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Alternative description of the 2D Blume-Capel model using Grassmann algebra

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Abstract. We use Grassmann algebra to study the phase transition of the 2D Blume-Capel from a fermionic point of view. This model presents a phase diagram, with a second order critical line which becomes first order through a tricritical point, and was used to model the phase transition in liquid mixtures of He³-He⁴. In particular, we are able to map the spin-1 system onto an effective fermionic action from which we obtain the exact mass of the theory. This effective action is actually an extension of the free fermion Ising action with an additional quartic interaction term. The effect of this term is to render the excitation spectrum of the fermions unstable at the tricritical point.

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1. Introduction

The Blume Capel (BC) model is a classical spin 1 model, originally introduced to qualitatively explain the phase transition in a mixture of He³-He⁴ [1, 2, 3]. Below a concentration of 67% in He³, the mixture undergoes a so-called λ transition: the two components separate through a first order phase transition and only He⁴ is superfluid. On a 2D lattice representing an Helium film, He atoms are modelled by a spin-like variable, according to the following rule: an He³ atom is associated to the value 0, whereas a He⁴ is represented by a classical Ising spin taking the value ± 1 . Within this framework, all the lattice sites are occupied either by an He³ or He⁴ atom.

The Blume-Capel model describes the behaviour of this ensemble of spins $\{\sigma\}$. In addition to the usual nearest-neighbour interaction, it includes the term $\Delta_0 \sum_{m,n} \sigma_{mn}^2 = \Delta_0 N_s$ to take into account a possible change in vacancies number. Δ_0 can be thought

as a chemical potential or crystal field. A simple analysis of the BC hamiltonian shows that this model presents a rather complex phase diagram in the plane (T, Δ_0) , where T is the temperature in the canonical ensemble [4]. In the limit $\Delta_0 \rightarrow -\infty$, the values $\sigma_{mn} = 0$ are excluded and the standard 2D Ising model is recovered, with its well-known second-order critical point at $(T, \Delta_0) = (T_c = 1/\text{arctanh}(\sqrt{2} - 1) \simeq 2.27, -\infty)$. At zero temperature, a simple energy argument shows that the ground state is the ordered state $|\sigma| = 1$ if $\Delta_0 < 2J$, and $|\sigma| = 0$ else. There is therefore a first order phase transition at $(T, \Delta_0) = (0, 2J)$, suggesting a change in the order of the transition at some tricritical point of the critical line. Mean field theory confirms this behaviour and provides a transition line in the plane (T, Δ_0) which is of second order for large negative Δ_0 . Beyond the tricritical point, the transition becomes first order. Precise numerical simulations have been performed to study precisely the phase diagram and to locate the tricritical point [5]. From a theoretical backgrounds, several approximations have been used such as mean field theory [6] and high temperature expansion [7]. Using correlation identities and Griffith's and Newman's inequalities, rigorous upper bounds for the critical temperature have also been obtained by Braga *et al* [8]. The aim of this article is to present a different analytical method, originally proposed by one of us for the 2D classical Ising model in the case of free fermions [9] and since then used to treat various problems around the 2D Ising model, such as finite size effects and boundary conditions [10], quenched disorder [11], boundary magnetic field [12, 13]. This method is based on the introduction of Grassmann variables directly in the partition function, in contrast with usual uses of Grassmann variables in this context [14]. It turns out to be particularly efficient to deal with the nearest-neighbour interactions in the 2D plane. As the additional crystal field term in the BC hamiltonian is local, we hoped that the method will be applicable as well in this context. We will see in the following that though it is not possible to compute exactly the partition function and thermodynamics quantities, our approach allows to derive in a controlled way the underlying fermionic lattice field theory. In the continuum limit the effective theory can be analysed in the low energy sector, leading to the exact equation of the critical line and to the effective interaction between fermions that is responsible for the existence of a tricritical point. An approximate method such as the Hartree-Fock method can be used to locate this point. It is interesting to note that a phase diagram with first order transition and tricritical point can be described not only with a bosonic Ginzburg-Landau theory [15], where the order parameter is a simple scalar, but also with fermionic variables.

The article is organized as follow: after presenting the BC Hamiltonian and the related partition function, we apply the fermionization procedure, leading to the exact fermionic action on the lattice. Then, from this result, we derive the effective action in the continuum and extract the exact mass. We then give the physical interpretation for the existence of a tricritical point on the phase diagram, by studying the fermionic stability of the critical line, and compare our results with recent numerical Monte Carlo simulations.

2. The 2D Blume-Capel model

2.1. Hamiltonian and partition function

The 2D Blume-Capel (BC) model is defined, on a square lattice of linear size L , via the following hamiltonian:

$$-\beta H = \sum_{m=1}^L \sum_{n=1}^L \left[K_1 \sigma_{mn} \sigma_{m+1n} + K_2 \sigma_{mn} \sigma_{mn+1} \right] + \Delta \sum_{m=1}^L \sum_{n=1}^L \sigma_{mn}^2. \quad (1)$$

In the previous expression $K_{1,2} = J_{1,2}/T$ represent the nearest-neighbours interactions along the bonds in the two different directions. The crystal field parameter $\Delta = -\beta \Delta_0$ plays the role of a chemical potential, responsible for the level splitting between $\sigma = 0$ and $\sigma = \pm 1$. The partition function Z of the BC model is obtained by summing up all possible configurations of the ensemble $\{\sigma_{mn} = 0, \pm 1\}$, $Z = \sum_{\sigma=0, \pm 1} e^{-\beta H} = \text{Tr}_{\{\sigma\}} e^{-\beta H}$. Using the property $\sigma_{mn}^2 \in \{0, 1\}$, it is easy to develop each Boltzmann factor appearing in the trace formula:

$$\exp(K_i \sigma \sigma') = 1 + \lambda_i \sigma \sigma' + \lambda'_i \sigma^2 \sigma'^2, \quad (2)$$

with

$$\lambda_i = \sinh K_i, \quad \lambda'_i = \cosh K_i - 1. \quad (3)$$

We thus obtain a representation of the partition function as a product of polynomials:

$$Z = \text{Tr}_{\{\sigma_{mn}=0, \pm 1\}} \left\{ \prod_{m=1}^L \prod_{n=1}^L e^{\Delta \sigma_{mn}^2} \left[(1 + \lambda_1 \sigma_{mn} \sigma_{m+1n} + \lambda'_1 \sigma_{mn}^2 \sigma_{m+1n}^2) \right. \right. \\ \left. \left. \times (1 + \lambda_2 \sigma_{mn} \sigma_{mn+1} + \lambda'_2 \sigma_{mn}^2 \sigma_{mn+1}^2) \right] \right\}. \quad (4)$$

This expression of the partition as a product of spin polynomials is the starting point of the fermionization procedure using Grassmann variables. As we will see below, there are different possible ways of introducing Grassmann variables directly in order to disconnect the previous interactions between the local spins. This method is useful to sum up the spin variables and obtain the fermionic action underlying the BC model. The partition function is then given by a path integral of this action, as we will see below.

2.2. Local spin decomposition

In the following we will use an alternative interpretation that turns out to be convenient for our derivation. Let us consider a 2D lattice of size L on which we put N_s Ising spins $S_{mn} = \pm 1$ on each site of coordinates (m, n) . If $N_s = L^2 = N$ all available sites are occupied (2D Ising model). If $N_s < L^2$, vacancies appear in the lattice: we can then define on each lattice site a new variable, ρ , that represents the occupation number taking two possible values $\rho = 0, 1$, so that $N_s = \sum_{m,n=1}^L \rho_{mn}$. The effective spin variable on each lattice site (m, n) is then

$$\sigma_{mn} = \rho_{mn} \times S_{mn} \in \{0, +1, -1\}. \quad (5)$$

The usual 2D Ising model is a particular case of this model, whom the configurations space is the subspace where $\rho_{mn} = 1$ for all (m, n) sites.

