_N\mathbb{E}_i(\Delta\xi^2) + 3e^{T/2}\sum_{k=i}^{N-1}(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}_i(\Delta f_k^2)\Gamma_k. \tag{3.17}$$

<u>*Final estimates for ΔZ_i.*</u> From (3.12), we have

$$\begin{aligned}\sum_{k=i}^{N-1}\Delta_k\mathbb{E}_i[|\Delta Z_k^R|^2]\Gamma_k &\leq \sum_{k=i}^{N-1} q\Gamma_{k+1}\big(\mathbb{E}_i[(\Delta Y_{k+1}^R)^2] - \mathbb{E}_i[(\mathbb{E}_k\Delta Y_{k+1}^R)^2]\big)\\ &\leq q\Gamma_N\mathbb{E}_i(\Delta\xi^2) + \sum_{k=i+1}^{N-1} q\Gamma_k\Big(\mathbb{E}_i[(\Delta Y_k^R)^2] - (1+\gamma_k\Delta_k)\mathbb{E}_i[(\mathbb{E}_k\Delta Y_{k+1}^R)^2]\Big).\end{aligned}$$

From (3.13), we have

$$\begin{aligned}&\mathbb{E}_i[(\Delta Y_k^R)^2] - (1+\gamma_k\Delta_k)\mathbb{E}_i[(\mathbb{E}_k\Delta Y_{k+1}^R)^2]\\ &\qquad\leq 3(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\Big[\mathbb{E}_i(\Delta f_k^2) + L_{\bar f_k}^2\mathbb{E}_i[(\Delta Y_{k+1}^R)^2] + L_{\bar f_k}^2\mathbb{E}_i[|\Delta Z_k^R|^2]\Big].\end{aligned}$$

Plugging this inequality into that above yields

$$\begin{aligned}&\sum_{k=i}^{N-1}\Delta_k\mathbb{E}_i[|\Delta Z_k^R|^2]\Gamma_k\\ &\leq q\Gamma_N\mathbb{E}_i(\Delta\xi^2) + 3\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k L_{\bar f_k}^2\mathbb{E}_i(|\Delta Z_k^R|^2)\Gamma_k\\ &\quad + 3\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k\mathbb{E}_i(\Delta f_k^2)\Gamma_k + 3\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k L_{\bar f_k}^2\mathbb{E}_i[(\Delta Y_{k+1}^R)^2]\Gamma_k.\end{aligned}$$

For γ_k and Δ_k as in the Proposition statement, we have $3q(\frac{1}{\gamma_k}+\Delta_k)L^2_{\bar{f}_k} \leq \frac{1}{2}$, and thus

$$\begin{aligned}
&\sum_{k=i}^{N-1} \Delta_k \mathbb{E}_i[|\Delta Z_k^R|^2]\Gamma_k \\
&\leq 2q\Gamma_N \mathbb{E}_i(\Delta\xi^2) + 6\sum_{k=i+1}^{N-1} q(\frac{1}{\gamma_k}+\Delta_k)\Delta_k \mathbb{E}_i(\Delta f_k^2)\Gamma_k + \sum_{k=i+1}^{N-1} \Delta_k \mathbb{E}_i[(\Delta Y_{k+1}^R)^2]\Gamma_k \\
&\leq (2q+Te^{T/2})\Gamma_N \mathbb{E}_i(\Delta\xi^2) + (6q+3Te^{T/2})\sum_{k=i+1}^{N-1}(\frac{1}{\gamma_k}+\Delta_k)\Delta_k \mathbb{E}_i(\Delta f_k^2)\Gamma_k,
\end{aligned}$$

where we have used the estimate (3.17) on ΔY in the last inequality. □

3.2. Projection errors for the ODP and MDP-based schemes

Projection on a closed convex subspace of $\mathbf{L}_2$.

Definition 3.2. *Let $\mathcal{S}$ be a non-empty closed convex subset of $\mathbf{L}_2(\mathcal{F}_T,\mathbb{P})$. Then, to any random variable $U \in \mathbf{L}_2(\mathcal{F}_T,\mathbb{P})$ we can associate $\mathcal{P}(U) \in \mathcal{S}$, the (unique) projection of U on $\mathcal{S}$, which satisfies $\mathbb{E}|U-\mathcal{P}(U)|^2 = \inf_{S\in\mathcal{S}} \mathbb{E}|U-S|^2$. For any $S \in \mathcal{S}$, we have*

$$\mathbb{E}\big((U-\mathcal{P}(U))(S-\mathcal{P}(U)\big) \leq 0. \tag{3.18}$$

$\mathcal{S}$ can be a finite dimensional vector space, i.e. $\mathcal{S} := \{S = S_0 + \sum_{k=1}^K \alpha_k p_k, (\alpha_k)_{1\leq k\leq K} \in \mathbb{R}^K\}$ for some $p_k \in \mathbf{L}_2$: this is our choice (with $S_0 = 0$) in Section 4. It can also be a convex ball of the form $\mathcal{S} = \{S = S_0 + \sum_{k=1}^K \alpha_k p_k, (\alpha_k)_{1\leq k\leq K} \in \mathbb{R}^K, \|\alpha\| \leq \rho\}$ ($\rho \geq 0$) where $\|\cdot\|$ is *a norm* in $\mathbb{R}^K$. When $\|\cdot\|$ is the Euclidean norm, we obtain the ridge regression [GL96, Section 12.1], whereas the ℓ_1-norm leads to the Lasso technique [Tib96], providing sparsity in the coefficients.

The projection operator $\mathcal{P}$ satisfies some simple but important properties:

- If $\mathcal{S}$ consists of $\mathcal{Q}$-measurable random variables, then $\mathcal{P}(U) = \mathcal{P}(\mathbb{E}(U|\mathcal{Q}))$. Indeed, $\mathcal{P}(U)$ is the minimizer over $S \in \mathcal{S}$ of $\mathbb{E}|U-S|^2 = \mathbb{E}|U-\mathbb{E}(U|\mathcal{Q})|^2 + \mathbb{E}|\mathbb{E}(U|\mathcal{Q}) - S|^2$.

- The operator $\mathcal{P}$ is 1-Lipschitz. Indeed, for any U_1, U_2 in $\mathbf{L}_2$, write

$$\begin{aligned}
\mathbb{E}|U_1-U_2|^2 =& \mathbb{E}|\mathcal{P}(U_1)-\mathcal{P}(U_2)|^2 + \mathbb{E}|U_1-\mathcal{P}(U_1)-(U_2-\mathcal{P}(U_2))|^2 \\
&+ 2\mathbb{E}\big((\mathcal{P}(U_1)-\mathcal{P}(U_2))(U_1-\mathcal{P}(U_1)-(U_2-\mathcal{P}(U_2))\big) \\
\geq& \mathbb{E}|\mathcal{P}(U_1)-\mathcal{P}(U_2)|^2 + \mathbb{E}|U_1-\mathcal{P}(U_1)-(U_2-\mathcal{P}(U_2))|^2 \\
\geq& \mathbb{E}|\mathcal{P}(U_1)-\mathcal{P}(U_2)|^2
\end{aligned}$$

  using (3.18) with $U = U_i$ and $S = \mathcal{P}(U_j)$, $1 \leq i \neq j \leq 2$.

Projection operators in the DP equations. In the following of this section, the conditional expectation operators in the DP equations will be replaced by projection operators. That is, for each $i \in \{0, \dots, N-1\}$, we consider $\mathcal{S}_i^Y, \mathcal{S}_i^{Z,1}, \dots, \mathcal{S}_i^{Z,q}$ that are non-empty closed convex subsets of $\mathbf{L}_2(\mathcal{F}_{t_i}, \mathbb{P})$. We let $\mathcal{P}_i^Y$ and $\mathcal{P}_i^{Z,1}, \dots, \mathcal{P}_i^{Z,q}$ be the related projection operators and denote the tensor projection $\mathcal{P}^Z := (\mathcal{P}^{Z,l}, \dots, \mathcal{P}^{Z,q})$. We sum up the above stated properties of the $\mathcal{P}_i$ $(= \mathcal{P}_i^Y$ or $\mathcal{P}_i^Z)$:

Lemma 3.2. *Let U and V be in $\mathbf{L}^2(\mathcal{F}_T, \mathbb{P})$. Then, we have*

a) $\mathcal{P}_i(U) = \mathcal{P}_i(\mathbb{E}_i(U))$,

b) $|\mathcal{P}_i(U) - \mathcal{P}_i(V)|_{\mathbf{L}_2} \leq |U - V|_{\mathbf{L}_2}$.

MDP scheme with projection. Using the above projection operators in the discrete BSDE (3.9), we obtain the following approximation scheme:

$$\begin{cases} \hat{Y}_i^R & = \mathcal{P}_i^Y \left(\xi + \sum_{k=i}^{N-1} f_k(\hat{Y}_{k+1}^R, \hat{Z}_k^R)\Delta_k \right), \\ \Delta_i \hat{Z}_{l,i}^R & = \mathcal{P}_i^{Z,l} \left([\xi + \sum_{k=i+1}^{N-1} f_k(\hat{Y}_{k+1}^R, \hat{Z}_k^R)\Delta_k][\Delta W_{l,i}]_w \right), \end{cases} \tag{3.19}$$

for $R \in [0, +\infty]$. The following theorem estimates the error between (Y^R, Z^R) and $(\hat{Y}^R, \hat{Z}^R)$.

Theorem 3.1. *Assume* $(\mathbf{A}_\xi)$ *and* $(\mathbf{A_F}$**-i-ii**$)$. *For a given $\gamma \in [0, +\infty)^N$ and $\Gamma_i := \prod_{k=0}^{i-1}(1 + \gamma_k \Delta_k)$, we define the weighted time-average of error projections on Y^R and Z^R as follows:*

$$\mathcal{E}_i^{\mathcal{P},Y}(\gamma) = \sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|Y_k^R - \mathcal{P}_k^Y(Y_k^R)|^2)\Gamma_k, \quad \mathcal{E}_i^{\mathcal{P},Z}(\gamma) = \sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|Z_k^R - \mathcal{P}_k^Z(Z_k^R)|^2)\Gamma_k.$$

For any $R \in [0, +\infty]$, any π and any $\gamma \in (0, +\infty)^N$ such that $24C_{3.15}(1+T)(1 \vee R_\pi)(\frac{1}{\gamma_k} + \Delta_k)\frac{L_f^2}{(T-t_k)^{1-\theta_L}} \leq 1$ for any $k < N$, we have for any $0 \leq i \leq N-1$

$$\mathbb{E}(|Y_i^R - \hat{Y}_i^R|^2)\Gamma_i \leq 2\mathbb{E}(|Y_i^R - \mathcal{P}_i^Y(Y_i^R)|^2)\Gamma_i + 2\mathcal{E}_{i+1}^{\mathcal{P},Y}(\gamma) + 2\mathcal{E}_i^{\mathcal{P},Z}(\gamma), \tag{3.20}$$

$$\sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|Z_k^R - \hat{Z}_k^R|^2)\Gamma_k \leq 4\mathcal{E}_{i+1}^{\mathcal{P},Y}(\gamma) + 4\mathcal{E}_i^{\mathcal{P},Z}(\gamma). \tag{3.21}$$

The choice $\gamma_k = 48C_{3.15}(1+T)(1 \vee R_\pi)\frac{L_f^2}{(T-t_k)^{1-\theta_L}}$ obviously implies $24C_{3.15}(1+T)(1 \vee R_\pi)(\frac{1}{\gamma_k} + \Delta_k)\frac{L_f^2}{(T-t_k)^{1-\theta_L}} \leq \frac{1}{2} + 24C_{3.15}(1+T)(1 \vee R_\pi)C_\pi L_f^2 \leq 1$ for N large enough (assuming $(\mathbf{A_F}$**-iii**$)$) and moreover, we derive the easy bounds

$$1 \leq \Gamma_i \leq \exp(\sum_{k=0}^{N-1} \gamma_k \Delta_k) \leq \exp(\int_0^T \frac{48C_{3.15}(1+T)(1 \vee R_\pi)L_f^2}{(T-t)^{1-\theta_L}} dt)$$

$$= \exp\Big(\frac{48C_{3.15}(1+T)(1\vee R_\pi)L_f^2 T^{\theta_L}}{\theta_L}\Big) := C_{3.22}, \tag{3.22}$$

which remains bounded π-uniformly as $N \to +\infty$ owing to (2.8). As a consequence, we obtain

Corollary 3.1. *Assume* $(\mathbf{A}_\xi)$ *and* $(\mathbf{A}_\mathbf{F})$ *. For any* $R \in [0,+\infty]$ *and for any* π *with* N *large enough (such that* $(1\vee R_\pi)C_\pi L_f^2 \le \frac{1}{48C_{3.15}(1+T)}$*), we have for any* $0 \le i \le N-1$

$$\mathbb{E}(|Y_i^R - \hat{Y}_i^R|^2) \le 2\mathbb{E}(|Y_i^R - \mathcal{P}_i^Y(Y_i^R)|^2) + 2C_{3.22}\big[\mathcal{E}_{i+1}^{\mathcal{P},Y}(0) + \mathcal{E}_i^{\mathcal{P},Z}(0)\big], \tag{3.23}$$

$$\sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|Z_k^R - \hat{Z}_k^R|^2) \le 4C_{3.22}\big[\mathcal{E}_{i+1}^{\mathcal{P},Y}(0) + \mathcal{E}_i^{\mathcal{P},Z}(0)\big]. \tag{3.24}$$

These estimates show how the error due to projections in the MDP scheme is controlled by the time-average of the projection errors on Y and Z. Moreover, we obtain similar estimates to the Bender-Denk scheme (compare [BD07, Theorem 11] with our estimates), but we avoid the Picard iterations.

