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Adaptive estimation of an additive regression function from weakly dependent data

Christophe Chesneau*, Jalal Fadili†and Bertrand Maillot*

Abstract

A d -dimensional nonparametric additive regression model with dependent observations is considered. Using the marginal integration technique and wavelets methodology, we develop a new adaptive estimator for a component of the additive regression function. Its asymptotic properties are investigated via the minimax approach under the \mathbb{L}_2 risk over Besov balls. We prove that it attains a sharp rate of convergence which turns to be the one obtained in the *i.i.d.* case for the standard univariate regression estimation problem.

Keywords and phrases: Additive regression, Adaptivity, Wavelets, Hard thresholding.

AMS 2000 Subject Classifications: 62G07, 62G20.

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1 Introduction

1.1 Problem statement

Let d be a positive integer, $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ be a $\mathbb{R} \times [0, 1]^d$ -valued stationary process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and ρ be a given real measurable function. The unknown regression function associated to $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ and ρ is defined by

$$g(\mathbf{x}) = \mathbb{E}(\rho(Y) | \mathbf{X} = \mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d.$$

In the additive regression model, the function g is considered to have an additive structure, i.e. there exist d unknown real measurable functions g_1, \dots, g_d and an unknown real number μ such that

$$g(\mathbf{x}) = \mu + \sum_{\ell=1}^d g_\ell(x_\ell). \quad (1.1)$$

For any $\ell \in \{1, \dots, d\}$, our goal is to estimate g_ℓ from n observations $(Y_1, \mathbf{X}_1), \dots, (Y_n, \mathbf{X}_n)$ of $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$.

1.2 Overview of previous work

When $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ is a *i.i.d.* process, this additive regression model becomes the standard one. In such a case, Stone in a series of papers [27, 28, 29] proved that g can be estimated with the same rate of estimation error as in the one-dimensional case. The estimation of the component g_ℓ has been investigated in several papers via various methods (kernel, splines, wavelets, etc.). See e.g. [4], [16], [18], [23, 24], [1], [2], [26], [32], [25] and [13].

In some applications, the *i.i.d.* assumption on the observations is too stringent. For this reason, some authors have explored the estimation of g_ℓ in the dependent case. When $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ is a strongly mixing process, this problem has been addressed by [5], [9], and results for continuous time processes under a strong mixing condition have been obtained by [10, 11]. In particular, they have developed non-adaptive kernel estimators for g_ℓ and studied its asymptotic properties.

1.3 Contributions

To the best of our knowledge, adaptive estimation of g_ℓ for dependent processes has been addressed only by [14]. The lack of results for adaptive estimation in this context motivates this work. To reach our goal, as in [32], we combine the marginal integration technique introduced by [22] with wavelet methods. We capitalize on wavelets to construct an adaptive thresholding estimator and show that it attains sharp rates of convergence under mild assumptions on the smoothness of the unknown function. By adaptive, it is meant that the parameters of the estimator do not depend on the parameter(s) of the dependent process nor on those of the smoothness class of the function. In particular, this leads to a simple and easily implementable estimator.

More precisely, our wavelet estimator is based on term-by-term hard thresholding. The idea of this estimator is simple: (i) we estimate the unknown wavelet coefficients of g_ℓ based on the observations; (ii) then we select the greatest ones and ignore the others; (iii) and finally we reconstruct the function estimate from the chosen wavelet coefficients on the considered wavelet basis. Adopting the minimax point of view under the \mathbb{L}_2 risk, we prove that our adaptive estimator attains a sharp rate of convergence over Besov balls which capture a variety of smoothness features in a function including spatially inhomogeneous behavior. The attained rate corresponds to the optimal one in the *i.i.d.* case for the univariate regression estimation problem (up to an extra logarithmic term).

1.4 Paper organization

The rest of the paper is organized as follows. Section 2 presents our assumptions on the model. In Section 3, we describe wavelet bases on $[0, 1]$, Besov balls and tensor product wavelet bases on $[0, 1]^d$. Our wavelet hard thresholding estimator is detailed in Section 4. Its rate of convergence under the \mathbb{L}_2 risk over Besov balls is established in Section 5. Section 6 provides a discussion of the relation of our result with respect to prior work. The proofs are detailed in Section 7.

2 Notations and assumptions

In this work, we assume the following on our model:

Assumptions on the variables.

- For any $i \in \{1, \dots, n\}$, we set $\mathbf{X}_i = (X_{1,i}, \dots, X_{d,i})$. We suppose that
 - for any $i \in \{1, \dots, n\}$, $X_{1,i}, \dots, X_{d,i}$ are identically distributed with the common distribution $\mathcal{U}([0, 1])$,
 - $\mathbf{X}_1, \dots, \mathbf{X}_n$ are identically distributed with the common known density f .
- We suppose that the following identifiability condition is satisfied: for any $\ell \in \{1, \dots, d\}$ and $i \in \{1, \dots, n\}$, we have

$$\mathbb{E}(g_\ell(X_{\ell,i})) = 0. \quad (2.1)$$

Strongly mixing assumption. Throughout this work, we use the strong mixing dependence structure on $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$. For any $m \in \mathbb{Z}$, we define the m -th strongly mixing coefficient of $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ by

$$\alpha_m = \sup_{(A,B) \in \mathcal{F}_{-\infty,0}^{(Y,\mathbf{X})} \times \mathcal{F}_{m,\infty}^{(Y,\mathbf{X})}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, \quad (2.2)$$

where $\mathcal{F}_{-\infty,0}^{(Y,\mathbf{X})}$ is the σ -algebra generated by $\dots, (Y_{-1}, \mathbf{X}_{-1}), (Y_0, \mathbf{X}_0)$ and $\mathcal{F}_{m,\infty}^{(Y,\mathbf{X})}$ is the σ -algebra generated by $(Y_m, \mathbf{X}_m), (Y_{m+1}, \mathbf{X}_{m+1}), \dots$.

We suppose that there exist two constants $\gamma > 0$ and $c > 0$ such that, for any integer $m \geq 1$,

$$\alpha_m \leq \gamma \exp(-cm). \quad (2.3)$$

Further details on strongly mixing dependence can be found in [3], [31], [12], [21] and [6].

Boundedness assumptions.

