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Christophe Sabot, Pierre Tarrès

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EDGE-REINFORCED RANDOM WALK, VERTEX-REINFORCED JUMP PROCESS AND THE SUPERSYMMETRIC HYPERBOLIC SIGMA MODEL

CHRISTOPHE SABOT AND PIERRE TARRES

ABSTRACT. Edge-reinforced random walk (ERRW), introduced by Coppersmith and Diaconis in 1986 [5], is a random process that takes values in the vertex set of a graph G , which is more likely to cross edges it has visited before. We show that it can be interpreted as an annealed version of the Vertex-reinforced jump process (VRJP), conceived by Werner and first studied by Davis and Volkov [7, 8], a continuous-time process favouring sites with more local time. We calculate, for any finite graph G , the limiting measure of the centred occupation time measure of VRJP, and interpret it as a supersymmetric hyperbolic sigma model in quantum field theory [13]. This enables us to deduce that VRJP is recurrent in any dimension for large reinforcement, using a localisation result of Disertori and Spencer [12].

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(G; \sim)$ be a nonoriented locally finite graph, and denote by $V(G)$ (resp. $E(G)$) its set of vertices (resp. edges). Let $(a_e)_{e \in E(G)}$ be a sequence of positive initial weights associated to each edge $e \in E(G)$.

Let $(X_n)_{n \in \mathbb{N}}$ be a random process that takes values in the set of vertices of G , and let $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ be the filtration of its past. For any $e \in E(G)$, $n \in \mathbb{N} \cup \{\infty\}$, let

$$(1.1) \quad Z_n(e) = \sum_{k=1}^n \mathbb{1}_{\{\{X_{k-1}, X_k\} = e\}} + a_e$$

be the number of crosses of e up to time n , plus the initial weight a_e .

Then $(X_n)_{n \in \mathbb{N}}$ is called Edge Reinforced Random Walk (ERRW) with starting point $x_0 \in G$ and weights $(a_e)_{e \in E(G)}$, if $X_0 = x_0$ and, for all $n \in \mathbb{N}$,

$$(1.2) \quad \mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{j \sim X_n} \frac{Z_n(\{X_n, j\})}{\sum_{k \sim X_n} Z_n(\{X_n, k\})}.$$

The Edge Reinforced Random Walk was introduced in 1986 by Diaconis [5]; on finite graphs it is a mixture of reversible Markov chains, and the mixing measure can be determined explicitly ([9], see also [14, 25]) which has applications in Bayesian statistics [11].

On infinite graphs, the research has so far focused on recurrence/transience criteria. On acyclic or directed graphs, the walk can be seen as a random walk in an *independent* random environment [23], and a recurrence/transience phase transition is observed on trees [2, 15, 23, 28]. In the case of infinite graphs with cycles, recurrence criteria and

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asymptotic estimates can be obtained on graphs of the form $\mathbb{Z} \times G$, G finite graph, and on a certain two-dimensional graph [18, 20, 22, 21, 26]. Recurrence or transience on \mathbb{Z}^k , $k \geq 2$, is still unresolved.

Also, this original ERRW model [5] has triggered a number of similar models of self-organization and learning behaviour; see detailed surveys by Davis [6], Merkl and Rolles [19], Pemantle [24], Tarrès [29] and Tóth [30], with different perspectives on the topic.

Our first result relates the ERRW to the Vertex-Jump Reinforced Process (VRJP), conceived by Werner and studied by Davis and Volkov [7, 8], Collevchio [3, 4] and Basdevant and Singh [1].

We call VRJP with weights $(W_e)_{e \in E(G)}$ a continuous-time process $(Y_t)_{t \geq 0}$ on $V(G)$, starting at time 0 at some vertex $x_0 \in V(G)$ and such that, if Y is at a vertex $x \in V(G)$ at time t , then, conditionally on $(Y_s, s \leq t)$, the process jumps to a neighbour y of x at rate $W_{\{x,y\}}L_y(t)$, where

$$L_y(t) := 1 + \int_0^t \mathbb{1}_{\{Y_s=y\}} ds.$$

The main results of the paper are the following. In Section 2, Theorem 1, we express the ERRW as the annealed version of the VRJP with independent gamma random conductances. Section 3 is dedicated to showing, in Theorem 2, that the VRJP is a mixture of time-changed Markov jump processes, with a computation the mixing law. In Section 6, we relate that mixing law with the supersymmetric hyperbolic sigma model introduced by Disertori, Spencer and Zirnbauer in [13] and prove recurrence of VRJP in any dimension for large reinforcement (cf Corollary 1), using a localization result in [12].

2. FROM ERRW TO VRJP.

It is convenient here to consider a time changed version of (Y_s) : consider the positive continuous additive functional of (Y_s)

$$A(s) = \int_0^s \frac{1}{L_{Y_u}(u)} du = \sum_{x \in V} \log(L_x(s)),$$

and the time changed process

$$X_t = Y_{A^{-1}(t)}.$$

Let $(T_x(t))$ be the local time of the process (X_t)

$$T_x(t) = \int_0^t \mathbb{1}_{\{X_u=x\}} du.$$

Lemma 1. *The inverse functional A^{-1} is given by*

$$A^{-1}(t) = \int_0^t e^{T_{X_u}(u)} du.$$

The law of the process X_t is described by the following: if at time t the process X_t is at the position i , then it jumps to a neighbor j of i at rate

$$W_{i,j} e^{T_i(t) + T_j(t)}.$$

Proof. First note that

$$T_x(A(s)) = \log(L_x(s)),$$

since

$$(T_x(A(s)))' = A'(s) \mathbb{1}_{\{X_{A(s)}=x\}} = \frac{1}{L_{Y_s(s)}} \mathbb{1}_{\{Y_s=x\}}.$$

Hence,

$$(A^{-1}(t))' = \frac{1}{A'(A^{-1}(t))} = L_{X_t}(A^{-1}(t)) = e^{T_{X_t}(t)},$$

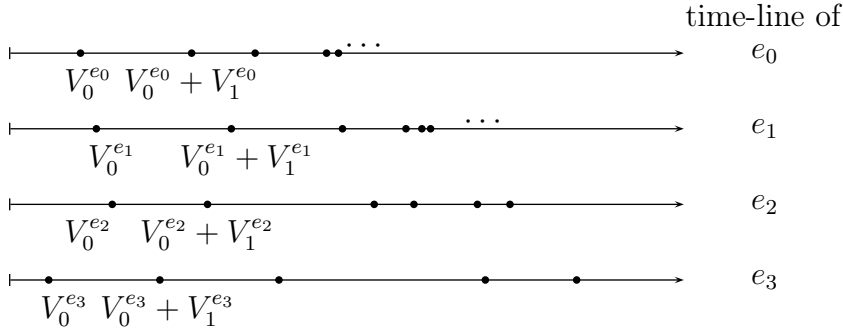
which yields the expression for A^{-1} . It remains to prove the last assertion:

$$\begin{aligned} \mathbb{P}(X_{t+dt} = j | \mathcal{F}_t) &= \mathbb{P}(Y_{A^{-1}(t+dt)} = j | \mathcal{F}_t) \\ &= W_{X_t, j} (A^{-1})'(t) L_j(A^{-1}(t)) dt \\ &= W_{i, j} e^{T_{X_t}(t)} e^{T_j(t)} dt \end{aligned}$$

□

In order to relate ERRW to VRJP, let us first define the following process $(\tilde{X}_t)_{t \in \mathbb{R}_+}$, initially introduced by Rubin, Davis and Sellke [6, 27], which we call here continuous-time ERRW with weights $(a_e)_{e \in E(G)}$ and starting at $\tilde{X}_0 := x_0$ at time 0:

- Let $(V_k^e)_{e \in E(G), k \in \mathbb{Z}_+}$ be a collection of independent exponential random variables with $\mathbb{E}V_k^e = (a_e + k)^{-1}$.
- Each edge e has its own clock, which only runs when the process $(\tilde{X}_t)_{t \geq 0}$ is adjacent to e .
- Each time an edge e has just been crossed, and at time 0, its clock sets up an alarm at distance V_k^e if e has been crossed k times so far (V_0^e at time 0).
- Each time an edge e sounds an alarm, \tilde{X}_t crosses it instantaneously.