Proof. (of Theorem 3.1). We first prove a weaker result on the global error, that is

$$\begin{aligned}\mathcal{E}_i(\gamma) &:= \sum_{k=i+1}^{N-1} \Delta_k \mathbb{E}(|Y_k^R - \hat{Y}_k^R|^2)\Gamma_k + \sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|Z_k^R - \hat{Z}_k^R|^2)\Gamma_k \\ &\le 4\mathcal{E}_{i+1}^{\mathcal{P},Y}(\gamma) + 4\mathcal{E}_i^{\mathcal{P},Z}(\gamma).\end{aligned} \tag{3.25}$$

We will also make use of the following intermediate process:

$$\begin{cases} \bar{Y}_i^R &= \mathbb{E}_i\Big[\xi + \sum_{k=i}^{N-1} f_k(\hat{Y}_{k+1}^R, \hat{Z}_k^R)\Delta_k\Big], \\ \Delta_i \bar{Z}_i^R &= \mathbb{E}_i\Big[(\xi + \sum_{k=i+1}^{N-1} f_k(\hat{Y}_{k+1}^R, \hat{Z}_k^R)\Delta_k)[\Delta W_i^\top]_w\Big]. \end{cases}$$

Observe that, from Lemma 3.2(a), one has the useful properties (for $i < N$)

$$\hat{Y}_i^R = \mathcal{P}_i^Y(\bar{Y}_i^R) \qquad \text{and} \qquad \hat{Z}_i^R = \mathcal{P}_i^Z(\bar{Z}_i^R). \tag{3.26}$$

Moreover, $(\bar{Y}^R, \bar{Z}^R)$ solves a discrete BSDE with truncated Brownian increments and data $(\xi, \bar{f}_k : (y,z) \mapsto f_k(\hat{Y}_k^R, \hat{Z}_k^R))$; the Lipschitz constant of $\bar{f}_k$ equals zero for all k. Using Cauchy's inequality and Lemma 3.2(b), we obtain

$$\begin{aligned}\mathcal{E}_i(\gamma) \le\ & 2\sum_{k=i+1}^{N-1} \Delta_k \mathbb{E}(|Y_k^R - \mathcal{P}_k^Y(Y_k^R)|^2)\Gamma_k + 2\sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|Z_k^R - \mathcal{P}_k^Z(Z_k^R)|^2)\Gamma_k \\ &+ 2\sum_{k=i+1}^{N-1} \Delta_k \mathbb{E}(|\mathcal{P}_k^Y(Y_k^R) - \hat{Y}_k^R|^2)\Gamma_k + 2\sum_{k=i}^{N-1} \Delta_k \mathbb{E}(|\mathcal{P}_k^Z(Z_k^R) - \hat{Z}_k^R|^2)\Gamma_k\end{aligned}$$

$$\leq 2\mathcal{E}_{i+1}^{\mathcal{P},Y}(\gamma) + 2\mathcal{E}_i^{\mathcal{P},Z}(\gamma) + 2\sum_{k=i+1}^{N-1}\Delta_k\mathbb{E}(|Y_k^R - \bar{Y}_k^R|^2)\Gamma_k + 2\sum_{k=i}^{N-1}\Delta_k\mathbb{E}(|Z_k^R - \bar{Z}_k^R|^2)\Gamma_k.$$

To bound the last two terms in the above inequality, we apply Proposition 3.1 on the BSDEs (Y^R, Z^R) and $(\bar{Y}^R, \bar{Z}^R)$ to get

$$\begin{aligned}
&2\sum_{k=i+1}^{N-1}\Delta_k\mathbb{E}(|Y_k^R - \bar{Y}_k^R|^2)\Gamma_k + 2\sum_{k=i}^{N-1}\Delta_k\mathbb{E}(|Z_k^R - \bar{Z}_k^R|^2)\Gamma_k \\
&\leq 6C_{3.15}\big(1 + \sum_{k=i+1}^{N-1}\Delta_k\big)\sum_{k=i}^{N-1}(\frac{1}{\gamma_k} + \Delta_k)\Delta_k\mathbb{E}\big(|f_k(Y_{k+1}^R, Z_k^R) - f_k(\hat{Y}_{k+1}^R, \hat{Z}_k^R)|^2\big)\Gamma_k \\
&\leq 12C_{3.15}(1+T)\sum_{k=i}^{N-1}(\frac{1}{\gamma_k} + \Delta_k)\Delta_k\frac{L_f^2}{(T-t_k)^{1-\theta_L}}\mathbb{E}\big(|Y_{k+1}^R - \hat{Y}_{k+1}^R|^2 + |Z_k^R - \hat{Z}_k^R)|^2\big)\Gamma_k \\
&\leq \frac{1}{2}\mathcal{E}_i(\gamma).
\end{aligned}$$

The penultimate inequality follows from $\Delta_k \leq R_\pi\Delta_{k+1}$ and the conditions on π and γ in the theorem statement. To sum up, we have obtained $\mathcal{E}_i(\gamma) \leq 2\mathcal{E}_{i+1}^{\mathcal{P},Y}(\gamma) + 2\mathcal{E}_i^{\mathcal{P},Z}(\gamma) + \frac{1}{2}\mathcal{E}_i(\gamma)$, which readily proves (3.25). This also implies (3.21).

We now prove (3.20). Proceeding similarly, we obtain

$$\begin{aligned}
&\mathbb{E}(|Y_i^R - \hat{Y}_i^R|^2)\Gamma_i \leq 2\mathbb{E}(|Y_i^R - \mathcal{P}_i^Y(Y_i^R)|^2)\Gamma_i + 2\mathbb{E}(|Y_i^R - \bar{Y}_i^R|^2)\Gamma_i \\
&\leq 2\mathbb{E}(|Y_i - \mathcal{P}_i^Y(Y_i^R)|^2)\Gamma_i \\
&\quad + 12C_{3.15}\sum_{k=i}^{N-1}(\frac{1}{\gamma_k} + \Delta_k)\Delta_k\frac{L_f^2}{(T-t_k)^{1-\theta_L}}\mathbb{E}\big(|Y_{k+1}^R - \hat{Y}_{k+1}^R|^2 + |Z_k^R - \hat{Z}_k^R)|^2\big)\Gamma_k \\
&\leq 2\mathbb{E}(|Y_i - \mathcal{P}_i^Y(Y_i^R)|^2)\Gamma_i + \frac{1}{2}\mathcal{E}_i(\gamma)
\end{aligned}$$

and the proof is complete using (3.25). □

Comparison with the ODP scheme with projection. The ODP equation associated with the MDP equation (3.19) is

$$\begin{cases} \check{Y}_N^R = \xi, \quad \check{Y}_k^R = \mathcal{P}_k^Y\left(\check{Y}_{k+1}^R + f_k(\check{Y}_{k+1}^R, \check{Z}_k^R)\Delta_k\right), \\ \Delta_k\check{Z}_{l,k}^R = \mathcal{P}_k^{Z,l}\left(\check{Y}_{k+1}^R[\Delta W_{l,k}]_w\right). \end{cases}$$

In this case, the ODP and MDP equations do not match up, because projection operators do not in general benefit from a tower law. This means that the error analysis would be fundamentally different, because (3.26) would no longer be true and one could not apply the stability result for discrete BSDEs. In fact, for uniform time-grid π ($\Delta_k = \frac{T}{N}$ for all k) one would need to multiply the $\mathcal{E}_{i+1}^{\mathcal{P},Y}(0)$ term in (3.23) and (3.24) by N for the ODP estimates. This is also observed in the error analysis of [LGW06, Theorem 2].

3.3. Application of a priori estimates to almost sure bounds

When the terminal condition is bounded, pointwise bounds on Y^R and Z^R are available. These bounds are used in Section 4.

Proposition 3.2 (*a.s.* upper bounds). *Assume* $(\mathbf{A}'_\xi\text{-}\mathbf{i})$ *and* $(\mathbf{A_F})$ *. For any* $R \in [0, +\infty]$ *and for any* π *with* N *large enough (such that* $C_\pi L_f^2 \leq \frac{1}{12q}$*), the following almost sure bounds on* Y_i^R *and* Z_i^R *apply:*

$$|Y_i^R| \leq C_y := C_{3.27}\Big(C_\xi + \frac{T^{\theta_c}}{\sqrt{4q(2\theta_c - \theta_L \wedge \theta_c)}} C_f\Big), \quad |Z_{l,i}^R| \leq C_{z,i} := \frac{C_y}{\sqrt{\Delta_i}}, \tag{3.27}$$

for $0 \leq i \leq N-1$ *and* $C_{3.27} = \exp\big(\frac{T}{4} + \frac{6q(1\vee L_f^2)}{\theta_L \wedge \theta_c}(T^{\theta_L} \vee 1)\big)$.

Observe that C_y and $C_{3.27}$ are uniform in i and $R \in [0, +\infty]$, and that they remain bounded as L_f and T go to 0 (as we naturally expect).

Proof. We derive the almost sure bounds from the global pointwise estimates in Proposition 3.1. To apply the results of this proposition, we take the pair $(0, 0)$ for the first discrete BSDE - the solution associated to the null driver and terminal condition - and (Y^R, Z^R) for the second discrete BSDE, which is given by the DP equation (3.9). From (3.16), for any Δ_i and γ_i such that $6q(\Delta_i + \frac{1}{\gamma_i})\frac{L_f^2}{(T-t_i)^{1-\theta_L}} \leq 1$, and recalling that $\lambda_i := \prod_{k=0}^{i-1}(1 + (\gamma_k + \frac{1}{2})\Delta_k)$, we have

$$\begin{aligned}(\Delta Y_i^R)^2 \leq (\Delta Y_i^R)^2 \lambda_i &\leq \lambda_N \mathbb{E}_i(\xi^2) + 3\sum_{k=i}^{N-1} \frac{(1+\gamma_k\Delta_k)}{\gamma_k}\lambda_k\Delta_k\mathbb{E}_i(f_k^2(0,0)) \\ &\leq \lambda_N\big(C_\xi^2 + 3C_f^2 \sum_{k=0}^{N-1} \frac{\Delta_k}{\gamma_k(T-t_k)^{2(1-\theta_c)}}\big).\end{aligned}$$

For N large enough, we have $C_\pi L_f^2 \leq \frac{1}{12q}$; additionally, we set

$$\gamma_k := 12q\frac{(1\vee T^{-\theta_L})(1\vee L_f^2)}{(T-t_k)^{1-\theta_L}}\big(\frac{T}{T-t_k}\big)^{\theta_L - \theta_L\wedge\theta_c} \geq \frac{12qL_f^2}{(T-t_k)^{1-\theta_L}}, \qquad 0 \leq k < N.$$

It follows $6q(\Delta_k + \frac{1}{\gamma_k})\frac{L_f^2}{(T-t_k)^{1-\theta_L}} \leq 1$. Easy computations similar to (3.22) give

$$\begin{aligned}\lambda_N &\leq \exp\big(\frac{T}{2} + 12q(1\vee T^{-\theta_L})(1\vee L_f^2)T^{\theta_L-\theta_L\wedge\theta_c}\int_0^T (T-t)^{\theta_L\wedge\theta_c - 1}dt\big) \\ &= \exp\big(\frac{T}{2} + \frac{12q(1\vee L_f^2)}{\theta_L\wedge\theta_c}(T^{\theta_L}\vee 1)\big),\end{aligned}$$

$$\sum_{k=0}^{N-1}\frac{\Delta_k}{\gamma_k(T-t_k)^{2(1-\theta_c)}} = \sum_{k=0}^{N-1}\frac{\Delta_k(1\wedge T^{\theta_L})T^{\theta_L\wedge\theta_c-\theta_L}}{12q(1\vee L_f^2)(T-t_k)^{(1-2\theta_c+\theta_L\wedge\theta_c)}}$$

$$\leq \frac{T^{2\theta_c}}{12q(2\theta_c - \theta_L \wedge \theta_c)}$$

(observing that $2\theta_c - \theta_L \wedge \theta_c \geq \theta_c > 0$). Combining the last three inequalities, we obtain the required upper bounds (3.27) on Y^R. The bound on Z_i^R is clear from the Cauchy-Schwartz inequality and the bound on Y_{i+1}^R. □

3.4. Impact of the threshold R

In the spirit of [LGW06], we compare the discrete BSDE (3.10) (or equivalently (3.9)) with $R < +\infty$, to (1.1) (or equivalently (1.4)). While unessential, we assume for simplicity that the terminal condition is bounded.

Proposition 3.3. *Assume* $(\mathbf{A}'_\xi\text{-}\mathbf{i})$ *and* $(\mathbf{A_F})$ *. For any $R \in [0, +\infty]$ and for any π with N large enough (such that $C_\pi L_f^2 \leq \frac{1}{12q}$), the following almost sure error bounds on $Y_i - Y_i^R$ and $Z_i - Z_i^R$ hold for any $0 \leq i < N$:*

$$|Y_i - Y_i^R| \leq C_y \exp\big(\frac{T}{8} + \frac{12qL_f^2}{\theta_L} T^{\theta_L}\big) \exp\big(-\frac{1}{4}R^2\big)\sqrt{N},$$

$$\big(\sum_{k=i}^{N-1} \mathbb{E}_i|Z_k - Z_k^R|^2 \Delta_k\big)^{\frac{1}{2}} \leq C_y \exp\big(\frac{12qL_f^2}{\theta_L} T^{\theta_L}\big)\big(8q + T\exp(\frac{T}{4})\big)^{\frac{1}{2}} \exp\big(-\frac{1}{4}R^2\big)\sqrt{N}.$$

For the proof, see [GT11]. Consequently, taking R in a logarithmic scale w.r.t. N is sufficient to make the threshold error negligible: for instance, taking $R = \sqrt{4(p + \frac{1}{2})\log(N+1)}$ (for $p \geq 0$) gives an error of magnitude $O(N^{-p})$.

3.5. Application of a priori bounds and Markov assumptions to additional smoothness properties

In this subsection, the Markovian assumptions $(\mathbf{A_X})$, $(\mathbf{A}'_\xi)$ and $(\mathbf{A}'_\mathbf{F})$ are in force. We demonstrate how additional smoothness conditions of the terminal condition Φ and the driver f_k strongly improve the smoothness properties of y_k^R and z_k^R. Increased smoothness is essential in the complexity analysis of numerical algorithms, as will be demonstrated in Section 4.3.

Lemma 3.3. *Assume $x \mapsto \Phi(x)$ is uniformly Lipschitz continuous with Lipschitz constant L_Φ, and that the Lipschitz property of the driver f_k is extended to the x component; i.e.,*

$$|f_k(x, y, z) - f_k(x', y', z')| \leq \frac{L_f}{(T - t_k)^{(1-\theta_L)/2}}(|x - x'| + |y - y'| + |z - z'|).$$

Furthermore, assume that, for any $x_1, x_2 \in \mathbb{R}^d$ and $0 \leq k \leq N-1$, two Markov chains $(X_i^{k,x_1})_{k \leq i \leq N}$ and $(X_i^{k,x_2})_{k \leq i \leq N}$ started at time t_k with values x_i ($i = 1, 2$) with the same transition probabilities as $(X_k)_k$ enjoy the property $\mathbb{E}[|X_i^{k,x_1} - X_i^{k,x_2}|^2] \leq C_X|x_1 - x_2|^2$ for some constant C_X.

Then, if $12qC_\pi L_f^2 \leq 1$ for all k, $x \mapsto y_k^R(x)$ is Lipschitz continuous uniformly in k and N.