- We suppose that $\rho \in \mathbb{L}_1(\mathbb{R}) \cap \mathbb{L}_\infty(\mathbb{R})$, i.e. there exist constants $C_1 > 0$ and C_2 (supposed known) such that

$$\int_{-\infty}^{\infty} |\rho(y)| dy \leq C_1, \quad (2.4)$$

$$\text{and } \sup_{y \in \mathbb{R}} |\rho(y)| \leq C_2. \quad (2.5)$$

- We suppose that there exists a known constant $c > 0$ such that

$$\inf_{\mathbf{x} \in [0,1]^d} f(\mathbf{x}) \geq c. \quad (2.6)$$

- For any $m \in \{1, \dots, n\}$, let $f_{(Y_0, \mathbf{X}_0, Y_m, \mathbf{X}_m)}$ be the density of $(Y_0, \mathbf{X}_0, Y_m, \mathbf{X}_m)$, $f_{(Y_0, \mathbf{X}_0)}$ the density of (Y_0, \mathbf{X}_0) and, for any $(y, \mathbf{x}, y_*, \mathbf{x}_*) \in \mathbb{R} \times [0, 1]^d \times \mathbb{R} \times [0, 1]^d$,

$$\begin{aligned} h_m(y, \mathbf{x}, y_*, \mathbf{x}_*) = \\ f_{(Y_0, \mathbf{X}_0, Y_m, \mathbf{X}_m)}(y, \mathbf{x}, y_*, \mathbf{x}_*) - f_{(Y_0, \mathbf{X}_0)}(y, \mathbf{x}) f_{(Y_0, \mathbf{X}_0)}(y_*, \mathbf{x}_*). \end{aligned} \quad (2.7)$$

We suppose that there exists a known constant $C > 0$ such that

$$\sup_{m \in \{1, \dots, n\}} \sup_{(y, \mathbf{x}, y_*, \mathbf{x}_*) \in \mathbb{R} \times [0,1]^d \times \mathbb{R} \times [0,1]^d} |h_m(y, \mathbf{x}, y_*, \mathbf{x}_*)| \leq C. \quad (2.8)$$

Such boundedness assumptions are standard for the estimation of g_ℓ from a strongly mixing process. See e.g. [10, 11].

3 Wavelets and Besov balls

3.1 Wavelet bases on $[0, 1]$

Let R be a positive integer. We consider an orthonormal wavelet basis generated by dilations and translations of the scaling and wavelet functions ϕ and ψ from the Daubechies family db_{2R} . In particular, ϕ and ψ have compact supports and unit \mathbb{L}_2 -norm, and ψ has R vanishing moments, i.e. for any $r \in \{0, \dots, R - 1\}$, $\int x^r \psi(x) dx = 0$.

Define the scaled and translated version of ϕ and ψ

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then, with an appropriate treatment at the boundaries, there exists an integer τ satisfying $2^\tau \geq 2R$ such that, for any integer $j_* \geq \tau$, the collection $\{\phi_{j_*,k}(\cdot), k \in \{0, \dots, 2^{j_*}-1\}; \psi_{j,k}(\cdot); j \in \mathbb{N}-\{0, \dots, j_*-1\}, k \in \{0, \dots, 2^j-1\}\}$, is an orthonormal basis of $\mathbb{L}_2([0, 1]) = \{h : [0, 1] \rightarrow \mathbb{R}; \int_0^1 h^2(x)dx < \infty\}$. See [7, 19].

Consequently, for any integer $j_* \geq \tau$, any $h \in \mathbb{L}_2([0, 1])$ can be expanded into a wavelet series as

$$h(x) = \sum_{k=0}^{2^{j_*}-1} \alpha_{j_*,k} \phi_{j_*,k}(x) + \sum_{j=j_*}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where

$$\alpha_{j,k} = \int_0^1 h(x) \phi_{j,k}(x) dx, \quad \beta_{j,k} = \int_0^1 h(x) \psi_{j,k}(x) dx. \quad (3.1)$$

3.2 Besov balls

As is traditional in the wavelet estimation literature, we will investigate the performance of our estimator by assuming that the unknown function to be estimated belongs to a Besov ball. The Besov norm for a function can be related to a sequence space norm on its wavelet coefficients. More precisely, let $M > 0$, $s \in (0, R)$, $p \geq 1$ and $q \geq 1$. A function h in $\mathbb{L}_2([0, 1])$ belongs to $\mathbf{B}_{p,q}^s(M)$ if, and only if, there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (3.1) satisfy

$$\left(\sum_{j=\tau}^{\infty} \left(2^{j(s+1/2-1/p)} \left(\sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \right)^{1/p} \right)^q \right)^{1/q} \leq M^*.$$

In this expression, s is a smoothness parameter and p and q are norm parameters. Besov spaces include many traditional smoothness spaces. For particular choices of s , p and q , Besov balls contain the standard Hölder and Sobolev balls. See [20].

3.3 Wavelet tensor product bases on $[0, 1]^d$

For the purpose of this paper, we will use compactly supported tensor product wavelet bases on $[0, 1]^d$ based on the Daubechies family. Let us briefly recall their construction. For any $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$, we construct a scaling function

$$\Phi(\mathbf{x}) = \prod_{v=1}^d \phi(x_v),$$

and $2^d - 1$ wavelet functions

$$\Psi_u(\mathbf{x}) = \begin{cases} \psi(x_u) \prod_{\substack{v=1 \\ v \neq u}}^d \phi(x_v) & \text{when } u \in \{1, \dots, d\}, \\ \prod_{v \in A_u} \psi(x_v) \prod_{v \notin A_u} \phi(x_v) & \text{when } u \in \{d+1, \dots, 2^d - 1\}, \end{cases}$$

where $(A_u)_{u \in \{d+1, \dots, 2^d - 1\}}$ forms the set of all non void subsets of $\{1, \dots, d\}$ of cardinality greater or equal to 2.

For any integer j and any $\mathbf{k} = (k_1, \dots, k_d)$, define the translated and dilated versions of Φ and Ψ_u as

$$\begin{aligned} \Phi_{j,\mathbf{k}}(\mathbf{x}) &= 2^{jd/2} \Phi(2^j x_1 - k_1, \dots, 2^j x_d - k_d), \\ \Psi_{j,\mathbf{k},u}(\mathbf{x}) &= 2^{jd/2} \Psi_u(2^j x_1 - k_1, \dots, 2^j x_d - k_d), \text{ for any } u \in \{1, \dots, 2^d - 1\}. \end{aligned}$$

Let $D_j = \{0, \dots, 2^j - 1\}^d$. Then, with an appropriate treatment at the boundaries, there exists an integer τ such that the collection

$$\{\Phi_{\tau,\mathbf{k}}, \mathbf{k} \in D_\tau; (\Psi_{j,\mathbf{k},u})_{u \in \{1, \dots, 2^d - 1\}}, j \in \mathbb{N} - \{0, \dots, \tau - 1\}, \mathbf{k} \in D_j\}$$

forms an orthonormal basis of $\mathbb{L}_2([0, 1]^d) = \{h : [0, 1]^d \rightarrow \mathbb{R}; \int_{[0, 1]^d} h^2(\mathbf{x}) d\mathbf{x} < \infty\}$.