Let τ_n be the n -th jump time of $(\tilde{X}_t)_{t \geq 0}$, with the convention that $\tau_0 := 0$.

Lemma 2. (Davis [6], Sellke [27]) *Let $(X_n)_{n \in \mathbb{N}}$ (resp. $(\tilde{X}_t)_{t \geq 0}$) be an ERRW (resp. continuous-time ERRW) with weights $(a_e)_e \in E(G)$, starting at some vertex $x_0 \in V(G)$. Then $(\tilde{X}_{\tau_n})_{n \geq 0}$ and $(X_n)_{n \geq 0}$ have the same distribution.*

Theorem 1. *Let $(\tilde{X}_t)_{t \geq 0}$ be a continuous-time ERRW with weights $(a_e)_{e \in E(G)}$. Then there exists a sequence of independent random variables $W_e \sim \text{Gamma}(a_e, 1)$, $e \in E(G)$, such that, conditionally on $(W_e)_{e \in E(G)}$, $(\tilde{X}_t)_{t \geq 0}$ has the same law as the time modification $(X_t)_{t \geq 0}$ of the VRJP with weights $(W_e)_{e \in E(G)}$; in other words, if \tilde{X} is at vertex x at time t , it jumps to a neighbour y of x with probability $W_{x,y} e^{T_x(t) + T_y(t)}$ dt .*

In particular, the ERRW $(X_n)_{n \geq 0}$ has the same law as the annealed law of the discrete time process associated with a VRJP in random independent conductances $W_e \sim \text{Gamma}(a_e)$.

Proof. For any $e \in E(G)$, define the simple birth process $\{N_t^e, t \geq 0\}$ with initial population size a_e , by

$$N_t^e := a_e + \sup \left\{ k \in \mathbb{N} \text{ s.t. } \sum_{i=0}^{k-1} V_i^e \leq t \right\}.$$

This process is sometimes called the Yule process: by a result of D. Kendall [16] (see also [29]), there exists $W_e := \lim N_t^e e^{-t}$, with distribution $\text{Gamma}(a_e, 1)$, such that, conditionally on W_e , $\{N_{f_{W_e}^e(t)}^e, t \geq 0\}$ is a Poisson process with unit parameter, where

$$f_W(t) := \log(1 + t/W);$$

hence N_e increases between times t and $t + dt$ with probability $W_e e^t dt = (f_{W_e}^{-1})'(t) dt$.

Let us now condition on $(W_e)_{e \in E(G)}$. If \tilde{X} is at vertex x at time t , it jumps to a neighbour y of x at rate $W_{x,y} e^{T_x(t) + T_y(t)}$. \square

3. THE MIXING MEASURE OF VRJP.

Next we study VRJP. Given fixed weights $(W_e)_{e \in E(G)}$, we denote by $(X_t)_{t \geq 0}$ the time modification of the VRJP defined in the previous Section starting at site $X_0 := i_0$ at time 0 and $(T_i(t))_{i \in V}$ its local time.

It is clear from the definition that the joint process $\Theta_t = (X_t, (T_i(t))_{i \in V})$ is a time continuous Markov process on the state space $V \times \mathbb{R}^V$ with generator \tilde{L} defined on C^∞ bounded functions by

$$\tilde{L}(f)(i, T) = \left(\frac{\partial}{\partial T_i} f \right) (i, T) + L(T)(f)(i), \quad \forall (i, T) \in V \times \mathbb{R}_+^V,$$

where $L(T)$ is the generator of the jump process on V at frozen T defined for $g \in \mathbb{R}^V$:

$$L(T)(g)(i) = \sum_{j \in V} W_{i,j} e^{T_i + T_j} (g(j) - g(i)), \quad \forall i \in V.$$

We denote by $\mathbb{P}_{x_0, T}$ the law of the Markov process with generator \tilde{L} starting from the initial state (x_0, T) .

By the strong Markov property, the law of $(X_t, T(t) - T)$ under $\mathbb{P}_{(x_0, T)}$ is equal to the law of the process starting from $(x_0, 0)$ with conductances

$$W_{i,j}^T = W_{i,j} e^{T_i + T_j}.$$

For simplicity, we let $\mathbb{P}_x := \mathbb{P}_{x, 0}$.

We show, in Proposition 1, that the centred occupation times converge a.s., and calculate the limiting measure in Theorem 2 i). In Theorem 2 ii) we show that the VRJP $(Y_s)_{s \geq 0}$ (as well as $(X_t)_{t \geq 0}$) is a mixture of time-changed Markov jump processes.

This limiting measure can be interpreted as a supersymmetric hyperbolic sigma model. We are grateful to a few specialists of field theory for their advice: Denis Perrot who mentioned that the limit measure of VRJP could be related to the sigma model, and Krzysztof Gawedski who pointed out reference [13], which actually mentions

a possible link of their model with ERRW, suggested by Kozma, Heydenreich and Sznitman, cf [13] Section 1.5.

Note that when G is a tree, if the edges are for instance oriented e towards the root, letting $V_e = e^{U_{\bar{e}} - U_e}$, the random variables (V_e) are independent and are distributed according to an inverse gaussian law. This was understood in previous works on VRJP [7, 8, 3, 4, 1].

Theorem 1 and Theorem 2 enable us to retrieve, in Section 5 the limiting measure of ERRWs, computed by Coppersmith and Diaconis in [5] (see also [14]) as the annealed measure arising from Theorem 1. This explains its renormalization constant, which had remained mysterious so far.

Proposition 1. *Suppose that G is finite and set $N = |V|$. The following limits exist \mathbb{P}_{i_0} a.s.*

$$U_i = \lim_{t \rightarrow \infty} T_i(t) - \frac{t}{N},$$

for all $i \in V$.

Theorem 2. *Suppose that G is finite and set $N = |V|$*

i) Under \mathbb{P}_{i_0} , $(U_i)_{i \in V}$ has the following density distribution on $\mathcal{H}_0 = \{(u_i), \sum u_i = 0\}$

$$(3.1) \quad \frac{1}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)},$$

where

$$H(W,u) = 2 \sum_{\{i,j\} \in E(V)} W_{i,j} \sinh^2 \left(\frac{1}{2}(u_i - u_j) \right)$$

and $D(W,u)$ is any diagonal minor of the $N \times N$ matrix $M(W,u)$ with coefficients

$$m_{i,j} = \begin{cases} W_{i,j} e^{u_i + u_j} & \text{if } i \neq j \\ -\sum_{k \in V} W_{i,k} e^{u_i + u_k} & \text{if } i = j \end{cases}$$

ii) Let $(U_i)_{i \in V}$ be a random variable in \mathcal{H}_0 distributed according to (3.1). Let (Z_t) be the Markov jump process starting at i_0 and with jump rates from i to j

$$\frac{1}{2} W_{i,j} e^{U_j - U_i}.$$

Let $(l_i(t))$ be the local times of the process Z at time t . Consider the positive continuous additive functional of Z

$$B(t) = \int_0^t \frac{1}{2} \frac{1}{\sqrt{1 + l_{Z_u}(u)}} du = \sum_{i \in V} \sqrt{1 + l_i(t)},$$

and the time changed process

$$\tilde{Y}_s = Z_{B^{-1}(s)}.$$

Then the annealed law of \tilde{Y} is the law of the VRJP $(Y_s)_{s \geq 0}$ with conductances $(W_{i,j})$. In particular, the discrete process associated with (Y_s) is a mixture of reversible Markov chains with conductances $W_{i,j} e^{U_i + U_j}$.

N.B.: 1) the density distribution is with respect to the Lebesgue measure on \mathcal{H}_0 which is $\prod_{i \in V \setminus \{j_0\}} du_i$ for any choice of j_0 in V . We simply write du for any of the $\prod_{i \in V \setminus \{j_0\}} du_i$.

2) The diagonal minors of the matrix $M(W, u)$ are all equal since the sum on any line or column of the coefficients of the matrix are null.

