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# Non-holonomic quantum control

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**Abstract.** In this paper, we review an open-loop evolution control method, called the nonholonomic control, based on the alternate application of only two physical perturbations for timings which play the role of adjustable control parameters. We present the algorithm which allows one to explicitly compute the pulse sequence achieving any arbitrarily prescribed unitary evolution in a nonholonomic system, *i.e.* a system subject to two physical perturbations which, together with the natural Hamiltonian of the system, span the entire Lie algebra  $\mathfrak{u}(N)$ . We moreover expose two extensions of our method to open quantum systems which, respectively, aim at preserving the information stored in the system and safely processing this information. The first is based on a generalization of the quantum Zeno effect, while the second is inspired by decoupling pulse techniques. The most important feature of the methods presented here is their universality: they indeed do not rely on any specific assumption on the system, which in particular is not bound to be a collection of two-level systems, or the error model considered, as is usually the case in the literature. Numerical and physical applications of our techniques are also provided.

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## 1. Introduction

Stimulated by the potential applications it may have in the future – *e.g.* in quantum chemistry, for controlling molecular dynamics and reactions, or in the very active field of quantum information processing [1], and greatly inspired by the already well-established mathematical framework of classical control, an intense theoretical activity has been developing on quantum control within the last few decades [2, 3]. Two main types of control scenarios exist, commonly referred to as open- and closed-loop controls. The latter implies that the system to be controlled is frequently measured and further

processed in a way which depends on the measurement outcome [4, 5, 6, 7]. Open-loop strategies merely consist in perturbing the system in a deterministic way without resorting to any information feedback.

In the context of open-loop quantum control, there exists a number of different tasks that have specific mathematical structures formulated in algebraic terms and require the development of specific algorithmic tools: the control of quantum evolution, of quantum state, or of density matrix. In the evolution control, the evolution operator of a closed system is designed through the application of a set of specifically tailored perturbations  $\sum_n \alpha_n(t) \hat{V}_n$ . The most general controllability requirement For the system to be controllable, the operators  $\hat{V}_n$ 's together with the natural Hamiltonian  $\hat{H}_0$  must span the whole Lie algebra  $\mathfrak{u}(N)$  and, when fulfilled, this requirement ensures that the evolution operator  $\hat{U}(t)$  is not bound to remain in the subgroup of natural evolutions  $\left\{ \exp\left(\frac{\hat{H}_0 t}{i\hbar}\right), t \in \mathbb{R} \right\}$  as for the uncontrolled case, but can take any value in  $U(N)$ , depending on how the  $\alpha_n(t)$ 's are chosen. The system is then relieved from its dynamical constraints (In terms of classical mechanics, such a system is said to have no *holonomic* constraints).

One can identify two qualitatively different situations. In the case when the Hamiltonian  $\hat{H}_0$ , the perturbations  $\hat{V}_n$ , and the available set of the parameters  $\alpha_n$  are such that by choosing  $\alpha_n(t)$  one can perform any infinitesimal transformation within an infinitesimally short interval of time, that is when the control is locally exerted over the entire Lie group, one can force the quantum system to follow any predetermined evolution. In the other case, when the parameters  $\alpha_n$  are such that within an infinitesimally short interval of time  $dt$  one cannot perform all the infinitesimal transformations, and in particular the transformation  $\mathbb{I} + i\hat{H}_0 dt$  inverse to the natural evolution suggested by  $\hat{H}_0$ , the complete control is still possible in the case of a compact Lie group. However this control cannot force the system to follow any predetermined trajectory, but just can ensure an evolution which coincides with the predetermined one in a set of equally spaced time intervals, that is stroboscopically. In this paper, we concentrate exclusively on the last case.

Many theoretical methods, based on optimization, have been put forward for explicitly identifying an appropriate set of functions  $\{\alpha_n(t)\}$  associated to a given set of perturbations  $\{\hat{V}_n\}$  and achieving a specific evolution target [8, 9, 10, 11, 12]. Most of these techniques rely on a known or intuitively guessed specific solution which can be further optimized with respect to an appropriate cost functional, through variational schemes: the straightforward exploration of the entire multidimensional space of the control parameters would indeed imply numerical work, the complexity of which grows exponentially with the dimensionality of the system. Other theoretical approaches [13, 14] use decompositions of the target evolution to build an appropriate pulse sequence by taking advantage of the specific algebraic properties of the system. This kind of constructive technique is, however, restricted to very specific systems, which present strong symmetry properties.

In this paper, we review a control method, called the nonholonomic control [15, 16, 17], which, by contrast, allows one to build pulse sequences achieving any unitary evolution of a generic closed quantum system, which requires neither any prior knowledge of the sought solution, nor any specific symmetry property of the system. We consider the simplest physical scenario, restricting the physical resources to the minimum, *i.e.* only two physical perturbations  $\{\hat{V}_a, \hat{V}_b\}$  which are alternately applied to the system during timings which play the role of adjustable control parameters. Given a specific evolution target, the appropriate control pulse sequence is derived via a two-step algorithm whose remarkable efficiency is due to interesting algebraic properties of random matrices and  $N^{\text{th}}$  roots of the identity.

Two important features of this method are its *universality* – it can be applied to any quantum system which exhibits the required algebraic properties, and its feasibility – two generic perturbations of the system, such as the interaction with two classical fields, will indeed generally free the system from its dynamical constraints. One of the most topical applications of our control method is the implementation of quantum gates as specific evolution operators of physical systems considered as candidates for a quantum computer. A third important feature is that the approach can be done robust with respect to a large class of uncontrolled external perturbations.

The reliability of quantum control – or information processing in the context of quantum computation, is threatened by the so-called quantum errors which result from the interaction of the system considered with its environment. These errors surely represent one of the most difficult challenges to be taken up on the route towards quantum computation. Inspired by classical error-correction, Shor was the first to propose to use redundant encoding of information in the computer's Hilbert space to actively protect it against a given set of errors [18]. The classical idea is to repeat information to make it more likely to be correctly transmitted. Quantum-mechanically, one appends to the information-carrying system an auxiliary system, called *ancilla*, and transfer – or *encode*, the information in a subspace of the full Hilbert space of the total system. Such a subspace is called a *quantum code* when errors act on it in a *detectable* and *correctable* way: after some time, one then can diagnose whether an error occurred on the information stored in the quantum code and, if needed, correct it through a recovery procedure [19, 1]. In the specific so-called non-degenerate case, errors map the code space onto orthogonal subspaces. This implies an inequality, called the quantum Hamming bound, on the number of correctable errors and the dimensionality of the total system. Note that adding ancilla introduces extra errors, which must also be taken into account. An important class of quantum codes are the stabilizer codes [20, 21]. Another, very specific, kind of quantum codes are the so-called decoherence-free subspaces [22, 23, 24, 25, 26], which present the property of being naturally immune to errors: the mere existence of such decoherence-free subspaces is, of course, not guaranteed and actually assumes very specific symmetry properties from the errors affecting the system. A unified theoretical presentation of a great variety of such passive and active quantum error correcting approaches has been recently

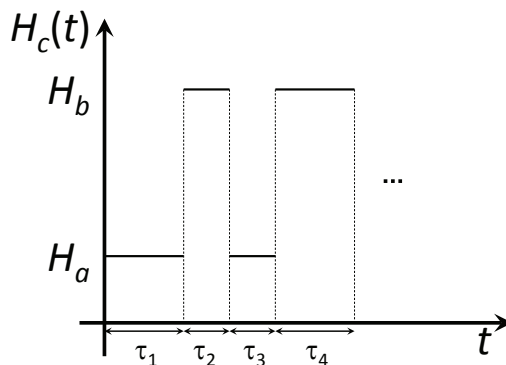
proposed in [27]. In the same spirit, topological protection methods take advantage of the high degree of symmetry of the system's Hamiltonian to safely encode information in its degenerate ground level [28, 29]. Alternative to the quantum code approach, the decoupling pulse strategy [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] was inspired by composite pulses, initially developed in NMR [42, 43, 44]. This technique basically consists in frequently perturbing the system in such a way that errors do not have the time to corrupt information before being averaged out.

Theoretical proposals for quantum codes or decoupling pulses usually rely on rather strong assumptions on the system and the error model: in general, they are developed for ensembles of two-level systems, affected by independent errors. Though quite reasonable for, *e.g.*, single-atom quantum computer scenarios in which one atom carries one qubit of information, this kind of assumptions is more questionable when qubits are actually distributed over ensembles [45, 46]. Moreover, based on symmetry properties of  $SU(2)$ , these methods fail to be translated to, *e.g.*, systems of qutrits. It is therefore interesting and necessary to develop alternative universal error-correcting/preventing approaches, which allow to deal with arbitrary systems affected by errors which *a priori* do not present any symmetry. In this spirit, we review here two techniques which were built as extensions of the nonholonomic evolution control method. The first one is inspired from both the quantum code approach and the quantum Zeno effect [47]. It basically consists in protecting the information encoded in an arbitrarily chosen subspace of the system's Hilbert space through frequently repeating a cycle of specifically designed steps, *i.e.* coding, decoding and measurement-induced projection of the system's state onto the initial code subspace [48, 49, 50]. The second one is a decoupling method which enables us to reliably apply an arbitrarily prescribed unitary transformation on the system's state, despite the presence of a (slow) environment [51]. Both methods required the development of specific algorithms, which are presented here.

The paper is structured as follows. In Sec. 2 we present the nonholonomic control technique for closed quantum systems – the principles as well as the algorithm used for explicitly calculating the control timings, and discuss a few applications in quantum computation. In Sec. 3, we present two extensions to coherence protection against first-order effects of the environment. For both techniques, the necessary algorithmic tools are explained and numerical examples are provided. Finally, we conclude in Sec. 4 and give possible perspectives of the nonholonomic control method.

## 2. Non-holonomic quantum control of closed systems

In this section we present the nonholonomic control method which enables one to perform an arbitrarily prescribed unitary transformation on a closed quantum system through alternately submitting it to two controlled perturbations. In the first subsection the principles of the technique are explained, as well as the algorithm for computing the interaction pulse timings. In the second subsection, we discuss possible applications of our method to quantum computation: in particular, we provide a numerical example



**Figure 1.** Two-valued pulsed control Hamiltonian  $H_c(t)$ .

involving an atomic qutrit, and suggest other uses of our technique in compound systems for quantum information processing.