In using this decomposition to sum up the spin degrees of freedom, we have to weight the various contributions in order not to count twice the contribution $\sigma = 0$. This comes from the fact that there are two different decompositions giving the same $\sigma = 0$. While the original averaging over the individual spin means

$$\sum_{\sigma=0,\pm 1} f(\sigma) = f(0) + f(+1) + f(-1), \quad (6)$$

the naive summation over independent states in the factorized form gives

$$\sum_{\rho=0,1;S=\pm 1} f(\rho S) = f(0) + f(0) + f(+1) + f(-1). \quad (7)$$

This may be corrected by introducing in the definition of the averaging the weight $\frac{1}{2}$ at $\rho = 0$, or factor $2^{\rho-1}$, which gives

$$\sum_{\rho=0,1;S=\pm 1} 2^{\rho-1} f(\rho S) = f(0) + f(+1) + f(-1). \quad (8)$$

The decomposition scheme $\rho_{mn} S_{mn}$ is close to the one of the site dilute Ising model [11, 16]. The case (7) means simply that there is a spin $S_{mn} = \pm 1$ at site $\rho_{mn} = 0$, which is not interacting with its nearest-neighbours. This empty site, by flipping over two states ± 1 under temperature fluctuations, will give however a contribution to the entropy, $\ln 2$ by empty site. The case (8) means that the site $\rho_{mn} = 0$ is really dilute, or empty, with no spin degree of freedom at it, even disconnected. This interpretation in term of dilute site variables is however quite different from the derivation of a quenched-impurity model where the set ρ_{mn} is fixed, and the average over these variables is perform after the logarithm of the partition function is taken. The present approach is closer to the annealed disorder case.

Another decomposition that we will not take into consideration in the following computations, but which appear to be convenient as well, relies on the \mathbb{Z}_2 gauge symmetry of the model. Indeed, only even terms of the polynomial for Z , which include variables $1, \sigma_{mn}^2$, but not σ_{mn} , will actually contribute to the sum, since $\sigma_{mn} \rightarrow 0$ under the averaging. Therefore, it is easy to see that Z will not be changed if we multiply each spin variable under the sum by a fixed factor taking values either 1 or -1 , which we denote as $S_{mn} = \pm 1$, so that the gauge symmetry $\sigma_{mn} \rightarrow \sigma_{mn} S_{mn}$ makes invariant $Z[\sigma_{mn}] \rightarrow Z[\sigma_{mn} S_{mn}]$. Now, though the above invariance under $\sigma_{mn} \rightarrow \sigma_{mn} S_{mn}$ holds already for a fixed set of $S_{mn} = \pm 1$, we can also average $Z[\sigma_{mn} S_{mn}]$ over $S_{mn} = \pm 1$ at each site, with a factor $\frac{1}{2}$ to compensate the doubling of the terms, leading to

$$Z = \sum_{\sigma_{mn}=0,\pm 1} Z[\sigma_{mn}] = \sum_{\sigma_{mn}=0,\pm 1} Z[\sigma_{mn} S_{mn}] = \frac{1}{2L^2} \sum_{S_{mn}=\pm 1} \sum_{\sigma_{mn}=0,\pm 1} Z[\sigma_{mn} S_{mn}]. \quad (9)$$

This symmetry outlined in the above transformations can also be used in what follows instead of the occupation number leading, of course, to the same result. As we will see later, it is however essential to use one of these two transformations to simplify the algebra .

3. Fermionization and lattice fermionic field theory

The expression of the partition function as a product of polynomials, as given by equation (4), is the starting point of the fermionization procedure. This procedure has first been introduced by one of us on the 2D Ising model [9]: it relies on interpreting each spin polynomial in (4) as the result of an integral over a set of Grassmann variables. Though the fermionization is slightly more difficult in the BC case, the idea remains the same. Before going into details, we remind in the following subsection basic features about Grassmann variables and integration/derivation definitions.

3.1. Grassmann variables

We remind here the essential features about Grassmann variables that are needed in the rest of the paper. More details can be found in [17, 18]. A Grassmann algebra \mathcal{A} of size N is a set of N anti-commuting objects $\{a_i\}_{i=1,N}$ satisfying:

$$\forall 1 \leq i, j \leq N, \quad a_i a_j = -a_j a_i,$$

which implies $a_i^2 = 0$.

Functions defined on such an algebra are particularly simple, they are polynomials with a finite degree. It is possible to define the notion of integration [17, 19] with the following rules:

$$\int da \, a = 1, \quad \int da \, 1 = 0, \tag{10}$$

and for any function $f(a)$,

$$\int da \, f(a) = \frac{\partial f(a)}{\partial a}. \tag{11}$$

With these definitions, Gaussian integrals are expressed by

$$\int \prod_{i=1}^N da_i^* da_i \exp \left(\sum_{i,j=1}^N a_i M_{ij} a_j^* \right) = \det M. \tag{12}$$

In the following we will also introduce trace operator acting on functions defined on a Grassmann set of variables. In order to easily identify them, such operators will be noted as \mathfrak{T} .

3.2. Fermionization trick

In the same spirit than for the 2D Ising model [9], we introduce two pairs of Grassmann variables, (a_{mn}, \bar{a}_{mn}) , and (b_{mn}, \bar{b}_{mn}) , to factorize the polynomials appearing in (4). Namely we use the relations

$$\begin{aligned} 1 + \lambda_1 \sigma_{mn} \sigma_{m+1n} + \lambda'_1 \sigma_{mn}^2 \sigma_{m+1n}^2 &= \int d\bar{a}_{mn} da_{mn} e^{(1+\lambda'_1 \sigma_{mn}^2 \sigma_{m+1n}^2) a_{mn} \bar{a}_{mn}} \\ &\times (1 + a_{mn} \sigma_{mn}) (1 + \lambda_1 \bar{a}_{mn} \sigma_{m+1n}), \end{aligned} \tag{13}$$

and

$$1 + \lambda_2 \sigma_{mn} \sigma_{mn+1} + \lambda'_2 \sigma_{mn}^2 \sigma_{mn+1}^2 = \int d\bar{b}_{mn} db_{mn} e^{(1+\lambda'_2 \sigma_{mn}^2 \sigma_{mn+1}^2) b_{mn} \bar{b}_{mn}} \times (1 + b_{mn} \sigma_{mn}) (1 + \lambda_2 \bar{b}_{mn} \sigma_{mn+1}). \quad (14)$$

For sake of simplicity we can introduce the following link factors

$$A_{mn} = 1 + a_{mn} \sigma_{mn}, \bar{A}_{m+1n} = 1 + \lambda_1 \bar{a}_{mn} \sigma_{m+1n}, \quad (15)$$

$$B_{mn} = 1 + b_{mn} \sigma_{mn}, \bar{B}_{mn+1} = 1 + \lambda_2 \bar{b}_{mn} \sigma_{mn+1}, \quad (16)$$

and the Grassmann local trace operators which associate to any function f on the Grassmann algebra

$$\mathfrak{Tr}_{(a_{mn})} [f] = \int d\bar{a}_{mn} da_{mn} e^{(1+\lambda'_1 \sigma_{mn}^2 \sigma_{m+1n}^2) a_{mn} \bar{a}_{mn}} f(a_{mn}, \bar{a}_{mn}), \quad (17)$$

$$\mathfrak{Tr}_{(b_{mn})} [f] = \int d\bar{b}_{mn} db_{mn} e^{(1+\lambda'_2 \sigma_{mn}^2 \sigma_{mn+1}^2) b_{mn} \bar{b}_{mn}} f(b_{mn}, \bar{b}_{mn}). \quad (18)$$

The Boltzmann weights then reads

$$1 + \lambda_1 \sigma_{mn} \sigma_{m+1n} + \lambda'_1 \sigma_{mn}^2 \sigma_{m+1n}^2 = \mathfrak{Tr}_{(a_{mn})} [A_{mn} \bar{A}_{m+1n}], \quad (19)$$

$$1 + \lambda_2 \sigma_{mn} \sigma_{mn+1} + \lambda'_2 \sigma_{mn}^2 \sigma_{mn+1}^2 = \mathfrak{Tr}_{(b_{mn})} [B_{mn} \bar{B}_{mn+1}]. \quad (20)$$

Introducing those Grassmann factors in the expression (4) of the partition function, we then obtain a mixed representation containing both spins and Grassmann variables. One should note that as $A_{mn}, \bar{A}_{mn}, B_{mn}, \bar{B}_{mn}$ are neither commuting or anti-commuting objects, the order in which it appears in the product is therefore important.

We then define the following trace operator :

$$\mathfrak{Tr}_{a,b} [f] = \int \prod_{m=1}^L \prod_{n=1}^L d\bar{a}_{mn} da_{mn} d\bar{b}_{mn} db_{mn} e^{\Delta \sigma_{mn}^2} f(a_{mn}, \bar{a}_{mn}, b_{mn}, \bar{b}_{mn}) \times \exp \left\{ \sum_{m=1}^L \sum_{n=1}^L [(1 + \lambda'_1 \sigma_{mn}^2 \sigma_{m+1n}^2) a_{mn} \bar{a}_{mn} + (1 + \lambda'_2 \sigma_{mn}^2 \sigma_{mn+1}^2) b_{mn} \bar{b}_{mn}] \right\}. \quad (21)$$

Note that all even power of the spin variables are put into the general measure of integrals, including the chemical potential term which is only local. The partition function is given by

$$Z = \text{Tr}_{\{\sigma\}} \mathfrak{Tr}_{a,b} \left[\prod_{n=1}^{\overrightarrow{L}} \left(\prod_{m=1}^{\overrightarrow{L}} ((A_{mn} \bar{A}_{m+1n})(B_{mn} \bar{B}_{mn+1})) \right) \right]. \quad (22)$$

At this stage the partition function appears as a double trace, over the spin degrees of freedom, with $\text{Tr}_{\{\sigma\}}$, and over the Grassmann variables, with $\mathfrak{Tr}_{a,b}$.