For any integer j_* such that $j_* \geq \tau$, a function $h \in \mathbb{L}_2([0, 1]^d)$ can be expanded into a wavelet series as

$$h(\mathbf{x}) = \sum_{\mathbf{k} \in D_{j_*}} \alpha_{j_*,\mathbf{k}} \Phi_{j_*,\mathbf{k}}(\mathbf{x}) + \sum_{u=1}^{2^d - 1} \sum_{j=j_*}^{\infty} \sum_{\mathbf{k} \in D_j} \beta_{j,\mathbf{k},u} \Psi_{j,\mathbf{k},u}(\mathbf{x}), \quad \mathbf{x} \in [0, 1]^d,$$

where

$$\alpha_{j,\mathbf{k}} = \int_{[0,1]^d} h(\mathbf{x})\Phi_{j,\mathbf{k}}(\mathbf{x})d\mathbf{x}, \quad \beta_{j,\mathbf{k},u} = \int_{[0,1]^d} h(\mathbf{x})\Psi_{j,\mathbf{k},u}(\mathbf{x})d\mathbf{x}. \quad (3.2)$$

4 The estimator

4.1 Wavelet coefficients estimator

The following proposition provides a wavelet decomposition of g_ℓ based on the ‘‘marginal integration’’ method (introduced by [22]) and a tensor product wavelet basis on $[0, 1]^d$.

Proposition 4.1 *Suppose that (2.1) holds. Then, for any $j_* \geq \tau$ and $\ell \in \{1, \dots, d\}$, we can write*

$$g_\ell(x) = \sum_{k=1}^{2^{j_*}-1} a_{j_*,k,\ell} \phi_{j_*,k}(x) + \sum_{j=j_*}^{\infty} \sum_{k=1}^{2^j-1} b_{j,k,\ell} \psi_{j,k}(x) - \mu, \quad x \in [0, 1],$$

where

$$a_{j,k,\ell} = a_{j,k_\ell,\ell} = 2^{-j(d-1)/2} \int_{[0,1]^d} g(\mathbf{x}) \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x})d\mathbf{x}, \quad (4.1)$$

$$b_{j,k,\ell} = b_{j,k_\ell,\ell} = 2^{-j(d-1)/2} \int_{[0,1]^d} g(\mathbf{x}) \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{x})d\mathbf{x}, \quad (4.2)$$

and $\mathbf{k}_{-\ell} = (k_1, \dots, k_{\ell-1}, k_{\ell+1}, \dots, k_d)$ and $D_j^* = \{0, \dots, 2^j - 1\}^{d-1}$.

Remark 4.1 *Due to the definitions of g and properties of $\Psi_{j,\mathbf{k},\ell}$, $b_{j,k,\ell}$ is nothing but the wavelet coefficient of g_ℓ , i.e.*

$$b_{j,k,\ell} = \int_0^1 g_\ell(x) \psi_{j,k}(x)dx = \beta_{j,k}. \quad (4.3)$$

Proposition 4.1 suggests that a first step to estimate g_ℓ should consist in estimating the unknown coefficients $a_{j,k,\ell}$ (4.1) and $b_{j,k,\ell}$ (4.2). To this end, we propose the following coefficients estimators

$$\hat{a}_{j,k,\ell} = \hat{a}_{j,k_\ell,\ell} = 2^{-j(d-1)/2} \frac{1}{n} \sum_{i=1}^n \frac{\rho(Y_i)}{f(\mathbf{X}_i)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{X}_i) \quad (4.4)$$

and

$$\widehat{b}_{j,k,\ell} = \widehat{b}_{j,k,\ell} = 2^{-j(d-1)/2} \frac{1}{n} \sum_{i=1}^n \frac{\rho(Y_i)}{f(\mathbf{X}_i)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{X}_i). \quad (4.5)$$

These estimators enjoy powerful statistical properties. Some of them are collected in the following propositions.

Proposition 4.2 (Unbiasedness) *Suppose that (2.1) holds. For any $j \geq \tau$, $\ell \in \{1, \dots, d\}$ and $k \in \{0, \dots, 2^j - 1\}$, $\widehat{a}_{j,k,\ell}$ and $\widehat{b}_{j,k,\ell}$ in (4.4) and (4.5) are unbiased estimators of $a_{j,k,\ell}$ and $b_{j,k,\ell}$ respectively.*

Proposition 4.3 (Moment inequality I) *Suppose that the assumptions of Section 2 hold. Let $j \geq \tau$ such that $2^j \leq n$, $k \in \{0, \dots, 2^j - 1\}$, $\ell \in \{1, \dots, d\}$. Then there exists a constant $C > 0$ such that*

$$\mathbb{E} \left((\widehat{a}_{j,k,\ell} - a_{j,k,\ell})^2 \right) \leq C \frac{1}{n}, \quad \mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \right) \leq C \frac{1}{n}.$$

Remark 4.2 *In the proof of Proposition 4.3, we only need to have the existence of two constants $C > 0$ and $q \in (0, 1)$ such that $\sum_{m=1}^n m^q \alpha_m^q \leq C < \infty$. This latter inequality is obviously satisfied by (2.3).*

Proposition 4.4 (Moment inequality II) *Under the same assumptions of Proposition 4.3, there exists a constant $C > 0$ such that*

$$\mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^4 \right) \leq C \frac{2^j}{n}.$$

Proposition 4.5 (Concentration inequality) *Suppose that the assumptions of Section 2 hold. Let $j \geq \tau$ such that $2^j \leq n/(\ln n)^3$, $k \in \{0, \dots, 2^j - 1\}$, $\ell \in \{1, \dots, d\}$ and $\lambda_n = (\ln n/n)^{1/2}$. Then there exist two constants $C > 0$ and $\kappa > 0$ such that*

$$\mathbb{P} \left(|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| \geq \kappa \lambda_n / 2 \right) \leq C \frac{1}{n^4}.$$

4.2 Hard thresholding estimator

We now turn to the estimator of g_ℓ from $\widehat{a}_{j,k,\ell}$ and $\widehat{b}_{j,k,\ell}$ as introduced in (4.4) and (4.5). Towards this goal, we will only keep the significant wavelet coefficients that are above a certain threshold according to the hard thresholding rule, and then reconstruct from these coefficients. In a compact form, this reads

$$\widehat{g}_\ell(x) = \sum_{k=0}^{2^\tau-1} \widehat{a}_{\tau,k,\ell} \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \widehat{b}_{j,k,\ell} \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| \geq \kappa \lambda_n\}} \psi_{j,k}(x) - \widehat{\mu}, \quad (4.6)$$

where

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^n \rho(Y_i). \quad (4.7)$$

In , j_1 is the resolution level satisfying $2^{j_1} = \lceil n/(\ln n)^3 \rceil$, κ is a large enough constant (the one in Proposition 4.5) and

$$\lambda_n = \sqrt{\frac{\ln n}{n}}.$$

Note that, due to the assumptions on the model, our wavelet hard thresholding estimator (4.2) is simpler than the one of [32].

5 Minimax upper-bound result

Theorem 5.1 below investigates the minimax rates of convergence attained by \widehat{g}_ℓ over Besov balls under the \mathbb{L}_2 risk.