### 2.1. The framework of the method

We consider a closed quantum system described by an  $N$ -dimensional Hilbert space, whose natural dynamics is governed by the unperturbed Hamiltonian  $\hat{H}_0$ . This system is alternately submitted to two constant perturbations  $\hat{V}_a$  and  $\hat{V}_b$ . The full Hamiltonian thus takes the following pulsed form

$$\begin{aligned}\hat{H}_c(t) &= \hat{H}_0 + \alpha(t)\hat{V}_a + [1 - \alpha(t)]\hat{V}_b \\ &= \alpha(t)\hat{H}_a + [1 - \alpha(t)]\hat{H}_b\end{aligned}$$

where  $\hat{H}_a \equiv \hat{H}_0 + \hat{V}_a$ ,  $\hat{H}_b \equiv \hat{H}_0 + \hat{V}_b$  and  $\alpha$  is a piecewise constant function from  $\mathbb{R}$  into  $\{0, 1\}$  (see figure 1). After a generic sequence of  $K$  pulses of respective durations  $\{\tau_{k=1, \dots, K}\}$ , the evolution operator takes the form

$$\hat{U}_c(\tau_1, \dots, \tau_K) = \exp\left\{\frac{\tau_K}{i\hbar}\hat{H}_K\right\} \times \dots \times \exp\left\{\frac{\tau_2}{i\hbar}\hat{H}_2\right\} \times \exp\left\{\frac{\tau_1}{i\hbar}\hat{H}_1\right\} \quad (1)$$

where  $\hat{H}_k = \hat{H}_a$  for odd  $k$ 's and  $\hat{H}_k = \hat{H}_b$  for even  $k$ 's.

It is a known result from Lie group theory that, when the operators  $\{i\hat{H}_a, i\hat{H}_b\}$  span the Lie algebra  $\mathfrak{u}(N)$  itself – the so-called Bracket Generation Condition (BGC), any unitary  $\hat{U}_d \in U(N)$  can be obtained as a *finite* product of elements of the single-parameter *semi*-groups  $\left\{\exp\left\{\frac{\tau}{i\hbar}\hat{H}_a\right\}, \tau \geq 0\right\}$  and  $\left\{\exp\left\{\frac{\tau}{i\hbar}\hat{H}_b\right\}, \tau \geq 0\right\}$  [52, 53, 3]. In other words, for any unitary evolution  $\hat{U}_d \in U(N)$ , there exists an integer  $K \geq 1$  and a  $K$ -dimensional time vector  $\vec{\tau} \in \mathbb{R}_+^{*K}$  such that  $\hat{U}_c(\tau_1, \dots, \tau_K) = \hat{U}_d$ . Note that the value of  $K$ , though uniformly bounded due to the compactness of  $U(N)$ , is not specified by the theorem. It is, however, reasonable to expect it to be of the same order as the number  $N^2$  of real parameters required to completely characterize a unitary matrix  $N \times N$ . Our goal is to be able, for any given  $\hat{U}_d$ , to exhibit a pulse sequence achieving the control objective  $\hat{U}_c(\tau_1, \dots, \tau_K) = \hat{U}_d$ .

The first step is to check whether the system is *nonholonomic*, *i.e.* controllable, that is whether BGC is satisfied by the two matrices  $\{i\hat{H}_a, i\hat{H}_b\}$ . A first way consists in iteratively calculating their commutators of increasing orders until they span  $\mathfrak{u}(N)$  [3]. When  $N$  increases this direct computation becomes, however, intractable. In that case, one can instead verify the following sufficient condition, suggested by V. Ka: the system becomes controllable, when the representative matrix of  $\hat{H}_b$  in the eigenbasis of  $\hat{H}_a$  has no off-diagonal zeros.

Once the BGC is checked, it is legitimate to look for a time vector  $\vec{\tau}$  achieving an arbitrarily prescribed transformation  $\hat{U}_d \in U(N)$  one wants to impose on the system. A straightforward way would consist in directly trying to minimize the function  $F_\kappa(\vec{s}) \equiv \left\| \exp\left\{\frac{s_\kappa}{i\hbar}\hat{H}_{a,b}\right\} \times \dots \times \exp\left\{\frac{s_2}{i\hbar}\hat{H}_b\right\} \times \exp\left\{\frac{s_1}{i\hbar}\hat{H}_a\right\} - \hat{U}_d \right\|$  for increasing time-vector lengths  $\kappa$ 's, stopping the procedure when the global minimum 0 is reached and setting  $K = \kappa_{min}$  and  $\vec{\tau} = \vec{s}_{min}$ . This approach is, however, very uneasy and unlikely to succeed since  $F_\kappa(\vec{s})$  exhibits many local minima. In the following, we present an alternative and systematic method consisting of two steps : (i) first, one determines a sequence of  $N^2$  pulses which achieves the identity transformation, (ii) then, one iteratively approaches a finite root of the desired transformation,  $\hat{U}_d^{1/n^*}$ , the full transformation  $\hat{U}_d$  being simply obtained by concatenating  $n^*$  times the same control sequence.

At the end of an  $N^2$ -control-pulse sequence, the overall evolution induced by  $\hat{H}_c(t)$  is

$$\hat{U}_c(\vec{\tau}) = \exp\left\{\frac{\tau_{N^2}}{i\hbar}\hat{H}_b\right\} \times \exp\left\{\frac{\tau_{N^2-1}}{i\hbar}\hat{H}_a\right\} \times \dots \times \exp\left\{\frac{\tau_1}{i\hbar}\hat{H}_a\right\}.$$

Our goal is first to determine an  $N^2$ -component time vector  $\vec{\tau}_0$  which achieves the identity, *i.e.*  $\hat{U}_c(\vec{\tau}_0) = \mathbb{I}$ . To this end, we look for an  $N$ -component time vector  $\vec{T}$  such that

$$\exp\left\{\frac{T_N}{i\hbar}\hat{H}_b\right\} \times \exp\left\{\frac{T_{N-1}}{i\hbar}\hat{H}_a\right\} \times \dots \times \exp\left\{\frac{T_1}{i\hbar}\hat{H}_a\right\} = e^{i\varphi/N} \times \hat{M}^{-1} \mathbb{I}^{1/N} \hat{M} \quad (2)$$

where  $e^{i\varphi/N}$  is an unimportant overall phase factor,  $\hat{M}$  is a  $N \times N$  unitary matrix and

$$\mathbb{I}^{1/N} \equiv \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & e^{i\frac{2\pi}{N}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i(N-1)\frac{2\pi}{N}} \end{bmatrix}$$

is an  $N^{\text{th}}$  root of the identity with nondegenerate spectrum. To compute  $\vec{T}$ , we take advantage of the following algebraic property: for a nondegenerate unitary matrix  $\mathcal{U} \in U(N)$  of characteristic polynomial  $\sum_{j=0}^N a_j X^j$ ,

$$\mathcal{U}^N = e^{i\varphi} \times \mathbb{I} \Leftrightarrow \sum_{j=0}^N |a_j|^2 \text{ takes its minimal value } 2. \quad (3)$$

To find  $\vec{T}$  we therefore only need to minimize the function  $F(\vec{T}) = \sum_{j=0}^N |a_j(\vec{T})|^2$  to 2, with respect to  $\vec{T}$ . Unexpectedly, minimizing  $F(\vec{T})$  turns out to be very easy. The reason for that has roots in the Random Matrix Theory : according to Dyson's law, the eigenvalues of random matrices tend to repel each other, and are thus very likely to be almost regularly distributed on the unit circle, as those of a nondegenerate  $N^{\text{th}}$  root of the identity. In other words, in the space of  $N \times N$  unitary matrices, matrices of the type  $e^{i\varphi/N} \hat{M}^{-1} \times \mathbb{I}^{1/N} \times \hat{M}$  are very abundant: a generic product of exponentials as in (2) is therefore often very close to an  $N^{\text{th}}$  root of the identity, which can easily be reached by small variations of the timings. Once the  $N$ -component vector  $\vec{T}$  is found, we concatenate  $\vec{T}$   $N$  times, thus getting the  $N^2$ -component vector  $\vec{\tau}_0$  such that

$$\begin{aligned} \hat{U}_c(\vec{\tau}_0) &= \left( \exp \left\{ \frac{T_N}{i\hbar} \hat{H}_b \right\} \times \exp \left\{ \frac{T_{N-1}}{i\hbar} \hat{H}_a \right\} \times \dots \times \exp \left\{ \frac{T_1}{i\hbar} \hat{H}_a \right\} \right)^N \\ &= \left( e^{i\varphi/N} \times \hat{M}^{-1} \times \mathbb{I}^{1/N} \times \hat{M} \right)^N \\ &= e^{i\varphi} \times \mathbb{I}. \end{aligned}$$

Forgetting the physically unimportant phase factor  $e^{i\varphi}$ , we hence have constructed a control pulse sequence which achieves the identity transformation.

Let us now show how, from  $\vec{\tau}_0$ , one can build a vector  $\vec{\tau}^*$  which achieves a finite  $(n^*)^{\text{th}}$  root of the desired transformation. We first put the target matrix  $\hat{U}_d$  under the form  $\hat{U}_d = \exp \left\{ \frac{1}{i\hbar} \hat{H}_d \right\}$ , and define  $\hat{U}_\epsilon = \exp \left\{ \frac{\epsilon}{i\hbar} \hat{H}_d \right\}$ . Obviously, one has  $\hat{U}_{\epsilon=0} = \mathbb{I}$  and  $\hat{U}_{\epsilon=1} = \hat{U}_d$ . For  $0 < \epsilon_1 \ll 1$ , one can use the linearized form for the target transformation  $\hat{U}_{\epsilon_1} = \mathbb{I} + \frac{\epsilon_1}{i\hbar} \hat{H}_d + o\left(\left|\frac{\epsilon_1}{\hbar}\right| \left\| \hat{H}_d \right\| \right)$ , very close to the identity. On the other hand, allowing for a slight variation of the time vector  $\vec{\tau}_0 \rightarrow \vec{\tau}_1 = \vec{\tau}_0 + \delta\vec{\tau}_0$ , one can write  $\hat{U}_c(\vec{\tau}_1)$  under the linearized form  $\hat{U}_c(\vec{\tau}_1) = \mathbb{I} + \vec{\nabla} \hat{U}_c(\vec{\tau}_0) \cdot \delta\vec{\tau}_0 + o\left(\left\| \delta\vec{\tau}_0 \right\| \left\| \hat{H}_a \right\| / \hbar, \left\| \delta\vec{\tau}_0 \right\| \left\| \hat{H}_b \right\| / \hbar\right)$ . If the  $N^2$  matrices  $\vec{\nabla} \hat{U}_c(\vec{\tau}_0) \cdot \vec{e}_k$ , where  $\vec{e}_k$  denotes the unit vector along the  $k^{\text{th}}$  component of the time vector  $\vec{\tau}$ , span the full vector space of  $N \times N$  anti-Hermitian matrices, one can easily find the time vector increment  $\delta\vec{\tau}_0$  such that  $\hat{U}_{\epsilon_1} \simeq \hat{U}_c(\vec{\tau}_0 + \delta\vec{\tau}_0)$ , that is  $\frac{\epsilon_1}{i\hbar} \hat{H}_d = \vec{\nabla} \hat{U}_c(\vec{\tau}_0) \cdot \delta\vec{\tau}_0$ , through standard techniques of linear algebra. Repeating the same operation from the new starting point  $\vec{\tau}_1$  with the new target transformation  $\hat{U}_{\epsilon_2} = \hat{U}_{\epsilon_1} + \frac{\epsilon_2 - \epsilon_1}{i\hbar} \times \hat{U}_{\epsilon_1} \times \hat{H}_d + o\left(\left|\frac{\epsilon_2 - \epsilon_1}{\hbar}\right| \left\| \hat{H}_d \right\| \right)$ , where  $|\epsilon_2 - \epsilon_1| \ll 1$ , one can compute the new time vector increment  $\delta\vec{\tau}_1$  such that  $\frac{\epsilon_2 - \epsilon_1}{i\hbar} \hat{U}_c(\vec{\tau}_1) \times \hat{H}_d = \vec{\nabla} \hat{U}_c(\vec{\tau}_1) \cdot \delta\vec{\tau}_1$ , provided that the  $N^2$  matrices  $\vec{\nabla} \hat{U}_c(\vec{\tau}_1) \cdot \vec{e}_k$  span the full vector space of  $N \times N$  anti-Hermitian matrices. One continues until the rank of the set  $\left\{ \vec{\nabla} \hat{U}_c(\vec{\tau}) \cdot \vec{e}_k, 1 \leq k \leq N^2 \right\}$  becomes less than  $N^2$  and therefore the time increment cannot be calculated any longer. This happens, say, at step  $l$ : setting  $n^* = 1/\epsilon_l$  - one can choose all  $\epsilon_k$ 's as inverses of decreasing integers, one finally gets a time vector  $\vec{\tau}^* \equiv \vec{\tau}_l$  for which  $\hat{U}_c(\vec{\tau}^*) = \hat{U}_d^{1/n^*}$ . One implements the desired transformation  $\hat{U}_d$  through concatenating the sequence of  $N^2$  pulses  $\{\tau_1^*, \tau_2^*, \dots, \tau_{N^2}^*\}$   $n^*$  times, thus achieving the result claimed by the general theorem, with  $K = n^* \times N^2$ .