Remark 1. *Remark that usually a result like ii) makes use of de Finetti's theorem: here, we provide a direct proof exploiting the explicit form of the density. In Section 5, we apply i) and ii) to give a new proof of Diaconis-Coppersmith formula including its de Finetti part.*

Remark 2. *The fact that (3.1) is a density is not at all obvious. Our argument is probabilistic: (3.1) is the law of the random variables (U_i) . It can also be explained directly as a consequence of supersymmetry, see (5.1) in [13].*

4. PROOF OF THE PROPOSITION 1 AND THEOREM 2

4.1. Proof of Proposition 1. We provide an interesting argument based on martingales, although there may be a more direct proof.

By a slight abuse of notation, we also use notation $L(T)$ for the $N \times N$ matrix $M(W^T, T)$ of that operator in the canonical basis. Let $\mathbb{1}$ be the $N \times N$ matrix with coefficients equal to 1, i.e. $\mathbb{1}_{i,j} = 1$ for all $i, j \in V$, and let I be the identity matrix.

Let us define, for all $T \in \mathbb{R}^V$,

$$Q(T) := \int_0^\infty \left(e^{uL(T)} - \frac{\mathbb{1}}{N} \right) du,$$

which exists since $e^{uL(T)}$ converges towards $\mathbb{1}/N$ at exponential rate.

Then $Q(T)$ is a solution of the Poisson equation for the Markov Chain $L(T)$, namely

$$L(T)Q(T) = Q(T)L(T) = I - \frac{\mathbb{1}}{N}.$$

Observe that $L(T)$ is symmetric, and thus $Q(T)$ as well.

For all $T \in \mathbb{R}^V$ and $i, j \in V$, let $E_i^T(\tau_j)$ denote the expectation of the first hitting time of site j for the continuous-time process with generator $L(T)$. Then

$$Q(T)_{i,j} = \frac{1}{N} E_i^T(\tau_j) + Q(T)_{j,j}$$

since, for all $j \in V$, $i \mapsto E_i^T(\tau_j)/N$ is a solution of the Poisson equation. As a consequence, $Q(T)_{j,j}$ is nonpositive for all j , using $\sum_{i \in V} Q(T)_{i,j} = 0$.

Let us fix $l \in V$. We want to study the asymptotics of $T_l(t) - t/N$ as $t \rightarrow \infty$:

$$\begin{aligned} T_l(t) - \frac{t}{N} &= \int_0^t \left(\mathbb{1}_{\{X_u=l\}} - \frac{1}{N} \right) du = \int_0^t (L(T(u))Q(T(u)))_{X_u,l} du \\ &= \int_0^t \tilde{L}(Q(\cdot, \cdot)_l)(X_u, T(u)) du - \int_0^t \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} du \\ (4.1) \quad &= Q(T(t))_{X_t,l} - Q(0)_{X_0,l} + M_l(t) - \int_0^t \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} du, \end{aligned}$$

where

$$M_l(t) := -Q(T(t))_{X_t,l} + Q(0)_{X_0,l} + \int_0^t \tilde{L}(Q(\cdot, \cdot)_l)(X_u, T(u)) du$$

is a martingale for all l . Recall that \tilde{L} is the generator of $(T(t), X_t)$.

The following lemma shows in particular the convergence of $Q(T(t))_{k,l}$ for all k, l , as t goes to infinity. It is a purely deterministic statement, which does not depend on the trajectory of the process X_t (as long as it only performs finitely many jumps in a finite time interval), but only on the added local time in W^T .

Lemma 3. *For all $k, l \in V$, $Q(T(t))_{k,l}$ converges as t goes to infinity, and*

$$\int_0^\infty \left| \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u, l} \right| du < \infty.$$

Proof. For all $k, l \in V$, let us compute $\frac{\partial}{\partial T_i} Q(T)_{k,l}$ for any $i, k, l \in V$: by differentiation of the Poisson equation,

$$\frac{\partial}{\partial T_i} Q(T)_{k,l} = - \left(Q(T) \left(\frac{\partial}{\partial T_i} L \right) Q(T) \right)_{k,l}.$$

Now, for any real function f on V ,

$$\frac{\partial}{\partial T_i} Lf(k) = \begin{cases} \sum_{j \sim i} W_{i,j}^T (f(j) - f(i)) & \text{if } k = i \\ W_{i,k}^T (f(i) - f(k)) & \text{if } k \sim i, k \neq i \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\frac{\partial}{\partial T_i} Lf(k) = \sum_{j \sim i} W_{i,j}^T (f(j) - f(i)) (\mathbb{1}_{\{i=k\}} - \mathbb{1}_{\{j=k\}})$$

and, therefore,

$$\begin{aligned} \frac{\partial}{\partial T_i} Q(T)_{k,l} &= \sum_{j \sim i} W_{i,j}^T (Q(T)_{k,i} - Q(T)_{k,j}) (Q(T)_{i,l} - Q(T)_{j,l}) \\ (4.2) \quad &= \sum_{j \sim i} W_{i,j}^T Q(T)_{k, \nabla_{i,j}} Q(T)_{\nabla_{i,j}, l} = \sum_{j \sim i} W_{i,j}^T Q(T)_{\nabla_{i,j}, k} Q(T)_{\nabla_{i,j}, l}, \end{aligned}$$

where we use the notation $f(\nabla_{i,j}) := f(j) - f(i)$ in the second equality, and the fact that $Q(T)$ is symmetric in the third one.

In particular, for all $l \in V$ and $t \geq 0$,

$$(4.3) \quad \frac{d}{dt} Q(T(t))_{l,l} = \frac{\partial}{\partial T_{X_t}} Q(T(t))_{l,l} = \sum_{j \sim X_t} W_{X_t, j} (Q(T(t))_{\nabla_{X_t, j}, l})^2.$$

Now recall that $Q(T(t))_{l,l}$ is nonpositive for all $t \geq 0$; therefore it must converge, and

$$\int_0^\infty \sum_{j \sim X_t} W_{X_t, j} (Q(T(t))_{\nabla_{X_t, j}, l})^2 = (Q(T(\infty)) - Q(0))_{l,l} < \infty.$$

The convergence of $Q(T(t))_{k,l}$ now follows from Cauchy-Schwarz inequality, using (4.2): for all $t \geq s$:

$$\begin{aligned} |(Q(T(t)) - Q(T(s)))_{k,l}| &= \int_s^t \sum_{j \sim X_u} W_{X_u, j}^T Q(T(u))_{\nabla_{X_u, j}, k} Q(T(u))_{\nabla_{X_u, j}, l} du \\ &\leq \sqrt{(Q(T(t)) - Q(T(s)))_{k,k}} \sqrt{(Q(T(t)) - Q(T(s)))_{l,l}} \end{aligned}$$

thus $Q(T(t))_{k,l}$ is Cauchy sequence, which converges as t goes to infinity. Now, using again Cauchy-Schwarz inequality

$$\begin{aligned}
& \int_0^\infty \left| \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u, l} \right| du \\
&= \int_0^\infty \left| \sum_{j \sim X_u} W_{X_u, j}^T Q(T(u))_{\nabla_{X_u, j}, X_u} Q(T(u))_{\nabla_{X_u, j}, l} \right| du \\
&\leq \sqrt{\sum_{k \in V} (Q(T(\infty)) - Q(T(0)))_{k, k}} \sqrt{(Q(T(\infty)) - Q(T(0)))_{l, l}},
\end{aligned}$$

which enables us to conclude. \square

We now show that $(M_l(t))_{t \geq 0}$ converges, which completes the proof: indeed, this implies that the size of the jumps in that martingale goes to 0 a.s., and therefore that $Q(T(t))_{X_t, l}$ must converge as well; then (4.1) enables us to conclude.