3.3. Spin summation

Up to now we only add extra variables to obtain the mixed expression (22). Further algebraic manipulations are necessary to simplify this expression, in particular to move

the link factors through the product, in order to put together those having the same indices (or same spin variable), *i.e.* A_{mn} , \bar{A}_{mn} , B_{mn} and \bar{B}_{mn} . Then, using the decomposition $\sigma_{mn} = \rho_{mn} S_{mn}$, a partial sum over the variables S_{mn} is possible, since the general trace $\mathfrak{Tr}_{a,b}$ depends only on the squares $\sigma_{mn}^2 = \rho_{mn}^2 S_{mn}^2 = \rho_{mn}^2$, therefore only on the ρ_{mn} s. To do so, we use the mirror-ordering introduced in the 2D Ising model [9], to move together the different link factors containing the same spin. Up to boundary terms irrelevant in the thermodynamic limit, and using the notation of the ordered products [9], this leads to

$$\begin{aligned}
Z &= \text{Tr}_{\sigma} \mathfrak{Tr}_{a,b} \left\{ \prod_{m=1}^L \prod_{n=1}^L \left[(A_{mn} \bar{A}_{m+1n})(B_{mn} \bar{B}_{mn+1}), \right] \right\}, \\
&= \text{Tr}_{\sigma} \mathfrak{Tr}_{a,b} \left\{ \prod_{n=1}^{\overrightarrow{L}} \left[\prod_{m=1}^{\overrightarrow{L}} \bar{B}_{mn} A_{mn} \bar{A}_{m+1n} \cdot \prod_{m=1}^{\overleftarrow{L}} B_{mn} \right] \right\}, \\
&= \text{Tr}_{\{\sigma\}} \mathfrak{Tr}_{a,b} \left[\prod_{n=1}^{\overrightarrow{L}} \left(\left(\prod_{m=1}^{\overrightarrow{L}} \bar{A}_{mn} \bar{B}_{mn} A_{mn} \right) \cdot \left(\prod_{m=1}^{\overleftarrow{L}} B_{mn} \right) \right) \right]. \tag{23}
\end{aligned}$$

In principle we should pay attention to boundary terms, which can actually be treated rigorously [10, 12, 13]. However, as we are interested only in the continuous limit of the BC model, this would lead to a more complicate and unnecessary analysis : we can neglected those boundary terms since they do not contribute to the thermodynamic limit.

In the case of the 2D Ising model, as $\sigma_{mn}^2 = 1$, we can explicitly perform the trace over the spins recursively at the junction of two products, directly from the last relation in (23). The situation is slightly different in the BC case, since $\sigma_{mn}^2 = 0, 1$ instead. As we said previously, the trace operator (21) contains terms $\sigma_{mn}^2 = \rho_{mn}^2 = \rho_{mn}$ that are coupled to the neighbouring sites. Therefore it is not possible to trace over the individual spins σ_{mn} directly, but only over the variables S_{mn} in a first stage. In other words, the ordering procedure on the link variables allows us to eliminate one part of the degrees of freedom.

At the junction of the two products in (23), we begin to perform the trace for $m = L$ or, in general,

$$\begin{aligned}
\frac{1}{2} \text{Tr}_{S_{mn}} \left[\bar{A}_{mn} \bar{B}_{mn} A_{mn} B_{mn} \right] &= \frac{1}{2} \sum_{S_{mn}=\pm 1} \bar{A}_{mn} \bar{B}_{mn} A_{mn} B_{mn}, \tag{24} \\
&= 1 + \left(a_{mn} b_{mn} + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} + (\lambda_1 \bar{a}_{m-1n} + \lambda_2 \bar{b}_{mn-1})(a_{mn} + b_{mn}) \right. \\
&\quad \left. + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} a_{mn} b_{mn} \right) \rho_{mn}, \\
&= \exp \left[\left(a_{mn} b_{mn} + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} + (\lambda_1 \bar{a}_{m-1n} + \lambda_2 \bar{b}_{mn-1})(a_{mn} + b_{mn}) \right) \rho_{mn} \right].
\end{aligned}$$

As this term is quadratic in Grassmann variables, it commutes with all other terms, and can therefore be move out from the ordered product. We then repeat the calculation

for $m = L - 1$ and so on. The partially traced partition function finally reads

$$Z = 2^N \text{Tr}_\rho \int \prod_{m,n=1}^L d\bar{a}_{mn} da_{mn} d\bar{b}_{mn} db_{mn} \exp \left[(\Delta + a_{mn} b_{mn} + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} \right. \\ \left. + (\lambda_1 \bar{a}_{m-1n} + \lambda_2 \bar{b}_{mn-1})(a_{mn} + b_{mn})) \rho_{mn} + (1 + \lambda'_1 \rho_{mn} \rho_{m+1n}) a_{mn} \bar{a}_{mn} \right. \\ \left. + (1 + \lambda'_2 \rho_{mn} \rho_{mn+1}) b_{mn} \bar{b}_{mn} \right]. \quad (25)$$

The trace over the ρ variables can be performed by decoupling them in the exponential term. Several methods are possible. One of them is to introduce another set of Grassmann link variables as we previously did to decouple the factors $\rho_{mn} \rho_{m+1n}$ and $\rho_{mn} \rho_{mn+1}$. It is possible however to avoid the introduction of new fields by using instead the following rescaling trick. It is achieved by a change of variables which leaves the integral invariant :

$$a_{mn} \rightarrow \frac{a_{mn}}{\rho_{mn}}, \quad b_{mn} \rightarrow \frac{b_{mn}}{\rho_{mn}} \quad \text{and} \quad (26)$$

$$da_{mn} \rightarrow \rho_{mn} da_{mn}, \quad db_{mn} \rightarrow \rho_{mn} db_{mn}. \quad (27)$$

The variable ρ_{mn} disappears in some places inside the exponential and the factors that were previously linked $\rho_{mn} \rho_{m+1n}$ and $\rho_{mn} \rho_{mn+1}$ are now decoupled. Also, the seemingly singular expressions like $\rho_{mn} \exp(a_{mn} \bar{a}_{mn} / \rho_{mn})$ are to be understood as $\rho_{mn} \exp(a_{mn} \bar{a}_{mn} / \rho_{mn}) = \rho_{mn} (1 + a_{mn} \bar{a}_{mn} / \rho_{mn}) = \rho_{mn} + a_{mn} \bar{a}_{mn}$. After shifting some indices, we obtain

$$Z = 2^N \text{Tr}_\rho \int \prod_{m,n=1}^L d\bar{a}_{mn} da_{mn} d\bar{b}_{mn} db_{mn} (\rho_{mn} + a_{mn} \bar{a}_{mn}) (\rho_{mn} + b_{mn} \bar{b}_{mn}) \quad (28) \\ \times \exp \left[\rho_{mn} (\Delta + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} + \lambda'_1 a_{m-1n} \bar{a}_{m-1n} + \lambda'_2 b_{mn-1} \bar{b}_{mn-1}) \right] \\ \times \exp \left[a_{mn} b_{mn} + (\lambda_1 \bar{a}_{m-1n} + \lambda_2 \bar{b}_{mn-1})(a_{mn} + b_{mn}) \right].$$

Now we can locally perform the sum over ρ_{mn} , using the rule (8), since we have to pay attention not to count twice the contribution $\sigma_{mn} = 0$. In that case we have, for the part depending on ρ_{mn} ,

$$\text{Tr}_{\rho_{mn}=0,1} \left\{ (\rho_{mn} + a_{mn} \bar{a}_{mn}) (\rho_{mn} + b_{mn} \bar{b}_{mn}) \quad (29) \right. \\ \left. \times \exp \left[\rho_{mn} (\Delta + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} + \lambda'_1 a_{m-1n} \bar{a}_{m-1n} + \lambda'_2 b_{mn-1} \bar{b}_{mn-1}) \right] \right\} \\ = \frac{1}{2} (a_{mn} \bar{a}_{mn} b_{mn} \bar{b}_{mn} + 2e^\Delta e^{G_{mn}}),$$

where G_{mn} is the quadratic form

$$G_{mn} = a_{mn} \bar{a}_{mn} + b_{mn} \bar{b}_{mn} + \lambda_1 \lambda_2 \bar{a}_{m-1n} \bar{b}_{mn-1} + \lambda'_1 a_{m-1n} \bar{a}_{m-1n} + \lambda'_2 b_{mn-1} \bar{b}_{mn-1}. \quad (30)$$