To finish this subsection, we evoke another possible parametrization of the evolution operator, *i.e.* by the amplitudes of the applied perturbations. To be more explicit, we now consider  $N^2$  pulses ( $k = 1, \dots, N^2$ ) of *equal* durations  $\tau$ , during which  $\hat{V}_a$  and  $\hat{V}_b$  are alternately applied to the system multiplied by different and controllable amplitudes  $\{\xi_{k=1, \dots, N^2}\}$ . The overall evolution operator now takes the following form

$$\hat{U}_c(\vec{\xi}) = \exp\left\{\frac{\tau}{i\hbar}(\hat{H}_0 + \xi_{N^2}\hat{V}_b)\right\} \times \dots \times \exp\left\{\frac{\tau}{i\hbar}(\hat{H}_0 + \xi_1\hat{V}_a)\right\}.$$

The same two-step procedure applies, as before. One first looks for the  $N$ -component amplitude vector  $\vec{\Xi}$  such that

$$\exp\left\{\frac{\tau}{i\hbar}(\hat{H}_0 + \Xi_N\hat{V}_b)\right\} \times \dots \times \exp\left\{\frac{\tau}{i\hbar}(\hat{H}_0 + \Xi_1\hat{V}_a)\right\} = \mathbb{I}^{1/N},$$

through minimizing to 2 the function  $F(\vec{\Xi}) = \sum_{j=0}^N |a_j(\vec{\Xi})|^2$  with respect to  $\vec{\Xi}$ , where  $\sum_{j=0}^N a_j(\vec{\Xi}) X^j$  is the characteristic polynomial of the  $N$ -exponential product. Concatenating  $\vec{\Xi}$   $N$  times yields the  $N^2$ -component amplitude vector  $\vec{\xi}_0$  which satisfies  $\hat{U}_c(\vec{\xi}) = \mathbb{I}$  up to a global phase factor. Secondly, one applies the same kind of successive approximations as in the previous paragraph, by iteratively varying the amplitude vector  $\vec{\xi}_0 \rightarrow \vec{\xi}_1 = \vec{\xi}_0 + \delta\vec{\xi}_0 \rightarrow \dots$  in order to successively achieve  $n^{\text{th}}$  roots of the desired transformation  $\hat{U}_d$  for decreasing  $n$ 's: one stops at step  $l$ , when the rank of the set  $\{\vec{\nabla}_{\xi} \hat{U}_c(\vec{\xi}_l) \cdot \vec{\varepsilon}_k, 1 \leq k \leq N^2\}$  becomes less than  $N^2$  ( $\vec{\varepsilon}_k$  denotes the unit vector in the direction of the  $k^{\text{th}}$  amplitude). Setting  $\vec{\xi}^* = \vec{\xi}_l$  and  $n^* = n_l$ , one finally gets  $[\hat{U}_c(\vec{\xi}^*)]^{n^*} = \hat{U}_d$ .

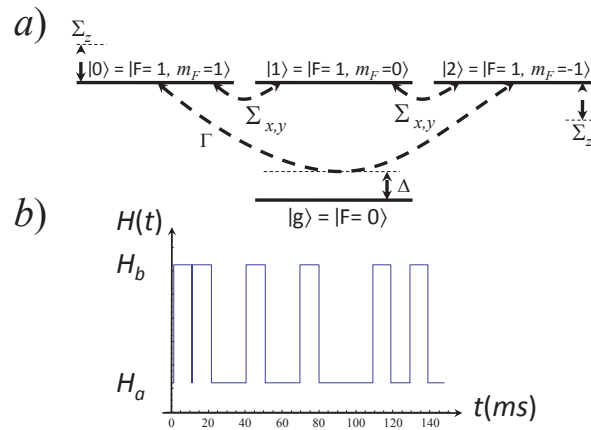
## 2.2. Applications to Quantum Computation

One of the basic prerequisites to Quantum Computation – the so-called DiVincenzo's criteria [54], is the ability to universally process the information encoded in the state of the system chosen as a computer, *i.e.* to induce any prescribed unitary evolution of the system chosen as a computer. Our technique precisely enables the experimenter to reach that goal, provided that he or she is able to alternately impose two different physical perturbations which check the BGC.

As an example we consider here an atomic qutrit consisting of a hydrogen atom, whose Hilbert space is restricted to three degenerate hyperfine states (see figure 2)

$$\begin{aligned} |0\rangle &\equiv |n=1; L=0; S=1/2; I=1/2; F=1, m_F=1\rangle, \\ |1\rangle &\equiv |n=1; L=0; S=1/2; I=1/2; F=1, m_F=0\rangle, \\ |2\rangle &\equiv |n=1; L=0; S=1/2; I=1/2; F=1, m_F=-1\rangle. \end{aligned}$$

If a static magnetic field  $\vec{B}_0 = (B_{0,x}, B_{0,y}, B_{0,z})$  is applied to the atom, the Zeeman Hamiltonian takes the form  $\hat{W}_Z = -\frac{g_e\mu_B}{\hbar}\vec{S} \cdot \vec{B}_0$  where  $g_e \simeq 2$  and  $\mu_B \simeq -9.274084 \times 10^{-24} J \cdot T^{-1}$  are the electronic gyromagnetic factor and Bohr magneton, respectively,



**Figure 2.** Control of an atomic qutrit : a) the three states of the hydrogen atom ground level, b) a pulse sequence achieving the desired transformation  $\hat{U}_d$  (see text).

and  $\hat{S}_l = \frac{\hbar}{2}\hat{\Sigma}_l$ , for  $l = x, y, z$ , with

$$\hat{\Sigma}_x \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\hat{\Sigma}_y \equiv \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\hat{\Sigma}_z \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Here, we implicitly assumed that the field is weak enough for the nonresonant couplings  $|0\rangle, |1\rangle, |2\rangle \longleftrightarrow |g\rangle = |n=1; L=0; S=1/2; I=1/2; F=0\rangle$  to be neglected. To be more explicit, we suppose that  $\mu_B B_{0,x,y,z} \ll \Delta E_{hf}$  where  $\Delta E_{hf} \simeq 1.42\text{GHz}$  is the hyperfine splitting of the ground level of the hydrogen atom.

Now, if one applies a harmonic time dependent magnetic field, oriented along the  $x$ -axis and whose frequency is slightly detuned by the amount  $\hbar\Delta$  from the transitions  $|0\rangle \leftrightarrow |g\rangle$  and  $|2\rangle \leftrightarrow |g\rangle$ , the effective resulting (second-order) Hamiltonian – obtained after adiabatically eliminating the essentially unpopulated ground state [55], couples the states  $|1\rangle$  and  $|3\rangle$  and can be put under the form  $\hat{W}' = \frac{(g_e \mu_B \mathcal{B})^2}{8\hbar\Delta} \times \hat{\Gamma}$ , with

$$\hat{\Gamma} \equiv \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

where  $\mathcal{B}$  denotes the amplitude of the magnetic field. Note that we do not take into account the electric field of the electromagnetic wave considered, since no electric couplings exist in the ground state hyperfine multiplicity. Furthermore, we again assume

that  $\mathcal{B}$  is weak enough, so that the nonresonance condition  $\frac{g_e \mu_B \mathcal{B}}{2\hbar\Delta} \ll 1$  is met. Finally, choosing two sets of values  $B_{0,x}^{(a)} = -0.203\text{mT}$ ,  $B_{0,y}^{(a)} = 0.101\text{mT}$ ,  $B_{0,z}^{(a)} = -0.051\text{mT}$ ,  $\mathcal{B}^{(a)} = 1.757\text{mT}$ ,  $\Delta^{(a)} = 0.892\text{GHz}$ , and  $B_{0,x}^{(b)} = -0.203\text{mT}$ ,  $B_{0,y}^{(b)} = 0.203\text{mT}$ ,  $B_{0,z}^{(b)} = 0.101\text{mT}$ ,  $\mathcal{B}^{(b)} = 1.015\text{mT}$ , and  $\Delta^{(b)} = -0.892\text{GHz}$  for the physical parameters, one gets two Hamiltonians  $\hat{H}_{a,b} \equiv -\frac{g_e \mu_B}{\hbar} \hat{S} \cdot \vec{B}_0^{(a,b)} + \frac{(g_e \mu_B \mathcal{B}^{(a,b)})^2}{8\hbar\Delta^{(a,b)}} \times \hat{\Gamma}$  which satisfy the BGC as can be numerically checked. As an example, we choose to achieve the target evolution

$$\hat{U}_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}.$$

Our algorithm provides us with a sequence of 13 pulses represented of figure 2, for a total control period of  $\simeq 148\mu\text{s}$ . Note that this system has been previously considered in our work [51] to illustrate another method, *i.e.* our coherence protection method via decoupling. The numerical results reported here are, however, new.

The nonholonomic control technique can also be applied to larger systems, including compound systems: in [17], for instance, it was used to design a sequence of tailored electromagnetic pulses achieving a CNOT quantum gate in a system of two Cesium atoms in dipole-dipole interaction.