Proof. Let (Y^1, Z^1) and (Y^2, Z^2) be discrete BSDEs from k to N with data $(\Phi(X_N^{k,x_j}), f_i(X_i^{k,x_j}, y, z))$ ($j = 1, 2$ resp.). Then $y_k^R(x_j) = Y_k^j$ and $z_k^R(x_j) = Z_k^j$ ($j = 1, 2$ resp.) hold almost surely. We use the result of (3.17), together with the choice $\gamma_k = 12qL_f^2(T - t_k)^{-(1-\theta_L)}$, to obtain

$$
\begin{aligned}
|y_k^R(x_1) - y_k^R(x_2)|^2 &\leq e^{T/2} C_\Gamma \mathbb{E}[|\Phi(X_N^{k,x_1}) - \Phi(X_N^{k,x_2})|^2] \\
&\quad + 3e^{T/2} C_\Gamma \sum_{i=k}^{N-1} (\Delta_i + \frac{1}{\gamma_i}) \mathbb{E}[|f_i(X_i^{k,x_1}, Y_{i+1}^1, Z_i^1) - f_i(, X_i^{k,x_2}, Y_{i+1}^1, Z_i^1)|^2] \Delta_i \\
&\leq e^{T/2} (\frac{T}{2q} + L_\Phi^2) C_X C_\Gamma |x_1 - x_2|^2 \qquad (3.28)
\end{aligned}
$$

where $C_\Gamma := e^{12qL_f^2 T^{\theta_L}/\theta_L}$ comes from the choice of γ_k; see (3.22) for details. □

We now show that, under assumptions of Lipschitz continuous data, the a.s. bound of Z does not suffer from the inverse dependency on the time increments.

Corollary 3.2. *Under the assumptions of Lemma 3.3 and additionally that for all $x \in \mathbb{R}^d$, the Markov chain $(X_i^{k,x})_{i \geq k}$ started at t_k with value x enjoys the property that $\mathbb{E}[|X_{k+1}^{k,x} - x|^2] \leq C_X \Delta_k$, then the function $x \mapsto z_k^R(x)$ is uniformly bounded, with a bound independent of the time increments.*

Proof. The result follows directly from Lemma 3.3:

$$
|z_k^R(x)|^2 = \frac{1}{\Delta_k^2} |\mathbb{E}\big[[\Delta W_k]_w (y_{k+1}^R(X_{k+1}^{k,x}) - y_{k+1}^R(x))\big]|^2 \leq e^{T/2} (\frac{T}{2} + qL_\Phi^2) C_X^2 C_\Gamma
$$

where we have used the fact that y_{k+1}^R is deterministic in the first equality, and the Cauchy-Schwartz inequality combined with (3.28) in the second inequality. □

The additional assumptions made in Lemma 3.3 and Corollary 3.2 are quite natural: they are satisfied by, for example, the Euler scheme for a jump-diffusion with bounded coefficients. Extensions to Lemma 3.3 and Corollary 3.2 to higher derivatives will be carried out in future work.

4. Empirical regression scheme

In this section, we approximate the projection scheme (3.19) using least-squares regression on simulated data. The details of the algorithm are made explicit in Section 4.1, and a full error analysis is undertaken in Section 4.2. Then the algorithm complexity is discussed in Section 4.3. Finally, Sections 4.4 and 4.5 are devoted to the proof of our main results.

4.1. Notation and algorithm

Markovian framework. In what follows, we always assume $(\mathbf{A_X})$, $(\mathbf{A}'_\xi)$ and $(\mathbf{A}'_\mathbf{F})$. This allows the representation

$$(Y_i^R, Z_i^R) := (y_i^R(X_i), z_i^R(X_i)) \tag{4.29}$$

for measurable, deterministic functions $y_i^R(\cdot)$ and $z_i^R(\cdot)$.

Samples. Let $\{(X_k^m)_{k\geq 0}\}_{m=1,\dots,M}$ denote M independent paths of the Markov chain, and $\{(\Delta W_k^m)_{k\geq 0}\}_{m=1,\dots,M}$ the independent increments of the Brownian Motion from which the Markov chain is generated. We denote the samples of the Markov chain at time k by $X_k^{1:M} := \{X_k^m\}_{m=1,\dots,M}$.
For function $\psi : \mathbb{R}^d \to \mathbb{R}$, we define the empirical norm with respect to the sample $X_k^{1:M}$ by

$$\|\psi\|_{k,M} := \big(\frac{1}{M}\sum_{m=1}^{M} |\psi(X_k^m)|^2\big)^{1/2}.$$

Basis functions. For each $l = 0,\dots,q$ and $k = 0,\dots,N-1$, we are given a finite number of deterministic *basis* functions $(p_{l,k}^i(.))_{1\leq i\leq K_{l,k}}$, where $p_{l,k}^i(.) : \mathbb{R}^d \to \mathbb{R}$ satisfies $\mathbb{E}[|p_{l,k}^i(X_k)|^2] < +\infty$. We write the functions as a column vector $p_{l,k}(.) = (p_{l,k}^1(.),\dots)^\top$, where $\top$ denotes the transpose operator. Without loss of generality, we assume that $M \geq K_{l,k}$. The random variables $(p_{l,k}^i(X_k))_{1\leq i\leq K_{l,k}}$ span a linear subspace of $\mathbf{L}_2(\mathcal{F}_{t_k},\mathbb{P})$, which is denoted $\mathcal{S}_k^Y$ if $l = 0$ and $\mathcal{S}_k^{Z,l}$ if $l \neq 0$ in the language of Subsection 3.2. The extension of our error estimates to more general closed convex subspaces will be considered in future research.
We write $p_{l,k}^m$ (resp. $p_{l,k}^{i,m}$) to mean $p_{l,k}(X_k^m)$ (resp. $p_{l,k}^i(X_k^m)$).

Least-squares problem. Instead of projections in the $\mathbf{L}_2(\mathbb{P})$-sense as in Subsection 3.2, we numerically compute empirical regressions using the samples. Generally speaking for given observation $X_k^{1:M}$ and response $S^{1:M} = (S^m)_{m=1,\dots,M}$, we aim at computing the best approximation of the response in the vector space generated by the basis functions $p_{l,k}$ w.r.t. the norm $\|.\|_{k,M}$: it is defined by $\alpha^\star \cdot p_{l,k}(.)$ where

$$\alpha^\star = \arg\inf_{\alpha\in\mathbb{R}^{K_{l,k}}} \|\alpha\cdot p_{l,k} - S\|_{k,M}^2. \tag{4.30}$$

Since colinearities may exist between basis functions, the above coefficient $\alpha^\star$ may be not unique and one must first clarify which solution to take. We take the Singular Value Decomposition (SVD in short) approach by taking the coefficient with minimal Euclidean norm (see Appendix A for details). We refer to this choice as the *SVD-optimal* coefficient. We now state basic properties related to least-squares regression with random observation $X_k^{1:M}$ that will be frequently used in this work.

Proposition 4.1. *Let $\alpha^\star$ be the SVD-optimal coefficient solving* $\arg\inf_{\alpha\in\mathbb{R}^{\kappa_{l,k}}} \|\alpha\cdot p_{l,k} - S\|^2_{k,M}$. *The following properties are satisfied:*

i) linearity: the mapping $S \mapsto \alpha^\star$ is linear.

ii) contraction property: $\|\alpha^\star \cdot p_{l,k}\|_{k,M} \leq \|S\|_{k,M}$.

iii) conditional expectation solution: assume that $(p^m_{l,k})_{m=1,\dots,M}$ is measurable with respect to the sub-σ-algebra $\mathcal{Q}$. Then the SVD-optimal coefficient associated to the response $\mathbb{E}(S|\mathcal{Q}) = (\mathbb{E}(S^m|\mathcal{Q}))_{m=1,\dots,M}$ is given by $\mathbb{E}(\alpha^\star|\mathcal{Q})$.

The proof is given in Appendix A.

Soft thresholds for approximate solutions. y^R_i and z^R_i are bounded by C_y and $C_{z,i}$, respectively, provided that N is large enough (see Proposition 3.2). We force the approximated solutions to satisfy these bounds: for $(y,z) := (y,(z_1,\cdots,z_q)) \in \mathbb{R}\times\mathbb{R}^q$ we define the soft thresholding

$$[y]_y = -C_y \vee y \wedge C_y, \quad [z_l]_z = -C_{z,i} \vee z_l \wedge C_{z,i}, \quad [z]_z = ([z_1]_z, \cdots, [z_q]_z). \tag{4.31}$$

In the notation for the z-threshold, we do not indicate that it depends on i because this is clear from the context.

Coefficients and solution approximations. We set $y^{R,M}_N(\cdot) := \Phi(\cdot)$. For $k < N$, we iteratively define the SVD-optimal coefficients

$$\begin{aligned}\alpha^M_{0,k} := \arg\min_\alpha \frac{1}{M}\sum_{m=1}^M \big(&\Phi(X^m_N) \\ &+ \sum_{i=k}^{N-1} f_i(X^m_i, y^{R,M}_{i+1}(X^m_{i+1}), z^{R,M}_i(X^m_i))\Delta_i - \alpha\cdot p^m_{0,k}\big)^2, \end{aligned}\tag{4.32}$$

$$\begin{aligned}\alpha^M_{l,k} := \arg\min_\alpha \frac{1}{M}\sum_{m=1}^M \big(&\frac{[\Delta W^m_{l,k}]_w}{\Delta_k}\big(\Phi(X^m_N) \\ &+ \sum_{i=k+1}^{N-1} f_i(X^m_i, y^{R,M}_{i+1}(X^m_{i+1}), z^{R,M}_i(X^m_i))\Delta_i\big) - \alpha\cdot p^m_{l,k}\big)^2. \end{aligned}\tag{4.33}$$

Then we define the following functions used to approximate y^R_k and z^R_k respectively

$$y^{R,M}_k(x) := [\alpha^M_{0,k}\cdot p_{0,k}(x)]_y, \qquad z^{R,M}_{l,k}(x) := [\alpha^M_{l,k}\cdot p_{l,k}(x)]_z, \tag{4.34}$$

where the thresholds $[\cdot]_y$ and $[\cdot]_z$ are defined in (4.31). Thanks to these thresholds, the random function

$$\Psi^{R,M}_k(x_k,\dots,x_N) := \Phi(x_N) + \sum_{i=k}^{N-1} f_i(x_i, y^{R,M}_{i+1}(x_{i+1}), z^{R,M}_i(x_i))\Delta_i \tag{4.35}$$

is bounded independently of the samples and of $(x_k, \dots, x_N)$; this property will be used repeatedly in the subsequent analysis.

Lemma 4.1. *Under* $(\mathbf{A_X})$, $(\mathbf{A}'_\xi)$ *and* $(\mathbf{A}'_\mathbf{F})$, *we have*

$$\sup_{0\le k\le N}\ \sup_{x_k\in\mathbb{R}^d,\dots,x_N\in\mathbb{R}^d} |\Psi_k^{R,M}(x_k,\dots,x_N)| \le C_\Psi$$

where $C_\Psi := C_\xi + L_f C_y T^{\frac{\theta_L}{2}}\big[\frac{2\sqrt{T}}{1+\theta_L} + \frac{\sqrt{q}\sqrt{N}}{\sqrt{\theta_L}}\big] + C_f\frac{T^{\theta_c}}{\theta_c}$.

Proof. From $(\mathbf{A}'_\xi\text{-}\mathbf{i})$, $(\mathbf{A_F}\text{-}\mathbf{i\text{-}ii})$, (4.34) and (4.31), we readily obtain

$$\begin{aligned}|\Psi_k^{R,M}(x_k,\dots,x_N)| &\le C_\xi + \sum_{i=0}^{N-1}\Big[\frac{L_f}{(T-t_i)^{\frac{1-\theta_L}{2}}}\big(C_y+\sqrt{q}\frac{C_y}{\sqrt{\Delta_i}}\big)\Delta_i + \frac{C_f}{(T-t_i)^{1-\theta_c}}\Delta_i\Big]\\ &\le C_\xi + L_f C_y\Big[\frac{T^{(1+\theta_L)/2}}{(1+\theta_L)/2} + \sqrt{q}\sqrt{N}\big(\sum_{i=0}^{N-1}\big(\frac{\sqrt{\Delta_i}}{(T-t_i)^{\frac{1-\theta_L}{2}}}\big)^2\big)^{1/2}\Big] + C_f\frac{T^{\theta_c}}{\theta_c}\end{aligned}$$

and the announced upper bound follows. □

4.2. Error analysis

In contrast to standard regression problems, a major difficulty for the error analysis is related to the non-independence of the random variables $\{\Phi(X_N^m) + \sum_{i=k}^{N-1} f_i(X_i^m, y_{i+1}^{R,M}(X_{i+1}^m), z_i^{R,M}(X_i^m))\Delta_i\}_{m=1,\dots,M}$ due to the interdependence of the random functions $(y_i^{R,M}(.), z^{R,M}(.))_{i=k,\dots,N-1}$. To deal with this, we follow the method of [LGW06], which uses methods from statistical learning, but introduce some important adaptations. In particular, we use intermediate processes in order to take advantage of the a priori results for discrete BSDEs, leading to important improvements in the error estimates. The subsection is organized as follows: first, we introduce the tools of statistical learning we require, the intermediate processes, and the local error terms; then we state a global error decomposition in terms of the local error terms in Theorem 4.1, which is the corner stone of our error analysis, and bound the local error terms in Theorem 4.2.

Ghost sample. In the upcoming error analysis, we employ the method of *symmetrization*, which is standard in statistical learning; see [Pol84] or [GKKW02]. This involves the introduction of paths of the Markov chain which are identically distributed but independent (*ghost*) to the original samples.
Let k be given. For each m, we denote by $(\tilde{X}_i^{k,m})_{i\ge k}$ an independent copy of the Markov chain $(X_i^m)_{i\ge k}$ starting at t_k with value X_k^m ($\tilde{X}_k^{k,m} = X_k^m$). Additionally, we denote by $\Delta\tilde{W}_k^{k,m}$ the ghost Brownian increment used to generate the Markov chain $(\tilde{X}_i^{k,m})_{i\ge k}$: it is independent of and identically distributed to ΔW_k^m. Conditionally on

$$\mathcal{F}_k^M := \sigma(X_i^m, \Delta W_{j-1}^m : i,j\le k, 1\le m\le M),$$

the ghost paths $\{(\tilde{X}_i^{k,m})_{i\geq k}, \Delta\tilde{W}_k^{k,m} : 1 \leq m \leq M\}$ are independent. Furthermore, we write $\mathbb{E}_k^M$ ($\mathbb{P}_k^M$) for the conditional expectation (probability) with respect to $\mathcal{F}_k^M$.