The last term in equation (29) can be easily written as an exponential :

$$a_{mn} \bar{a}_{mn} b_{mn} \bar{b}_{mn} + 2e^\Delta e^{G_{mn}} = 2e^\Delta e^{G_{mn}} \left(1 + \frac{1}{2} a_{mn} \bar{a}_{mn} b_{mn} \bar{b}_{mn} e^{-\Delta - G_{mn}} \right), \quad (31) \\ = 2e^\Delta \exp \left(G_{mn} + \frac{1}{2} a_{mn} \bar{a}_{mn} b_{mn} \bar{b}_{mn} e^{-\Delta - G_{mn}} \right).$$

Just to simplify the comparison with the 2D Ising model, we can rescale some Grassmann variables using the following transformation:

$$(1 + \lambda'_1)\bar{a}_{mn} \rightarrow \bar{a}_{mn}, \quad (1 + \lambda'_2)\bar{b}_{mn} \rightarrow \bar{b}_{mn}. \quad (32)$$

Introducing $t_i = \tanh K_i = \lambda_i/(1 + \lambda'_i)$, we obtain the final result of this section, namely the representation of the 2D BC model as a fermionic integral :

$$Z = 2^N (\cosh K_1 \cosh K_2)^N e^{N\Delta} \int \prod_{m,n=1}^L d\bar{a}_{mn} da_{mn} d\bar{b}_{mn} db_{mn} \quad (33)$$

$$\exp \left(a_{mn}\bar{a}_{mn} + b_{mn}\bar{b}_{mn} + a_{mn}b_{mn} + (t_1\bar{a}_{m-1n} + t_2\bar{b}_{mn-1})(a_{mn} + b_{mn}) \right. \\ \left. + t_1t_2\bar{a}_{m-1n}\bar{b}_{mn-1} + g_0a_{mn}\bar{a}_{mn}b_{mn}\bar{b}_{mn} e^{-\gamma_1a_{m-1n}\bar{a}_{m-1n} - \gamma_2b_{mn-1}\bar{b}_{mn-1} - t_1t_2\bar{a}_{m-1n}\bar{b}_{mn-1}} \right),$$

where we have introduced the following constants

$$g_0 = \frac{e^{-\Delta}}{2 \cosh K_1 \cosh K_2}, \quad t_i = \tanh K_i, \quad \gamma_i = 1 - \frac{1}{\cosh K_i} = 1 - \sqrt{1 - t_i^2}. \quad (34)$$

We can recognize in (33) the Ising action

$$\mathcal{S}_{\text{Ising}} = \sum_{m,n=1}^L a_{mn}\bar{a}_{mn} + b_{mn}\bar{b}_{mn} + a_{mn}b_{mn} \\ + (t_1\bar{a}_{m-1n} + t_2\bar{b}_{mn-1})(a_{mn} + b_{mn}) + t_1t_2\bar{a}_{m-1n}\bar{b}_{mn-1}, \quad (35)$$

with an additional interaction part which is a polynomial of degree 8 in Grassmann variables (after expanding the exponential):

$$\mathcal{S}_{\text{int}} = g_0 \sum_{m,n=1}^L a_{mn}\bar{a}_{mn}b_{mn}\bar{b}_{mn} e^{-\gamma_1a_{m-1n}\bar{a}_{m-1n} - \gamma_2b_{mn-1}\bar{b}_{mn-1} - t_1t_2\bar{a}_{m-1n}\bar{b}_{mn-1}}. \quad (36)$$

The BC model differs from the Ising model by the interaction term (36), which is not quadratic, therefore it is not solvable in the sense of free fermions. When $g_0 = 0$, *i.e.* in the limit $\Delta \rightarrow \infty$, the gap between the two degenerate states $\sigma = \pm 1$ and the singlet $\sigma = 0$ becomes so large that the model reduces effectively to the 2D Ising model.

In the following, for simplification, we will only consider the homogeneous coupling case, where $t_1 = t_2 = t$ and $\gamma_1 = \gamma_2 = \gamma = 1 - \sqrt{1 - t^2}$, and we will note $\mathcal{S} = \mathcal{S}_{\text{Ising}} + \mathcal{S}_{\text{int}}$ the total action.

3.4. Partial bosonization

The previous action contains two pairs of Grassmann variables per site. This can not be reduced to one pair (minimal action) unlike the Ising model, where half of the variables are irrelevant in the sense they do not contribute to the critical behaviour. Here the 2 pairs are coupled together by Eq. (36). However, as we will see in the following, it is possible to recover the minimal Ising action [11] using bosonic variables. In the previous interaction action (36), it is indeed tempting to replace the products $a_{mn}\bar{a}_{mn}$

and $b_{mn}\bar{b}_{mn}$, which look similar to occupation number operators, by the commuting variables

$$\rho_{mn} = a_{mn}\bar{a}_{mn}, \quad \eta_{mn} = b_{mn}\bar{b}_{mn}, \quad (37)$$

$$\rho_{mn}\rho_{m'n'} = \rho_{m'n'}\rho_{mn}, \quad \text{and } \rho_{mn}^2 = 0. \quad (38)$$

This new variables are nilpotent (as Grassmann variables) but commuting: that is why in the following we will abusively call them (hard core) "bosons".

This allows us to reduce the degree of the polynomials in Grassmann variables by a factor 2 each time the replacement is performed, even if terms like $\bar{a}_{m-1n}\bar{b}_{mn-1}$ in (36) can not be replaced. We will show below that we can write down an action containing one pair of Grassmann variables and one pair of bosonic ones per site. To do so, we introduce the following Dirac distribution for any function f of $a_{mn}\bar{a}_{mn}$ or $b_{mn}\bar{b}_{mn}$:

$$f(a_{mn}\bar{a}_{mn}) = \int d\rho_{mn}d\bar{\rho}_{mn}f(\rho_{mn}) \exp[\bar{\rho}_{mn}(\rho_{mn} + a_{mn}\bar{a}_{mn})], \quad (39)$$

and

$$f(b_{mn}\bar{b}_{mn}) = \int d\eta_{mn}d\bar{\eta}_{mn}f(\eta_{mn}) \exp[\bar{\eta}_{mn}(\eta_{mn} + b_{mn}\bar{b}_{mn})]. \quad (40)$$

Applying (39,40) directly in (33), we finally obtain the following expression for the action

$$\begin{aligned} \mathcal{S} = \sum_{m,n} & \left[a_{mn}\bar{a}_{mn} + b_{mn}\bar{b}_{mn} + t^2\bar{a}_{m-1n}\bar{b}_{mn-1} + a_{mn}b_{mn} + t(\bar{a}_{m-1n} + \bar{b}_{mn-1})(a_{mn} + b_{mn}) \right] \\ & + g_0\rho_{mn}\eta_{mn} \left[1 - \gamma(\rho_{m-1n} + \eta_{mn-1}) + \gamma^2\rho_{m-1n}\eta_{mn-1} - t^2\bar{a}_{m-1n}\bar{b}_{mn-1} \right] \\ & + \bar{\rho}_{mn}(\rho_{mn} + a_{mn}\bar{a}_{mn}) + \bar{\eta}_{mn}(\eta_{mn} + b_{mn}\bar{b}_{mn}). \end{aligned} \quad (41)$$

We can now integrate over the a_{mn} s and b_{mn} s, and replace formally, for convenience, the variables \bar{a}_{mn} by a_{mn} and \bar{b}_{mn} by \bar{a}_{mn} . We obtain

$$\begin{aligned} \mathcal{S} = \sum_{mn=1}^L & t^2 a_{m-1n}\bar{a}_{mn-1} + t(a_{m-1n} + \bar{a}_{mn-1})[\bar{a}_{mn}(1 + \bar{\eta}_{mn}) - a_{mn}(1 + \bar{\rho}_{mn})] \\ & + \bar{a}_{mn}a_{mn}(1 + \bar{\eta}_{mn})(1 + \bar{\rho}_{mn}) + \bar{\rho}_{mn}\rho_{mn} + \bar{\eta}_{mn}\eta_{mn} \\ & + g_0\rho_{mn}\eta_{mn} \left[1 - \gamma(\rho_{m-1n} + \eta_{mn-1}) + \gamma^2\rho_{m-1n}\eta_{mn-1} - t^2 a_{m-1n}\bar{a}_{mn-1} \right]. \end{aligned} \quad (42)$$