Let us compute the quadratic variation of the martingale $(M_l(t))_{t \geq 0}$ at time t :

$$\begin{aligned}
& \left(\frac{d}{d\varepsilon} \mathbb{E} \left((M_l(T(t+\varepsilon)) - M_l(t))^2 | \mathcal{F}_t \right) \right)_{\varepsilon=0} \\
&= \left(\frac{d}{d\varepsilon} \mathbb{E} \left((Q(T(t+\varepsilon))_{X_{t+\varepsilon}, l} - Q(T(t))_{X_t, l})^2 | \mathcal{F}_t \right) \right)_{\varepsilon=0} \\
&= R(T(t))_{X_t, l}
\end{aligned}$$

where, for all $(i, l, T) \in V \times V \times \mathbb{R}^V$, we let

$$R(T)_{i, l} := \tilde{L}(Q^2(\cdot)_{\cdot, l})(i, T) - 2Q(T)_{i, l} \tilde{L}(Q(\cdot)_{\cdot, l})(i, T);$$

here $Q^2(T)$ denotes the matrix with coefficients $(Q(T)_{i, j})^2$, rather than $Q(T)$ composed with itself. But

$$\begin{aligned}
\tilde{L}(Q^2(\cdot)_{\cdot, l})(i, T) &= 2(Q(T))_{i, l} \left(\frac{\partial}{\partial T_i} Q(T) \right)_{i, l} + (L(T)Q^2(T))_{i, l} \\
Q(T)_{i, l} \tilde{L}(Q(\cdot)_{\cdot, l})(i, T) &= (Q(T))_{i, l} \left(\frac{\partial}{\partial T_i} Q(T) \right)_{i, l} + Q(T)_{i, l} (L(T)Q(T))_{i, l},
\end{aligned}$$

so that

$$\begin{aligned}
R(T)_{i, l} &= L(T)(Q^2(T))_{i, l} - 2Q(T)_{i, l} (L(T)Q(T))_{i, l} \\
&= \sum_{j \sim i} W_{i, j}^T ((Q(T)_{j, l})^2 - (Q(T)_{i, l})^2) - 2Q(T)_{i, l} \sum_{j \sim i} W_{i, j}^T (Q(T)_{j, l} - Q(T)_{i, l}) \\
&= \sum_{j \sim i} W_{i, j}^T (Q(T)_{\nabla_{i, j}, l})^2 = \frac{\partial}{\partial T_i} Q(T)_{l, l},
\end{aligned}$$

using (4.2) in the last equality. Thus

$$\langle M_l, M_l \rangle_\infty = \int_0^\infty \frac{d}{du} Q(u)_{l, l} du = Q(T(\infty))_{l, l} - Q(0)_{l, l} \leq -Q(0)_{l, l} < \infty.$$

Therefore $(M_l(t))_{t \geq 0}$ is a martingale bounded in L^2 , which converges a.s.

Remark 3. *Once we know that $T_i(t) - t/N$ converges we know that $T(\infty) = \infty$, $Q(T(\infty))_{l, l} = 0$, and that the last inequality is in fact an equality. Hence, we have the equality $\langle M_l, M_l \rangle_\infty = -Q(0)_{l, l}$.*

4.2. **Proof of Theorem 2 i).** We consider, for $i_0 \in V$, $T \in \mathbb{R}^V$, $\lambda \in \mathcal{H}_0$

$$(4.4) \quad \Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{u_{i_0}} e^{i \langle \lambda, u \rangle} \phi(W^T, u) du,$$

where

$$(4.5) \quad \phi(W^T, U) = e^{-H(W^T, U)} \sqrt{D(W^T, U)},$$

and $W_{i,j}^T = W_{i,j} e^{T_i + T_j}$. We will prove that

$$\frac{1}{\sqrt{2\pi}^{N-1}} \Psi(i_0, T, \lambda) = \mathbb{E}_{i_0, T} (e^{i \langle \lambda, U \rangle}),$$

for all $i_0 \in V$, $T \in \mathbb{R}^V$.

Lemma 4. *The function Ψ is solution of the Feynman-Kac equation*

$$i\lambda_{i_0} \Psi(i_0, T, \lambda) + (\tilde{L}\Psi)(i_0, T, \lambda) = 0.$$

Proof. Let $\bar{T}_i = T_i - \frac{1}{N} \sum_{j \in V} T_j$. With the change of variables $\tilde{u}_i = u_i + \bar{T}_i$, we obtain

$$(4.6) \quad \Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{\tilde{u}_{i_0} - \bar{T}_{i_0}} e^{i \langle \lambda, \tilde{u} - \bar{T} \rangle} \phi(W^T, \tilde{u} - \bar{T}) d\tilde{u}$$

Remark now that $H(W^T, \tilde{u} - \bar{T}) = H(W^T, \tilde{u} - T)$ since $H(W^T, u)$ only depends on the differences $u_i - u_j$. We observe that the coefficients of the matrix $M(W^T, u)$ only contain terms of the form $W_{i,j} e^{u_i + T_i + u_j + T_j}$, hence

$$\sqrt{D(W^T, \tilde{u} - \bar{T})} = e^{\frac{N-1}{N} \sum_j T_j} \sqrt{D(W, \tilde{u})}.$$

Finally, $\langle \lambda, \bar{T} \rangle = \langle \lambda, T \rangle$ since $\lambda \in \mathcal{H}_0$. This implies that

$$(4.7) \quad \Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{\sum_j T_j} e^{\tilde{u}_{i_0} - T_{i_0}} e^{i \langle \lambda, \tilde{u} - T \rangle} e^{-H(W^T, \tilde{u} - T)} \sqrt{D(W, \tilde{u})} d\tilde{u}.$$

We have

$$\begin{aligned} & \frac{\partial}{\partial T_{i_0}} H(W^T, \tilde{u} - T) \\ &= \frac{\partial}{\partial T_{i_0}} \left(2 \sum_{\{i,j\} \in E} W_{i,j} e^{T_i + T_j} \sinh^2 \left(\frac{1}{2} (\tilde{u}_i - \tilde{u}_j - T_i + T_j) \right) \right) \\ &= 2 \sum_{j \sim i_0} W_{i_0, j} e^{T_{i_0} + T_j} \left(\sinh^2 \left(\frac{1}{2} (\tilde{u}_{i_0} - \tilde{u}_j - T_{i_0} + T_j) \right) - \frac{1}{2} \sinh(\tilde{u}_{i_0} - \tilde{u}_j - T_{i_0} + T_j) \right) \\ &= \sum_{j \sim i_0} W_{i_0, j} e^{T_{i_0} + T_j} (e^{-\tilde{u}_{i_0} + \tilde{u}_j + T_{i_0} - T_j} - 1) \\ &= e^{-(\tilde{u}_{i_0} - T_{i_0})} L(T) (e^{\tilde{u} - T})(i_0). \end{aligned}$$

Hence,

$$\begin{aligned} & -\frac{\partial}{\partial T_{i_0}} \Psi(i_0, T, \lambda) \\ &= \int_{\mathcal{H}_0} (i\lambda_{i_0} e^{\tilde{u}_{i_0} - T_{i_0}} + L(T) (e^{\tilde{u} - T})(i_0)) e^{\sum_j T_j} e^{i \langle \lambda, \tilde{u} - T \rangle} e^{-H(W^T, \tilde{u} - T)} \sqrt{D(W, \tilde{u})} d\tilde{u} \\ &= i\lambda_{i_0} \Psi(i_0, T, \lambda) + (L(T)\Psi)(i_0, T, \lambda). \end{aligned}$$

This gives

$$(\tilde{L}\Psi)(i_0, T, \lambda) = -i\lambda_{i_0}\Psi(i_0, T, \lambda).$$

□

Since Ψ is a solution of the Feynman-Kac equation we have that for all $t > 0$

$$\Psi(i_0, T, \lambda) = \mathbb{E}_{i_0, T} \left(e^{i\langle \lambda, \bar{T}(t) \rangle} \Psi(X_t, T(t), \lambda) \right),$$

where we recall that $\bar{T}_i(t) = T_i(t) - t/N$. When, t tends to infinity $T_i(t)$ tends to $+\infty$. We need first to prove that $\Psi(X_t, T(t), \lambda)$ is dominated and that \mathbb{P}_{i_0} a.s.

$$(4.8) \quad \lim_{t \rightarrow \infty} \Psi(X_t, T(t), \lambda) = \sqrt{2\pi}^{N-1}.$$