Extra coefficients. To analyze the convergence, we make use of coefficients calculated using the ghost paths:

$$\begin{aligned}\tilde{\alpha}_{0,k}^M := \arg\min_\alpha \frac{1}{M}\sum_{m=1}^M \big(\Phi(\tilde{X}_N^{k,m}) \\ + \sum_{i=k}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R,M}(\tilde{X}_{i+1}^{k,m}), z_i^{R,M}(\tilde{X}_i^{k,m}))\Delta_i - \alpha\cdot p_{0,k}^m\big)^2, \qquad (4.36)\end{aligned}$$

$$\begin{aligned}\tilde{\alpha}_{l,k}^M := \arg\min_\alpha \frac{1}{M}\sum_{m=1}^M \big(\frac{[\Delta\tilde{W}_{l,k}^{k,m}]_w}{\Delta_k}\big(\Phi(\tilde{X}_N^{k,m}) \\ + \sum_{i=k+1}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R,M}(\tilde{X}_{i+1}^{k,m}), z_i^{R,M}(\tilde{X}_i^{k,m}))\Delta_i\big) - \alpha\cdot p_{l,k}^m\big)^2. \qquad (4.37)\end{aligned}$$

In addition, we need the following coefficients, also calculated with the ghost paths but with the functions y^R and z^R, from the Markov representation (4.29) of (Y^R, Z^R), in the place of $y^{R,M}$ and $z^{R,M}$:

$$\begin{aligned}\tilde{\beta}_{0,k}^M := \arg\min_\alpha \frac{1}{M}\sum_{m=1}^M \big(\Phi(\tilde{X}_N^{k,m}) \\ + \sum_{i=k}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R}(\tilde{X}_{i+1}^{k,m}), z_i^{R}(\tilde{X}_i^{k,m}))\Delta_i - \alpha\cdot p_{0,k}^m\big)^2, \qquad (4.38)\end{aligned}$$

$$\begin{aligned}\tilde{\beta}_{l,k}^M := \arg\min_\alpha \frac{1}{M}\sum_{m=1}^M \big(\frac{[\Delta\tilde{W}_{l,k}^{k,m}]_w}{\Delta_k}\big(\Phi(\tilde{X}_N^{k,m}) \\ + \sum_{i=k+1}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R}(\tilde{X}_{i+1}^{k,m}), z_i^{R}(\tilde{X}_i^{k,m}))\Delta_i\big) - \alpha\cdot p_{l,k}^m\big)^2. \qquad (4.39)\end{aligned}$$

Intermediate processes. Let $(X_i^{k,x})_{k\leq i\leq N}$ be a Markov chain starting at t_i with value x with the same transition probabilities as X. We generate the following intermediate (sample dependent) functions:

$$\bar{y}_k^{R,M}(x) := \int \Psi_k^{R,M}(x, x_{k+1}, \ldots, x_N)\mu^x(\mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_N)\ , \qquad (4.40)$$

$$\Delta_k \bar{z}_k^{R,M}(x) := \int [w]_w^\top \Psi_{k+1}^{R,M}(x_{k+1}, \ldots, x_N)\mu^{x,W}(\mathrm{d}w, \mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_N) \qquad (4.41)$$

where μ^x is the law of $(X_{k+1}^{k,x}, \ldots, X_N^{k,x})$ and $\mu^{x,W}$ the law of $(\Delta W_k, X_{k+1}^{k,x}, \ldots, X_N^{k,x})$. The following lemma is derived from the Markov property and the independence of the ghost sample; it is proved in [GT11].

Lemma 4.2. *With the current notation and assumptions, for all m we have*

$$\bar{y}_k^{R,M}(X_k^m) = \mathbb{E}_N^M\big[\Phi(\tilde{X}_N^{k,m}) + \sum_{i=k}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R,M}(\tilde{X}_{i+1}^{k,m}), z_i^{R,M}(\tilde{X}_i^{k,m}))\Delta_i\big],$$

$$\Delta_k \bar{z}_{l,k}^{R,M}(X_k^m) = \mathbb{E}_N^M\big[[\Delta\tilde{W}_{l,k}^m]_w\big(\Phi(\tilde{X}_N^{k,m}) + \sum_{i=k+1}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R,M}(\tilde{X}_{i+1}^{k,m}), z_i^{R,M}(\tilde{X}_i^{k,m}))\Delta_i\big)\big].$$

Local error terms. For given *accuracy parameters* $\varepsilon^{\cdot}_{\cdot,A}, \varepsilon^{\cdot}_{\cdot,B}, \varepsilon^{\cdot}_{\cdot,C} \in (0,+\infty)^{2N}$, we define the events:

$$A_k^{Y,M} := \big\{\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M)\cdot p_{0,k}\|_{k,M}^2 > \varepsilon_{k,A}^Y\big\}, \tag{4.42}$$

$$A_k^{Z,M} := \big\{\exists l \in \{1,\dots,q\} \text{ s.t. } \|(\alpha_{l,k}^M - \tilde{\alpha}_{l,k}^M)\cdot p_{l,k}\|_{k,M}^2 > \varepsilon_{k,A}^Z\big\},$$

$$B_k^{Y,M} := \big\{\varepsilon_{k,B}^Y + 2\mathbb{E}_N^M[(\bar{y}_k^{R,M}(X_k) - y_k^R(X_k))^2] < \|\bar{y}_k^{R,M} - y_k^R\|_{k,M}^2\big\}, \tag{4.43}$$

$$B_k^{Z,M} := \big\{\exists l \in \{1,\dots,q\} \text{ s.t. } \varepsilon_{k,B}^Z + 2\mathbb{E}_N^M[(\bar{z}_{l,k}^{R,M}(X_k) - z_{l,k}^R(X_k))^2] < \|\bar{z}_{l,k}^{R,M} - z_{l,k}^R\|_{k,M}^2\big\},$$

$$C_k^{Y,M} := \big\{\varepsilon_{k,C}^Y + 2\|y_k^{R,M} - y_k^R\|_{k,M}^2 < \mathbb{E}_N^M[(y_k^{R,M}(X_k) - y_k^R(X_k))^2]\big\}, \tag{4.44}$$

$$C_k^{Z,M} := \big\{\exists l \in \{1,\dots,q\} \text{ s.t. } \varepsilon_{k,C}^Z + 2\|z_{l,k}^{R,M} - z_{l,k}^R\|_{k,M}^2 < \mathbb{E}_N^M[(z_{l,k}^{R,M}(X_k) - z_{l,k}^R(X_k))^2]\big\}.$$

These six events are *large deviation events*. In Theorem 4.2, we show that their probabilities are exponentially small under appropriate choice of the accuracy parameters; the exponent ought to depend on the complexity of the class of functions spanned by $p_{l,k}$, on Δ_k, C_y, C_Ψ, R and M. We also consider

$$T_{1,k}^{Y,M} := \mathbb{E}\big(\inf_\alpha \|\alpha\cdot p_{0,k} - y_k^R\|_{k,M}^2\big), \tag{4.45}$$

$$T_{1,k}^{Z,M} := \sum_{l=1}^q \mathbb{E}\big(\inf_\alpha \|\alpha\cdot p_{l,k} - z_{l,k}^R\|_{k,M}^2\big), \tag{4.46}$$

$$T_{2,k}^{Y,M} := \mathbb{E}\big\|(\tilde{\alpha}_{0,k}^M - \mathbb{E}_N^M[\tilde{\alpha}_{0,k}^M])\cdot p_{0,k}\big\|_{k,M}^2, \tag{4.47}$$

$$T_{2,k}^{Z,M} := \sum_{l=1}^q \mathbb{E}\big\|(\tilde{\alpha}_{l,k}^M - \mathbb{E}_N^M[\tilde{\alpha}_{l,k}^M])\cdot p_{l,k}\big\|_{k,M}^2. \tag{4.48}$$

Equations (4.45) - (4.48) have standard interpretation in regression theory: the two first terms are square bias terms (best approximation error of the basis functions), while the last two are variance terms (statistical errors).

Error decomposition. We now state the main results of the global error analysis. Similarly to Corollary 3.1, these global errors read as time-average of local errors.

Theorem 4.1. *Assume* $(\mathbf{A_X})$, $(\mathbf{A}'_\xi)$ *and* $(\mathbf{A}'_\mathbf{F})$. *Define*

$$\mathcal{E}_k^Y(R,M) := T_{1,k}^{Y,M} + 3T_{2,k}^{Y,M} + 3\varepsilon_{k,A}^Y + 12C_\Psi^2\mathbb{P}(A_k^{Y,M})$$

$$+ 3\varepsilon^Y_{k,B} + 6(C_y^2 + C_\Psi^2)\mathbb{P}(B_k^{Y,M}) + \frac{1}{4}\varepsilon^Y_{k,C} R_\pi + C_y^2 R_\pi \mathbb{P}(C_k^{Y,M}),$$

$$\mathcal{E}_k^Z(R,M) := T_{1,k}^{Z,M} + 3T_{2,k}^{Z,M} + 3q\varepsilon^Z_{k,A} + 12\frac{C_\Psi^2 R^2}{\Delta_k}\mathbb{P}(A_k^{Z,M})$$
$$+ 3q\varepsilon^Z_{k,B} + 6q\frac{(C_y^2 + C_\Psi^2 R^2)}{\Delta_k}\mathbb{P}(B_k^{Z,M}) + \frac{1}{4}q\varepsilon^Z_{k,C} + q\frac{C_y^2}{\Delta_k}\mathbb{P}(C_k^{Z,M}).$$

For any $R \in [0,+\infty)$ and any π such that $C_\pi L_f^2(R_\pi \vee 1) \le (288C_{3.15}(1+T))^{-1}$, we have, for all $0 \le i \le N-1$, that

$$\mathbb{E}\|y_i^R - y_i^{R,M}\|_{i,M}^2 \le \mathcal{E}_i^Y(R,M) + 2C_{3.22}^6\big(\sum_{k=i+1}^{N-1}\Delta_k\mathcal{E}_k^Y(R,M) + \sum_{k=i}^{N-1}\Delta_k\mathcal{E}_k^Z(R,M)\big),$$

$$\sum_{k=i}^{N-1}\Delta_k\mathbb{E}\|z_k^R - z_k^{R,M}\|_{k,M}^2 \le 2C_{3.22}^6\big(\sum_{k=i+1}^{N-1}\Delta_k\mathcal{E}_k^Y(R,M) + \sum_{k=i}^{N-1}\Delta_k\mathcal{E}_k^Z(R,M)\big).$$

Theorem 4.2. *Under the assumptions of Theorem 4.1, the bias and variance terms are bounded as follows:*

$$T_{1,k}^{Y,M} \le \min_\alpha \mathbb{E}|y_k^R(X_k) - \alpha\cdot p_{0,k}(X_k)|^2, \qquad T_{2,k}^{Y,M} \le C_\Psi^2\frac{K_{0,k}}{M},$$
$$T_{1,k}^{Z,M} \le \sum_{l=1}^q \min_\alpha \mathbb{E}|z_{l,k}^R(X_k) - \alpha\cdot p_{l,k}(X_k)|^2, \qquad T_{2,k}^{Z,M} \le \frac{C_\Psi^2 R^2}{\Delta_k}\sum_{l=1}^q\frac{K_{l,k}}{M}.$$

For the large deviation events, we have

$$\mathbb{P}(A_k^{Y,M}) \le 2K_{0,k}\exp\big(-\frac{M\varepsilon^Y_{k,A}}{72C_\Psi^2 K_{0,k}}\big)$$
$$\prod_{i=k}^{N-1} 3^{(q+1)}\Delta_i^{2(K_{0,i}+1)}\big(\frac{96K_{0,k}L_f^2T^{1+\theta_L}(q+1)C_y^2}{\theta_L\varepsilon^Y_{k,A}\Delta_i}\big)^{2\sum_{l=0}^q(K_{l,i}+1)},$$

$$\mathbb{P}(A_k^{Z,M}) \le \sum_{l=1}^q 2K_{l,k}\exp\big(-\frac{M\Delta_k\varepsilon^Z_{k,A}}{72C_\Psi^2R^2K_{l,k}}\big)$$
$$\prod_{i=k}^{N-1} 3^{(q+1)}\Delta_i^{2(K_{0,i}+1)}\big(\frac{96K_{l,k}L_f^2T^{1+\theta_L}(q+1)C_y^2R^2}{\theta_L\varepsilon^Z_{k,A}\Delta_i\Delta_k}\big)^{2\sum_{l'=0}^q(K_{l',i}+1)},$$

$$\mathbb{P}(B_k^{Y,M}) \le 4\exp\big(-\frac{M\varepsilon^Y_{k,B}}{60(C_\Psi+C_y)^2}\big)$$
$$\prod_{i=k}^{N-1} 3^{(q+1)}\Delta_i^{K_{0,i}+1}\big(\frac{132L_fT^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi+C_y)C_y}{\varepsilon^Y_{k,B}\sqrt{\Delta_i}}\big)^{2\sum_{l=0}^q(K_{l,i}+1)},$$

$$\mathbb{P}(B_k^{Z,M}) \le 4\ q\exp\big(-\frac{M\varepsilon^Z_{k,B}\Delta_k}{60(C_\Psi R+C_y)^2}\big)$$
$$\prod_{i=k}^{N-1} 3^{(q+1)}\Delta_i^{K_{0,i}+1}\big(\frac{132L_fT^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi R+C_y)C_y}{\varepsilon^Z_{k,B}\sqrt{\Delta_i}\sqrt{\Delta_k}}\big)^{2\sum_{l=0}^q(K_{l,i}+1)},$$

$$\mathbb{P}(C_k^{Y,M}) \leq 12 \exp\big(-\frac{M\varepsilon_{k,C}^Y}{507C_y^2}\big)\ \big(\frac{1056C_y^2}{5\varepsilon_{k,C}^Y}\big)^{2(K_{0,k}+1)},$$

$$\mathbb{P}(C_k^{Z,M}) \leq 12 \exp\big(-\frac{M\varepsilon_{k,C}^Y\Delta_k}{507C_y^2}\big)\sum_{l=1}^{q}\big(\frac{1056C_y^2}{5\varepsilon_{k,C}^Z\Delta_k}\big)^{2(K_{l,k}+1)},$$

provided that $\varepsilon_{k,A}^Y, \varepsilon_{k,A}^Z, \varepsilon_{k,B}^Y, \varepsilon_{k,B}^Z, \varepsilon_{k,C}^Y, \varepsilon_{k,C}^Z$ *are small enough in the sense that*

$$0 < \varepsilon_{k,A}^Y \leq \big(1 \wedge \min_{i=k\dots N-1}\Delta_i^{-1}\big)K_{0,k}\frac{9C_y^2L_f^2T^{1+\theta_L}(q+1)}{2\theta_L},$$

$$0 < \varepsilon_{k,A}^Z \leq \big(1 \wedge \min_{i=k\dots N-1}\Delta_i^{-1}\big)\big(\min_{l=1\dots q}K_{l,k}\big)\frac{R^2}{\Delta_k}\ \frac{9C_y^2L_f^2T^{1+\theta_L}(q+1)}{2\theta_L},$$

$$0 < \varepsilon_{k,B}^Y \leq \big(1 \wedge \min_{i=k\dots N-1}\Delta_i^{-1/2}\big)15L_fT^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi + C_y)C_y,$$

$$0 < \varepsilon_{k,B}^Z \leq \big(1 \wedge \min_{i=k\dots N-1}\Delta_i^{-1/2}\big)\frac{15L_fT^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi R + C_y)C_y}{\sqrt{\Delta_k}},$$

$$0 < \varepsilon_{k,C}^Y \leq 24C_y^2, \qquad 0 < \varepsilon_{k,C}^Z \leq 24C_y^2\Delta_k^{-1/2}.$$

4.3. Algorithm complexity

The error analysis of Section 4.2 shows us that the numerical parameters may play multiple and often contradictory roles in the convergence of the scheme: the higher the number N of steps, the smaller the discretization error but the larger effect for the propagation of errors through the DP equation; the higher the dimension of the function spaces for the empirical regression, the better the approximation accuracy (the *bias* term in regression) but the larger the statistical error (the *variance* term); the higher the number of simulations, the smaller the statistical error, but the more computational work to be done. We now demonstrate how the results of Theorems 4.1 and 4.2 can be used to optimize these parameters by means of an error vs. computational work (*complexity*) analysis. Moreover, we show how extra smoothness of the Markov functions can bring about substantial improvements in the complexity analysis.