Theorem 5.1 *Let $\ell \in \{1, \dots, d\}$. Suppose that the assumptions of Section 2 hold. Let \widehat{g}_ℓ be the estimator given in (4.2). Suppose that $g_\ell \in \mathbf{B}_{p,q}^s(M)$ with $q \geq 1$, $\{p \geq 2 \text{ and } s \in (0, R)\}$ or $\{p \in [1, 2) \text{ and } s \in (1/p, R)\}$. Then there exists a constant $C > 0$ such that*

$$\mathbb{E} \left(\int_0^1 (\widehat{g}_\ell(x) - g_\ell(x))^2 dx \right) \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}.$$

The proof of Theorem 5.1 is based on a suitable decomposition of the \mathbb{L}_2 risk and the statistical properties of (4.4) and (4.5) summarized in Propositions 4.2, 4.3, 4.4 and 4.5 above.

6 Relation to prior work

The rate $(\ln n/n)^{2s/(2s+1)}$ is, up to an extra logarithmic term, known to be the optimal one for the standard one-dimensional regression model with uniform random design. See e.g. [15] and [30].

Theorem 5.1 provides an “adaptive contribution” to the results of [5], [9] and [10, 11]. Furthermore, if we confine ourselves to the *i.i.d.* case, we recover a similar result to [32, Theorem 3] but without the condition $s > \max(d/2, d/p)$. The price to pay is more restrictive assumptions on the model (ρ is bounded from above, the density of \mathbf{X} is known, etc.). Additionally, our estimator has a more straightforward and friendly implementation than the one in [32].

7 Proofs

In this section, the quantity C denotes any constant that does not depend on j , k and n . Its value may change from one term to another and may depend on ϕ or ψ .

7.1 Technical results on wavelets

Proof of Proposition 4.1. Because of (2.5), we have $g \in \mathbb{L}_2([0, 1]^d)$. For any $j_* \geq \tau$, we can expand g on our wavelet-tensor product basis as

$$g(\mathbf{x}) = \sum_{\mathbf{k} \in D_{j_*}} \alpha_{j_*, \mathbf{k}} \Phi_{j_*, \mathbf{k}}(\mathbf{x}) + \sum_{u=1}^{2^d-1} \sum_{j=j_*}^{\infty} \sum_{\mathbf{k} \in D_j} \beta_{j, \mathbf{k}, u} \Psi_{j, \mathbf{k}, u}(\mathbf{x}), \quad \mathbf{x} \in [0, 1]^d \quad (7.1)$$

where

$$\alpha_{j, \mathbf{k}} = \int_{[0, 1]^d} g(\mathbf{x}) \Phi_{j, \mathbf{k}}(\mathbf{x}) d\mathbf{x}, \quad \beta_{j, \mathbf{k}, u} = \int_{[0, 1]^d} g(\mathbf{x}) \Psi_{j, \mathbf{k}, u}(\mathbf{x}) d\mathbf{x}.$$

Moreover, using the “marginal integration” method based on (2.1), we can write

$$g_\ell(x_\ell) = \int_{[0, 1]^{d-1}} g(\mathbf{x}) \prod_{\substack{v=1 \\ v \neq \ell}}^d dx_v - \mu, \quad x_\ell \in [0, 1]. \quad (7.2)$$

Since $\int_0^1 \phi_{j,k}(x)dx = 2^{-j/2}$ and $\int_0^1 \psi_{j,k}(x)dx = 0$, observe that

$$\int_{[0,1]^{d-1}} \Phi_{j^*,\mathbf{k}}(\mathbf{x}) \prod_{\substack{v=1 \\ v \neq \ell}}^d dx_v = 2^{-j^*(d-1)/2} \phi_{j^*,k_\ell}(x_\ell)$$

and

$$\int_{[0,1]^{d-1}} \Psi_{j,\mathbf{k},u}(\mathbf{x}) \prod_{\substack{v=1 \\ v \neq \ell}}^d dx_v = \begin{cases} 2^{-j(d-1)/2} \psi_{j,k_\ell}(x_\ell) & \text{if } u = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, putting (7.1) in (7.2) and writing $x = x_\ell$, we obtain

$$g_\ell(x) = \sum_{\mathbf{k} \in D_{j^*}} 2^{-j^*(d-1)/2} \alpha_{j^*,\mathbf{k}} \phi_{j^*,k_\ell}(x) + \sum_{j=j^*}^{\infty} \sum_{\mathbf{k} \in D_j} 2^{-j(d-1)/2} \beta_{j,\mathbf{k},\ell} \psi_{j,k_\ell}(x) - \mu.$$

Or, equivalently,

$$g_\ell(x) = \sum_{k=1}^{2^{j^*}-1} a_{j^*,k,\ell} \phi_{j^*,k}(x) + \sum_{j=j^*}^{\infty} \sum_{k=1}^{2^j-1} b_{j,k,\ell} \psi_{j,k}(x) - \mu,$$

where

$$a_{j,k,\ell} = a_{j,k_\ell,\ell} = 2^{-j(d-1)/2} \int_{[0,1]^d} g(\mathbf{x}) \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) d\mathbf{x}$$

and

$$b_{j,k,\ell} = b_{j,k_\ell,\ell} = 2^{-j(d-1)/2} \int_{[0,1]^d} g(\mathbf{x}) \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{x}) d\mathbf{x}.$$

Proposition 4.1 is proved. ■

Proposition 7.1 *For any $\ell \in \{1, \dots, d\}$, $j \geq \tau$ and $k = k_\ell \in \{0, \dots, 2^j - 1\}$, set*

$$h_{j,k}^{(1)}(\mathbf{x}) = \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}), \quad h_{j,k}^{(2)}(\mathbf{x}) = \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{x}), \quad \mathbf{x} \in [0, 1]^d.$$

Then there exists a constant $C > 0$ such that, for any $a \in \{1, 2\}$,

$$\sup_{\mathbf{x} \in [0,1]^d} |h_{j,k}^{(a)}(\mathbf{x})| \leq C2^{jd/2}, \quad \int_{[0,1]^d} |h_{j,k}^{(a)}(\mathbf{x})| d\mathbf{x} \leq C2^{-j/2}2^{j(d-1)/2}$$

and

$$\int_{[0,1]^d} (h_{j,k}^{(a)}(\mathbf{x}))^2 d\mathbf{x} = 2^{j(d-1)}.$$

Proof:

- Since $\sup_{x \in [0,1]} |\phi_{j,k}(x)| \leq C2^{j/2}$ and $\sup_{x \in [0,1]} \sum_{k=0}^{2^j-1} |\phi_{j,k}(x)| \leq C2^{j/2}$, we obtain

$$\sup_{\mathbf{x} \in [0,1]^d} |h_{j,k}^{(1)}(\mathbf{x})| = \left(\sup_{x \in [0,1]} |\phi_{j,k}(x)| \right) \left(\sup_{x \in [0,1]} \sum_{k=0}^{2^j-1} |\phi_{j,k}(x)| \right)^{d-1} \leq C2^{jd/2}.$$

- Using $\int_0^1 |\phi_{j,k}(x)| dx = C2^{-j/2}$, we obtain

$$\begin{aligned} \int_{[0,1]^d} |h_{j,k}^{(1)}(\mathbf{x})| d\mathbf{x} &\leq \left(\int_0^1 |\phi_{j,k}(x)| dx \right) \left(\sum_{k=0}^{2^j-1} \int_0^1 |\phi_{j,k}(x)| dx \right)^{d-1} \\ &= C2^{-j/2}2^{j(d-1)/2}. \end{aligned}$$