We have, denoting by \mathcal{T} the set of spanning trees of G ,

$$(4.9) \quad \begin{aligned} e^{u_{i_0}} \phi(W^T, u) &= e^{u_{i_0}} e^{-H(W^T, u)} \sqrt{\sum_{T \in \mathcal{T}} \prod_{e \in T} W_e^T e^{u_{\underline{e}} + u_{\bar{e}}}} \\ &\leq e^{N \max_{i \in V} |u_i|} e^{-\frac{1}{2} \sum_{\{i, j\} \in V} W_{i, j}^T (u_i - u_j)^2} \sqrt{D(W^T, 0)} \\ &\leq \left(\sum_{i \in V} e^{Nu_i} + e^{-Nu_i} \right) e^{-\frac{1}{2} \sum_{\{i, j\} \in V} W_{i, j}^T (u_i - u_j)^2} \sqrt{D(W^T, 0)} \end{aligned}$$

This is a gaussian integrand: for any real a and $i_0 \in V$,

$$\begin{aligned} &\int_{\mathcal{H}_0} e^{au_{i_0}} e^{-\frac{1}{2} \sum_{\{i, j\} \in V} W_{i, j}^T (u_i - u_j)^2} \sqrt{D(W^T, 0)} du \\ &= e^{-\frac{1}{2} a^2 Q(T)_{i_0, i_0}} \int e^{-\frac{1}{2} \langle U - aQ(T)_{i_0, \cdot}, L(T)(U - aQ(T)_{i_0, \cdot}) \rangle} \sqrt{D(W^T, 0)} du \\ &= e^{-\frac{1}{2} a^2 Q(T)_{i_0, i_0}} (2\pi)^{(N-1)/2}. \end{aligned}$$

where $Q(T)$ is defined at the beginning of Section 4.1. Therefore for all $i_0 \in V$, $(T_i) \in \mathbb{R}^V$

$$|\Psi(i_0, T, \lambda)| \leq 2 \sum_{i \in V} (2\pi)^{(N-1)/2} e^{-\frac{1}{2} N^2 Q(T)_{i, i}},$$

But $W^{T(t)}$ is increasing in t , hence

$$|\Psi(X_t, T(t), \lambda)| \leq 2 \sum_{i \in V} (2\pi)^{(N-1)/2} e^{-\frac{1}{2} N^2 Q(0)_{i, i}},$$

for all $t \geq 0$. Let us prove now (4.8). We have

$$\begin{aligned} &\Psi(X_t, T(t), \lambda) \\ &= \int e^{i\langle \lambda, u \rangle} e^{u_{X_t}} e^{-2 \sum_{\{i, j\} \in V} W_{i, j}^{T(t)} \sinh^2(\frac{1}{2}(u_i - u_j))} \sqrt{D(W^{T(t)}, u)} du \\ &= \int e^{i\langle \lambda, u \rangle} e^{u_{X_t}} e^{-2 \sum_{\{i, j\} \in V} e^{2t/N} W_{i, j}^{\bar{T}(t)} \sinh^2(\frac{1}{2}(u_i - u_j))} \sqrt{D(W^{\bar{T}(t)}, u)} e^{(N-1)t/N} du. \end{aligned}$$

Changing to variables $\tilde{u}_i = e^{t/N} u_i$, we deduce that $\Psi(X_t, T(t), \lambda)$ equals

$$\int e^{i\langle \lambda, e^{-t/N} \tilde{u} \rangle} e^{-t/N \tilde{u}_{X_t}} e^{-2 \sum_{\{i, j\} \in V} W_{i, j}^{\bar{T}(t)} e^{2t/N} \sinh^2(\frac{1}{2} e^{-t/N} (\tilde{u}_i - \tilde{u}_j))} \sqrt{D(W^{\bar{T}(t)}, e^{-t/N} \tilde{u})} d\tilde{u}.$$

Since $\lim_{t \rightarrow \infty} \bar{T}_i(t) = U_i$, the integrand converges pointwise to the Gaussian integrand

$$e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^U (\tilde{u}_i - \tilde{u}_j)^2} \sqrt{D(W^U, 0)},$$

whose integral is $\sqrt{2\pi}^{N-1}$. Consider $\bar{U}_i = \sup_{t \geq 0} \bar{T}_i(t)$ and $\underline{U}_i = \inf_{t \geq 0} \bar{T}_i(t)$. Proceeding as in (4.9) the integrand is dominated for all t by

$$\begin{aligned} & e^{Ne^{-t/N} \max_{i \in V} |\tilde{u}_i|} e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^{\bar{T}(t)} (\tilde{u}_i - \tilde{u}_j)^2} \sqrt{D(W^{\bar{T}(t)}, 0)} \\ & \leq \left(\sum_{i \in V} e^{N\tilde{u}_i} + e^{-N\tilde{u}_i} \right) e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^U (\tilde{u}_i - \tilde{u}_j)^2} \sqrt{D(W^U, 0)}. \end{aligned}$$

which is integrable, which yields (4.8) by dominated convergence.

4.3. Proof of Theorem 2 ii). The same change of variables as in (4.6) and (4.7), applied to $T_i = \log \lambda_i$, implies that, for any $j_0 \in V$ and $(\lambda_i)_{i \in V}$ positive reals, letting $\bar{\lambda}_i = \lambda_i / (\prod_j \lambda_j)^{1/N}$,

$$\frac{1}{\sqrt{2\pi}^{N-1}} \int_{\mathcal{H}_0} e^{u_{j_0} - \log(\bar{\lambda}_{j_0})} e^{-\frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} \lambda_i \lambda_j \left(e^{\frac{1}{2}(u_j - u_i)} \sqrt{\frac{\lambda_i}{\lambda_j}} - e^{\frac{1}{2}(u_j - u_i)} \sqrt{\frac{\lambda_j}{\lambda_i}} \right)^2} \sqrt{D(W, u)}$$

is the density of a probability measure, which we call ν^{λ, j_0} . Remark that this density can be rewritten as

$$\frac{1}{\sqrt{2\pi}^{N-1}} e^{u_{j_0} - \log(\bar{\lambda}_{j_0})} e^{-\frac{1}{2} \sum_{i,j} W_{i,j} (\lambda_i^2 e^{u_j - u_i} - \lambda_i \lambda_j)} \sqrt{D(W, u)}$$

Let (U_i) be a random variable distributed according to (3.1). Let (Z_t) be the Markov jump process starting at i_0 and with jump rates from i to j

$$\frac{1}{2} W_{i,j} e^{U_j - U_i}.$$

Let $(\mathcal{F}_t^Z)_{t \geq 0}$ be the filtration generated by Z , and let E_i^U be the quenched law of the process Z starting at i . We denote by $(l_k(t))_{k \in V}$ the vector of local times of the process Z at time t .

Let us first prove that the law of U conditioned on \mathcal{F}_t^Z is

$$(4.10) \quad \mathcal{L}(U | \mathcal{F}_t^Z) = \nu^{\lambda(t), Z_t},$$

where $\lambda_i(t) = \sqrt{1 + l_i(t)}$. Indeed, let $t > 0$: if $\tau_1, \dots, \tau_{K(t)}$ denote the jumping times of the Markov process Z_t up to time t , then for any positive test function,

$$\begin{aligned} & E_{i_0}^U \left(\psi(\tau_1, \dots, \tau_{K(t)}, Z_{\tau_1}, \dots, Z_{\tau_{K(t)}}) \right) = \\ & \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_k} \left(\prod_{l=0}^{k-1} \frac{1}{2} W_{i_l, i_{l+1}} \right) \int_{[0, t]^k} \psi((t_j), (i_j)) e^{U_{i_k} - U_{i_0}} e^{-\frac{1}{2} \sum_{l=0}^{k-1} \left(\sum_{j \sim i_l} W_{i_l, j} e^{U_j - U_{i_l}} \right) (t_{l+1} - t_l)} dt_1 \dots dt_k \end{aligned}$$