For simplicity, we assume that the time-grid is uniform: $\Delta_i = T/N$.

Assume that the function y_i (defined in (1.3)) is of class $C_b^{\kappa+1+\eta}$ uniformly in i, meaning that y_i is uniformly bounded and $\kappa+1$-continuously differentiable ($\kappa \geq 0$), with bounded derivatives, and the $\kappa+1$-th derivatives are η-Hölder continuous ($\eta \in (0,1]$). Additionally, assume that z_i is $C_b^{\kappa+\eta}$. These enhanced assumptions are natural: in the continuous time case, see [CD11] for a recent account, y_i inherits from the smoothness of the terminal condition and the driver, and z_i is once less differentiable that y_i. We already see in Lemma 3.3 and Corollary 3.2 that these assumptions are viable under the assumption of Lipschitz continuity for the terminal condition and the driver. The bounds on the functions y_i and z_i and their derivatives are assumed to be uniform in N.

This improves the bounds in Theorems 4.1 and 4.2, because one can remove the the dependence of the constant C_Ψ on N.

Let us make the squared global errors $\mathbb{E}\|y_i^R - y_i^{R,M}\|_{i,M}^2$ and $\sum_{k=i}^{N-1} \Delta_k \mathbb{E}\|z_k^R - z_k^{R,M}\|_{k,M}^2$ be of magnitude $(N^{-2\theta_{\text{conv}}})$ where $\theta_{\text{conv}} > 0$. The parameter θ_{conv} can be the convergence order of the time discretization scheme: in the case of Lipschitz continuous f and Φ in a diffusion setting, $\theta_{\text{conv}} = \frac{1}{2}$ (see [LGW06, Theorem 1]). When the forward component is simulated with a strong error of order 1, then one can achieve $\theta_{\text{conv}} = 1$ (see [GL07, Theorems 7 and 8]). It is sufficient due to Theorem 4.1 to tune the parameters (basis functions, number of simulations) so that each term in $\mathcal{E}_i^Y(R,M)$ and $\mathcal{E}_i^Z(R,M)$ be $O(N^{-2\theta_{\text{conv}}})$.

For the basis functions, we take local polynomials defined on disjoint hypercubes $(\mathcal{H}_n)_{n=1,\dots,K_{l,k}}$ $(l = 0,\dots,q)$ with edge length δ_y (for y) and δ_z (for z_l). The union of these hypercubes is of the form $[-\bar{R},\bar{R}]^d$ for each component $y, z_1, \dots, z_q$. The degree of local polynomials is $\kappa+1$ for y, and κ for z_l. We denote by x_n the center of the n-th hypercube $\mathcal{H}_n$.

In the following, c is a positive constant that does not depend on N and may change from line to line; c is assumed to be large enough for the arguments to be consistent.

Bias terms. Because of Proposition 3.3, we can replace y^R and z^R by y and z (that is $R = +\infty$) in the expression of $T_{1,i}^{Y,M}$ and $T_{1,i}^{Z,M}$ by choosing $R = \sqrt{4(\theta_{\text{conv}} + \frac{1}{2})\log(N+1)}$. It adds an extra squared error $O(N^{-2\theta_{\text{conv}}})$. The projection error $\min_\alpha \mathbb{E}|y_i(X_i) - \alpha \cdot p_{0,i}(X_i)|^2$ is equal to

$$\begin{aligned}
&\mathbb{E}|y_i(X_i)\mathbf{1}_{|X_i|_\infty > \bar{R}}|^2 + \sum_{n=1}^{K_{0,i}} \min_\alpha \mathbb{E}|y_i(X_i) - \alpha \cdot p_{0,i}(X_i)|^2 \mathbf{1}_{X_i \in \mathcal{H}_n} \\
&\qquad \le |y_i|_\infty^2 \mathbb{P}(|X_i|_\infty > \bar{R}) + \sum_{n=1}^{K_{0,i}} c|y_i|_{\kappa+1+\eta}^2 (\delta_y^{\kappa+1+\eta})^2 \mathbb{P}(X_i \in \mathcal{H}_n) \\
&\qquad \le |y_i|_\infty^2 \mathbb{P}(|X_i|_\infty > \bar{R}) + c|y_i|_{\kappa+1+\eta}^2 (\delta_y^{\kappa+1+\eta})^2
\end{aligned}$$

where we have used a Taylor expansion on each set $\mathcal{H}_n$ and taken the local polynomials to be equal to the first terms of the expansion. Assume additionally that X_i has exponential moments (uniformly in i), i.e. for some $\lambda > 0$, $\sup_{N\ge1}\sup_{0\le i\le N} \mathbb{E}(e^{\lambda|X_i|_\infty}) < +\infty$, so that the choice $\bar{R} = 2\theta_{\text{conv}}\lambda^{-1}\log(N+1)$ is sufficient to ensure $\mathbb{P}(|X_i| > \bar{R}) = O(N^{-2\theta_{\text{conv}}})$. Hence, the choice $\delta_y = cN^{-\frac{\theta_{\text{conv}}}{\kappa+1+\eta}}$ ensures that $T_{1,i}^{Y,M} = O(N^{-2\theta_{\text{conv}}})$. With similar arguments for the z_l components, we have to choose $\delta_z = cN^{-\frac{\theta_{\text{conv}}}{\kappa+\eta}}$. Thus the sizes of the vector spaces are $K_{0,i} = cN^{d\frac{\theta_{\text{conv}}}{\kappa+1+\eta}}\log^d(N+1)$ and $K_{l,i} = cN^{d\frac{\theta_{\text{conv}}}{\kappa+\eta}}\log^d(N+1)$.

Variance terms. Making $T_{2,i}^{Z,M}$ of order $N^{-2\theta_{\text{conv}}}$ implies $M = cN^{1+2\theta_{\text{conv}}}K_{l,i}\log(N+1) = cN^{1+2\theta_{\text{conv}}+d\frac{\theta_{\text{conv}}}{\kappa+\eta}}\log^{d+1}(N+1)$; this dominates the requirements on M for $T_{2,i}^{Y,M}$.

Large deviation events. We set $\varepsilon^Y_{k,A}, \varepsilon^Y_{k,B}, \varepsilon^Y_{k,C}, \varepsilon^Z_{k,A}, \varepsilon^Z_{k,B}, \varepsilon^Z_{k,C}$ equal to $N^{-2\theta_{\text{conv}}}$. In order to make the probability upper bound exponentially small a quick look at Theorem 4.2 shows that the strongest constraint comes from $\mathbb{P}(A_i^{Z,M})$ which imposes $c(NK_{l,i}\log(N+1)+\log(N+1)) = \frac{M}{N^{1+2\theta_{\text{conv}}}K_{l,i}\log(N+1)}$. This condition on M is much stronger than the previous one, ensuring that the variance terms have the right magnitude; the requirement of having more simulations seems to be the price to pay for having a single set of paths and interdependent regression problems. To fulfill this condition, take

$$M = cK_{l,i}^2 N^{2+2\theta_{\text{conv}}}\log^2(N+1) = cN^{2+2\theta_{\text{conv}}+2d\frac{\theta_{\text{conv}}}{\kappa+\eta}}\log^{2+2d}(N+1).$$

Complexity analysis. Due to the properties of hypercubes (disjoint intervals), the final computational cost $\mathcal{C}$ (counting the elementary operations) is of order MN, that is

$$\mathcal{C} = cN^{3+2\theta_{\text{conv}}+2d\frac{\theta_{\text{conv}}}{\kappa+\eta}}\log^{2+2d}(N+1).$$

Equivalently, the global error, as a function of complexity and ignoring log factors, is

$$N^{-\theta_{\text{conv}}} \leq c\,\mathcal{C}^{\frac{-\theta_{\text{conv}}}{3+2\theta_{\text{conv}}+2d\frac{\theta_{\text{conv}}}{\kappa+\eta}}} = c\,\mathcal{C}^{\frac{-1}{2(1+\frac{3}{2\theta_{\text{conv}}}+\frac{d}{\kappa+\eta})}}.$$

This analysis shows that the smaller the parameter $\frac{3}{2\theta_{\text{conv}}}+\frac{d}{\kappa+\eta}$, the quicker the convergence. There are several numerically significant implications of this:

- The higher the smoothness of the solution, the better the convergence. This may motivate first solving the BSDE without driver and then solve the BSDE difference (which in general gives a smoother problem, see [GM10]) - see our discussion about proxys in the Section 2.
- The higher the dimension, the worse the convergence. This is the usual curse of dimensionality.
- The better the discretization error (θ_{conv} large), the better the convergence. This motivates the development of high-order discretization schemes for BSDEs.

If we apply the same analysis to [LGW06, Theorem 2] for $\kappa+\eta \geq 1$ and $\theta_{\text{conv}} = 1/2$, we obtain that the error is of order $\mathcal{C}^{-\frac{1}{2(4+\frac{2d}{\kappa+1+\eta})}}$ for the ODP. In contrast, the MDP has error of order $\mathcal{C}^{-\frac{1}{2(4+\frac{d}{\kappa+\eta})}}$. This implies that, for sufficiently large N, the MDP performs better than the ODP for $\kappa+\eta > 1$, at least in the theoretical framework given in this section. For $\kappa+\eta = 1$, the performance is the same. In the future, we will undertake numerical comparisons between ODP and MDP to test this theoretical analysis.

4.4. Proof of Theorem 4.1

From $[y_k^R]_y = y_k^R$ and the Lipschitz property of $[\cdot]_y$, it follows that

$$\mathbb{E}\|y_k^R - y_k^{R,M}\|_{k,M}^2 \le \mathbb{E}\|y_k^R - \alpha_{0,k}^M \cdot p_{0,k}\|_{k,M}^2.$$

For any m,

$$\mathbb{E}_N^M\Big[\Phi(\tilde{X}_N^{k,m}) + \sum_{i=k}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^R(\tilde{X}_{i+1}^{k,m}), z_i^R(\tilde{X}_i^{k,m}))\Delta_i\Big] = y_k^R(\tilde{X}_k^{k,m}) = y_k^R(X_k^m)$$

is clear from the definition of y^R in (4.29). Hence, owing to (4.38) and Proposition 4.1(iii) (applied with $\mathcal{Q} = \mathcal{F}_N^M$), $\mathbb{E}_N^M[\tilde{\beta}_{0,k}^M]$ is the SVD-minimizer of $\|y_k^R - \alpha \cdot p_{0,k}\|_{k,M}^2$. We now apply Pythagoras' theorem and take expectations to obtain

$$\mathbb{E}\|y_k^R - y_k^{R,M}\|_{k,M}^2 \le T_{1,k}^{Y,M} + \mathbb{E}\|(\alpha_{0,k}^M - \mathbb{E}_N^M[\tilde{\beta}_{0,k}^M]) \cdot p_{0,k}\|_{k,M}^2.$$

To decompose the last term above, we introduce the coefficients $\tilde{\alpha}_{0,k}^M$ and $\mathbb{E}_N^M(\tilde{\alpha}_{0,k}^M)$:

$$\begin{aligned}&\mathbb{E}\|(\alpha_{0,k}^M - \mathbb{E}_N^M[\tilde{\beta}_{0,k}^M]) \cdot p_{0,k}\|_{k,M}^2 \\ &\quad \le 3\mathbb{E}\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M) \cdot p_{0,k}\|_{k,M}^2 + 3T_{2,k}^{Y,M} + 3\mathbb{E}\|(\mathbb{E}_N^M[\tilde{\beta}_{0,k}^M - \tilde{\alpha}_{0,k}^M]) \cdot p_{0,k}\|_{k,M}^2.\end{aligned}$$

The first term on the r.h.s. is estimated using the event $A_k^{Y,M}$ in (4.42). To do this, we first need to obtain an almost sure bound on the integrand. Indeed, from Proposition 4.1(i), $\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M$ is the SVD-optimal coefficient of the least-squares problem w.r.t. the $\|\cdot\|_{k,M}$-norm associated to the differences of the responses of $\alpha_{0,k}^M$ and $\tilde{\alpha}_{0,k}^M$. Proposition 4.1(ii) then applies to give $\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M) \cdot p_{0,k}\|_{k,M}^2 \le 4C_\Psi^2$. Using this upper bound, we now have

$$\mathbb{E}\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M) \cdot p_{0,k}\|_{k,M}^2 \le \varepsilon_{k,A}^Y + 4C_\Psi^2 \mathbb{P}(A_k^{Y,M}). \tag{4.49}$$

To handle $\mathbb{E}\|(\mathbb{E}_N^M[\tilde{\beta}_{0,k}^M - \tilde{\alpha}_{0,k}^M]) \cdot p_{0,k}\|_{k,M}^2$, observe that $\mathbb{E}_N^M[\tilde{\beta}_{0,k}^M - \tilde{\alpha}_{0,k}^M]$ is the SVD-optimal coefficient of the least-squares problem related to the response

$$\begin{aligned}&\mathbb{E}_N^M\Big[\Phi(\tilde{X}_N^{k,m}) + \sum_{i=k}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^R(\tilde{X}_{i+1}^{k,m}), z_i^R(\tilde{X}_i^{k,m}))\Delta_i\Big] \\ &- \mathbb{E}_N^M\Big[\Phi(\tilde{X}_N^{k,m}) + \sum_{i=k}^{N-1} f_i(\tilde{X}_i^{k,m}, y_{i+1}^{R,M}(\tilde{X}_{i+1}^{k,m}), z_i^{R,M}(\tilde{X}_i^{k,m}))\Delta_i\Big] \\ &= y_k^R(X_k^m) - \bar{y}_k^{R,M}(X_k^m) \quad \text{(by Lemma 4.2).}\end{aligned}$$

By the contraction property (item (ii) of Proposition 4.1),

$$\begin{aligned}\mathbb{E}\|(\mathbb{E}_N^M[\tilde{\beta}_{0,k}^M - \tilde{\alpha}_{0,k}^M]) \cdot p_{0,k}\|_{k,M}^2 &\le \mathbb{E}\|y_k^R - \bar{y}_k^{R,M}\|_{k,M}^2 \\ &\le \varepsilon_{k,B}^Y + 2\mathbb{E}(y_k^R(X_k) - \bar{y}_k^{R,M}(X_k))^2 + 2(C_y^2 + C_\Psi^2)\mathbb{P}(B_k^{Y,M}).\end{aligned}$$