- Since, for any $(u_{\mathbf{k}})_{\mathbf{k} \in D_j}$, $\int_{[0,1]^d} \left(\sum_{\mathbf{k} \in D_j} u_{\mathbf{k}} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right)^2 d\mathbf{x} = \sum_{\mathbf{k} \in D_j} u_{\mathbf{k}}^2$, we obtain

$$\int_{[0,1]^d} (h_{j,k}^{(1)}(\mathbf{x}))^2 d\mathbf{x} = \int_{[0,1]^d} \left(\sum_{\mathbf{k} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right)^2 d\mathbf{x} = 2^{j(d-1)}.$$

Proceeding in a similar fashion, using $\sup_{x \in [0,1]} |\psi_{j,k}(x)| \leq C2^{j/2}$,

$$\int_0^1 |\psi_{j,k}(x)| dx = C2^{-j/2} \text{ and, for any } (u_{\mathbf{k}})_{\mathbf{k} \in D_j}, \int_{[0,1]^d} \left(\sum_{\mathbf{k} \in D_j} u_{\mathbf{k}} \Psi_{j,\mathbf{k},\ell}(\mathbf{x}) \right)^2 d\mathbf{x} = \sum_{\mathbf{k} \in D_j} u_{\mathbf{k}}^2, \text{ we obtain the same results for } h_{j,k}^{(2)}.$$

This ends the proof of Proposition 7.1. ■

7.2 Statistical properties of the coefficients estimators

Proof of Proposition 4.2. We have

$$\begin{aligned}
\mathbb{E}(\widehat{a}_{j,k,\ell}) &= 2^{-j(d-1)/2} \mathbb{E} \left(\frac{\rho(Y_1)}{f(\mathbf{X}_1)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{X}_1) \right) \\
&= 2^{-j(d-1)/2} \mathbb{E} \left(\mathbb{E}(\rho(Y_1) | \mathbf{X}_1) \frac{1}{f(\mathbf{X}_1)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{X}_1) \right) \\
&= 2^{-j(d-1)/2} \mathbb{E} \left(\frac{g(\mathbf{X}_1)}{f(\mathbf{X}_1)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{X}_1) \right) \\
&= 2^{-j(d-1)/2} \int_{[0,1]^d} \frac{g(\mathbf{x})}{f(\mathbf{x})} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \\
&= 2^{-j(d-1)/2} \int_{[0,1]^d} g(\mathbf{x}) \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) d\mathbf{x} = a_{j,k,\ell}.
\end{aligned}$$

Proceeding in a similar fashion, we prove that $\mathbb{E}(\widehat{b}_{j,k,\ell}) = b_{j,k,\ell}$. ■

Proof of Proposition 4.3. For the sake of simplicity, for any $i \in \{1, \dots, n\}$, set

$$Z_i = \frac{\rho(Y_i)}{f(\mathbf{X}_i)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{X}_i).$$

Thanks to Proposition 4.2, we have

$$\mathbb{E}((\widehat{a}_{j,k,\ell} - a_{j,k,\ell})^2) = \mathbb{V}(\widehat{a}_{j,k,\ell}) = 2^{-j(d-1)} \frac{1}{n^2} \mathbb{V} \left(\sum_{i=1}^n Z_i \right). \quad (7.3)$$

An elementary covariance decomposition gives

$$\begin{aligned}
\mathbb{V} \left(\sum_{i=1}^n Z_i \right) &= n\mathbb{V}(Z_1) + 2 \sum_{v=2}^n \sum_{u=1}^{v-1} \text{Cov}(Z_v, Z_u) \\
&\leq n\mathbb{V}(Z_1) + 2 \left| \sum_{v=2}^n \sum_{u=1}^{v-1} \text{Cov}(Z_v, Z_u) \right|. \quad (7.4)
\end{aligned}$$

Using (2.5), (2.6) and Proposition 7.1, we have

$$\begin{aligned}
\mathbb{V}(Z_1) &\leq \mathbb{E}(Z_1^2) \leq \frac{\sup_{y \in \mathbb{R}} \rho^2(y)}{\inf_{\mathbf{x} \in [0,1]^d} f(\mathbf{x})} \mathbb{E} \left(\frac{1}{f(\mathbf{X}_1)} \left(\sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{X}_1) \right)^2 \right) \\
&\leq C \int_{[0,1]^d} \frac{1}{f(\mathbf{x})} \left(\sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right)^2 f(\mathbf{x}) d\mathbf{x} \\
&= C \int_{[0,1]^d} \left(\sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right)^2 d\mathbf{x} = C 2^{j(d-1)}. \tag{7.5}
\end{aligned}$$

It follows from the stationarity of $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ and $2^j \leq n$ that

$$\left| \sum_{v=2}^n \sum_{u=1}^{v-1} \text{Cov}(Z_v, Z_u) \right| = \left| \sum_{m=1}^n (n-m) \text{Cov}(Z_0, Z_m) \right| \leq R_1 + R_2, \tag{7.6}$$

where

$$R_1 = n \sum_{m=1}^{2^j-1} |\text{Cov}(Z_0, Z_m)|, \quad R_2 = n \sum_{m=2^j}^n |\text{Cov}(Z_0, Z_m)|.$$

It remains to bound R_1 and R_2 .

- (i) **Bound for R_1 .** Let, for any $(y, \mathbf{x}, y_*, \mathbf{x}_*) \in \mathbb{R} \times [0, 1]^d \times \mathbb{R} \times [0, 1]^d$, $h_m(y, \mathbf{x}, y_*, \mathbf{x}_*)$ be (2.7). Using (2.8), (2.4) and Proposition 7.1, we

obtain

$$\begin{aligned}
& |\text{Cov}(Z_0, Z_m)| \\
&= \left| \int_{-\infty}^{\infty} \int_{[0,1]^d} \int_{-\infty}^{\infty} \int_{[0,1]^d} h_m(y, \mathbf{x}, y_*, \mathbf{x}_*) \times \right. \\
&\quad \left. \left(\frac{\rho(y)}{f(\mathbf{x})} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \frac{\rho(y_*)}{f(\mathbf{x}_*)} \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}_*) \right) dy d\mathbf{x} dy_* d\mathbf{x}_* \right| \\
&\leq \int_{-\infty}^{\infty} \int_{[0,1]^d} \int_{-\infty}^{\infty} \int_{[0,1]^d} |h_m(y, \mathbf{x}, y_*, \mathbf{x}_*)| \times \\
&\quad \left| \frac{\rho(y)}{f(\mathbf{x})} \right| \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right| \left| \frac{\rho(y_*)}{f(\mathbf{x}_*)} \right| \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}_*) \right| dy d\mathbf{x} dy_* d\mathbf{x}_* \\
&\leq C \left(\int_{-\infty}^{\infty} |\rho(y)| dy \right)^2 \left(\int_{[0,1]^d} \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right| d\mathbf{x} \right)^2 \leq C 2^{-j} 2^{j(d-1)}.
\end{aligned}$$