The next operation is to notice that we can further integrate over the variables η_{mn} and $\bar{\eta}_{mn}$, using the integration rules (40), since

$$\begin{aligned} \int d\eta_{mn}d\bar{\eta}_{mn} \exp \left[\bar{\eta}_{mn}(\eta_{mn} - t\bar{a}_{mn}(a_{m-1n} + \bar{a}_{mn-1}) + \bar{a}_{mn}a_{mn}(1 + \bar{\rho}_{mn})) \right] f(\eta_{mn}) = \\ f[-t\bar{a}_{mn}(a_{m-1n} + \bar{a}_{mn-1}) + \bar{a}_{mn}a_{mn}(1 + \bar{\rho}_{mn})]. \end{aligned} \quad (43)$$

We could also have chosen to integrate over the ρ_{mn} s and $\bar{\rho}_{mn}$ s instead. We obtain finally

$$\begin{aligned} \mathcal{S} = \bar{a}_{mn}a_{mn} + t^2 a_{m-1n}\bar{a}_{mn-1} + t(a_{mn} - \bar{a}_{mn})(a_{m-1n} + \bar{a}_{mn-1}) \\ + \bar{\rho}_{mn}\rho_{mn} + \bar{\rho}_{mn} \left[\bar{a}_{mn} - t(a_{m-1n} + \bar{a}_{mn-1}) \right] a_{mn} \\ + g_0\rho_{mn}Q_{mn} \left[1 - \gamma(\rho_{m-1n} + Q_{mn-1}) + \gamma^2\rho_{m-1n}Q_{mn-1} - t^2 a_{m-1n}\bar{a}_{mn-1} \right], \end{aligned} \quad (44)$$

with

$$Q_{mn} = \bar{a}_{mn}[a_{mn}(1 + \bar{\rho}_{mn}) - t(a_{m-1n} + \bar{a}_{mn-1})]. \quad (45)$$

We recognize in the first line of the previous expression (44) the minimal action for the Ising model with only one pair of Grassmann variables [11]

$$\mathcal{S}_{\text{Ising}} = \bar{a}_{mn}a_{mn} + t^2 a_{m-1n} \bar{a}_{mn-1} + t(a_{mn} - \bar{a}_{mn})(a_{m-1n} + \bar{a}_{mn-1}). \quad (46)$$

The rest of the action describes the interaction between the fermions and bosons.

$$\begin{aligned} \mathcal{S}_{\text{int}} = & \bar{\rho}_{mn}\rho_{mn} + \bar{\rho}_{mn} \left[\bar{a}_{mn} - t(a_{m-1n} + \bar{a}_{mn-1}) \right] a_{mn} + g_0 \rho_{mn} Q_{mn} \left[1 - \gamma(\rho_{m-1n} + Q_{mn-1}) \right. \\ & \left. + \gamma^2 \rho_{m-1n} Q_{mn-1} - t^2 a_{m-1n} \bar{a}_{mn-1} \right]. \end{aligned} \quad (47)$$

It is easy to check that when $g_0 = 0$ the boson variables can be integrated out.

We would like to end this section commenting the previous results. We finally obtained a lattice field theory containing the same number of "fermions" (a, \bar{a}) and "bosons" ($\rho, \bar{\rho}$). Physically, this means that it is indeed possible to describe the system with fermionic variables for the states $\sigma = \pm 1$ and bosonic ones for the third state $\sigma = 0$. In the limit $\Delta_0 \rightarrow -\infty$, the system is completely described in terms of fermions. While lowering Δ_0 , an interaction between fermions is added. Beyond a value Δ_{0t} , fermions form bosonic pairs : in the limit $\Delta_0 \rightarrow +\infty$, all the fermions condense into bosons, leading to an purely bosonic systems. In this interpretation, the tricritical point may be interpreted as the particular point on the critical line where the interaction is such that an additional symmetry between fermions and bosons appears. This would correspond to super-symmetry appearing in the conformal field theory describing tricritical Ising model [20]. To our knowledge there is no evidence of super-symmetry derived directly from a lattice model : action (44) could be a good way to see how super-symmetry emerges from a lattice model. Of course all we said so far is only speculative : we are currently studying it in more details, to confirm or infirm this hypothesis.

4. Effective action in the continuum limit

In the 2D Ising model case, the fermionic action on the lattice is quadratic, and the Grassmann integrals equivalent to expression (33) with $g_0 = 0$ can actually be computed exactly. The situation for the 2D BC model is less simple, as the interaction (36) contains terms of order up to 8th. The Grassmann integral leading to the partition function can no longer be decomposed into elements of size independent of the system size : in this sense the 2D BC model is not integrable. However it is still possible to extract physical information by taking the continuous limit of the action (33) and analysing it using tools from field theory.

4.1. Effective 2nd order fermionic field theory

We would like to obtain an effective theory up to order 2 in momentum k , from the previous calculations, in order to analyse the critical behaviour of the model. In the Ising

model, the critical behaviour is given, in the continuous limit, by a mass-less theory. In the following, we will see how to compute the mass of the model. In the infrared limit, the spectrum is given by expanding the action in second order in the momentum k . The "stiffness" of the Ising model, or the coefficient in front of the term k^2 , is always a strictly positive coefficient. Here in the BC model, we will show that it can vanish at the tricritical point, rendering the spectrum unstable and changing the nature of the singularity.

Instead of integrating over the variables η_{mn} and $\bar{\eta}_{mn}$ as in Eq. (44), we proceed by integrating first over $\bar{\eta}_{mn}$ and $\bar{\rho}_{mn}$ from Eq. (42)

$$Z_{\Delta} = 2^N \cosh(K)^{2N} e^{N\Delta} \int \prod_{m,n} dc_{mn} d\bar{c}_{mn} d\rho_{mn} d\eta_{mn} [\bar{c}_{mn}c_{mn} + \rho_{mn}q_{mn} + \eta_{mn}\bar{q}_{mn} + \rho_{mn}\eta_{mn}] \times \exp(\mathcal{S}_{\text{Ising}} + \mathcal{S}_{\text{int}}),$$

where

$$\mathcal{S}_{\text{int}} = g_0 \sum_{m,n} \rho_{mn} \eta_{mn} [(1 - \gamma\rho_{m-1n})(1 - \gamma\eta_{mn-1}) - t^2 c_{m-1n} \bar{c}_{mn-1}],$$

and

$$\bar{q}_{mn} = \bar{c}_{mn}c_{mn} + t c_{mn}(c_{m-1n} + \bar{c}_{mn-1}), \quad q_{mn} = \bar{c}_{mn}c_{mn} - t \bar{c}_{mn}(c_{m-1n} + \bar{c}_{mn-1}). \quad (48)$$

It is also useful to notice that $q_{mn}\bar{q}_{mn} = 0$. The action $\mathcal{S}_{\text{Ising}}$ is given by the relation (46). The previous integral involves the product of monomials such as $\bar{c}_{mn}c_{mn} + \rho_{mn}q_{mn} + \eta_{mn}\bar{q}_{mn} + \rho_{mn}\eta_{mn}$ which can not be written as a single exponential. However, when integrating over ρ_{mn} and η_{mn} , it is easy to realize that these terms *roughly* impose the following substitution in the action \mathcal{S}_{int} :

$$\rho_{mn}\eta_{mn} \rightarrow \bar{c}_{mn}c_{mn} \quad (49)$$

$$\rho_{mn} \rightarrow \bar{q}_{mn} \quad (50)$$

$$\eta_{mn} \rightarrow q_{mn}. \quad (51)$$

In one sense, this product can be considered as an approximate Dirac function for the variables ρ_{mn} and η_{mn} . However it is *not* exact: when expanding the exponential of \mathcal{S}_{int} it appears terms that couple to each other to give $\bar{c}_{mn}c_{mn}$ and not $\bar{q}_{mn}q_{mn} = 0$ as given by the substitution rules. For example terms such as

$$(g_0\rho_{m+1n}\eta_{m+1n}\gamma\rho_{mn}) \times (g_0\rho_{mn+1}\eta_{mn+1}\gamma\eta_{mn}) \quad (52)$$

leads to a contribution equal to

$$g_0^2 \gamma^2 \bar{c}_{mn}c_{mn} \bar{c}_{m+1n}c_{m+1n} \bar{c}_{mn+1}c_{mn+1}. \quad (53)$$

Therefore there are more terms in the final effective action $\mathcal{S}_{\text{eff}}(c_{mn}, \bar{c}_{mn})$ than in the one resulting from the substitution rules. However, we would like to obtain an effective action up to order 2 in momentum k , in order to study the stability of the free fermion spectrum which gives basically a $\log(k^2)$ contribution in the free energy for the Ising model at criticality. The terms that contribute to the second order in momentum are, from the free quadratic part Eq. 46 and in the continuous limit, combinations of $c\partial_x c$ or

$\bar{c}\partial_x c$, and $\bar{c}\partial_y \bar{c}$ or $c\partial_y \bar{c}$. We indeed expect that the effective action will contain quartic contributions such as $\bar{c}c\partial_i \bar{c}\partial_j c$, with $i, j = x, y$ at the lowest order. This term is order 4 in Grassmann variables and 2 in derivatives. The expansion of the exponential of such terms will give corrective coefficients to the k^2 behaviour, and thus change the order of the transition if such renormalized coefficient vanishes. We also have to consider not only the direct substitution of the variables with the rules given above, but also the possible corrective terms that can lead to the same order. Also we should drop terms which contain a ratio of number of derivatives to the number of Grassmann variables higher strictly than $1/2$.