Bringing together the thus far obtained results yields

$$\begin{aligned}\mathbb{E}\|y_k^R - y_k^{R,M}\|_{k,M}^2 \leq\ & T_{1,k}^{Y,M} + 3\varepsilon_{k,A}^Y + 12C_\Psi^2\mathbb{P}(A_k^{Y,M}) \\ & + 3T_{2,k}^{Y,M} + 3\varepsilon_{k,B}^Y + 6(C_y^2 + C_\Psi^2)\mathbb{P}(B_k^{Y,M}) \\ & + 6\mathbb{E}[(y_k^R(X_k) - \bar{y}_k^{R,M}(X_k))^2]. \end{aligned} \tag{4.50}$$

We can perform analogous calculations for the Z component (replacing C_Ψ by $C_\Psi \frac{R}{\sqrt{\Delta_k}}$ and C_y by $\frac{C_y}{\sqrt{\Delta_k}}$), obtaining

$$\begin{aligned}\mathbb{E}\|z_k^R - z_k^{R,M}\|_{k,M}^2 \leq\ & T_{1,k}^{Z,M} + 3q\varepsilon_{k,A}^Z + 12q\frac{C_\Psi^2 R^2}{\Delta_k}\mathbb{P}(A_k^{Z,M}) \\ & + 3T_{2,k}^{Z,M} + 3q\varepsilon_{k,B}^Z + 6q\frac{(C_y^2 + C_\Psi^2 R^2)}{\Delta_k}\mathbb{P}(B_k^{Z,M}) \\ & + 6\mathbb{E}[|z_k^R(X_k) - \bar{z}_k^{R,M}(X_k)|^2]. \end{aligned} \tag{4.51}$$

Observe that here, we rely on truncated Brownian increments in order to *a.s.* upper bound $\|(\alpha_{l,k}^M - \tilde{\alpha}_{l,k}^M)\cdot p_{l,k}\|_{k,M}^2$ by $4\frac{C_\Psi^2 R^2}{\Delta_k}$. We can use the same reasoning as in Lemma 4.2 to show that

$$\begin{aligned}\bar{y}_k^{R,M}(X_k) =& \mathbb{E}[\Phi(X_N) + \sum_{i=k}^{N-1} f_i(X_i, y_{i+1}^{R,M}(X_{i+1}), z_i^{R,M}(X_i))\Delta_i \mid \mathcal{F}_N^M \vee \mathcal{F}_{t_k}], \\ \Delta_k \bar{z}_{l,k}^{R,M}(X_k) =& \mathbb{E}\big[[\Delta W_{l,k}]_w\big(\Phi(X_N) \\ & + \sum_{i=k+1}^{N-1} f_i(X_i, y_{i+1}^{R,M}(X_{i+1}), z_i^{R,M}(X_i))\Delta_i\big) \mid \mathcal{F}_N^M \vee \mathcal{F}_{t_k}\big], \end{aligned}$$

which implies that $(\bar{y}_k^{R,M}(X_k), \bar{z}_k^{R,M}(X_k))_{k\geq 0}$ solves a discrete BSDE with data $(\Phi(X_N), \bar{f}_i(y,z) = f_i(X_i, y_{i+1}^{R,M}(X_{i+1}), z_i^{R,M}(X_i)))$. We now apply the stability result from Proposition 3.1 (w.r.t. filtration $(\mathcal{F}_N^M \vee \mathcal{F}_{t_k})_{0\leq k\leq N}$), taking the first discrete BSDE to be $(y_k^R(X_k), z_k^R(X_k))_{k\geq 0}$ and the second to be $(\bar{y}_k^{R,M}(X_k), \bar{z}_k^{R,M}(X_k))_{k\geq 0}$ ($L_{\bar{f}_k} = 0$). Combined with the local Lipschitz continuity of f_k and a choice of $\gamma \in (0, +\infty)^N$ such that

$$144(R_\pi \vee 1)C_{3.15}(1+T)(\frac{1}{\gamma_k} + \Delta_k)\frac{L_f^2}{(T-t_k)^{1-\theta_L}} \leq 1, \qquad (0 \leq k < N), \tag{4.52}$$

we obtain the bound

$$\begin{aligned}&\sum_{k=i+1}^{N-1} \Delta_k \mathbb{E}[|y_k^R(X_k) - \bar{y}_k^{R,M}(X_k)|^2]\Gamma_k + \sum_{k=i}^{N-1} \Delta_k \mathbb{E}[|z_k^R(X_k) - \bar{z}_k^{R,M}(X_k)|^2]\Gamma_k \\ &\leq 6C_{3.15}(1+T)\sum_{k=i}^{N-1}(\frac{1}{\gamma_k} + \Delta_k)\Delta_k \frac{L_f^2}{(T-t_k)^{1-\theta_L}}\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}\big[|y^R_{k+1}(X_{k+1}) - y^{R,M}_{k+1}(X_{k+1})|^2 + |z^R_k(X_k) - z^{R,M}_k(X_k)|^2\big]\Gamma_k \\
&\leq \frac{1}{12}\mathcal{E}_{i,N}(\gamma, R, M) \qquad (4.53)\\
&+ \frac{1}{24}\sum_{k=i}^{N-1}\Delta_k\big(1_{k+1<N}[\varepsilon^Y_{k+1,C} + 4C_y^2\mathbb{P}(C^{Y,M}_{k+1})] + q\varepsilon^Z_{k,C} + 4q\frac{C_y^2}{\Delta_k}\mathbb{P}(C^{Z,M}_k)\big)\Gamma_k
\end{aligned}
$$

where

$$
\mathcal{E}_{i,N}(\gamma, R, M) := \sum_{k=i+1}^{N-1}\Delta_k\mathbb{E}\|y^R_k - y^{R,M}_k\|^2_{k,M}\Gamma_k + \sum_{k=i}^{N-1}\Delta_k\mathbb{E}\|z^R_k - z^{R,M}_k\|^2_{k,M}\Gamma_k.
$$

Using (4.53) together with (4.50) and (4.51), it readily follows that

$$
\begin{aligned}
\mathcal{E}_{i,N}(\gamma, R, M) \leq &\sum_{k=i+1}^{N-1}\Delta_k\{T^{Y,M}_{1,k} + 3\varepsilon^Y_{k,A} + 12C_\Psi^2\mathbb{P}(A^{Y,M}_k) + 3T^{Y,M}_{2,k} + 3\varepsilon^Y_{k,B} \\
&\quad + 6(C_y^2 + C_\Psi^2)\mathbb{P}(B^{Y,M}_k)\}\Gamma_k \\
&+ \sum_{k=i}^{N-1}\Delta_k\{T^{Z,M}_{1,k} + 3q\varepsilon^Z_{k,A} + 12q\frac{C_\Psi^2R^2}{\Delta_k}\mathbb{P}(A^{Z,M}_k) + 3T^{Z,M}_{2,k} + 3q\varepsilon^Z_{k,B} \\
&+ 6q\frac{(C_y^2 + C_\Psi^2R^2)}{\Delta_k}\mathbb{P}(B^{Z,M}_k)\}\Gamma_k + \frac{1}{2}\mathcal{E}_{i,N}(\gamma, R, M) \\
&+ \frac{1}{4}\sum_{k=i}^{N-1}\Delta_k\{1_{k+1<N}[\varepsilon^Y_{k+1,C} + 4C_y^2\mathbb{P}(C^{Y,M}_{k+1})] + q\varepsilon^Z_{k,C} + 4q\frac{C_y^2}{\Delta_k}\mathbb{P}(C^{Z,M}_k)\}\Gamma_k.
\end{aligned}
$$

The choice $\gamma_k = 288(R_\pi \vee 1)C_{3.15}(1+T)\frac{L_f^2}{(T-t_k)^{1-\theta_L}}$ leads to $1 \leq \Gamma_i \leq C^6_{3.22}$ and then for N large enough (such that $C_\pi L_f^2(R_\pi \vee 1) \leq \frac{1}{288C_{3.15}(1+T)}$), the condition (4.52) is satisfied and we obtain the announced estimate on the Z component. Now, applying the same arguments used in (4.53) directly to $\mathbb{E}[|y^R_i(X_i) - y^{R,M}_i(X_i)|^2]$ in (4.50), we obtain

$$
\begin{aligned}
\mathbb{E}\|y^R_i - y^{R,M}_i\|^2_{i,M}\Gamma_i \leq &\mathcal{E}^Y_i(R, M)\Gamma_i + \frac{1}{2}\mathcal{E}_{i,N}(\gamma, R, M) \\
&+ \sum_{k=i+1}^{N-1}\Delta_k\mathcal{E}^Y_k(R, M)\Gamma_k + \sum_{k=i}^{N-1}\Delta_k\mathcal{E}^Z_k(R, M)\Gamma_k
\end{aligned}
$$

and we easily complete the proof for the Y component. □

4.5. Proof of Theorem 4.2

In each of the following subsections, we prove the bounds for the local error terms given in Theorem 4.2.

4.5.1. Bias/variance terms

The bounds on squared bias terms $T_{1,k}^{Y,M}$ and $T_{1,k}^{Z,M}$ are straightforward. For the variance terms $T_{2,k}^{Y,M}$ and $T_{2,k}^{Z,M}$, we use the same arguments as [GKKW02, pp.186–187] or [LGW06, Proposition 4]. □

4.5.2. Large deviation events

In the proofs below we use *covering* techniques to allow us to apply exponential inequalities to the large deviation events. We recall few definitions here for the benefit of the reader; for a fuller account, see [GKKW02, Chapter 9].
If $\mathcal{G}$ is a class of functions from $\mathbb{R}^d$ to $\mathbb{R}$ and $x^{1:M} = \{x^m\}_{m=1\dots M}$ are M points of $\mathbb{R}^d$, an *ε-cover* ($\varepsilon > 0$) of $\mathcal{G}$ w.r.t. the $\mathbf{L}_p(p \geq 1)$-empirical norm $\|g\|_M = \left(\frac{1}{M}\sum_{m=1}^M |g(x^m)|^p\right)^{\frac{1}{p}}$ is a finite collection of functions $g_1, \dots, g_n \in \mathcal{G}$ such that for any $g \in \mathcal{G}$, we can find a $j \in \{1, \cdots, n\}$ such that $\|g - g_j\|_M \leq \varepsilon$. The smallest integer n for which an ε-cover exists is called the *ε-covering number* and denoted by $\mathcal{N}_p(\varepsilon, \mathcal{G}, x^{1:M})$; we usually consider an ε-cover with minimal number of elements. In the following the points $x^{1:M}$ are possibly random. More generally, we may consider ε-cover w.r.t. the $\mathbf{L}_p$-norm of a probability measure ν (instead of the empirical measure associated to $x^{1:M}$): the related covering number is then denoted $\mathcal{N}_p(\varepsilon, \mathcal{G}, \nu)$.

Bounds on $\mathbb{P}(A_k^{Y,M})$ and $\mathbb{P}(A_k^{Z,M})$. We only prove the bound related to $A_k^{Y,M}$; the proof for $A_k^{Z,M}$ is analogous. We use a similar method to [LGW06] but with some important differences.
Define the following sets of functions

$$\begin{cases} [\mathcal{S}_N^Y]_y := \{\Phi\}, \qquad [\mathcal{S}_k^Y]_y := \left\{[\alpha \cdot p_{0,k}]_y : \alpha \in \mathbb{R}^{K_{0,k}}\right\}, \\ [\mathcal{S}_k^{Z,l}]_z := \left\{[\alpha \cdot p_{l,k}]_z : \alpha \in \mathbb{R}^{K_{l,k}}\right\} \qquad (1 \leq l \leq q), \end{cases} \tag{4.54}$$

$$\mathcal{G}_k := \Big\{(x_k, \dots, x_N) \in \bigotimes_{i=k}^N \mathbb{R}^d \mapsto \Phi(x_N) + \sum_{i=k}^{N-1} f_i(x_i, \psi_{i+1}(x_{i+1}), \eta_i(x_i))\Delta_i \\ : \psi_i \in [\mathcal{S}_i^Y]_y, \quad \eta_{l,i} \in [\mathcal{S}_i^{Z,l}]_z\Big\}$$

for $k = N-1, \dots, 0$. These definitions originate from the observation that $y_k^{R,M} \in [\mathcal{S}_k^Y]_y$, $z_{l,k}^{R,M} \in [\mathcal{S}_k^{Z,l}]_z$ and $\Psi_k^{R,M} \in \mathcal{G}_k$. By the same arguments as in Lemma 4.1, every element in $\mathcal{G}_k$ is bounded by C_Ψ. Define

$$G_k^m := \Psi_k^{R,M}(X_k^m, \dots, X_N^m), \qquad \tilde{G}_k^m := \Psi_k^{R,M}(\tilde{X}_k^{k,m}, \dots, \tilde{X}_N^{k,m}).$$

Similarly, for any $G \in \mathcal{G}_k$, we write G^m or $\tilde{G}^m$ for the function evaluated along the m-th sample path or its ghost path. Firstly, observe that

$$\mathbb{P}\big(\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M) \cdot p_{0,k}\|_{k,M}^2 > \varepsilon_{k,A}^Y\big) = \mathbb{E}\big(\mathbb{P}_k^M\big(\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M) \cdot p_{0,k}\|_{k,M}^2 > \varepsilon_{k,A}^Y\big)\big).$$

Conditionally on $\mathcal{F}_k^M$, we can assume that the basis functions are orthonormalized w.r.t. $\langle\cdot,\cdot\rangle_{k,M}$ and that the coefficients $\alpha_{0,k}^M$ and $\tilde{\alpha}_{0,k}^M$ are computed for the orthonormalized functions. Nevertheless, the dimension of the vector space w.r.t. $\langle\cdot,\cdot\rangle_{k,M}$ may be smaller than $K_{0,k}$; we denote this empirical dimension of the vector space by $K_{0,k}^M$. Then

$$\|(\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M)\cdot p_{0,k}\|_{k,M}^2 = |\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M|^2_{\mathbb{R}^{K_{0,k}^M}}.$$