Therefore

$$R_1 \leq C n 2^{-j} 2^{j(d-1)} 2^j = C n 2^{j(d-1)}. \quad (7.7)$$

(ii) **Bound for R_2 .** By the Davydov inequality for strongly mixing processes (see [8]), for any $q \in (0, 1)$, we have

$$\begin{aligned}
& |\text{Cov}(Z_0, Z_m)| \leq 10 \alpha_m^q \left(\mathbb{E} \left(|Z_0|^{2/(1-q)} \right) \right)^{1-q} \\
&\leq 10 \alpha_m^q \left(\frac{\sup_{y \in \mathbb{R}} |\rho(y)|}{\inf_{\mathbf{x} \in [0,1]^d} f(\mathbf{x})} \sup_{\mathbf{x} \in [0,1]^d} \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right| \right)^{2q} \left(\mathbb{E}(Z_0^2) \right)^{1-q}.
\end{aligned}$$

By (2.5), (2.6) and Proposition 7.1, we have

$$\begin{aligned}
\frac{\sup_{y \in \mathbb{R}} |\rho(y)|}{\inf_{\mathbf{x} \in [0,1]^d} f(\mathbf{x})} \sup_{\mathbf{x} \in [0,1]^d} \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right| &\leq C \sup_{\mathbf{x} \in [0,1]^d} \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Phi_{j,\mathbf{k}}(\mathbf{x}) \right| \\
&\leq C 2^{jd/2}.
\end{aligned}$$

By (7.5), we have

$$\mathbb{E}(Z_0^2) \leq C 2^{j(d-1)}.$$

Therefore

$$|\text{Cov}(Z_0, Z_m)| \leq C2^{qj}2^{j(d-1)}\alpha_m^q.$$

Observe that $\sum_{m=1}^{\infty} m^q \alpha_m^q = \gamma^q \sum_{m=1}^{\infty} m^q \exp(-cqm) < \infty$. Hence

$$R_2 \leq Cn2^{qj}2^{j(d-1)} \sum_{m=2^j}^n \alpha_m^q \leq Cn2^{j(d-1)} \sum_{m=2^j}^n m^q \alpha_m^q \leq Cn2^{j(d-1)}. \quad (7.8)$$

Putting (7.6), (7.7) and (7.8) together, we have

$$\left| \sum_{v=2}^n \sum_{u=1}^{v-1} \text{Cov}(Z_v, Z_u) \right| \leq Cn2^{j(d-1)}. \quad (7.9)$$

Combining (7.3), (7.4), (7.5) and (7.9), we obtain

$$\mathbb{E} \left((\widehat{a}_{j,k,\ell} - a_{j,k,\ell})^2 \right) \leq C2^{-j(d-1)} \frac{1}{n^2} n2^{j(d-1)} = C \frac{1}{n}.$$

Proceeding in a similar fashion, we prove that

$$\mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \right) \leq C \frac{1}{n}.$$

This ends the proof of Proposition 4.3. ■

Proof of Proposition 4.4. It follows from (2.5), (2.6) and Proposition 7.1 that

$$\begin{aligned} |\widehat{b}_{j,k,\ell}| &\leq 2^{-j(d-1)/2} \frac{1}{n} \sum_{i=1}^n \frac{|\rho(Y_i)|}{|f(\mathbf{X}_i)|} \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{X}_i) \right| \\ &\leq 2^{-j(d-1)/2} \frac{\sup_{y \in \mathbb{R}} |\rho(y)|}{\inf_{\mathbf{x} \in [0,1]^d} f(\mathbf{x})} \sup_{\mathbf{x} \in [0,1]^d} \left| \sum_{\mathbf{k}_{-\ell} \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{x}) \right| \\ &\leq C2^{-j(d-1)/2} 2^{jd/2} = C2^{j/2}. \end{aligned}$$

Because of (2.5), we have $\sup_{\mathbf{x} \in [0,1]^d} |g(\mathbf{x})| \leq C$. It follows from Proposition 7.1 that

$$\begin{aligned}
|b_{j,k,\ell}| &\leq 2^{-j(d-1)/2} \int_{[0,1]^d} |g(\mathbf{x})| \left| \sum_{\mathbf{k}-\ell \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{x}) \right| d\mathbf{x} \\
&\leq C 2^{-j(d-1)/2} \int_{[0,1]^d} \left| \sum_{\mathbf{k}-\ell \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{x}) \right| d\mathbf{x} \\
&\leq C 2^{-j(d-1)/2} 2^{-j} 2^{jd/2} = C 2^{-j/2}.
\end{aligned} \tag{7.10}$$

Hence

$$|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| \leq |\widehat{b}_{j,k,\ell}| + |b_{j,k,\ell}| \leq C 2^{j/2}. \tag{7.11}$$

It follows from (7.11) and Proposition 4.3 that

$$\mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^4 \right) \leq C 2^j \mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \right) \leq C \frac{2^j}{n}.$$

The proof of Proposition 4.4 is complete. ■

Proof of Proposition 4.5. Let us first state a Bernstein inequality for exponentially strongly mixing process.

Lemma 7.1 ([17]) *Let $\gamma > 0$, $c > 0$ and $(Y_i)_{i \in \mathbb{Z}}$ be a stationary process with the m -th strongly mixing coefficient α_m (2.2). Let n be a positive integer, $h : \mathbb{R} \rightarrow \mathbb{C}$ be a measurable function and, for any $i \in \mathbb{Z}$, $U_i = h(Y_i)$. We assume that $\mathbb{E}(U_1) = 0$ and there exists a constant $M > 0$ satisfying $|U_1| \leq M$. Then, for any $m \in \{1, \dots, [n/2]\}$ and $\lambda > 0$, we have*

$$\mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \lambda \right) \leq 4 \exp \left(- \frac{\lambda^2 n}{16(D_m/m + \lambda M m/3)} \right) + 32 \frac{M}{\lambda} n \alpha_m,$$

where $D_m = \max_{l \in \{1, \dots, 2m\}} \mathbb{V} \left(\sum_{i=1}^l U_i \right)$.

We now apply this lemma by setting for any $i \in \{1, \dots, n\}$,

$$U_i = 2^{-j(d-1)/2} \frac{\rho(Y_i)}{f(\mathbf{X}_i)} \sum_{\mathbf{k}-\ell \in D_j^*} \Psi_{j,\mathbf{k},\ell}(\mathbf{X}_i) - b_{j,k,\ell}.$$

Then we can write

$$\widehat{b}_{j,k,\ell} - b_{j,k,\ell} = \frac{1}{n} \sum_{i=1}^n U_i.$$

So

$$\mathbb{P} \left(|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| \geq \kappa \lambda_n / 2 \right) = \mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \kappa \lambda_n / 2 \right),$$

where U_1, \dots, U_n are identically distributed, depend on $(Y_i, \mathbf{X}_i)_{i \in \mathbb{Z}}$ satisfying (2.3),

- by Proposition 4.2, we have $\mathbb{E}(U_1) = 0$,
- using arguments similar to Proposition 4.3 with l instead of n , we prove that

$$\mathbb{V} \left(\sum_{i=1}^l U_i \right) \leq Cl.$$

$$\text{Hence } D_m = \max_{l \in \{1, \dots, 2m\}} \mathbb{V} \left(\sum_{i=1}^l U_i \right) \leq Cm.$$

- proceeding in a similar fashion to (7.11), we obtain $|U_1| \leq C2^{j/2}$.