As the number of its elementary subsystems grows, the direct control of a compound system becomes, however, computationally harder and finally intractable: for instance, the control of an  $N$ -qubit quantum computer requires the control of  $\sim 4^N$  physical parameters. To overcome this difficulty, one can, as suggested in [16], construct quantum devices from fully controlled elementary cells, of relatively low dimensionality, comprising a few information-carrying particles, typically three. The spatial arrangement of these cells as well as the way they connect to each other must be designed for the specific computational task the device has to perform. As an example, it was shown in [16] how to perform universal computation in a 9-qubit quantum processor made of three triads of Rydberg atoms as elementary cells.

### 3. Coherence protection and the nonholonomic control

Let us consider an  $N$ -level quantum system  $\mathcal{S}$ , of Hilbert space  $\mathcal{H}$ , interacting with its environment  $\mathcal{E}$  plus an additional classical noise, and submitted to a time-dependent control Hamiltonian, designed by the experimenter. The total Hamiltonian of the system ( $\mathcal{S} + \mathcal{E}$ ) can thus be put under the form

$$\hat{H}(t) = \left[ \hat{H}_c(t) + \hat{\Omega}(t) \right] \otimes \mathbb{I}_{\mathcal{E}} + \mathbb{I}_{\mathcal{S}} \otimes \hat{H}_{\mathcal{E}} + \hat{V}_{\mathcal{S}\mathcal{E}}$$

where  $\hat{H}_c$  represents the time-dependent control Hamiltonian,  $\hat{\Omega}(t)$  is the classical noise – as, *e.g.*, the interaction with a parasitic classical field or inaccuracies in the control Hamiltonian  $\hat{H}_c$ ,  $\hat{H}_{\mathcal{E}}$  is the natural Hamiltonian of  $\mathcal{E}$  and  $\hat{V}_{\mathcal{S}\mathcal{E}}$  is the interaction

Hamiltonian between  $\mathcal{S}$  and  $\mathcal{E}$ . The operators  $\hat{\Omega}$  and  $\hat{V}_{\mathcal{S}\mathcal{E}}$  can be expanded on a basis of Hermitian traceless generators of  $\mathfrak{su}(N)$  of norm one, denoted by  $\{\hat{G}_{m=1,\dots,N^2-1}\}$ ,

$$\begin{aligned}\hat{\Omega}(t) \otimes \mathbb{I}_{\mathcal{E}} &= \sum_{1 \leq m \leq M} \varepsilon_m(t) \hat{G}_m \otimes \mathbb{I}_{\mathcal{E}} \\ \hat{V}_{\mathcal{S}\mathcal{E}} &= \sum_{1 \leq m \leq M} \zeta_m \hat{G}_m \otimes \hat{E}_m\end{aligned}$$

where the  $\hat{E}_m$ 's denote Hermitian traceless operators of norm one acting on the environment's Hilbert space, the  $\varepsilon_m$ 's are real functions of time, and the  $\zeta_m$ 's are real numbers. Note that the number  $M$  of terms in the expansion checks  $M \leq N^2 - 1$ , since not all the generators  $\hat{G}_m$  are always necessary in the decomposition. Note also that, as the  $\hat{G}_m$ 's and the  $\hat{E}_m$ 's have been assumed of norm one, the strength of the interactions is fully characterized by the  $\varepsilon_m$ 's and the  $\zeta_m$ 's. We shall moreover assume that these quantities are bounded: there exists a finite real number  $\Gamma$  such that  $\forall m, |\varepsilon_m|, |\zeta_m| \lesssim \hbar\Gamma$ .

In the interaction picture relative to the Hamiltonian  $\hat{H}_c(t) \otimes \mathbb{I}_{\mathcal{E}} + \mathbb{I}_{\mathcal{S}} \otimes \hat{H}_{\mathcal{E}}$ , the evolution operator of the system ( $\mathcal{S} + \mathcal{E}$ ) satisfies

$$\begin{cases} i\hbar\partial_t \tilde{U}(t) = \left[ \sum_{1 \leq m \leq M} \tilde{G}_m(t) \otimes \left( \varepsilon_m(t) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(t) \right) \right] \tilde{U}(t) \\ \tilde{U}(0) = \mathbb{I} \end{cases}$$

where we introduced the (unit-normed) operators in the interaction picture  $\tilde{G}_m(t) \equiv \hat{U}_c^\dagger(t) \times \hat{G}_m \times \hat{U}_c(t)$  and  $\tilde{E}_m(t) \equiv \hat{U}_{\mathcal{E}}^\dagger(t) \times \hat{E}_m \times \hat{U}_{\mathcal{E}}(t)$ , with  $\hat{U}_c(t) \equiv T \left[ \exp \left\{ \frac{1}{i\hbar} \int_0^t \hat{H}_c(s) ds \right\} \right]$  and  $\hat{U}_{\mathcal{E}}(t) \equiv T \left[ \exp \left\{ \frac{1}{i\hbar} \int_0^t \hat{H}_{\mathcal{E}}(s) ds \right\} \right]$ . Up to the second order in  $\frac{t}{\hbar} \times \sum_m (|\varepsilon_m| + |\zeta_m|) \lesssim 2M\Gamma t \equiv t/\tau_Z$ , where  $\tau_Z^{-1} \equiv 2M\Gamma$  is called the Zeno range, the evolution operator takes the form

$$\begin{aligned}\tilde{U}(t) &\simeq \mathbb{I} + \frac{1}{i\hbar} \sum_m \int_0^t \tilde{G}_m(s) \otimes \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) ds \\ &+ \left( \frac{1}{i\hbar} \right)^2 \sum_{m,p} \int_0^t ds \int_0^s ds' \tilde{G}_m(s) \tilde{G}_p(s') \\ &\otimes \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) \times \left( \varepsilon_p(s') \mathbb{I}_{\mathcal{E}} + \zeta_p \tilde{E}_p(s') \right) \\ &+ o(t^2/\tau_Z^2).\end{aligned}\tag{4}$$

The reliability of storage/processing of information in the system  $\mathcal{S}$  is threatened by the terms of order  $n \geq 1$  in the interaction picture, which represent a discrepancy from the controlled evolution  $\hat{U}_c(t)$ .

At the end of this section, we discuss the relevance of correcting all orders of the perturbation series, and show that, indeed, in most cases of interest the first-order error-correction only needs to be considered. In the following two subsections, we focus on two strategies for suppressing the first-order term, which use ideas from the nonholonomic control technique. The first one is inspired by the quantum Zeno effect. It consists in stabilizing a subspace of the system's Hilbert space – which, therefore, becomes

effectively decoherence-free, by the frequent repetition of a coding-decoding-incomplete measurement procedure. The coding-decoding steps are physically implemented via two symmetric pulse sequences involving only two different control Hamiltonians, as in the closed-system evolution control technique exposed in the previous section. Here, however, the full control of the evolution operator is not required. As a consequence, fewer control parameters, *i.e.* fewer pulses are needed: the coding and decoding sequences can thus be made quite short. The second strategy we present is more ambitious, since it aims not only at completely decoupling the system from the classical noise and the environment, but also at performing an arbitrarily prescribed evolution on its state. We show that this can only be achieved through a pulsed control Hamiltonian which takes at least three different values, one of which may be zero. We also suggest how to construct an appropriate sequence from the two-valued control pulse sequence one would get for achieving the same control objective for the system  $\mathcal{S}$  considered closed, via the technique presented in the previous section.

The main feature of both methods is their *universality*: they can indeed be applied to any system exhibiting the required minimal algebraic properties and, in particular, not only to systems of qubits, as it is the case for most of the techniques proposed in the literature. Another important point about these methods is that they employ minimal physical resources in terms of control Hamiltonian: for the first one, we only need two controlled perturbations, three for the second one.

### 3.1. Multidimensional Zeno Effect and Coherence Protection

In this subsection, following the main idea of the quantum Zeno effect, we investigate how repeated measurements together with pulse sequences enable one to annul the first-order term in (4). To be more explicit, we present a strategy which makes it possible to stabilize a subspace of the system's Hilbert space, therefore making it effectively decoherence-free. We want to emphasize that, here, we present our method in a more general context than in previous works. In [48, 49, 50], we indeed focused on unitary errors corresponding to, *e.g.*, the action of parasitic classical fields. But, as we shall see in the following, the same method holds for the generic model described in the introduction of this section.

*3.1.1. The principle of the method* The quantum Zeno effect takes place in a system which is frequently measured in its (necessarily known) initial state: if the time interval between two projective measurements is small enough, the evolution of the system is blocked with a very high probability, equal to 1 in the (unphysical) limit of continuous measurements. As we shall now see, the same kind of idea can be used to stabilize the information stored in a multidimensional subspace of a system's Hilbert space.

Let us first assume there exists an observable  $\mathcal{O}$  of the system  $\mathcal{S}$ , for which the  $I(\geq 2)$ -dimensional subspace  $\mathcal{C} \subset \mathcal{H}$ , of projector  $\hat{P}$ , is an eigenspace. For simplicity, we shall assume that  $\mathcal{C}^\perp$  is also an eigenspace of  $\mathcal{O}$ , of projector  $\hat{Q} = \mathbb{I}_{\mathcal{S}} - \hat{P}$  (considering

more than just two eigenspaces for the observable  $\mathcal{O}$  would not make the demonstrations more difficult, but more cumbersome). Moreover, we suppose the total system ( $\mathcal{S} + \mathcal{E}$ ) is initially in the state  $\hat{\rho}(0) \otimes \hat{\rho}_e$ , such that  $\hat{P} \times \hat{\rho}(0) = \hat{\rho}(0) \times \hat{P} = \hat{\rho}(0)$ . If no control is imposed on  $\mathcal{S}$ , *i.e.*  $\hat{H}_c \equiv 0$  which implies that  $\hat{H}_0 \equiv 0$ , the evolution operator (4) for ( $\mathcal{S} + \mathcal{E}$ ) at time  $\tau \lesssim \tau_Z$  takes the form

$$\begin{aligned} \tilde{U}(\tau) &\simeq \mathbb{I} + \frac{1}{i\hbar} \sum_m \hat{G}_m \otimes \left[ \int_0^\tau \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) ds \right] \\ &- \frac{1}{\hbar^2} \sum_{m,p} \hat{G}_m \hat{G}_p \otimes \left[ \int_0^\tau ds \int_0^s ds' \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) \right. \\ &\quad \left. \times \left( \varepsilon_p(s') \mathbb{I}_{\mathcal{E}} + \zeta_p \tilde{E}_p(s') \right) \right] + o(\tau^2/\tau_Z^2) \end{aligned}$$

and the state at time  $\tau$  is  $\tilde{U}(\tau) \times \tilde{\rho}(0) \otimes \tilde{\rho}_e \times \tilde{U}(\tau)^\dagger$ . Note that, for sake of consistency, we introduced the notation  $\tilde{\rho}(0) \otimes \tilde{\rho}_e$  for the initial state in the interaction picture relative to  $\hat{H}_c$ , which of course coincides with  $\hat{\rho}(0) \otimes \hat{\rho}_e$ . Measuring the observable  $\mathcal{O}$  yields the new (mixed) state

$$\begin{aligned} &\left( \hat{P} \otimes \mathbb{I}_{\mathcal{E}} \right) \tilde{U}(\tau) \left( \hat{P} \otimes \mathbb{I}_{\mathcal{E}} \right) \left( \tilde{\rho}(0) \otimes \tilde{\rho}_e \right) \left( \hat{P} \otimes \mathbb{I}_{\mathcal{E}} \right) \tilde{U}(\tau)^\dagger \left( \hat{P} \otimes \mathbb{I}_{\mathcal{E}} \right) \\ &+ \left( \hat{Q} \otimes \mathbb{I}_{\mathcal{E}} \right) \tilde{U}(\tau) \left( \hat{Q} \otimes \mathbb{I}_{\mathcal{E}} \right) \left( \tilde{\rho}(0) \otimes \tilde{\rho}_e \right) \left( \hat{Q} \otimes \mathbb{I}_{\mathcal{E}} \right) \tilde{U}(\tau)^\dagger \left( \hat{Q} \otimes \mathbb{I}_{\mathcal{E}} \right) \end{aligned}$$

whose first-order term in  $\tau/\tau_Z$  is

$$\frac{1}{i\hbar} \sum_m \left[ \hat{P} \hat{G}_m \hat{P} \otimes \int_0^\tau \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) ds, \tilde{\rho}(0) \otimes \tilde{\rho}_e \right].$$