After some algebra, these following terms contribute to the effective action :

$$\begin{aligned} \mathcal{S}_{\text{eff}} = \mathcal{S}_{\text{Ising}} + g_0 \sum_{m,n} \bar{c}_{mn} c_{mn} [(1 - \gamma \bar{q}_{m-1n})(1 - \gamma q_{mn-1}) - t^2 c_{m-1n} \bar{c}_{mn-1}] \\ + g_0^2 \gamma^2 \sum_{m,n} \bar{c}_{mn} c_{mn} \bar{c}_{m+1n} c_{m+1n} \bar{c}_{mn+1} c_{mn+1} + \dots \end{aligned} \quad (54)$$

In the continuous limit we replace c_{mn} by $c = c(x, y)$ and \bar{c}_{mn} by $\bar{c} = \bar{c}(x, y)$, with the rules $c_{m-1n} = c - \partial_x c + \partial_{xx} c/2! + \dots$. For example, $q_{mn} \rightarrow q = \bar{c}c(1 - t) + t\bar{c}(\partial_x c + \partial_y \bar{c})$ and $\bar{q}_{mn} \rightarrow \bar{q} = \bar{c}c(1 - t) - tc(\partial_x c + \partial_y \bar{c})$. The free Ising part gives simply

$$\mathcal{S}_{\text{Ising}} = \int dx dy [(t^2 + 2t - 1)c\bar{c} + t(t + 1)\bar{c}\partial_x c - t(t + 1)c\partial_y \bar{c} - tc\partial_x c + t\bar{c}\partial_y \bar{c}]. \quad (55)$$

In the continuous limit, the last term of Eq. (54) give $\bar{c}c\partial_x \bar{c}\partial_x c\partial_y \bar{c}\partial_y c$, which is order 4 in derivatives and 6 in Grassmann variables. The ratio of these numbers is $2/3$ which is higher than $1/2$, and therefore this term can be dropped, as explained above. The term in factor of g_0 in Eq. (54) contain \bar{q}_{m-1n} and q_{mn-1} which need to be expanded up to the order 2 in derivatives, with $\partial_{xx} \bar{q} = 2(1 - t)\partial_x \bar{c}\partial_x c$, and $\partial_{yy} q = 2(1 - t)\partial_y \bar{c}\partial_y c$. The effective action finally can be written in the continuous limit as

$$\begin{aligned} \mathcal{S}_{\text{eff}} = \mathcal{S}_{\text{Ising}} \\ + \int dx dy \left\{ g_0 \bar{c}c + g_0 \bar{c}c [t(t + 2\gamma)\partial_y \bar{c}\partial_x c + \gamma(1 - t)(\partial_x c\partial_x \bar{c} + \partial_y c\partial_y \bar{c})] \right\} + \dots \end{aligned} \quad (56)$$

In the following we use this effective action to obtain information on the phase diagram of the BC model.

5. Spectrum analysis and phase diagram

In this section we analyse the critical properties of the effective action (56) and the low energy spectrum. In particular we develop a physical argument for the existence of a tricritical point on the phase diagram from a fermionic action.

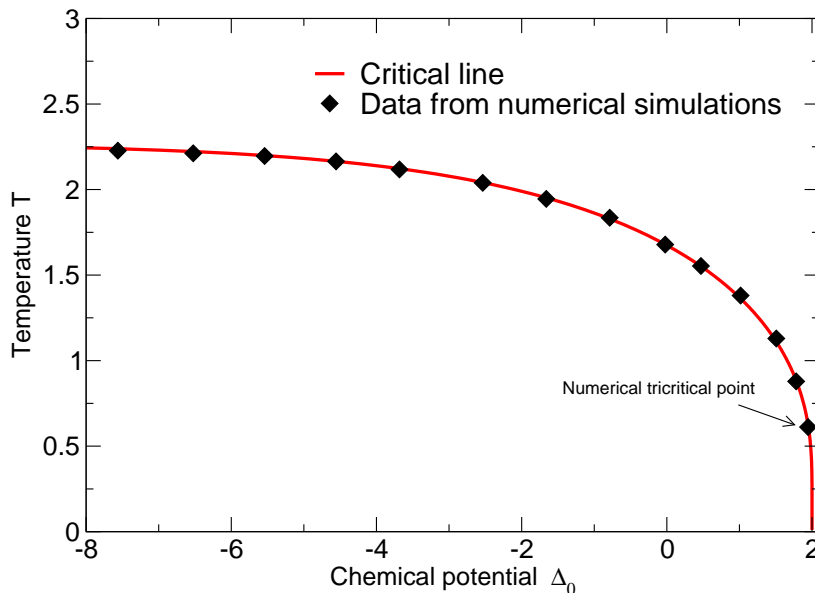


Figure 1. Comparison between the critical line (57) (plain line) and numerical results from Monte Carlo simulations by da Silva *et al.* [5] (dots). The last numerical point (arrow) is the tricritical point identified by da Silva *et al.*.

5.1. Critical line

From Eqs. (55) and (55), we can easily identify the mass $M_0 = -t^2 - 2t + 1 + g_0$ in factor of the term $\bar{c}c$, and which vanishes on the critical line $M_0 = 0$,

$$\tanh^2\left(\frac{1}{T}\right) + 2 \tanh\left(\frac{1}{T}\right) - 1 = \frac{e^{\frac{\Delta_0}{T}}}{2 \cosh^2\left(\frac{1}{T}\right)}. \quad (57)$$

On this line, only derivative contributions remain in Eq.(55) which give the free fermion kinetic terms, and the ones in Eq.(56) give the interaction between the singlet level and the Ising doublet at the quartic level. The critical line given by (57) is plotted in figure 1, and compared with recent Monte Carlo simulations by da Silva *et al.* [5]. The agreement between numerical simulations and our results is very good, the mass of the system (57) being exact. Note that our results is also compatible with exact upper bond obtained by Braga *et al.* [8].

5.2. Tricritical point : Hartree-Fock analysis

The main physical feature of the 2D BC model is the existence of a tricritical point, on the critical line. Below this point, the phase transition goes from second order to first order: the tricritical point is characterized by a change in the nature of the singularity. This change should be seen in the BC spectrum from (56). In this section, we analysis

the effect of the quartic terms, on the critical line $g_0 = t^2 + 2t - 1$ by considering the effect of the quartic interaction on the stability of the free fermion spectrum.

The Ising part can be easily written in the Fourier space, after having defined the following transformations

$$c(\mathbf{r}) = \frac{1}{L} \sum_{\mathbf{k}} c_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}) \text{ and } \bar{c}(\mathbf{r}) = \frac{1}{L} \sum_{\mathbf{k}} \bar{c}_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{r}). \quad (58)$$

Using these transformations, the Ising part is almost diagonalized, in particular

$$\mathcal{S}_{\text{Ising}} = \sum_{\mathbf{k} \in S} it(t+1)(k_x - k_y)(\bar{c}_{\mathbf{k}} c_{\mathbf{k}} - \bar{c}_{-\mathbf{k}} c_{-\mathbf{k}}) + 2itk_x c_{\mathbf{k}} c_{-\mathbf{k}} + 2itk_y \bar{c}_{\mathbf{k}} \bar{c}_{-\mathbf{k}}, \quad (59)$$

where S is the set of Fourier modes that correspond to half of the Brillouin zone, *i.e.* we choose \mathbf{k} in S such that $-\mathbf{k}$ is not included in S (to avoid repetition of modes in the different sums above), and such that the couples of modes $(\mathbf{k}, -\mathbf{k})$ fill up the Brillouin zone exactly once. The quartic term can be written in the Fourier space as

$$\mathcal{S}_{\text{int}} = \frac{1}{L^2} \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4} V(\mathbf{k}_2, \mathbf{k}_4) c_{\mathbf{k}_1} c_{\mathbf{k}_2} \bar{c}_{\mathbf{k}_3} \bar{c}_{\mathbf{k}_4}, \quad (60)$$

with the potential

$$\begin{aligned} V(\mathbf{k}_2, \mathbf{k}_4) &= -\alpha k_2^x k_4^y + \beta(k_2^x k_4^x + k_2^y k_4^y) \\ \alpha &= g_0 t(t + 2\gamma) \text{ and } \beta = g_0 \gamma(1 - t). \end{aligned} \quad (61)$$

Up to now we only expressed the action in the Fourier space, without further approximations. In order to see if a tricritical point appears in this approach, we use a mean field like approximation, similar to the Hartree-Fock method [21, 22]. To do so, we decompose the fourth order interacting terms into sums of quadratic terms with coefficients to be determined self-consistently. These coefficients are actually two point correlation functions.