with the convention $t_{k+1} = t$. Hence, for any test function G ,

$$\begin{aligned} & \mathbb{E} (G(U) | \mathcal{F}_t^Z) \\ &= \frac{\int_{\mathcal{H}_0} G(u) e^{u z_t} e^{-H(W,u) - \frac{1}{2} \sum_{i \in V} (\sum_{j \sim i} W_{i,j} e^{u_j - u_i}) l_i(t)} \sqrt{D(W,u)} du}{\int_{\mathcal{H}_0} e^{u z_t} e^{-H(W,u) - \frac{1}{2} \sum_{i \in V} (\sum_{j \sim i} W_{i,j} e^{u_j - u_i}) l_i(t)} \sqrt{D(W,u)} du} \\ &= \frac{\sqrt{2\pi}^{-(N-1)} \int_{\mathcal{H}_0} G(u) e^{u z_t - \log \bar{\lambda}_{Z_t}} e^{-\frac{1}{2} \sum_{i,j} W_{i,j} (\lambda_i(t)^2 e^{u_j - u_i} - \lambda_i(t) \lambda_j(t))} \sqrt{D(W,u)} du}{\sqrt{2\pi}^{-(N-1)} \int_{\mathcal{H}_0} e^{u z_t - \log \bar{\lambda}_{Z_t}} e^{-\frac{1}{2} \sum_{i,j} W_{i,j} (\lambda_i(t)^2 e^{u_j - u_i} - \lambda_i(t) \lambda_j(t))} \sqrt{D(W,u)} du}; \end{aligned}$$

recall that $(\lambda_i(t))_{i \in V}$ are independent of u , and that $\lambda_i(t) = \sqrt{1 + l_i(t)}$. The denominator is 1 since it is the integral of the density of $\nu^{\lambda(t), Z_t}$. This proves (4.10).

Subsequently, by (4.10), conditioned on (\mathcal{F}_t^Z) , if the process Z is at i at time t , then it jumps to a neighbour j of i with rate

$$\frac{1}{2} W_{i,j} \mathbb{E}^{\nu^{\lambda(t), i}} (e^{U_j - U_i}) = \frac{1}{2} W_{i,j} \frac{\bar{\lambda}_j(t)}{\bar{\lambda}_i(t)} = \frac{1}{2} W_{i,j} \frac{\lambda_j(t)}{\lambda_i(t)}.$$

In order to conclude, we now compute the corresponding rate for \tilde{Y} : by definition,

$$B'(t) = \frac{1}{2} \frac{1}{\sqrt{1 + l_{Z_t}(t)}}.$$

Therefore, similarly as in the proof of Lemma 1,

$$\begin{aligned} \mathbb{P} (\tilde{Y}_{s+ds} = j | \mathcal{F}_t^Z) &= \mathbb{P} (\tilde{Z}_{B^{-1}(s+ds)} = j | \mathcal{F}_t^Z) \\ &= \frac{1}{2} W_{Y_s, j} \frac{1}{B'(B^{-1}(s))} \frac{\lambda_j(B^{-1}(s))}{\lambda_{Y_s}(B^{-1}(s))} ds \\ &= W_{Y_s, j} \lambda_j(B^{-1}(s)) ds. \end{aligned}$$

Let $(L_i(s))$ be the local time of the process \tilde{Y} . Then

$$(L_i(B(t)))' = B'(t) \mathbb{1}_{\{Y_{B(t)} = i\}} = \frac{1}{2} (1 + l_i(t))^{-\frac{1}{2}} \mathbb{1}_{\{Z_t = i\}}.$$

This implies $L_i(B(t)) = \sqrt{1 + l_i(t)} - 1$ and

$$\mathbb{P} (\tilde{Y}_{s+ds} = j | \mathcal{F}_t^Z) = W_{Y_s, j} (1 + L_j(s)) ds$$

This means that the annealed law of \tilde{Y} is the law of a VRJP with conductances $(W_{i,j})$.

5. BACK TO DIACONIS-COPPERSMITH FORMULA

It follows from de Finetti's theorem for Markov chains [10] that the law of the ERRW is a mixture of reversible Markov chains; its mixing measure was explicitly described by Coppersmith and Diaconis ([5], see also [14, 25]).

Theorems 1 and 2 enable us to retrieve this so-called Coppersmith-Diaconis formula, including its de Finetti part: they imply that the ERRW $(X_n)_{n \in \mathbb{N}}$ follows the annealed law of a reversible Markov chain in a random conductance network $x_{i,j} = W_{i,j} e^{U_i + U_j}$ where $W_e \sim \text{Gamma}(a_e)$, $e \in E$, are independent random variables and, conditioned on W , the random variables (U_i) are distributed according to the law (3.1).

Let us compute the law it induces on the random variables (x_e) . The random variable (x_e) is only significant up to a scaling factor, hence we consider a 0-homogeneous bounded measurable test function ϕ ; by Theorem 2,

$$\begin{aligned} & \mathbb{E}(\phi((x_e))) \\ &= \frac{1}{\sqrt{2\pi}^{N-1}} \int_{\mathbb{R}_+^E \times \mathcal{H}_0} \phi(x) \left(\prod_{e \in E} \frac{1}{\Gamma(a_e)} W_e^{a_e} e^{-W_e} \right) e^{u_{i_0}} \sqrt{D(W, u)} e^{-H(W, u)} \frac{dW}{W} du \end{aligned}$$

where we write $\frac{dW}{W} = \prod_{e \in E} \frac{dW_e}{W_e}$. Changing to coordinates $\bar{u}_i = u_i - u_{i_0}$ yields

$$C(a) \int_{\mathbb{R}_+^E \times \mathbb{R}^{V \setminus \{i_0\}}} \phi(x) \left(\prod_{e \in E} W_e^{a_e} e^{-W_e} \right) e^{-\sum_{i \neq i_0} \bar{u}_i} \sqrt{D(W, \bar{u})} e^{-H(W, \bar{u})} \frac{dW}{W} d\bar{u}$$

with $d\bar{u} = \prod_{i \neq i_0} d\bar{u}_i$ and $C(a) = \frac{1}{\sqrt{2\pi}^{N-1}} \left(\prod_{e \in E} \frac{1}{\Gamma(a_e)} \right)$. But

$$-\sum_{e \in E} W_e - H(W, \bar{u}) = -\frac{1}{2} \sum_{\{i, j\} \in E} W_{i, j} e^{\bar{u}_i + \bar{u}_j} (e^{-2\bar{u}_j} + e^{-2\bar{u}_i}).$$

The change of variables

$$\left((x_{i, j} = W_{i, j} e^{\bar{u}_i + \bar{u}_j})_{\{i, j\} \in E}, (v_i = e^{-2\bar{u}_i})_{i \in V \setminus \{i_0\}} \right),$$

with $v_{i_0} = 1$ implies

$$-\sum_{e \in E} W_e - H(W, \bar{u}) = -\frac{1}{2} \sum_{i \in V} v_i x_i,$$

where $x_i = \sum_{j \sim i} x_{i, j}$, and $\mathbb{E}(\phi((x_e)))$ is equal to the integral

$$C'(a) \int \phi(x) \left(\prod_{e \in E} x_e^{a_e} \right) \left(\prod_{i \in V} v_i^{(a_i+1)/2} \right) v_{i_0}^{-\frac{1}{2}} \sqrt{D(x)} e^{-\frac{1}{2} \sum_{i \in V} v_i x_i} \left(\prod_{e \in E} \frac{dx_e}{x_e} \right) \left(\prod_{i \neq i_0} \frac{dv_i}{v_i} \right),$$

with $a_i = \sum_{j \sim i} a_{i, j}$, $D(x)$ is determinant of any diagonal minor of the $N \times N$ matrix

$$m_{i, j} = \begin{cases} x_{i, j} & \text{if } i \neq j \\ -\sum_{k \sim i} x_{i, k} & \text{if } i = j \end{cases}$$

and

$$C'(a) = \frac{2^{-N+1}}{\sqrt{2\pi}^{N-1}} \left(\prod_{e \in E} \frac{1}{\Gamma(a_e)} \right).$$