Assuming the  $\hat{G}_m$ 's act orthogonally on the subspace  $\mathcal{C} \subset \mathcal{H}$ , *i.e.*

$$\forall m \leq M, \quad \hat{P} \times \hat{G}_m \times \hat{P} = 0, \quad (5)$$

washes out this first-order contribution. The state of ( $\mathcal{S} + \mathcal{E}$ ) is therefore given by  $\tilde{\rho}(0) \otimes \tilde{\rho}_e$  up to second-order terms in  $\tau/\tau_Z$ . Repeating the same sequence of operations  $n$  times, and tracing out the environment to get the reduced state of the system  $\mathcal{S}$  at time  $T = n\tau$ , one obtains

$$\tilde{\rho}(T) = \tilde{\rho}(0) + n\tau^2 \sum_{m,p} \chi_{m,p}^2 \hat{G}_m \hat{P} \tilde{\rho}(0) \hat{P} \hat{G}_p - \xi_{m,p}^2 \left\{ \hat{P} \hat{G}_m \hat{G}_p \hat{P}, \tilde{\rho}(0) \right\}_+ + o(n\tau^2/\tau_Z^2)$$

where

$$\begin{aligned} \chi_{m,p}^2 &\equiv \frac{1}{\hbar^2 \tau^2} \text{Tr} \left[ \tilde{\rho}_e \left\{ \int_0^\tau \left( \varepsilon_p(s) \mathbb{I}_{\mathcal{E}} + \zeta_p \tilde{E}_p(s) \right) ds \right\} \right. \\ &\quad \left. \times \left\{ \int_0^\tau \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) ds \right\} \right] \lesssim (2\Gamma)^2 \\ \xi_{m,p}^2 &\equiv \frac{1}{\hbar^2 \tau^2} \int_0^\tau ds \int_0^s ds' \text{Tr} \left[ \tilde{\rho}_e \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) \right. \\ &\quad \left. \times \left( \varepsilon_p(s') \mathbb{I}_{\mathcal{E}} + \zeta_p \tilde{E}_p(s') \right) \right] \lesssim (2\Gamma)^2 \end{aligned}$$

When  $n$  tends to infinity with fixed  $T$  (*i.e.*  $\tau \rightarrow 0$ ),

$$\begin{aligned} \tilde{\rho}(n\tau) &\sim \tilde{\rho}(0) + \frac{T^2}{n} \left\{ \sum_{m,p} \chi_{m,p}^2 \hat{G}_m \hat{P} \tilde{\rho}(0) \hat{P} \hat{G}_p \right. \\ &\quad \left. - \xi_{m,p}^2 \left\{ \hat{P} \hat{G}_m \hat{G}_p \hat{P}, \tilde{\rho}(0) \right\}_+ \right\} + o\left(\frac{T^2}{n\tau_Z^2}\right) \\ &\rightarrow \tilde{\rho}(0), \end{aligned}$$

which means that the information initially stored in  $\mathcal{C} \subset \mathcal{H}$  is preserved through the frequently repeated measurement of  $\mathcal{O}$ .

The situation we have just considered is, however, rather unlikely. It is indeed pretty strong to assume that the  $\hat{G}_m$ 's act orthogonally on  $\mathcal{C}$  and, at the same time, that  $\mathcal{C}$  is an eigenspace of some observable of the system. It is, however, sufficient to suppose that *two* distinct  $I$ -dimensional subspaces  $\mathcal{C}, \tilde{\mathcal{C}}$  exist, each of which presents one of these properties, *i.e.*  $\mathcal{C}$ , is an eigenspace of an observable  $\mathcal{O}$ , while  $\tilde{\mathcal{C}}$  exhibits the orthogonality property. Since these two subspaces have the same dimension, they are isomorphic and there exists a unitary transformation such that  $\tilde{\mathcal{C}} = \hat{C}\mathcal{C}$ . The orthogonality conditions (5) on the subspace  $\tilde{\mathcal{C}}$  can be translated into an equivalent set of conditions on the coding matrix  $\hat{C}$ , *i.e.*

$$\forall m, \hat{P} \times \hat{C}^\dagger \times \hat{G}_m \times \hat{C} \times \hat{P} = 0. \quad (6)$$

We now apply a control Hamiltonian  $H_c(t)$  which is non zero only for a very short amount of time  $\tau_c \ll \tau_Z$  at the beginning and the end of each period  $\tau$ , *i.e.* just before and just after every single measurement. The time  $\tau_c$  is assumed so short that one can neglect the accumulated action due to the classical noise and the interaction with the environment during the control pulse sequences. We moreover assume that  $H_c(t)$  is designed in such a way that it implements appropriate coding and decoding operations  $\hat{C}$  and  $\hat{C}^\dagger$  (we shall show later how to proceed technically). Taking the orthogonality conditions (6) into account, one can repeat the same calculations as above and finally get the reduced state of the system  $\mathcal{S}$  at time  $T = n\tau$ , that is after  $n$  cycles coding-decoding-measurement

$$\begin{aligned} \tilde{\rho}(T) &= \tilde{\rho}(0) + n\tau^2 \sum_{m,p} \left( \chi_{m,p}^2 \hat{C}^\dagger \hat{G}_m \hat{C} \hat{P} \tilde{\rho}(0) \hat{P} \hat{C}^\dagger \hat{G}_p \hat{C} \right. \\ &\quad \left. - \xi_{m,p}^2 \left\{ \hat{P} \hat{C}^\dagger \hat{G}_m \hat{G}_p \hat{C} \hat{P}, \tilde{\rho}(0) \right\}_+ \right) + o(n\tau^2/\tau_Z^2). \end{aligned}$$

When  $n$  tends to infinity with fixed  $T$  (*i.e.*  $\tau \rightarrow 0$ ), we again have  $\tilde{\rho}(n\tau) \rightarrow \tilde{\rho}(0)$ . Going back into the original picture, the initial state  $\hat{\rho}(0) = \hat{P} \times \hat{\rho}(0) \times \hat{P}$  has therefore been applied the overall controlled evolution  $U_c(T_c) = (C \times C^\dagger)^n = \mathbb{I}$ , while the first-order contribution of the noise and interaction with the environment has been washed away. In other words, the information initially stored in  $\mathcal{C}$  has been preserved.

To finish this paragraph, we consider the situation when  $\mathcal{S}$  is a compound system comprising two subsystems, the information carrying subsystem  $\mathcal{I}$  and the ancilla  $\mathcal{A}$ , whose respective Hilbert spaces are  $\mathcal{H}_I$  of dimension  $I$  and  $\mathcal{H}_A$  of dimension  $A$ . Let

us assume the ancilla is initially prepared in a state  $|\alpha\rangle$ , which is a non-degenerate eigenstate of some observable  $\mathcal{O}_A$  of the ancilla. From what we have just shown above, if there exists an entangling unitary matrix  $\hat{C}$  acting on  $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_A$  such that  $\forall m, \hat{P}_\alpha \times \hat{C}^\dagger \times \hat{G}_m \times \hat{C} \times \hat{P}_\alpha = 0$ , where  $\hat{P}_\alpha \equiv \hat{\mathbb{I}} \otimes |\alpha\rangle\langle\alpha|$  is the projector onto the  $I$ -dimensional eigensubspace of  $\mathcal{O}_A \mathcal{C} = \mathcal{H}_I \otimes \text{Span}[|\alpha\rangle]$ , the information initially stored in  $\mathcal{C}$  can be protected with very high probability by the frequent repetition of the following procedure: (i) transfer of the information from  $\mathcal{C}$  to  $\tilde{\mathcal{C}}$  through the application of  $\hat{C}$ , (ii) free evolution for a short period of time  $\tau \lesssim \tau_Z$ , (iii) decoding via the application of  $\hat{C}^\dagger$  and finally (iv) projection onto  $\mathcal{C}$  via the measurement of the ancilla in its initial state  $|\alpha\rangle$ .

A lower bound on the dimension  $A$  of the ancilla's Hilbert space  $\mathcal{H}_A$  can be obtained by merely comparing the dimension of the Hilbert space  $\mathcal{H}_I \otimes \mathcal{H}_A$ , *i.e.*  $\dim \mathcal{H}_I \otimes \mathcal{H}_A = I \times A$ , with the dimension of the union  $\tilde{\Sigma}$  of the subspace  $\tilde{\mathcal{C}}$  with all the subspaces  $\tilde{\mathcal{C}}_m \equiv \hat{G}_m \tilde{\mathcal{C}}$  induced by the action of the different  $\hat{G}_m$ 's on  $\tilde{\mathcal{C}} \equiv \hat{C}\mathcal{C}$ , *i.e.*  $\tilde{\Sigma} \equiv \tilde{\mathcal{C}} \cup \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2 \cup \dots \cup \tilde{\mathcal{C}}_M$ . The largest possible dimension for  $\tilde{\Sigma}$  is obtained when all the  $\tilde{\mathcal{C}}_m$ 's are orthogonal not only to  $\tilde{\mathcal{C}}$  – they are by definition of the matrix  $\hat{C}$ , but also to each other: then  $\tilde{\Sigma} \equiv \tilde{\mathcal{C}} \oplus \tilde{\mathcal{C}}_1 \oplus \dots \oplus \tilde{\mathcal{C}}_M$  and  $\dim \tilde{\Sigma} = I \times (M + 1)$ . Finally, as  $\tilde{\Sigma} \subset \mathcal{H}_I \otimes \mathcal{H}_A$ , one has  $\dim \tilde{\Sigma} \leq \dim \mathcal{H}_I \otimes \mathcal{H}_A$  and

$$A \geq M + 1. \quad (7)$$

This condition, called the Hamming bound, clearly prevents us from correcting the full basis of the  $(N^2 - 1)$  traceless Hermitian generators of  $\mathfrak{su}(N)$ , since one always has  $M + 1 = N^2 = I^2 \times A^2 > A$ .