Lemma 7.1 applied with the random variables U_1, \dots, U_n , $\lambda = \kappa \lambda_n / 2$, $\lambda_n = (\ln n / n)^{1/2}$, $m = u \ln n$ with $u > 0$ (chosen later), $M = C2^{j/2}$, $2^j \leq n / (\ln n)^3$ and (2.3) gives

$$\begin{aligned} & \mathbb{P} \left(|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| \geq \kappa \lambda_n / 2 \right) \\ & \leq C \left(\exp \left(-C \frac{\kappa^2 \lambda_n^2 n}{1 + \kappa \lambda_n m M} \right) + \frac{M}{\lambda_n} n \exp(-cm) \right) \\ & \leq C \left(\exp \left(-C \frac{\kappa^2 \ln n}{1 + \kappa u 2^{j/2} \ln n (\ln n / n)^{1/2}} \right) + \frac{2^{j/2}}{(\ln n / n)^{1/2}} n \exp(-cu \ln n) \right) \\ & \leq C \left(n^{-C\kappa^2 / (1 + \kappa u)} + n^{1 - cu} \right). \end{aligned}$$

Therefore, for large enough κ and u , we have

$$\mathbb{P} \left(|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| \geq \kappa \lambda_n / 2 \right) \leq C \frac{1}{n^4}.$$

This ends the proof of Proposition 4.5. ■

7.3 Proof of Theorem 5.1

Using Proposition 4.1, we have

$$\begin{aligned}
& \widehat{g}_\ell(x) - g_\ell(x) \\
&= \sum_{k=0}^{2^\tau-1} (\widehat{\alpha}_{\tau,k,\ell} - \alpha_{\tau,k,\ell}) \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} (\widehat{b}_{j,k,\ell} \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| \geq \kappa \lambda_n\}} - b_{j,k,\ell}) \psi_{j,k}(x) \\
&- \sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} b_{j,k,\ell} \psi_{j,k}(x) - (\widehat{\mu} - \mu).
\end{aligned}$$

Using the elementary inequality: $(x+y)^2 \leq 2(x^2+y^2)$, $(x,y) \in \mathbb{R}^2$, and the orthonormality of the wavelet basis, we have

$$\mathbb{E} \left(\int_0^1 (\widehat{g}_\ell(x) - g_\ell(x))^2 dx \right) \leq 2(T + U + V + W), \quad (7.12)$$

where

$$\begin{aligned}
T &= \mathbb{E}((\widehat{\mu} - \mu)^2), & U &= \sum_{k=0}^{2^\tau-1} \mathbb{E}((\widehat{\alpha}_{\tau,k,\ell} - \alpha_{\tau,k,\ell})^2), \\
V &= \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left((\widehat{b}_{j,k,\ell} \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| \geq \kappa \lambda_n\}} - b_{j,k,\ell})^2 \right), & W &= \sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} b_{j,k,\ell}^2.
\end{aligned}$$

- (i) **Bound for T .** We proceed as in the proof of Proposition 4.3. By (2.1), we have $\mathbb{E}(\rho(Y_1)) = \mu$. Thanks to the stationarity of $(Y_i)_{i \in \mathbb{Z}}$, we have

$$T = \mathbb{V}(\widehat{\mu}) \leq \frac{1}{n} \mathbb{V}(\rho(Y_1)) + 2 \frac{1}{n} \sum_{m=1}^n |\text{Cov}(\rho(Y_0), \rho(Y_m))|.$$

Using (2.5), the Davydov inequality (see [8]) and (2.3), we obtain

$$T \leq C \frac{1}{n} \left(1 + \sum_{m=1}^n \alpha_m^q \right) \leq C \frac{1}{n} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \quad (7.13)$$

- (ii) **Bound for U .** Using Proposition 4.3, we obtain

$$U \leq C 2^\tau \frac{1}{n} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \quad (7.14)$$

(iii) **Bound for W .** For $q \geq 1$ and $p \geq 2$, we have $g_\ell \in \mathbf{B}_{p,q}^s(M) \subseteq \mathbf{B}_{2,\infty}^s(M)$. Hence, by (4.3),

$$W \leq C \sum_{j=j_1+1}^{\infty} 2^{-2js} \leq C 2^{-2j_1s} \leq C \left(\frac{(\ln n)^3}{n} \right)^{2s} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}.$$

For $q \geq 1$ and $p \in [1, 2)$, we have $g_\ell \in B_{p,q}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$. Since $s > 1/p$, we have $s + 1/2 - 1/p > s/(2s+1)$. So, by (4.3),

$$\begin{aligned} W &\leq C \sum_{j=j_1+1}^{\infty} 2^{-2j(s+1/2-1/p)} \leq C 2^{-2j_1(s+1/2-1/p)} \\ &\leq C \left(\frac{(\ln n)^3}{n} \right)^{2(s+1/2-1/p)} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \end{aligned}$$

Hence, for $q \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > 1/p\}$, we have

$$W \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \quad (7.15)$$

(iv) **Bound for V .** We have

$$V = V_1 + V_2 + V_3 + V_4, \quad (7.16)$$

where

$$V_1 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| \geq \kappa \lambda_n\}} \mathbf{1}_{\{|b_{j,k,\ell}| < \kappa \lambda_n / 2\}} \right),$$

$$V_2 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| \geq \kappa \lambda_n\}} \mathbf{1}_{\{|b_{j,k,\ell}| \geq \kappa \lambda_n / 2\}} \right),$$

$$V_3 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(b_{j,k,\ell}^2 \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| < \kappa \lambda_n\}} \mathbf{1}_{\{|b_{j,k,\ell}| \geq 2\kappa \lambda_n\}} \right)$$

and

$$V_4 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left(b_{j,k,\ell}^2 \mathbf{1}_{\{|\widehat{b}_{j,k,\ell}| < \kappa \lambda_n\}} \mathbf{1}_{\{|b_{j,k,\ell}| < 2\kappa \lambda_n\}} \right).$$