Let e_0 be a fixed edge, we normalize the conductance to be 1 at e_0 by changing to variables

$$\left(\left(y_e = \frac{x_e}{x_{e_0}} \right)_{e \neq e_0}, (z_i = x_{e_0} v_i)_{i \in V} \right),$$

with $y_{e_0} = 1$. Now, observe that

$$\left(\prod_{e \in E} \frac{dx_e}{x_e} \right) \left(\prod_{i \neq i_0} \frac{dv_i}{v_i} \right) = \left(\prod_{e \in E, e \neq e_0} \frac{dy_e}{y_e} \right) \left(\prod_{i \in V} \frac{dz_i}{z_i} \right).$$

We deduce that $\mathbb{E}(\phi((x_e)))$ equals the integral

$$C(a) \int_{\mathbb{R}_+^V \times \mathbb{R}_+^{E \setminus \{e_0\}}} \phi(y) \left(\prod_{e \in E} y_e^{a_e} \right) \left(\prod_{i \in V} z_i^{a_i/2} \right) z_{i_0}^{-\frac{1}{2}} \sqrt{D(y)} e^{-\frac{1}{2} \sum_{i \in V} z_i y_i} \left(\frac{dy}{y} \right) \left(\frac{dz}{z} \right),$$

with $\frac{dy}{y} = \prod_{e \neq e_0} \frac{dy_e}{y_e}$ and $\frac{dz}{z} = \prod_{i \in V} \frac{dz_i}{z_i}$. Therefore, integrating over the variables z_i

$$\mathbb{E}(\phi((x_e))) = C'''(a) \int_{\mathbb{R}_+^{E \setminus \{e_0\}}} \phi(y) y_{i_0}^{\frac{1}{2}} \left(\frac{\prod_{e \in E} y_e^{a_e}}{\prod_{i \in V} y_i^{(a_i+1)/2}} \right) \sqrt{D(y)} \left(\frac{dy}{y} \right),$$

where

$$C'''(a) = \frac{2^{1-N-\sum_{e \in E} a_e} \Gamma(a_{i_0}/2) \prod_{i \neq i_0} \Gamma((a_i+1)/2)}{\pi^{(N-1)/2} \prod_{e \in E} \Gamma(a_e)}$$

which is Diaconis-Coppersmith formula: the extra term $(|E|-1)!$ in [14, 11] arises from the normalization of $(x_e)_{e \in E}$ on the simplex $\Delta = \{\sum x_e = 1\}$ (see Section 2.2 [11]).

6. THE SUPERSYMMETRIC HYPERBOLIC SIGMA MODEL

We first relate VRJP to the supersymmetric hyperbolic sigma model studied in Disertori, Spencer and Zirnbauer [13, 12]. For notational purposes, we restrict our attention to the d -dimensional lattice, that is, our graph is \mathbb{Z}^d with $x \sim y$ if $|x-y|_1 = 1$. Let us add a vertex δ connected to 0, that is, consider the graph with vertices $\mathbb{Z}^d \cup \{\delta\}$ with an extra edge $\{\delta, 0\}$.

Assume that we are given positive conductances on the network : in order to be closer to [13, 12], we denote by $\beta_{x,y} = W_{x,y}$ the conductances on the edge $\{x,y\}$, if $x, y \in \mathbb{Z}^d$ and $\varepsilon = W_{0,\delta}$ the conductance on the edge $\{0,\delta\}$. Note that VRJP on \mathbb{Z}^d and on $\mathbb{Z}^d \cup \{\delta\}$ are easy to compare.

For any connected finite subset $V \subseteq \mathbb{Z}^d$ containing 0, let $\mu_V^{\varepsilon,\beta}$ be a generalization of the measure studied in [12] (see (1.1)-(1.7) in that paper), namely

$$\begin{aligned} d\mu_V^{\varepsilon,\beta}(t) &:= \prod_{i \in V} \frac{dt_j}{\sqrt{2\pi}} e^{-\sum_{k \in V} t_k} e^{-F_V(\nabla t)} e^{-M_V^\varepsilon(t)} \sqrt{\det A_V^\varepsilon} \\ &= \prod_{i \in V} \frac{dt_j}{\sqrt{2\pi}} e^{-F_V(\nabla t)} e^{-M_V^\varepsilon(t)} \sqrt{\det D_V^\varepsilon} \end{aligned}$$

where $A_V^\varepsilon = A^\varepsilon$ and $D_V^\varepsilon = D^\varepsilon$ are defined by, for all $i, j \in V$, by

$$A_{ij}^\varepsilon = e^{t_i} D_{ij}^\varepsilon e^{t_j} = \begin{cases} 0 & |i-j| > 1 \\ -\beta_{ij} e^{t_i+t_j} & |i-j| = 1 \\ \sum_{j \sim i} \beta_{ij} e^{t_i+t_j} + \varepsilon e^{t_0} \mathbb{1}_{i=0} & i = j \end{cases}$$

$$F_V(\nabla t) := \sum_{i,j \in \Lambda, \{i,j\} \in E} \beta_{ij} (\cosh(t_i - t_j) - 1)$$

$$M_V^\varepsilon(t) := \varepsilon (\cosh t_0 - 1).$$

The fact that $\mu_V^{\varepsilon,\beta}$ is a probability measure can be seen as a consequence of supersymmetry (see (5.1) in [13]).

Set $\tilde{V} = V \cup \{\delta\}$. Let us again use notation $(U_i)_{i \in \tilde{V}}$ for the limiting centred occupation times of VRJP on \tilde{V} starting at δ , and consider the change of variables,

from \mathcal{H}_0 into \mathbb{R}^V , which maps u_i to $t_i := u_i - u_\delta$. Then, by Theorem 2, for any test function ϕ ,

$$\begin{aligned} \mathbb{E}_\delta^W(\phi(U - U_\delta)) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathcal{H}_0} \phi(u - u_\delta) e^{u_\delta} e^{-H(W,u)} \sqrt{D(W,u)} du \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\mathcal{H}_0} \phi(t) e^{-\sum_{i \neq \delta} t_i} e^{-H(W,t)} \sqrt{D(W,t)} \left(\prod_{i \neq \delta} dt_i \right) \\ &= \mu_V^{\varepsilon, \beta}(\phi(t)), \end{aligned}$$

which means that $U - U_\delta$ is distributed according to $\mu_V^{\varepsilon, \beta}$. Indeed, let ι be the canonical injection $\mathbb{R}^V \rightarrow \mathbb{R}^{\tilde{V}}$; then A_V^ε is the restriction to $V \times V$ of the matrix $M(W, \iota(t))$ (which is on $\tilde{V} \times \tilde{V}$) (so that $\det A_V^\varepsilon = D(W, \iota(t))$), and $F_V(\nabla t) + M_V^\varepsilon(t) = H(W, \iota(t))$.

Set, for all $\beta > 0$,

$$I_\beta := \sqrt{\beta} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-\beta(\cosh t - 1)},$$

which is strictly increasing in β .

Let β_c^d be defined as the unique solution to the equation

$$I_{\beta_c^d} e^{\beta_c^d(2d-2)}(2d-1) = 1$$

for all $d > 1$, $\beta_c^d := \infty$ if $d = 1$.

If the parameters β_e are constant over all edges e , equal to β , then Theorem 2 in [12] readily implies that VRJP over any graph \mathbb{Z}^d is recurrent for $\beta < \beta_c^d$ (i.e. for large reinforcement).

Theorem 3 (Disertori, Spencer [12], Theorem 2). *Suppose that $\beta_e = \beta$ for all e , and that $0 < \beta < \beta_c^d$. Then there exists a constant $C_0 > 0$ such that, for all $x \in \mathbb{Z}^d$,*

$$\mu_\Lambda^{\varepsilon, \beta}(e^{t_x/2}) \leq C_0 [I_\beta e^{\beta(2d-2)}(2d-1)]^{|x|},$$

the inequality being independent of the size of the finite connected subset $\Lambda \subseteq \mathbb{Z}^d$ containing 0.