*3.1.2. Implementation of the coding matrix via nonholonomic control for low-dimension systems* The question now is: given a subspace  $\mathcal{C}$  and a set of operators  $\{\hat{G}_{m=1,\dots,M}\}$ , how can one compute and physically implement the ‘‘coding matrix’’  $\hat{C}$  satisfying (6)? As a unitary matrix on  $\mathcal{H}_I \otimes \mathcal{H}_A$ ,  $\hat{C}$  can, if it exists, *i.e.* when (7) is checked, be implemented via the nonholonomic control technique. Provided that two physical perturbations  $\hat{V}_a, \hat{V}_b$  can be alternately applied to the system, which satisfy the BGC on the whole Hilbert space  $\mathcal{H}_I \otimes \mathcal{H}_A$  – note that, here,  $\hat{H}_0 \equiv 0$  so that  $\hat{H}_{a,b} = \hat{V}_{a,b}$ , it is legitimate to look for the coding matrix under the form

$$\hat{U}(\vec{t}) = \exp\left\{\frac{t_{n_c}}{i\hbar}\hat{H}_{n_c}\right\} \times \dots \times \exp\left\{\frac{t_2}{i\hbar}\hat{H}_2\right\} \times \exp\left\{\frac{t_1}{i\hbar}\hat{H}_1\right\} \quad (8)$$

where  $\hat{H}_k \equiv \hat{V}_a$  for odd  $k$ 's and  $\hat{H}_k \equiv \hat{V}_b$  for even  $k$ 's, and search the time vector  $\vec{t}$  such that  $\hat{U}(\vec{t})$  meets (6). It is important to note that, here, the number  $n_c$  of required parameters can be chosen of the same order as that of the conditions set by (6) on the terms of  $\hat{U}(\vec{t})$ , which is less than  $N^2 = I^2 \times A^2$ . The problem we face is actually much less demanding in terms of physical resources than the usual complete evolution control dealt with in the previous section, and a specific algorithm must therefore be developed to solve it.



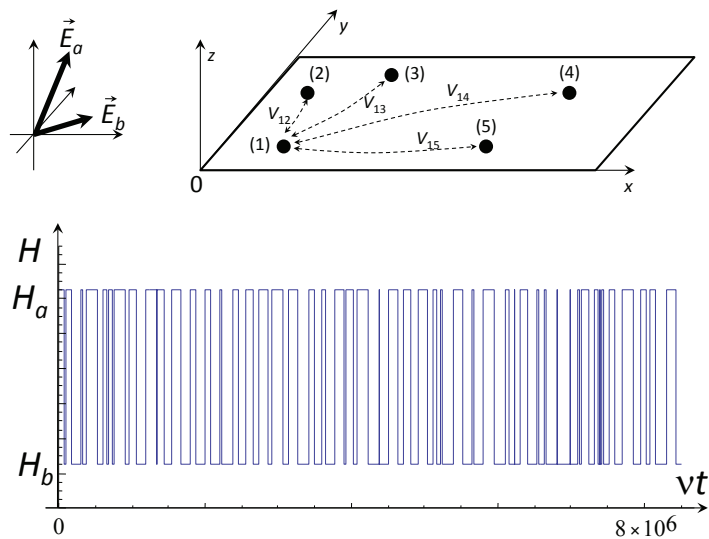
The method we propose is actually a mixture of two algorithms. The first one allows us to explicitly compute an appropriate coding matrix  $\hat{C}$ . We start from an arbitrarily chosen unitary matrix  $\hat{C}_0$  which is then iteratively slightly varied, under the prescription that it remains unitary,  $\hat{C}_0 \rightarrow \hat{C}_1 = \hat{C}_0 + \delta\hat{C}_0 \rightarrow \hat{C}_2 = \hat{C}_1 + \delta\hat{C}_1 \rightarrow \dots$ . At each step, the increment  $\delta\hat{C}$  is calculated through minimizing the quantity  $F(\{\delta_{m=1,\dots,M}\}) \equiv \sum_m \left\| \hat{C} \times \hat{P} + \delta_m \hat{G}_m \times \hat{C} \times \hat{P} \right\|^2$  with respect to the  $\delta_m$ 's, and setting  $\delta\hat{C} = \frac{1}{2} \sum_{m=1}^M \delta_m \hat{G}_m \hat{C}$ . Conditions (6) are always better satisfied by the successive  $\hat{C}_i$ , and, ultimately, completely checked by  $\hat{C} \equiv \hat{C}_\infty$ . Our algorithm converges rapidly, and only a few steps are needed.

We are, however, less interested in the explicit expression of the coding matrix than in getting a physical way to implement it. We thus propose to mix the iterative algorithm we have just described above with the nonholonomic control approach to construct a time vector  $\vec{t}$  such that  $\hat{U}(\vec{t})$  is an appropriate coding matrix. One first chooses an arbitrary time vector  $\vec{t}_0$  – the order of magnitude of the timings is given by the natural time scale of the perturbations. The matrix  $\hat{U}(\vec{t}_0)$  plays the role of  $\hat{C}_0$  for the previous algorithm. Then, one slightly varies the time vector  $\vec{t}_0 \rightarrow \vec{t}_1 = \vec{t}_0 + \delta\vec{t}_0$ , the time increment being determined by  $\delta\hat{C}_0 = \vec{\nabla}\hat{U}(\vec{t}_0) \cdot \delta\vec{t}_0$ . One repeats the same operation to determine  $\delta\vec{t}_1$  such that  $\delta\hat{C}_1 = \vec{\nabla}\hat{U}(\vec{t}_1) \cdot \delta\vec{t}_1$  and set  $\vec{t}_2 = \vec{t}_1 + \delta\vec{t}_1$ , then we compute  $\delta\vec{t}_2$  from  $\delta\hat{C}_2 = \vec{\nabla}\hat{U}(\vec{t}_2) \cdot \delta\vec{t}_2$ , etc. In this way, one forces the unitary matrix  $\hat{U}(\vec{t})$  to follow the path  $\hat{C}_0 \rightarrow \hat{C}_1 \rightarrow \hat{C}_2 \rightarrow \dots$ , while iteratively building up the desired time vector. Finally,  $\hat{U}(\vec{t}_\infty) = \hat{C}_\infty = \hat{C}$ . We shall not present more details about our search algorithm here, but refer to [48] for more information.

As an example, we consider as a toy model the protection of one out of five qubits, arranged in a random two-dimensional array, against all the  $M = 15$  single-qubit errors

$$\begin{aligned} & \hat{\sigma}_x \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}, \\ & \hat{\sigma}_y \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}, \\ & \dots, \\ & \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \hat{\sigma}_y, \\ & \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \hat{\sigma}_z. \end{aligned}$$

as we are allowed by the dimensionality of the ancillary subspace  $A = 2^4 = 16$ , according to the Hamming bound. The model Hamiltonian we choose comprises an interaction part among the qubits  $\hat{V}_{int} = \frac{1}{2} \sum_{i \neq j} \sum_{k=x,y,z} \frac{\alpha_k}{R_{ij}^3} \hat{\sigma}_k^{(i)} \hat{\sigma}_k^{(j)}$  and the (collective) interaction of the qubits with an external field  $\sum_{k=x,y,z} \zeta_k \hat{\Sigma}_k$ , where  $\hat{\Sigma}_k \equiv \sum_i \hat{\sigma}_k^{(i)}$ . Here  $R_{ij}$  denotes the distance between qubits  $i$  and  $j$ , the  $\alpha_k$ 's are constants characterizing the dipole-dipole-like interactions between two qubits, and  $\zeta_k$  denotes the coupling of a qubit to the  $k$ -component of the exterior field. Running our algorithm for two different values of the total Hamiltonian  $\hat{H}_{a,b} = \hat{V}_{int} + \sum_{k=x,y,z} \zeta_k^{(a,b)} \hat{\Sigma}_k$ , with  $\left| \frac{\alpha_k}{R_{ij}^3} \right| \sim \left| \zeta_k^{(a,b)} \right| \sim h\nu$ , where  $\nu$  is the typical frequency of the system, we find an 100-pulse sequence, reproduced in figure 3, achieving an appropriate coding matrix. The timings range from  $4 \times 10^3 \times \nu^{-1}$  to  $1.6 \times 10^5 \times \nu^{-1}$ . Assuming that the whole control Hamiltonian can be reversed, including



**Figure 3.** Protection of one out of five qubits against single-qubit errors. The qubits are arranged on a surface, interact via a dipole-dipole-like Hamiltonian and are alternately submitted to two different control fields  $\vec{E}^{(a,b)}$  responsible for the perturbations  $\hat{V}_{a,b}$ . A typical control pulse sequence achieving an appropriate coding matrix is reproduced on the figure.

the interactions among qubits, the decoding matrix can be obtained by merely running the same sequence of pulses backwards with opposite Hamiltonians  $-H_{a,b}$ . If this is not possible, the full control algorithm of the previous section must be run again, with the matrix  $\hat{C}^\dagger$  as the target evolution.

The example we have just addressed is completely new, but we also considered other applications and examples in previous works. For instance, we exhibited a coding matrix able to protect two qubits out of seven against 21 single-qubit errors plus 10 collective errors. This is indeed an important feature of our method that it does not rely on any specific symmetry of the errors, contrary to the independent error-model very often assumed in the literature about error-correction, but can protect information against any operator set  $\{\hat{G}_{m \leq M}\}$  provided (7) is satisfied. We also addressed more realistic specific physical situations, *e.g.* the protection of one qubit of information, stored in the spin of the external electron of a rubidium atom [48], its orbital part playing the role of the ancilla, as well as the protection of two qubits of information in a seven-spin molecule [49].

*3.1.3. Zeno coherence protection via random coding for large systems* The method presented above is adapted to low-dimensional systems. As the dimension increases, however, the algorithm we used to compute and implement the exact coding matrix becomes heavier and heavier, and is finally intractable for large dimensions. Fortunately, for high-dimensional compound systems, affected by physically reasonable error-inducing interactions, by which we mean the independent interaction of each subsystem

with the environment plus binary interactions between subsystems, one can alternatively use random coding to check conditions (6) to a satisfactorily good accuracy, and therefore cancel the first-order effects in (4).