- **Bounds for V_1 and V_3 .** The following inclusions hold:

$$\begin{aligned} & \left\{ |\widehat{b}_{j,k,\ell}| < \kappa\lambda_n, |b_{j,k,\ell}| \geq 2\kappa\lambda_n \right\} \subseteq \left\{ |\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| > \kappa\lambda_n/2 \right\}, \\ & \left\{ |\widehat{b}_{j,k,\ell}| \geq \kappa\lambda_n, |b_{j,k,\ell}| < \kappa\lambda_n/2 \right\} \subseteq \left\{ |\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| > \kappa\lambda_n/2 \right\} \\ & \text{and } \left\{ |\widehat{b}_{j,k,\ell}| < \kappa\lambda_n, |b_{j,k,\ell}| \geq 2\kappa\lambda_n \right\} \subseteq \left\{ |b_{j,k,\ell}| \leq 2|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| \right\}. \end{aligned}$$

So

$$\max(V_1, V_3) \leq C \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \mathbf{1}_{\{|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| > \kappa\lambda_n/2\}} \right).$$

Applying the Cauchy-Schwarz inequality and using Propositions 4.4, 4.5 and $2^j \leq n$, we have

$$\begin{aligned} & \mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \mathbf{1}_{\{|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| > \kappa\lambda_n/2\}} \right) \\ & \leq \left(\mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^4 \right) \right)^{1/2} \left(\mathbb{P} \left(|\widehat{b}_{j,k,\ell} - b_{j,k,\ell}| > \kappa\lambda_n/2 \right) \right)^{1/2} \\ & \leq C \left(\frac{2^j}{n} \right)^{1/2} \left(\frac{1}{n^4} \right)^{1/2} \leq C \frac{1}{n^2}. \end{aligned}$$

Therefore

$$\max(V_1, V_3) \leq C \frac{1}{n^2} \sum_{j=\tau}^{j_1} 2^j \leq C \frac{1}{n^2} 2^{j_1} \leq C \frac{1}{n} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)} \quad (7.17)$$

- **Bound for V_2 .** Using Proposition 4.3, we obtain

$$\mathbb{E} \left((\widehat{b}_{j,k,\ell} - b_{j,k,\ell})^2 \right) \leq C \frac{1}{n} \leq C \frac{\ln n}{n}.$$

Hence

$$V_2 \leq C \frac{\ln n}{n} \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbf{1}_{\{|b_{j,k,\ell}| > \kappa\lambda_n/2\}}.$$

Let j_2 be the integer defined by

$$2^{j_2} = \left\lceil \left(\frac{n}{\ln n} \right)^{1/(2s+1)} \right\rceil. \quad (7.18)$$

We have

$$V_2 \leq V_{2,1} + V_{2,2},$$

where

$$V_{2,1} = C \frac{\ln n}{n} \sum_{j=\tau}^{j_2} \sum_{k=0}^{2^j-1} \mathbf{1}_{\{|b_{j,k,\ell}| > \kappa \lambda_n / 2\}}$$

and

$$V_{2,2} = C \frac{\ln n}{n} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} \mathbf{1}_{\{|b_{j,k,\ell}| > \kappa \lambda_n / 2\}}.$$

We have

$$V_{2,1} \leq C \frac{\ln n}{n} \sum_{j=\tau}^{j_2} 2^j \leq C \frac{\ln n}{n} 2^{j_2} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}.$$

For $q \geq 1$ and $p \geq 2$, we have $g_\ell \in B_{p,q}^s(M) \subseteq \mathbf{B}_{2,\infty}^s(M)$. So, by (4.3),

$$\begin{aligned} V_{2,2} &\leq C \frac{\ln n}{n \lambda_n^2} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} b_{j,k,\ell}^2 \leq C \sum_{j=j_2+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_2 s} \\ &\leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \end{aligned}$$

For $q \geq 1$, $p \in [1, 2)$ and $s > 1/p$, using (4.3), $\mathbf{1}_{\{|b_{j,k,\ell}| > \kappa \lambda_n / 2\}} \leq C |b_{j,k,\ell}|^p / \lambda_n^p = C |\beta_{j,k}|^p / \lambda_n^p$ and $(2s+1)(2-p)/2 + (s+1/2-1/p)p = 2s$, we have

$$\begin{aligned} V_{2,2} &\leq C \frac{\ln n}{n \lambda_n^p} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \leq C \left(\frac{\ln n}{n} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{-j(s+1/2-1/p)p} \\ &\leq C \left(\frac{\ln n}{n} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p)p} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > 1/p\}$, we have

$$V_2 \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \quad (7.19)$$

• **Bound for V_4 .** We have

$$V_4 \leq \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} b_{j,k,\ell}^2 \mathbf{1}_{\{|b_{j,k,\ell}| < 2\kappa \lambda_n\}}.$$

Let j_2 be the integer (7.18). Then

$$V_4 \leq V_{4,1} + V_{4,2},$$

where

$$V_{4,1} = \sum_{j=\tau}^{j_2} \sum_{k=0}^{2^j-1} b_{j,k,\ell}^2 \mathbf{1}_{\{|b_{j,k,\ell}| < 2\kappa\lambda_n\}}, \quad V_{4,2} = \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} b_{j,k,\ell}^2 \mathbf{1}_{\{|b_{j,k,\ell}| < 2\kappa\lambda_n\}}.$$

We have

$$V_{4,1} \leq C \sum_{j=\tau}^{j_2} 2^j \lambda_n^2 = C \frac{\ln n}{n} \sum_{j=\tau}^{j_2} 2^j \leq C \frac{\ln n}{n} 2^{j_2} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}.$$

For $q \geq 1$ and $p \geq 2$, we have $g_\ell \in B_{p,q}^s(M) \subseteq \mathbf{B}_{2,\infty}^s(M)$. Hence, by (4.3),

$$V_{4,2} \leq \sum_{j=j_2+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_2 s} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}.$$

For $q \geq 1$, $p \in [1, 2)$ and $s > 1/p$, using (4.3), $b_{j,k,\ell}^2 \mathbf{1}_{\{|b_{j,k,\ell}| < 2\kappa\lambda_n\}} \leq C \lambda_n^{2-p} |b_{j,k,\ell}|^p = C \lambda_n^{2-p} |\beta_{j,k}|^p$ and $(2s+1)(2-p)/2 + (s+1/2 - 1/p)p = 2s$, we have

$$\begin{aligned} V_{4,2} &\leq C \lambda_n^{2-p} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p = C \left(\frac{\ln n}{n} \right)^{(2-p)/2} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \\ &\leq C \left(\frac{\ln n}{n} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{-j(s+1/2-1/p)p} \\ &\leq C \left(\frac{\ln n}{n} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p)p} \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \end{aligned}$$

Thus, for $q \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > 1/p\}$, we have

$$V_4 \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \quad (7.20)$$

It follows from (7.16), (7.17), (7.19) and (7.20) that

$$V \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}. \quad (7.21)$$

Combining (7.12), (7.13), (7.14), (7.15) and (7.21), we have, for $q \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > 1/p\}$,

$$\mathbb{E} \left(\int_0^1 (\widehat{g}_\ell(x) - g_\ell(x))^2 dx \right) \leq C \left(\frac{\ln n}{n} \right)^{2s/(2s+1)}.$$

The proof of Theorem 5.1 is complete. ■

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