Furthermore, the coefficients $\alpha_{0,k}^M$ and $\tilde{\alpha}_{0,k}^M$ have simple expressions and we obtain

$$\begin{aligned}
\mathbb{P}_k^M(|\alpha_{0,k}^M - \tilde{\alpha}_{0,k}^M|^2_{\mathbb{R}^{K_{0,k}^M}} > \varepsilon_{k,A}^Y) &= \mathbb{P}_k^M\big(\sum_{i=1}^{K_{0,k}^M}\big|\frac{1}{M}\sum_{m=1}^M p_{0,k}^{i,m}(G_k^m - \tilde{G}_k^m)\big|^2 > \varepsilon_{k,A}^Y\big)\\
&\le \sum_{i=1}^{K_{0,k}^M}\mathbb{P}_k^M\big(\big|\frac{1}{M}\sum_{m=1}^M p_{0,k}^{i,m}(G_k^m - \tilde{G}_k^m)\big|^2 > \frac{\varepsilon_{k,A}^Y}{K_{0,k}^M}\big)\\
&\le \sum_{i=1}^{K_{0,k}^M}\mathbb{P}_k^M\big(\big|\frac{1}{M}\sum_{m=1}^M p_{0,k}^{i,m}(G_k^m - \tilde{G}_k^m)\big|^2 > \frac{\varepsilon_{k,A}^Y}{K_{0,k}}\big)\\
&\le \sum_{i=1}^{K_{0,k}^M}\mathbb{P}_k^M\big(\exists G\in\mathcal{G}_k : \big|\frac{1}{M}\sum_{m=1}^M p_{0,k}^{i,m}(G^m - \tilde{G}^m)\big|^2 > \frac{\varepsilon_{k,A}^Y}{K_{0,k}}\big)\\
&= \sum_{i=1}^{K_{0,k}^M}\mathbb{P}_k^M\big(\exists G\in\mathcal{G}_k : \big|\frac{1}{M}\sum_{m=1}^M p_{0,k}^{i,m}U^m(G^m - \tilde{G}^m)\big|^2 > \frac{\varepsilon_{k,A}^Y}{K_{0,k}}\big)
\end{aligned}$$

where U^m are independent random variables uniformly distributed on $\{-1,+1\}$ that are also independent of everything else. The last equality follows by an invariance of the $\mathbb{P}_k^M$-distribution of $G^m - \tilde{G}^m$ under multiplication by -1.
We now introduce a particular cover for the set $\mathcal{G}_k$. For a function $g:\mathbb{R}^d\mapsto\mathbb{R}$, we define the squared $\mathbf{L}_2$ empirical norm

$$\|g\|_{i,M,\tilde{M}}^2 := \frac{1}{2M}\sum_{m=1}^M\big(|g(X_i^m)|^2 + |g(\tilde{X}_i^{k,m})|^2\big).$$

Then for $\varepsilon>0$ (the value of which is chosen later), denote by $[\mathcal{S}_i^Y(\varepsilon)]_y$ and $[\mathcal{S}_i^{Z,l}(\varepsilon)]_z$ ε-covers for $[\mathcal{S}_i^Y]_y$ and $[\mathcal{S}_i^{Z,l}]_z$, respectively, w.r.t. the norm $\|\cdot\|_{i,M,\tilde{M}}$. Take

$$\begin{aligned}
\mathcal{G}_k(\varepsilon) := \big\{(x_k,\dots,x_N)\mapsto\Phi(x_N) + \sum_{i=k}^{N-1} f_i(x_i, g_{i+1}^Y(x_{i+1}), g_i^Z(x_i))\Delta_i\\
:\ g_i^Y\in[\mathcal{S}_i^Y(\varepsilon)]_y,\ g_{l,i}^Z\in[\mathcal{S}_i^{Z,l}(\varepsilon)]_z\big\}.
\end{aligned}$$

Since elements of $[\mathcal{S}_i^Y(\varepsilon)]_y$ and $[\mathcal{S}_i^{Z,l}(\varepsilon)]_z$ are bounded by C_y and $C_{z,i}$, elements of $\mathcal{G}_k(\varepsilon)$ are bounded by C_Ψ.
For every $G \in \mathcal{G}_k$, there exists a $G_\varepsilon \in \mathcal{G}_k(\varepsilon)$ such that

$$
\begin{aligned}
&\frac{1}{2M}\sum_{m=1}^{M}\big\{|G^m - G_\varepsilon^m|^2 + |\tilde{G}^m - \tilde{G}_\varepsilon^m|^2\big\} \\
&= \frac{1}{2M}\sum_{m=1}^{M}\big\{\big|\sum_{i=k}^{N-1}\Delta_i\big(f_i(X_i^m, \psi_{i+1}(X_{i+1}^m), \eta_i(X_i^m)) - f_i(X_i^m, g_{i+1}^Y(X_{i+1}^m), g_i^Z(X_i^m))\big)\big|^2 \\
&\quad + \big|\sum_{i=k}^{N-1}\Delta_i\big(f_i(\tilde{X}_i^{k,m}, \psi_{i+1}(\tilde{X}_{i+1}^{k,m}), \eta_i(\tilde{X}_i^{k,m})) - f_i(\tilde{X}_i^{k,m}, g_{i+1}^Y(\tilde{X}_{i+1}^{k,m}), g_i^Z(\tilde{X}_i^{k,m}))\big)\big|^2\big\} \\
&\le \frac{1}{2M}\sum_{m=1}^{M} 2T\sum_{i=k}^{N-1}\frac{\Delta_i L_f^2}{(T-t_i)^{1-\theta_L}}\big\{|\psi_{i+1}(X_{i+1}^m) - g_{i+1}^Y(X_{i+1}^m)|^2 + |\eta_i(X_i^m) - g_i^Z(X_i^m)|^2 \\
&\qquad\qquad + |\psi_{i+1}(\tilde{X}_{i+1}^{k,m}) - g_{i+1}^Y(\tilde{X}_{i+1}^{k,m})|^2 + |\eta_i(\tilde{X}_i^{k,m}) - g_i^Z(\tilde{X}_i^{k,m})|^2\big\} \\
&\le 2\frac{T^{1+\theta_L}}{\theta_L}L_f^2(q+1)\varepsilon^2
\end{aligned}
$$

where we have used $(\mathbf{A_F})$. Taking $\varepsilon^2 = \frac{\theta_L \varepsilon_{k,A}^Y}{18K_{0,k}L_f^2T^{1+\theta_L}(q+1)}$ and $G_\varepsilon \in \mathcal{G}_k(\varepsilon)$ as above, we easily obtain

$$
\begin{aligned}
\big|\frac{1}{M}\sum_{m=1}^{M} p_{0,k}^{i,m}U^m(G^m - \tilde{G}^m)\big|^2 &\le 3\big|\frac{1}{M}\sum_{m=1}^{M} p_{0,k}^{i,m}U^m(G_\varepsilon^m - \tilde{G}_\varepsilon^m)\big|^2 \\
&\quad + 3\big(\frac{1}{M}\sum_{m=1}^{M}|p_{0,k}^{i,m}|^2\big)\big(\frac{1}{M}\sum_{m=1}^{M}\big\{|G^m - G_\varepsilon^m|^2 + |\tilde{G}_\varepsilon^m - \tilde{G}^m|^2\big\}\big) \\
&\le 3\big|\frac{1}{M}\sum_{m=1}^{M} p_{0,k}^{i,m}U^m(G_\varepsilon^m - \tilde{G}_\varepsilon^m)\big|^2 + \frac{2}{3}\frac{\varepsilon_{k,A}^Y}{K_{0,k}}
\end{aligned}
$$

where we take advantage of the orthonormality property of $p_{0,k}^i$. Then it follows

$$
\begin{aligned}
&\mathbb{P}_k^M\big(\exists G \in \mathcal{G}_k : \big|\frac{1}{M}\sum_{m=1}^{M} p_{0,k}^{i,m}U^m(G^m - \tilde{G}^m)\big|^2 > \frac{\varepsilon_{k,A}^Y}{K_{0,k}}\big) \\
&\le \mathbb{P}_k^M\big(\exists G_\varepsilon \in \mathcal{G}_k(\varepsilon) : \big|\frac{1}{M}\sum_{m=1}^{M} p_{0,k}^{i,m}U^m(G_\varepsilon^m - \tilde{G}_\varepsilon^m)\big| > \frac{1}{3}\big(\frac{\varepsilon_{k,A}^Y}{K_{0,k}}\big)^{\frac{1}{2}}\big) \\
&\le \mathbb{E}_k^M|\mathcal{G}_k(\varepsilon)|\max_{G_\varepsilon \in \mathcal{G}_k(\varepsilon)}\tilde{\mathbb{P}}^M\big(\big|\frac{1}{M}\sum_{m=1}^{M} p_{0,k}^{i,m}U^m(G_\varepsilon^m - \tilde{G}_\varepsilon^m)\big| > \frac{1}{3}\big(\frac{\varepsilon_{k,A}^Y}{K_{0,k}}\big)^{\frac{1}{2}}\big) \\
&\le 2\mathbb{E}_k^M|\mathcal{G}_k(\varepsilon)|\exp\big(-\frac{2M}{\frac{1}{M}\sum_{m=1}^{M}|4C_\Psi p_{0,k}^{i,m}|^2}\times\frac{\varepsilon_{k,A}^Y}{9K_{0,k}}\big) = 2\mathbb{E}_k^M|\mathcal{G}_k(\varepsilon)|\exp\big(-\frac{M\varepsilon_{k,A}^Y}{72C_\Psi^2K_{0,k}}\big)
\end{aligned}
$$

where $|\mathcal{G}_k(\varepsilon)|$ denotes the number of elements of $\mathcal{G}_k(\varepsilon)$, $\tilde{\mathbb{P}}^M$ is the conditional probability with respect to $\mathcal{F}_N^M \vee \sigma(\tilde{X}_i^{k,m} : i \geq k, m = 1, \ldots, M)$, and the final inequality follows from Hoeffding's inequality [GKKW02, Lemma A.3].
It remains to bound $|\mathcal{G}_k(\varepsilon)|$, which is equal to the product of the ε-covering numbers of $[\mathcal{S}_i^Y(\varepsilon)]_y$ $(i = k+1, \ldots, N-1)$ and $[\mathcal{S}_i^{Z,l}(\varepsilon)]_z$ $(l = 1, \ldots, q, i = k, \ldots, N-1)$ w.r.t. $\|\cdot\|_{i,M,\tilde{M}}$. From [GKKW02, inequality (9.22) pp.153], we have

$$\mathcal{N}_2(\varepsilon, [\mathcal{S}_i^Y(\varepsilon)]_y, [X_i^{1:M}, \tilde{X}_i^{k,1:M}]) \leq 3\big(2e(\frac{2C_y}{\varepsilon})^2 \log(3e(\frac{2C_y}{\varepsilon})^2)\big)^{K_{0,i}+1} \leq 3\big(\frac{16C_y^2}{3\varepsilon^2}\big)^{2(K_{0,i}+1)}$$

for any $\varepsilon \leq C_y/2$; the second inequality follows from the concavity of $\log(\cdot)$:

$$\forall x \geq 16, \quad 2e\,x\log(3e\,x) \leq 2e\,x[\log(48e) + \frac{3e\,x - 48e}{48e}] \leq e\frac{1+\log(48)}{8}x^2 \leq (\frac{4}{3}x)^2.$$

A similar inequality holds for $[\mathcal{S}_i^{Z,l}(\varepsilon)]_z$ by replacing C_y by $C_y/\sqrt{\Delta_i}$ and $K_{0,i}$ and $K_{l,i}$. Finally, we obtain

$$\begin{aligned}|\mathcal{G}_k(\varepsilon)| &\leq \prod_{i=k}^{N-1} 3^{(q+1)}\big(\frac{16C_y^2}{3\varepsilon^2}\big)^{2(K_{0,i}+1)}\big(\frac{16C_y^2}{3\varepsilon^2\Delta_i}\big)^{2\sum_{l=1}^q(K_{l,i}+1)} \\ &= \prod_{i=k}^{N-1} 3^{(q+1)}\Delta_i^{2(K_{0,i}+1)}\big(\frac{96K_{0,k}L_f^2T^{1+\theta_L}(q+1)C_y^2}{\theta_L\varepsilon_{k,A}^Y\Delta_i}\big)^{2\sum_{l=0}^q(K_{l,i}+1)}\end{aligned}$$

provided that $\varepsilon^2 = \frac{\theta_L\varepsilon_{k,A}^Y}{18K_{0,k}L_f^2T^{1+\theta_L}(q+1)} \leq \frac{C_y^2}{4} \wedge \frac{C_y^2}{4\Delta_k} \wedge \cdots \wedge \frac{C_y^2}{4\Delta_{N-1}}$. Gather different inequalities and bound $K_{0,k}^M$ by $K_{0,k}$ to derive the announced result. □

Bounds on $\mathbb{P}(B_k^{Y,M})$ *and* $\mathbb{P}(B_k^{Z,M})$. Again, we only detail the proof for $B_k^{Y,M}$. We define the set of functions

$$\begin{aligned}\mathcal{G}_k' := \big\{x \in \mathbb{R}^d \mapsto \int \big(\Phi(x_N) + \sum_{i=k}^{N-1} f_i(x_i, \psi_{i+1}(x_{i+1}), \eta_i(x_i))\Delta_i\big)\mu^x(\mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_N) \\ - y_k^R(x) : \quad \psi_i \in [\mathcal{S}_i^Y]_y, \quad \eta_{l,i} \in [\mathcal{S}_i^{Z,l}]_z\big\}\end{aligned}$$

where μ^x is the law of $(X_{k+1}^{k,x}, \ldots, X_N^{k,x})$, and $[\mathcal{S}_k^Y]_y$ and $[\mathcal{S}_k^{Z,l}]_z$ are defined in (4.54). Notice that all functions in $\mathcal{G}_k'$ are bounded by $C_\Psi + C_y$. In view of the definition (4.40), $\bar{y}_k^{R,M} - y_k^R$ belongs to the set $\mathcal{G}_k'$. Since the expectation $\mathbb{E}_N^M[|\bar{y}_k^{R,M}(X_k) - y_k^R(X_k)|^2]$ integrates only w.r.t. the law of X_k, we clearly have

$$\begin{aligned}\mathbb{P}(B_k^{Y,M}) &= \mathbb{P}\big(\varepsilon_{k,B}^Y + 2\mathbb{E}_N^M[|\bar{y}_k^{R,M}(X_k) - y_k^R(X_k)|^2] < \|\bar{y}_k^{R,M} - y_k^R\|_{k,M}^2\big) \\ &\leq \mathbb{P}\big(\exists G \in \mathcal{G}_k' : \varepsilon_{k,B}^Y + 2\mathbb{E}[G(X_k)^2] < \|G\|_{k,M}^2\big). \qquad (4.55)\end{aligned}$$

The latter probability is equal to $\mathbb{P}\big(\exists G \in \mathcal{G}'_k : \frac{\|G\|^2_{k,M} - \mathbb{E}[G(X_k)^2]}{2\varepsilon^Y_{k,B} + \|G\|^2_{k,M} + \mathbb{E}[G(X_k)^2]} > \frac{1}{3}\big)$ which can be estimated by applying Lemma 1 in Appendix B: it gives

$$\mathbb{P}(B^{Y,M}_k) \leq 4\mathbb{E}\big(\mathcal{N}_1\big(\frac{2\varepsilon^Y_{k,B}}{15}, [\mathcal{G}'_k]^2, X^{1:M}_k\big)\big) \exp\big(-\frac{M\varepsilon^Y_{k,B}}{60(C_\Psi + C_y)^2}\big),$$

where we use the short notation $[\mathcal{G}'_k]^2 := \{g^2 : g \in \mathcal{G}'_k\}$. Thus, it remains to upper bound the $\mathbf{L}_1$ ε-covering number of $[\mathcal{G}'_k]^2$ and, for this, we exhibit a particular cover. Write $\mu^x_i(\mathrm{d}x_i)$ for the law of $X^{k,x}_i$; then define the probability measure

$$\nu^M_i(\mathrm{d}x_i) := \frac{1}{M}\sum_{m=1}^M \mu_i^{X^m_k}(\mathrm{d}x_i)$$

and denote by $[\mathcal{S}^Y_i(\varepsilon')]_y$ (resp. $[\mathcal{S}^{Z,l}_i(\varepsilon')]_z$) a $\mathbf{L}_1(\nu^M_i)$ ε'-cover of $[\mathcal{S}^Y_i]_y$ (resp $[\mathcal{S}^{Z,l}_i]_z$) where $\varepsilon' = \frac{\varepsilon}{4L_f T^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi + C_y)}$. We claim that

$$\begin{aligned}[\mathcal{G}'_k(\varepsilon)]^2 :=& \big\{x \in \mathbb{R}^d \mapsto \big[\int \big(\Phi(x_N) + \sum_{i=k}^{N-1} f_i(x_i, g^Y_{i+1}(x_{i+1}), g^Z_i(x_i))\Delta_i\big)\mu^x(\mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_N) \\ &- y^R_k(x)\big]^2 : \quad g^Y_i \in [\mathcal{S}^Y_i(\varepsilon')]_y, \quad g^Z_{l,i} \in [\mathcal{S}^{Z,l}_i(\varepsilon')]_z\big\}\end{aligned}$$