Corollary 1. *For $0 < \beta < \beta_c^d$, the VRJP on \mathbb{Z}^d with constant conductance β comes back to 0 infinitely often.*

Proof. We consider the VRJP on \mathbb{Z}^d with an extra point δ connected to 0 only, and conductances $W_{x,y} = \beta$ and $W_{0,\delta} = 1$. The recurrence of this process is equivalent to the recurrence of the VRJP on \mathbb{Z}^d itself.

On finite size box V , we know from Theorem 2 that $(Y_n)_{n \in \mathbb{N}}$, the discrete-time process associated with $(Y_s)_{s \geq 0}$, is a mixture of reversible Markov chains with conductances $c_{x,y} = \beta e^{t_x + t_y}$, where $(t_x)_{x \in V}$ has law $\mu_V^{1, \beta}$.

Now Theorem 3 implies that $\mu_V^{1, \beta}((c_e/c_{\delta,0})^{1/4})$ decreases exponentially with the distance from e to 0: indeed, by Cauchy-Schwarz inequality,

$$\begin{aligned} \mu_V^{1, \beta}((c_{x,y}/c_{\delta,0})^{1/4}) &\leq \left[\mu_V^{1, \beta}(e^{t_x/2}) \mu_V^{1, \beta}(e^{(t_y - t_0)/2}) \right]^{1/2} \\ &\leq C \left[\mu_V^{1, \beta}(e^{t_x/2}) \mu_V^{1, \beta}(e^{\frac{1}{2}(\cosh(t_0) - 1)} e^{t_y/2}) \right]^{1/2} \\ &\leq 2C \left[\mu_V^{1, \beta}(e^{t_x/2}) \mu_V^{1/2, \beta}(e^{t_y/2}) \right]^{1/2} \end{aligned}$$

for some $C > 0$ such that $|z| \leq 4 \log C + \cosh(z) - 1$.

This implies, similarly as in [22] Section 9, that the probability to leave the ball of radius n before coming back to 0 is exponentially decreasing. A simple argument (see [17]) then ensures that (Y_n) comes back infinitely often to 0. \square

By Theorems 1 and 2, ERRW corresponds to the case where $(\beta_e)_{e \in E(G)}$ are independent random variables with Gamma(a) distribution for some parameter $a > 0$: it is natural to try to infer a similar result for a small enough.

This requires an extension of Theorem 3 for general weights $(\beta_e)_{e \in E(G)}$: we propose one in the following Proposition 2, in the same line of proof as in [12].

Proposition 2. *Let Γ_x be the set of non-intersecting paths from 0 to a vertex x in V . For all $x \in V$,*

$$\mu_V^{\varepsilon, \beta} (e^{t_x/2}) \leq \sqrt{\varepsilon} \sum_{\gamma \in \Gamma_x} e^{\sum_{\{i,j\} \in E, i \in \Lambda_\gamma, j \notin \Lambda_\gamma} \beta_{ij}} I_\varepsilon \prod_{e \in \gamma} I_{\beta_e}.$$

where Λ_γ and Λ_γ^c are respectively the set of vertices in the path and its complement.

We can then sum up the upper bound from this result over the random variables $(\beta_e)_{e \in E(G)}$, assuming they are random i.i.d. and $\mathbb{E}(e^{\beta_{x,y}}) < \infty$: this implies recurrence of VRJP in the i.i.d. random environment $\beta_e \sim \text{Gamma}(a, \mu)$ for any $\mu > 1$ and a small enough, but does not cover the case $\mu = 1$ of the ERRW.

Proof. (Proposition 2) Let us define, for all $\Lambda \subseteq V$,

$$d\nu_\Lambda^{\varepsilon, \beta}(t) := \prod_{i \in \Lambda} \frac{dt_j}{\sqrt{2\pi}} e^{-F_\Lambda(\nabla t)} e^{-M_\Lambda^\varepsilon(t)},$$

which is not a probability measure in general.

Let Γ_x be the set of non-intersecting paths from 0 to x . For notational purposes, any element γ in Γ_x is defined here as the set of non-oriented edges in the path. We let Λ_γ and Λ_γ^c be respectively the set of vertices in the path and its complement.

First observe that, similarly to (3.1)-(3.4) in [12], Lemma 2 in that paper implies

$$\det D^\varepsilon = \varepsilon e^{-t_x} \sum_{\gamma \in \Gamma_x} \prod_{e \in \gamma} \beta_e \det D_{\Lambda_\gamma^c}^{\tilde{\varepsilon}},$$

where

$$\tilde{\varepsilon}_i := \varepsilon_i + \sum_{k \in \Lambda_\gamma, \{i,k\} \in E} \beta_{\{i,k\}} e^{t_k}.$$

Let us define, similarly as in (2.12) and (2.14) in [12],

$$\begin{aligned} Z_{\Lambda_\gamma^c}^\gamma(t_\gamma) &:= \nu_{\Lambda_\gamma^c}^{\tilde{\varepsilon}} \left(\sqrt{\det D_{\Lambda_\gamma^c}^{\tilde{\varepsilon}}} e^{-F_{\partial\gamma}(\nabla t)} \right) \\ F_{\partial\gamma}(\nabla t) &:= \sum_{\{j,k\} \in E, j \in \Lambda_\gamma, k \notin \Lambda_\gamma} \beta_{ij} (\cosh(t_j - t_k) - 1). \end{aligned}$$

Now

$$(6.1) \quad \begin{aligned} \mu_V^{\varepsilon,\beta} (e^{t_x/2}) &= \nu_V^{\varepsilon,\beta} \left(\sqrt{\det D_V^\varepsilon e^{t_x}} \right) = \sqrt{\varepsilon} \nu_V^{\varepsilon,\beta} \left(\sqrt{\sum_{\gamma \in \Gamma_x} \prod_{e \in \gamma} \beta_e \det D_{\Lambda_\gamma}^\varepsilon} \right) \\ &\leq \sqrt{\varepsilon} \sum_{\gamma \in \Gamma_x} \prod_{e \in \gamma} \sqrt{\beta_e} \nu_{\Lambda_\gamma}^\varepsilon \left(Z_{\Lambda_\gamma}^\gamma(t_\gamma) \right), \end{aligned}$$

where we use in the second equality that $\det_{\Lambda_\gamma} D_V^\varepsilon = \det D_{\Lambda_\gamma}^\varepsilon$, and in the inequality that, for all $\gamma \in \Gamma_x$,

$$d\nu_V^\varepsilon(t) = d\nu_{\Lambda_\gamma}^\varepsilon(t) d\nu_{\Lambda_\gamma}^\varepsilon(t) e^{-F_{\partial\gamma}(\nabla t)}.$$

Now $Z_{\Lambda_\gamma}^\gamma(t_\gamma)$ approximates the normalization constant

$$Z_{\Lambda_\gamma}^\varepsilon = 1 = \nu_{\Lambda_\gamma}^\varepsilon \left(\sqrt{\det D_{\Lambda_\gamma}^\varepsilon} \right),$$

with the difference that, in the former, the measure considered is ν^ε instead of ν^ε , and there is a multiplicative factor $e^{-F_{\partial\gamma}(\nabla t)}$ (which is helpful, since we aim to upper bound $Z_{\Lambda_\gamma}^\gamma(t_\gamma)$). The following lemma, which adapts Lemma 3 [12], provides an upper bound of $Z_{\Lambda_\gamma}^\gamma(t_\gamma)$; its proof is very similar, and is left to the reader.

Lemma 5. *For any configuration of $\{t_k$ s.t. $k \in \Lambda_\gamma\}$, $Z_{\Lambda_\gamma}^\gamma(t_\gamma) \leq e^{\sum_{\{i,j\} \in E, i \in \Lambda_\gamma, j \notin \Lambda_\gamma} \beta_{ij}}$.*

We combine (6.1) and Lemma 5 to conclude. \square

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UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN, CNRS UMR 5208, 43, BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: `sabot@math.univ-lyon1.fr`

UNIVERSITÉ PAUL SABATIER, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, CNRS UMR 5219, 118 ROUTE DE NARBONNE, TOULOUSE CEDEX 9, FRANCE

E-mail address: `pierre.tarres@math.univ-toulouse.fr`