To be more specific, let us consider an  $n$ -qubit compound system comprising an  $i$ -qubit information-carrying system  $\mathcal{I}$  ( $I = 2^i$ ), and an  $a$ -qubit ancilla  $\mathcal{A}$  ( $A = 2^a$ ). We assume the error-inducing Hamiltonian takes the form

$$\begin{aligned} \hat{H}(t) = & \sum_{\alpha=x,y,z} \sum_{k=1,\dots,n} \hat{\sigma}_{\alpha}^{(k)} \otimes \left( \varepsilon_{\alpha}^{(k)}(t) \hat{\mathbb{I}}_{\mathcal{E}} + \varsigma_{\alpha}^{(k)} \hat{E}_{\alpha}^{(k)} \right) \\ & + \frac{1}{2} \sum_{\alpha,\beta} \sum_{k \neq l} \chi_{\alpha,\beta}^{k,l} \hat{\sigma}_{\alpha}^{(k)} \hat{\sigma}_{\beta}^{(l)} \otimes \hat{\mathbb{I}}_{\mathcal{E}} \end{aligned}$$

comprising the  $3n$  independent interactions of each qubit with a classical parasitic field and the environment, plus the  $9n(n-1)/2$  binary interactions among qubits. We therefore have  $M = 3n + \frac{9n(n-1)}{2} = \frac{3n(3n-1)}{2} \sim \frac{9n^2}{2}$  different error operators  $\hat{G}_m$ 's (*i.e.* the  $\sigma_{\alpha}^{(k)}$ 's and the products  $\sigma_{\alpha}^{(k)} \sigma_{\beta}^{(l)}$ ). Moreover, we assume that all the couplings have the same order of magnitude  $\hbar/T_2$ , whence the Zeno range here is given by  $\tau_Z = T_2/M \simeq T_2/n^2$ .

Let us first note that the matrices  $\hat{G}_m$ 's expressed in the natural (decoupled) basis are quite sparse. Typically, they contain  $2^n$  nonzero terms, of order 1. Accordingly, in the same decoupled basis, the matrices  $\hat{P}\hat{G}_m\hat{P}$  typically have  $2^i$  terms of order 1. If we now apply a generic random unitary matrix  $\hat{C}$  at the beginning of the free evolution period and its inverse  $\hat{C}^{\dagger}$  at the end, we will smear out the nonzero terms over all  $2^n$  states of the entire system. In other words, the matrices  $\hat{C}^{\dagger}\hat{G}_m\hat{C}$  will now have  $\sim 2^{2n}$  terms of the same order  $\sim 2^n/2^{2n} = 2^{-n}$ , while the matrices  $\hat{P}\hat{C}^{\dagger}\hat{G}_m\hat{C}\hat{P}$  will have  $\sim 2^{2i}$  nonzero terms of the same order  $\sim 2^{-n}$ . When  $n \rightarrow +\infty$  for a fixed  $i$  (*i.e.* when  $a \rightarrow +\infty$ ), the conditions (6) are perfectly met. It is, however, not enough for the correction method to work. Indeed, when  $n \rightarrow +\infty$ , the number  $M$  of errors to be considered also goes to infinity which could possibly result in a non zero accumulated action of the total error Hamiltonian: fortunately, since  $M$  is only polynomial in  $n$ , the overall action indeed tends to zero when  $n \rightarrow +\infty$ .

As in the low-dimensional case, we suggest to physically implement the random coding matrix  $\hat{C}$  via the alternate application of two Hamiltonians  $\hat{H}_a, \hat{H}_b$  checking the BGC. But, here, there is no need for a precise design of the pulse sequence: a generic choice of a sequence of control timings  $t_1, t_2, \dots, t_R$  of a reasonably short length  $R \sim n$  will result in a completely generic random encoding transformation

$$\hat{C} = e^{\frac{t_R}{i\hbar} \hat{H}_R} \times e^{\frac{t_{R-1}}{i\hbar} \hat{H}_{R-1}} \times \dots \times e^{\frac{t_1}{i\hbar} \hat{H}_1}$$

where, as usual,  $\hat{H}_k = \hat{H}_a$  for odd  $k$ 's and  $\hat{H}_k = \hat{H}_b$  for even  $k$ 's. The decoding matrix  $\hat{C}^{\dagger}$  is simply obtained by running the same sequence of pulses backwards with the Hamiltonians  $-\hat{H}_a, -\hat{H}_b$ . It is important to underline that, though the precise choice of timings  $t_k$  is not necessary to implement an appropriate coding matrix  $\hat{C}$ , it is, however, critical for our method to be able to apply the exact inverse transformation.

As a first example, the method of random coding was applied to the protection of two qubits carried by two nuclear spins in a seven-atom molecule [50]. In the future, it should also allow to deal with much larger systems.

### 3.2. Universal dynamical decoupling from slow noise via the nonholonomic control

The method presented in the previous subsection allows one to protect the information stored in a system through adding a large enough ancilla and repeatedly applying a coding-decoding ancilla measurement cycle to the compound system. The coding and decoding steps are relatively short, compared to the nonholonomic pulse sequences required to solve the complete evolution control problem. The error-prevention is, however, based on a strong redundancy of the information to protect, due to the ancilla-adding.

In this subsection, we aim at fully decoupling and controlling the evolution of an information-carrying system, without resorting to such redundancy, but only through nonholonomic pulse sequences.

*3.2.1. The general framework of the method* Our objective is to design a control Hamiltonian  $\hat{H}_c(t)$  such that, at the end of the control period  $[0, T_c]$ , the first-order of (4) vanishes while the controlled evolution takes an arbitrarily prescribed value  $\hat{U}_d$ , a priori different from the identity,

$$\begin{aligned} \hat{U}_c(T_c) &= \hat{U}_d \quad (9) \\ \forall m, \int_0^t \hat{U}_c^\dagger(s) \times \hat{G}_m \times \hat{U}_c(s) \otimes \left( \varepsilon_m(s) \mathbb{I}_{\mathcal{E}} + \zeta_m \tilde{E}_m(s) \right) ds &= 0. \end{aligned}$$

Assuming  $\varepsilon_m(s)$  and  $\tilde{E}_m(s)$  vary only very slightly on the control period, the second condition boils down to

$$\forall m, \int_0^t \hat{U}_c^\dagger(s) \times \hat{G}_m \times \hat{U}_c(s) ds = 0. \quad (10)$$

(9,10) can be considered as the starting point of most of the noise-tolerant quantum control methods, *i.e.* dynamical decoupling and composite pulses.

Our first goal is to determine the minimal resources required, in terms of Hamiltonian, to reach the double objective given by (9,10). The first naive attempt one could be tempted to make is to use for  $\hat{H}_c(t)$  the same two-valued pulsed form as in Sec. II. A NoGo theorem [51] however claims that universal control and protection cannot always be simultaneously performed in the two-valued control Hamiltonian approach. As we shall now show, (9,10) indeed cannot be universally satisfied by a two-valued pulse sequence for all  $\hat{G}_m$ 's and  $\hat{U}_d$ 's. To show this, we explicitly construct for any two-valued Hamiltonian a noise operator which cannot be universally eliminated. Before giving the details of the proof, let us introduce a few useful definitions and notations. We first define the operators  $\hat{C}_n \equiv \frac{1}{2} \left[ \hat{H}_a + \hat{H}_b, \left( \hat{H}_b - \hat{H}_a \right)^n \right]$  and the operators

$$\hat{U}_{n \geq 1} \equiv \exp \left\{ \frac{t_n}{i\hbar} \hat{H}_n \right\} \times \dots \times \exp \left\{ \frac{t_2}{i\hbar} \hat{H}_b \right\} \times \exp \left\{ \frac{t_1}{i\hbar} \hat{H}_a \right\}$$

$$\hat{U}_0 \equiv \mathbb{I}.$$

We introduce the superoperators  $\hat{\mathcal{H}}_k$  whose action on any operator  $\hat{X}$  is given by  $\mathcal{H}_k \hat{X} \equiv [\hat{H}_k, \hat{X}]$ . We can thus write  $e^{+\varepsilon \hat{H}_k} \hat{X} e^{-\varepsilon \hat{H}_k} = \sum_{l=0}^{+\infty} \frac{\varepsilon^l}{l!} \mathcal{H}_k^l \hat{X}$ . Since  $[\hat{H}_a, (\hat{H}_b - \hat{H}_a)^n] = [\hat{H}_b, (\hat{H}_b - \hat{H}_a)^n] = [\hat{H}_k, (\hat{H}_b - \hat{H}_a)^n]$  for every  $k$ , we can moreover set  $\hat{C}_n \equiv \mathcal{H}_k (\hat{H}_b - \hat{H}_a)^n$ . We can now start the proof by evaluating the first-order effect of  $\hat{C}_n$ , as noise source, according to (10):

$$\begin{aligned} & \int_0^t \hat{U}_c^\dagger(s) \times \hat{C}_n \times \hat{U}_c(s) ds &= \\ & \sum_{k=1}^K \int_0^{t_k} \hat{U}_{k-1}^\dagger e^{is\hat{H}_k} \mathcal{H}_k (\hat{H}_b - \hat{H}_a)^n e^{-is\hat{H}_k} \hat{U}_{k-1} ds &= \\ & \sum_{k=1}^K \hat{U}_{k-1}^\dagger \int_0^{t_k} \sum_{l=0}^{+\infty} \frac{(is)^l}{l!} \mathcal{H}_k^{l+1} (\hat{H}_b - \hat{H}_a)^n \hat{U}_{k-1} ds &= \\ & -i \sum_{k=1}^K \hat{U}_{k-1}^\dagger \left[ e^{it_k \hat{H}_k} (\hat{H}_b - \hat{H}_a)^n e^{-it_k \hat{H}_k} - (\hat{H}_b - \hat{H}_a)^n \right] \hat{U}_{k-1} = \\ & -i \sum_{k=1}^K \hat{U}_k^\dagger (\hat{H}_b - \hat{H}_a)^n \hat{U}_k + i \sum_{k=1}^K \hat{U}_{k-1}^\dagger (\hat{H}_b - \hat{H}_a)^n \hat{U}_{k-1} &= \\ & -i \hat{U}_c^\dagger(T_c) (\hat{H}_b - \hat{H}_a)^n \hat{U}_c(T_c) + i (\hat{H}_b - \hat{H}_a)^n. \end{aligned}$$

The contribution we have just obtained does not depend on the specific timing parameters  $t_k$ 's but only on the final operation  $\hat{U}(T_c) = \hat{U}_d$ . Moreover, when  $\hat{U}_d$  and  $(\hat{H}_b - \hat{H}_a)$  do not commute, this contribution is nonzero at least for  $n = 1$  and possibly for other  $n > 1$ . Note that  $1 \leq \text{rank} [\hat{C}_n, n > 0] \leq N - 1$ . Hence, for any Hamiltonian successively taking the two values  $\hat{H}_a$  and  $\hat{H}_b$ , we have found a noise source  $\hat{C}_1 = \frac{1}{2} [\hat{H}_a + \hat{H}_b, \hat{H}_b - \hat{H}_a] \neq 0$  which cannot be universally eliminated for all operations. This completes the proof: universal control and universal protection cannot always be simultaneously performed in the two-valued control Hamiltonian approach.