The interaction can be decoupled in different ways. For example, considering the terms contributing to the Ising action, we may take account of the averages $\langle \bar{c}_{\mathbf{k}} c_{\mathbf{k}} \rangle$, $\langle \bar{c}_{-\mathbf{k}} c_{-\mathbf{k}} \rangle$, $\langle c_{\mathbf{k}} c_{-\mathbf{k}} \rangle$ and $\langle \bar{c}_{\mathbf{k}} \bar{c}_{-\mathbf{k}} \rangle$. There are also three different ways to decouple the interacting term, since $c_{\mathbf{k}_1}$ can be paired with either $c_{\mathbf{k}_2}$, $\bar{c}_{\mathbf{k}_3}$ or $\bar{c}_{\mathbf{k}_4}$. For example,

$$c_{\mathbf{k}_1} c_{\mathbf{k}_2} = \langle c_{\mathbf{k}_1} c_{\mathbf{k}_2} \rangle + (c_{\mathbf{k}_1} c_{\mathbf{k}_2} - \langle c_{\mathbf{k}_1} c_{\mathbf{k}_2} \rangle) \equiv \langle c_{\mathbf{k}_1} c_{\mathbf{k}_2} \rangle + \delta_{c_1 c_2}, \quad (62)$$

where $\delta_{c_1 c_2}$ is assume to be a small fluctuation. In this case, from Eq. (59), the average is non zero only for $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$ or $-\mathbf{k}$, with $\mathbf{k} \in S$. We can pair the other terms by writing the action in the three different possible ways that are compatible with Eq. (59), and by using the fermionic rules

$$\begin{aligned} \mathcal{S}_{\text{int}} &= \frac{1}{L^2 g} \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4} V(\mathbf{k}_2, \mathbf{k}_4) \left[(\langle c_{\mathbf{k}_1} c_{\mathbf{k}_2} \rangle + \delta_{c_1 c_2}) (\langle \bar{c}_{\mathbf{k}_3} \bar{c}_{\mathbf{k}_4} \rangle + \delta_{\bar{c}_3 \bar{c}_4}) \right. \\ &\quad \left. - (\langle c_{\mathbf{k}_1} \bar{c}_{\mathbf{k}_3} \rangle + \delta_{c_1 \bar{c}_3}) (\langle c_{\mathbf{k}_2} \bar{c}_{\mathbf{k}_4} \rangle + \delta_{c_2 \bar{c}_4}) + (\langle c_{\mathbf{k}_1} \bar{c}_{\mathbf{k}_4} \rangle + \delta_{c_1 \bar{c}_4}) (\langle c_{\mathbf{k}_2} \bar{c}_{\mathbf{k}_3} \rangle + \delta_{c_2 \bar{c}_3}) \right], \end{aligned} \quad (63)$$

with $g = 3$. The next step is to discard terms that are proportional to the squares of fluctuations δ^2 , and keep the others. After some algebra, we obtain the mean-field quadratic operator for the interaction term

$$\begin{aligned} \mathcal{S}_{\text{int}} = \frac{1}{L^2 g} \sum_{\mathbf{k}, \mathbf{k}' \in S} & 4c_{\mathbf{k}}c_{-\mathbf{k}}\langle \bar{c}_{\mathbf{k}'}\bar{c}_{-\mathbf{k}'} \rangle V(\mathbf{k}, \mathbf{k}') + 4\bar{c}_{\mathbf{k}}\bar{c}_{-\mathbf{k}}\langle c_{\mathbf{k}'}c_{-\mathbf{k}'} \rangle V(\mathbf{k}', \mathbf{k}) \\ & + \bar{c}_{\mathbf{k}}c_{\mathbf{k}} \left(\langle \bar{c}_{\mathbf{k}'}c_{\mathbf{k}'} \rangle v(\mathbf{k}, \mathbf{k}') + \langle \bar{c}_{-\mathbf{k}'}c_{-\mathbf{k}'} \rangle v(\mathbf{k}, -\mathbf{k}') \right) \\ & + \bar{c}_{-\mathbf{k}}c_{-\mathbf{k}} \left(\langle \bar{c}_{\mathbf{k}'}c_{\mathbf{k}'} \rangle v(-\mathbf{k}, \mathbf{k}') + \langle \bar{c}_{-\mathbf{k}'}c_{-\mathbf{k}'} \rangle v(-\mathbf{k}, -\mathbf{k}') \right), \end{aligned} \quad (64)$$

where we have defined the potential

$$v(\mathbf{k}, \mathbf{k}') = -V(\mathbf{k}, \mathbf{k}) - V(\mathbf{k}', \mathbf{k}') + V(\mathbf{k}, \mathbf{k}') + V(\mathbf{k}', \mathbf{k}). \quad (65)$$

In the previous expressions, there are three different kinds of quantities that contribute to the action, for example sums involving quantities like $\sum_{\mathbf{k}} \bar{c}_{\mathbf{k}}c_{\mathbf{k}}$, $\sum_{\mathbf{k}} \bar{c}_{\mathbf{k}}c_{\mathbf{k}}k_i$ or $\sum_{\mathbf{k}} \bar{c}_{\mathbf{k}}c_{\mathbf{k}}k_i k_j$, with $i, j = x, y$. The first term gives a contribution to the total mass, the second one corresponds to current operators, and the third one can be thought as a dispersion energy tensor. Considering the symmetries of the Ising part, and the fact that the action should be invariant by a dilation factor at criticality, we may only take account of the current operators. We define therefore the following unknown parameters, for fermions and pairs of fermions or "Cooper pairs" ($i = x, y$)

$$\begin{aligned} t_i &= \frac{i}{2L^2} \sum_{\mathbf{k} \in S} (\langle \bar{c}_{\mathbf{k}}c_{\mathbf{k}} \rangle - \langle \bar{c}_{-\mathbf{k}}c_{-\mathbf{k}} \rangle) k_i, \\ u_i &= \frac{i}{L^2} \sum_{\mathbf{k} \in S} \langle c_{\mathbf{k}}c_{-\mathbf{k}} \rangle k_i, \quad \bar{u}_i = \frac{i}{L^2} \sum_{\mathbf{k} \in S} \langle \bar{c}_{\mathbf{k}}\bar{c}_{-\mathbf{k}} \rangle k_i. \end{aligned} \quad (66)$$

From the previous discussion, we can drop the first two terms in the potential $v(\mathbf{k}, \mathbf{k}')$ defined in Eq. (65), since we already assume that only currents are kept as parameters along the critical line. In this case, it is easy to rewrite, from the property $v(\mathbf{k}, \mathbf{k}') = -v(-\mathbf{k}, \mathbf{k}') = -v(\mathbf{k}, -\mathbf{k}')$, the effective mean field action (64) as :

$$\begin{aligned} \mathcal{S}_{\text{int}} = \frac{1}{g} \sum_{\mathbf{k} \in S} & 4ic_{\mathbf{k}}c_{-\mathbf{k}}[(\alpha\bar{u}_y - \beta\bar{u}_x)k_x - \beta\bar{u}_y k_y] + 4i\bar{c}_{\mathbf{k}}\bar{c}_{-\mathbf{k}}[-\beta u_x k_x + (\alpha u_x - \beta u_y)k_y] \\ & + 2i(\bar{c}_{\mathbf{k}}c_{\mathbf{k}} - \bar{c}_{-\mathbf{k}}c_{-\mathbf{k}})[(\alpha t_y - 2\beta t_x)k_x + (\alpha t_x - 2\beta t_y)k_y]. \end{aligned} \quad (67)$$