ε-covers $[\mathcal{G}'_k]^2$ in the $\mathbf{L}_1$-norm w.r.t. the sample $X^{1:M}_k$. Indeed, for any $G \in \mathcal{G}'_k$ there exists $G_\varepsilon \in \mathcal{G}'_k(\varepsilon)$ such that

$$\begin{aligned}&\frac{1}{M}\sum_{m=1}^M |G(X^m_k)^2 - G_\varepsilon(X^m_k)^2| \\ &\leq \frac{1}{M}\sum_{m=1}^M |G(X^m_k) - G_\varepsilon(X^m_k)| 2(C_\Psi + C_y) \\ &\leq \frac{2}{M}\sum_{m=1}^M \sum_{i=k}^{N-1} \Delta_i \frac{(C_\Psi + C_y)L_f}{(T-t_i)^{\frac{1-\theta_L}{2}}} \\ &\qquad \times \int \big\{|\psi_{i+1}(x_{i+1}) - g^Y_{i+1}(x_{i+1})| + |\eta_i(x_i) - g^Z_i(x_i)|\big\}\mu^{X^m_k}(\mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_N) \\ &\leq 2\sum_{i=k}^{N-1} \Delta_i \frac{(C_\Psi + C_y)L_f}{(T-t_i)^{\frac{1-\theta_L}{2}}}(q+1)\varepsilon' \leq 2L_f \frac{T^{\frac{1+\theta_L}{2}}}{\frac{1+\theta_L}{2}}(q+1)(C_\Psi + C_y)\varepsilon' \leq \varepsilon,\end{aligned}$$

where we have used $(\mathbf{A_F})$. Furthermore, following the method for the bound of $\mathbb{P}(A^{Y,M}_k)$, we derive (for any $\varepsilon' \leq C_y/2$)

$$\mathcal{N}_1(\varepsilon', [\mathcal{S}^Y_i]_y, \nu^M_i) \leq 3\big(2e\frac{2C_y}{\varepsilon'}\log(3e\frac{2C_y}{\varepsilon'})\big)^{K_{0,i}+1} \leq 3\big(\frac{22C_y}{5\varepsilon'}\big)^{2(K_{0,i}+1)} \tag{4.56}$$

using $2e\,x\log(3e\,x) \le (\frac{11x}{5})^2$ for $x \ge 4$. Similarly, $\mathcal{N}_1(\varepsilon', [\mathcal{S}_i^{Z,l}]_z, \nu_i^M) \le 3\big(\frac{22C_y}{5\varepsilon'\sqrt{\Delta_i}}\big)^{2(K_{l,i}+1)}$ for $\varepsilon' \le C_y/(2\sqrt{\Delta_i})$. Finally, we obtain

$$\begin{aligned}\mathcal{N}_1\big(\frac{2\varepsilon_{k,B}^Y}{15}, [\mathcal{G}_k']^2, X_k^{1:M}\big) &\le \prod_{i=k}^{N-1} 3^{(q+1)}\big(\frac{22C_y}{5\varepsilon'}\big)^{2(K_{0,i}+1)}\big(\frac{22C_y}{5\varepsilon'\sqrt{\Delta_i}}\big)^{2\sum_{l=1}^q(K_{l,i}+1)}\\ &= \prod_{i=k}^{N-1} 3^{(q+1)}\Delta_i^{K_{0,i}+1}\big(\frac{132L_fT^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi+C_y)C_y}{\varepsilon_{k,B}^Y\sqrt{\Delta_i}}\big)^{2\sum_{l=0}^q(K_{l,i}+1)}\end{aligned}$$

if $\varepsilon' = \frac{\varepsilon_{k,B}^Y}{30L_fT^{\frac{1+\theta_L}{2}}(q+1)(C_\Psi+C_y)} \le \frac{C_y}{2} \wedge \frac{C_y}{2\sqrt{\Delta_k}} \wedge \cdots \wedge \frac{C_y}{2\sqrt{\Delta_{N-1}}}$. □

Bounds on $\mathbb{P}(C_k^{Y,M})$ *and* $\mathbb{P}(C_k^{Z,M})$. We detail the estimate only for $\mathbb{P}(C_k^{Y,M})$. Define $\mathcal{G}_k'' := \{x \mapsto g(x) - y_k^R(x) : g \in [\mathcal{S}_k^Y]_y\}$. The elements of $\mathcal{G}_k''$ are absolutely bounded by $2C_y$. Similarly to (4.55), observe that

$$\begin{aligned}\mathbb{P}(C_k^{Y,M}) &\le \mathbb{P}\big(\exists G \in \mathcal{G}_k'' : \mathbb{E}[G(X_k)^2] > \frac{2}{M}\sum_{m=1}^M G(X_k^m)^2 + \varepsilon_{k,C}^Y\big)\\ &= \mathbb{P}\big(\exists G \in \mathcal{G}_k'' : \frac{\mathbb{E}[G(X_k)^2] - \|G\|_{k,M}^2}{2\varepsilon_{k,C}^Y + \|G\|_{k,M}^2 + \mathbb{E}[G(X_k)^2]} > \frac{1}{3}\big)\\ &\le 4\mathbb{E}\big(\mathcal{N}_1\big(\frac{\varepsilon_{k,C}^Y}{12}, [\mathcal{G}_k'']^2, X_k^{1:M}\big)\big)\exp\big(-\frac{M\varepsilon_{k,C}^Y}{507C_y^2}\big)\end{aligned}$$

where the second inequality follows from the second relation of Lemma 1 in Appendix B. Observe that $\mathcal{N}_1\big(\frac{\varepsilon_{k,C}^Y}{12}, [\mathcal{G}_k'']^2, X_k^{1:M}\big) \le \mathcal{N}_1\big(\frac{\varepsilon_{k,C}^Y}{48C_y}, [\mathcal{S}_k^Y]_y, X_k^{1:M}\big)$. (4.56) is valid also for the empirical measure associated to $X_k^{1:M}$ and we obtain

$$\mathbb{P}(C_k^{Y,M}) \le 12\big(\frac{1056C_y^2}{5\varepsilon_{k,C}^Y}\big)^{2(K_{0,k}+1)}\exp\big(-\frac{M\varepsilon_{k,C}^Y}{507C_y^2}\big)$$

for $\frac{\varepsilon_{k,C}^Y}{48C_y} \le \frac{C_y}{2}$.

Appendix A. SVD-optimal coefficients

Let us study the set of coefficients $\alpha \in \mathbb{R}^{K_{l,k}}$ minimizing $\|\alpha\cdot p_{l,k} - S\|_{k,M}^2$ using the Singular Value Decomposition (see [GL96, Theorem 2.5.2 and Theorem 5.5.1]). The SVD of the $M \times K_{l,k}$-matrix $P_{l,k} = (p_{l,k}^{i,m})_{m,i}$ (with $M \ge K_{l,k}$) writes

$$P_{l,k} = UP_{l,k}'V^\top \quad \text{with} \quad P_{l,k}' = \begin{pmatrix} \sigma_1 & & 0\\ 0 & \ddots & \sigma_{K_{l,k}}\\ 0 & \cdots & 0\end{pmatrix}$$

where U and V are two unitary matrices respectively of size $M \times M$ and $K_{l,k} \times K_{l,k}$, and $\sigma_1 \geq \cdots \geq \sigma_{K_{l,k}} \geq 0$. If $P_{l,k}$ is a full rank matrix ($\sigma_{K_{l,k}} > 0$), the set of minimizers of $\|\alpha \cdot p_{l,k} - S\|_{k,M}^2$ reduces to a single element, whereas there are infinitely many minimizers for $\mathrm{rank}(P_{l,k}) < K_{l,k}$. Our choice of SVD-optimal solution consists of taking the element with minimal Euclidean norm which is given by

$$\alpha^\star = V \begin{pmatrix} \cdots \\ \mathbf{1}_{\sigma_i > 0} \frac{(U^\top S)_i}{\sigma_i} \\ \cdots \end{pmatrix}. \tag{A.1}$$

Proof. (of Proposition 4.1) The linearity property is clear from (A.1). The Pythagoras decomposition yields, for any $\alpha \in \mathbb{R}^M$, that $\|\alpha \cdot p_{l,k} - S\|_{k,M}^2 = \|\alpha^\star \cdot p_{l,k} - S\|_{k,M}^2 + \|(\alpha^\star - \alpha) \cdot p_{l,k}\|_{k,M}^2$. Taking $\alpha = 0$ gives statement ii). To prove iii), it is enough to use the linear relation (A.1) and to observe that $U, V, (\sigma_i)_i$ are $\mathcal{Q}$-measurable. □

Appendix B. Upper bound of a deviation probability, uniform over a class of functions

For the definition of the covering number $\mathcal{N}_1(\dots)$ used below, we refer to the notation of Subsection 4.5.2.

Lemma 1. *Let $\mathcal{G}$ be a countable set of functions $g : \mathbb{R}^d \mapsto [0, B]$ with $B > 0$. Let $X, X^1, \dots, X^M$ ($M \geq 1$) be i.i.d. $\mathbb{R}^d$ valued random variables. For any $\alpha > 0$ and $\varepsilon \in (0, 1)$ one has*

$$\mathbb{P}\big(\sup_{g \in \mathcal{G}} \frac{\frac{1}{M}\sum_{m=1}^M g(X^m) - \mathbb{E}[g(X)]}{\alpha + \frac{1}{M}\sum_{m=1}^M g(X^m) + \mathbb{E}[g(X)]} > \varepsilon\big) \leq 4\mathbb{E}\big(\mathcal{N}_1\big(\frac{\alpha\varepsilon}{5}, \mathcal{G}, X^{1:M}\big)\big) \exp\big(-\frac{3\varepsilon^2 \alpha M}{40B}\big),$$

$$\mathbb{P}\big(\sup_{g \in \mathcal{G}} \frac{\mathbb{E}[g(X)] - \frac{1}{M}\sum_{m=1}^M g(X^m)}{\alpha + \frac{1}{M}\sum_{m=1}^M g(X^m) + \mathbb{E}[g(X)]} > \varepsilon\big) \leq 4\mathbb{E}\big(\mathcal{N}_1\big(\frac{\alpha\varepsilon}{8}, \mathcal{G}, X^{1:M}\big)\big) \exp\big(-\frac{6\varepsilon^2 \alpha M}{169B}\big).$$

For the proof, see [GT11].

References

[BD07] C. Bender and R. Denk. A forward scheme for backward SDEs. *Stochastic Processes and their Applications*, 117(12):1793–1823, 2007.

[BGM09] E. Benhamou, E. Gobet, and M. Miri. Smart expansion and fast calibration for jump diffusion. *Finance and Stochastics*, 13(4):563–589, 2009.

[BT04] B. Bouchard and N. Touzi. Discrete time approximation and Monte Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111:175–206, 2004.

[CD11] D. Crisan and F. Delarue. Sharp gradient bounds for solutions of degenerate semi-linear partial differential equations. *Submitted, available at* `http://hal.archives-ouvertes.fr/hal-00599543/fr/`, 2011.

[DG06] F. Delarue and G. Guatteri. Weak existence and uniqueness for forward-backward SDEs. *Stochastic processes and their applications*, 116(12):1712–1742, 2006.

[EPQ97] N. El Karoui, S.G. Peng, and M.C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[GGG11] C. Geiss, S. Geiss, and E. Gobet. Generalized fractional smoothness and L_p-variation of BSDEs with non-Lipschitz terminal condition. *To appear in Stochastic Processes and their Applications*, 2011.

[GKKW02] L. Gyorfi, M. Kohler, A. Krzyzak, and H. Walk. *A distribution-free theory of nonparametric regression*. Springer Series in Statistics, 2002.

[GL96] G. Golub and C.F. Van Loan. *Matrix computations*. Baltimore, MD: The Johns Hopkins Univ. Press. xxvii, 694 p. , third edition, 1996.

[GL07] E. Gobet and C. Labart. Error expansion for the discretization of backward stochastic differential equations. *Stochastic Processes and their Applications*, 117(7):803 – 829, 2007.

[GL10] E. Gobet and C. Labart. Solving BSDE with adaptive control variate. *SIAM Numerical Analysis*, 48(1):257–277, 2010.

[GLW05] E. Gobet, J.P. Lemor, and X. Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Annals of Applied Probability*, 15(3):2172–2202, 2005.

[GM10] E. Gobet and A. Makhlouf. L_2-time regularity of BSDEs with irregular terminal functions. *Stochastic Processes and their Applications*, 120:1105–1132, 2010.

[GT11] E. Gobet and P. Turkedjiev. Complementary proofs to "Approximation of discrete BSDE using least-squares regression". *Available at* `http://hal.archives-ouvertes.fr/hal-00642656/fr/`, 2011.

[LGW06] J.P. Lemor, E. Gobet, and X. Warin. Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli*, 12(5):889–916, 2006.

[LS01] F. Longstaff and E.S. Schwartz. Valuing American options by simulation: A simple least squares approach. *The Review of Financial Studies*, 14:113–147, 2001.

[Mos10] T. Moseler. *A Picard-type iteration for backward stochastic differential equations: convergence and importance sampling*. PhD thesis, Fachbereich Mathematik und Statistik der Universität Konstanz, 2010.

[Pol84] D. Pollard. *Convergence of stochastic processes*. Springer Series in Statistics. Springer-Verlag, New York, 1984.

[Ric10] A. Richou. *Étude théorique et numérique des équations différentielles stochastiques rétrogrades*. PhD thesis, Université Rennes 1, 2010.

[Tib96] R. Tibshirani. Regression shrinkage and selection via the Lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, 58(1):267–288, 1996.

[Tur12] P. Turkedjiev. *In preparation*. PhD thesis, Humboldt Universität zu Berlin, 2012.

[Zha04] J. Zhang. A numerical scheme for BSDEs. *Ann. Appl. Probab.*, 14(1):459–488, 2004.