Now the question is how to design a decoupling/control pulse sequence: in the following, we suggest a way to construct an appropriate three-valued pulse sequence, through simply adding “waiting periods” to a two-valued pulse scheme achieving the control objective (9) obtained via the technique exposed in Sec. 2. Suppose we have a sequence of timings  $\{t_1, t_2, \dots, t_K\}$  which yields the desired evolution  $\hat{U}_d$

$$\hat{U}_d = \exp \left\{ \frac{t_K}{i\hbar} \hat{H}_b \right\} \times \dots \times \exp \left\{ \frac{t_2}{i\hbar} \hat{H}_b \right\} \times \exp \left\{ \frac{t_1}{i\hbar} \hat{H}_a \right\}.$$

The first-order contribution  $\hat{\mathcal{G}}_m$  of the operator  $\hat{G}_m$  over the control period can be put under the form  $\hat{\mathcal{G}}_m = \sum_{k=1}^K \hat{U}_{k-1}^\dagger \hat{g}_{m,k} \hat{U}_{k-1}$  where  $\hat{g}_{m,k} \equiv \int_0^{t_k} \exp \left( -\frac{s}{i\hbar} \hat{H}_k \right) \hat{G}_m \exp \left( \frac{s}{i\hbar} \hat{H}_k \right) ds$ . If we now allow for waiting periods  $\tau_{k=1, \dots, K}$  between each pulse, during which the Hamiltonian is set to zero – this corresponds to a

third value  $\hat{H}_c = 0$  of the control Hamiltonian, the overall controlled evolution is not changed, but the first-order contribution of the operator  $\hat{G}_m$  over the control period  $\left[0, \sum_{k=1}^K (t_k + \tau_k)\right]$  now takes the form  $\delta\hat{\mathcal{G}}_m(\{\tau_k\}) \equiv \hat{\mathcal{G}}_m + \sum_{k=1}^K \tau_k \times (\hat{U}_k^\dagger \hat{G}_m \hat{U}_k)$  which is a linear function of the waiting times. Assuming that, for all  $m = 1, \dots, M$ , each set of operators  $\left\{\hat{U}_k^\dagger \hat{G}_m \hat{U}_k, \quad k = 1, \dots, K\right\}$  spans the entire space of  $N \times N$  traceless Hermitian operators, then it is possible to find sets of parameters  $\left\{\tau_k^{(m)}\right\}$  which, independently, annul the linear combinations  $\delta\hat{\mathcal{G}}_m$ , that is  $\delta\hat{\mathcal{G}}_m\left(\left\{\tau_k^{(m)}\right\}\right) = 0$ . For the same set of parameters  $\{\tau_k\}$  to simultaneously annul all the quantities  $\delta\hat{\mathcal{G}}_m(\{\tau_k\})$ ,  $m = 1, \dots, M$ , it seems reasonable to impose that  $K$  must be of the same order as  $M \times N^2$ , that is, for  $M = N^2 - 1$  corresponding to the protection against all possible error operators  $\hat{G}_m$ ,  $K$  must be of the order of  $N^4$ .

Hence, to compute the appropriate control sequence, we suggest to apply the following two-step algorithm. First, given two Hamiltonians  $\hat{H}_a$  and  $\hat{H}_b$  which satisfy the BGC, we compute the  $L \simeq N^2$  timings  $\{t_{i=1, \dots, L}^*\}$  which achieve  $\hat{U}_d^{\frac{1}{N^2}}$  according to the method described in Sec. II. The same sequence is concatenated  $N^2$  times to yield the  $K = L \times N^2 \simeq N^4$  timings  $\{t_{k=1, \dots, K}\}$  which yield  $\hat{U}_d$ . Then, we look for the waiting times  $\{\tau_k\}$  by direct minimization to zero of the functional

$$F(\{\tau_k\}) \equiv \sum_{m=1}^M \left\| \delta\hat{\mathcal{G}}_m(\{\tau_k\}) \right\|^2$$

constructed as the sum of the norms of the actions of all possible errors accumulated during the control period. In practice, since the waiting times must be positive,  $K$  should be chosen slightly larger than  $N^4$  and the use of linear programming methods or other minimization techniques is required. Of course, any such technique will only find an all-positive solution if one exists. If this is not the case, however, one can simply repeat the algorithm with different initial timings and try again.

*3.2.2. Control and protection of an atomic qutrit* As an example, we applied our method to the same system as considered in Sec. II, *i.e.* an atomic qutrit consisting of three degenerate hyperfine states of a hydrogen atom's ground level [51]. The goal was to apply the same evolution operator as previously

$$\hat{U}_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

to the system while completely decoupling it from any arbitrary, though slow enough, classical noise and/or environment. Running our algorithm, we find a control sequence consisting in 208 pulses, whose durations range from  $2.5\mu\text{s}$  to  $690\mu\text{s}$  for a total period of control  $T_c \simeq 6.58\text{ms}$ . Though the pulse timings and magnetic field intensities are experimentally feasible, the time duration required for the whole gate may seem quite long; it must, however, be recalled that we are working with the hydrogen atom ground

state which is very stable indeed. If noises are too large – and therefore their accumulated action on the whole sequence too big, higher levels may also be considered, which could offer shorter gate timescales.

### 3.3. Higher-order correction

In this last subsection, we address the question of error-correction beyond the first order of the perturbation theory. On first thoughts, it seems desirable to try and correct all the orders of the perturbative expansion of the evolution operator. High-order correction, however, has a cost, that is the lengthening of the control pulse sequence, which, in turn, also increases the errors accumulated in the system. The necessary compromise one has to find between these two contradictory effects sets an upper limit on the order of correction one should consider to effectively reduce the impact of the errors to its minimum. In the following paragraphs, we propose a quantitative study of these different aspects.

To begin with, let us assume that the first-order error contribution in (4) has been suppressed thanks to the control Hamiltonian  $\hat{H}_c(t)$  as described in the previous subsection: the second-order contribution is now the leading term in the expansion, hence the most dangerous, and must therefore be compensated for. Still assuming the classical noise and environment evolve slowly enough on the control period, we must therefore add  $(N^2 - 1)^2$  matrix constraints on  $\hat{H}_c(t)$

$$\forall (m, n), \quad \int_0^{T_c} dt \int_0^t ds \hat{U}_c^\dagger(t) \hat{G}_m \hat{U}_c(t) \hat{U}_c^\dagger(s) \hat{G}_n \hat{U}_c(s) = 0$$

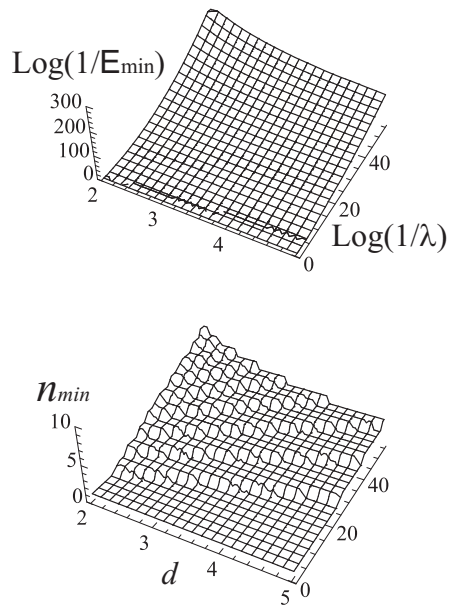
which are equivalent to  $N^2 \times (N^2 - 1)^2$  scalar equations. Together with the control objective and the first-order conditions (9,10), this imposes  $\sim N^6$  requirements on the control sequence.

In the same way, assuming errors are suppressed up to the  $n^{\text{th}}$  order, by a control sequence of length  $K \sim N^{2(n+1)}$ , the remaining error contribution is of order  $(n+1)$  and given by  $\mathcal{E}_n \sim \left(\frac{T_c}{\tau_Z}\right)^{n+1} \times \frac{1}{(n+1)!}$  where the control period can be estimated from the average duration  $\bar{t}$  of one pulse, *i.e.*  $T_c \simeq K \times \bar{t} \sim N^{2(n+1)} \times \bar{t}$ , whence

$$\mathcal{E}_n \sim \lambda^{n+1} \times \frac{N^{2(n+1)^2}}{(n+1)!}$$

where  $\lambda \equiv \frac{\bar{t}}{\tau_Z}$  corresponds to the typical action of the errors in units of  $\hbar$ , and can also be viewed as the error probability amplitude during one control step.

To be more specific, let us consider a system of  $d$  qubits, for which  $N = 2^d$ . Given  $\lambda$  and  $d$ , the remaining error  $\mathcal{E}_n$  reaches its minimum  $\mathcal{E}_{\min}(\lambda, d)$  for a certain order  $n = n_{\min}(\lambda, d)$ . Figure 4 shows  $\mathcal{E}_{\min}(\lambda, d)$  and the corresponding order of error-correction  $n_{\min}(\lambda, d)$  as functions of  $\lambda$  and  $d$ . On this figure, one sees, in particular, that, for a system of more than three qubits, the second-order correction becomes valuable only for  $\lambda \lesssim e^{-20}$ , *i.e.*  $\bar{t} \lesssim 10^{-9} \times \tau_Z$ . The control step duration cannot be less than  $10^{-9}$ s, corresponding to the limit of feasible commutation times for the



5

**Figure 4.** Minimum error (up) and the corresponding order of the perturbation (down) as functions of the typical accumulated error during the control step  $\lambda$  and the dimension of the Hilbert space of the system  $N = 2^d$ .

current laser devices. Conversely, for the second-order to be meaningful, the Zeno range  $\tau_Z$  must be larger than 1s or, equivalently, the accumulated action over one second must be less than  $\hbar$ . This requirement is very strong: basically it boils down to demanding that the system is almost perfectly isolated! Our quantitative study therefore shows that, for systems of more than three qubits, second- and higher-order error-correction is not relevant, since it would rather strengthen than correct the error contribution.

#### 4. Conclusion

In this paper, we reviewed an open-loop control method, called the nonholonomic control technique, which allows one to impose an arbitrarily prescribed unitary evolution on a closed system subject to only two controlled perturbations checking the BGC. The general algorithm we developed to explicitly compute the appropriate pulse sequence achieving a specific control objective in a given specific physical situation was presented together with numerical examples, demonstrating the universality of our approach.

We also presented two extensions of our method which enable one to protect information storage and processing from the unwanted effects of the interaction with the environment (quantum errors). These two techniques aim at cancelling the first-order action of uncontrolled quantum and classical arbitrary noises by frequently and strongly perturbing the system either through measurement – quantum-Zeno-effect-like protection method, or by strong and short decoupling pulses. Again, the strength of our methods consists in their generality and universality: no specific assumption is required either on the structure and algebraic properties of the system considered or on the errors



which affect the system, as was shown by the numerical examples we chose to provide.

The perspectives of our work of course concern the field of quantum information where our control techniques could help performing safe quantum gates in physical systems which are currently considered for quantum computing implementations: single-atom systems and atomic ensembles, quantum dots, rare-earth ionic crystals, trapped polar molecules, etc. We would also like to study whether/how they could help enhancing physical phenomena such as, *e.g.*, the Rydberg blockade in atomic ensembles. Finally, we want to investigate how, combined with measurements, our method may enable one to impose any arbitrarily chosen  $SL(N)$  transformation on an  $N$ -dimensional system.

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