We make the further assumption that, by symmetry invariance in the space, there exists a solution satisfying $\bar{u}_y = u_x$, $\bar{u}_x = u_y$ and $t_x = -t_y$:

$$\begin{aligned} \mathcal{S}_{\text{int}} = \frac{1}{g} \sum_{\mathbf{k} \in S} & 4ic_{\mathbf{k}}c_{-\mathbf{k}}[(\alpha u_x - \beta u_y)k_x - \beta u_x k_y] + 4i\bar{c}_{\mathbf{k}}\bar{c}_{-\mathbf{k}}[-\beta u_x k_x + (\alpha u_x - \beta u_y)k_y] \\ & - 2i(\bar{c}_{\mathbf{k}}c_{\mathbf{k}} - \bar{c}_{-\mathbf{k}}c_{-\mathbf{k}})(k_x - k_y)(\alpha + 2\beta)t_x. \end{aligned} \quad (68)$$

The total effective action can finally be written as

$$\begin{aligned} \mathcal{S}_{\text{eff}} = \sum_{\mathbf{k} \in S} & i \left[t(t+1) - \frac{2}{g}(\alpha + 2\beta)t_x \right] (k_x - k_y)(\bar{c}_{\mathbf{k}}c_{\mathbf{k}} - \bar{c}_{-\mathbf{k}}c_{-\mathbf{k}}) \\ & + i\frac{4}{g} \left[\left(\frac{g}{2}t + (\alpha u_x - \beta u_y) \right) k_x - \beta u_x k_y \right] c_{\mathbf{k}}c_{-\mathbf{k}} + i\frac{4}{g} \left[-\beta u_x k_x + \left(\frac{g}{2}t + (\alpha u_x - \beta u_y) \right) k_y \right] \bar{c}_{\mathbf{k}}\bar{c}_{-\mathbf{k}}, \end{aligned} \quad (69)$$

or in a more compact form as

$$\mathcal{S}_{\text{eff}} = \sum_{\mathbf{k} \in S} ic(k_x - k_y)(\bar{c}_{\mathbf{k}}c_{\mathbf{k}} - \bar{c}_{-\mathbf{k}}c_{-\mathbf{k}}) + 2i(ak_x - bk_y)c_{\mathbf{k}}c_{-\mathbf{k}} + 2i(-bk_x + ak_y)\bar{c}_{\mathbf{k}}\bar{c}_{-\mathbf{k}}, \quad (70)$$

with the following coefficients

$$\begin{aligned} a &= t + 2\frac{\alpha u_x - \beta u_y}{g}, \\ b &= 2\beta\frac{u_x}{g}, \\ c &= t(t+1) - 2t_x\frac{\alpha + 2\beta}{g}. \end{aligned} \quad (71)$$

The partition function can then be written as a product over the Fourier modes $Z = \prod_{\mathbf{k} \in S} Z_{\mathbf{k}}$, with

$$Z_{\mathbf{k}} = k^2[A + B \sin 2\theta], \quad (72)$$

θ being the angle of the vector \mathbf{k} , and

$$A = c^2 - 4ab, \quad B = -c^2 + 2(a^2 + b^2). \quad (73)$$

We assume that $|A|$ is larger than $|B|$ on the second order critical line, until the tricritical point is reached, where eventually $A^2 = B^2$. Indeed, the expression (72) is valid only if the elements $A + B \sin 2\theta$ are all positive, which is the case only if $A^2 > B^2$. This will be checked using numerical analysis. Beyond this point, the effective action is unstable and has to be modified to incorporate further corrections. In a φ^6 Ginzburg-Landau theory describing a first order transition, this will be equivalent to say that at the tricritical point, not only the mass coefficient of the φ^2 term vanishes, but also the coefficient of the φ^4 term.

The parameters t_x , u_x and u_y are determined self-consistently from the definitions Eqs. (66). In the continuous limit, these reduce to

$$\begin{aligned} t_x &= \frac{c}{4\pi} \int_0^\pi d\theta \frac{1 - \sin 2\theta}{A + B \sin 2\theta}, \\ u_x &= \frac{1}{2\pi} \int_0^\pi d\theta \frac{a \sin 2\theta - b}{A + B \sin 2\theta}, \quad u_y = \frac{1}{2\pi} \int_0^\pi d\theta \frac{a - b \sin 2\theta}{A + B \sin 2\theta}. \end{aligned} \quad (74)$$

After computing the trigonometric integrals, we obtain the relations

$$\begin{aligned} t_x &= \frac{c}{4B} \left(-1 + (A + B) \frac{\text{sign}(A)}{\sqrt{A^2 - B^2}} \right), \\ u_x &= \frac{1}{2B} \left(a - (aA + bB) \frac{\text{sign}(A)}{\sqrt{A^2 - B^2}} \right), \\ u_y &= \frac{1}{2B} \left(-b + (bA + aB) \frac{\text{sign}(A)}{\sqrt{A^2 - B^2}} \right). \end{aligned} \quad (75)$$

To find the tricritical point, we proceed the following way. For each temperature T , we solve the consistency equations for t_x , u_x and u_y with the value of Δ_0 given by the critical line (57). The self-consistent values are then plug into the coefficients $A(T)$ and $B(T)$. We then plot $A(T)^2 - B(T)^2$ as a function of T , as shown on figure 2. By doing so we

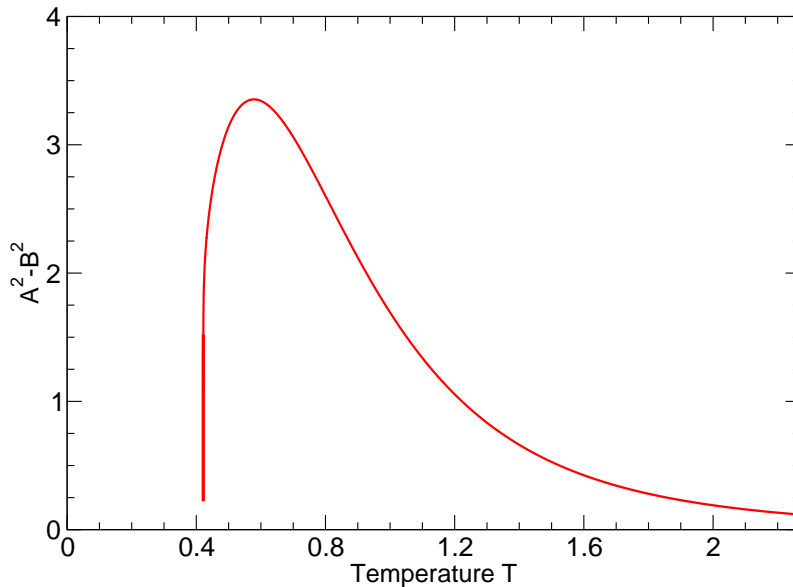


Figure 2. Stability of the spectrum: coefficient $A(T)^2 - B(T)^2$ as a function of T . The temperature where $A(T)^2 - B(T)^2 = 0$ gives the location of the tricritical point.

find a tricritical point approximately located at $(T_t, \Delta_{0,t}) \simeq (0.42158, 1.9926)$. Monte Carlo simulations [5] gives a $T_t \simeq 0.61$. As expected the mean-field like treatment of the underlying field theory underestimates the fluctuations, rendering the second order critical line more stable at lower temperature, as we approach $(T = 0, \Delta_0 = 2)$. Stronger fluctuations can be simulated by lowering the value of g , which increases (lowers) the value of T_t (Δ_0) respectively. Instead of $g = 3$, taking $g = 2.5$ for example leads to a $T_t \simeq 0.48$, closer to the Monte Carlo result. This can be achieved precisely by incorporating more diagrams in the computation of the effective free energy [22]. Also, due to the fact we are in a region near $(T = 0, \Delta_0 = 2)$ where the change in temperature is large compare to the change of Δ_0 (the slope is vertical at this point as seen in figure 1), it is more difficult to obtain a precise value of T_t with a mean-field treatment.

6. Conclusion

In this paper, we have considered the physics of the BC model as a fermionic field theory. Using Grassmann algebra, we have shown that the model can be transformed as an exact fermionic action on the discrete lattice. This action can be reduced, in the continuum limit, to an effective field theory which is a modified Ising action with a quartic interaction. From there we have extracted the exact mass of the model, and analysed the effect of the quartic term on the stability of the free fermion spectrum.

In particular, the *stiffness* of the excitation spectrum (the coefficient in front of the k^2 term as we expand the dispersion relation in momentum variable) vanishes at a singular point we have identified as the tricritical point. A Hartree-Fock analysis gave an approximate location of this point on the phase diagram which can be compared to the numerical results of Monte Carlo simulations. The precise location of the point could be achieved by taking into account further diagrams contributing to the effective free energy. However, we have shown the existence of a singular point by studying the stability of the kinetic spectrum where the nature of the transition is changed. The main result of this paper is the possibility to study precisely first-order transition driven systems from a fermionic point of view using Grassmann algebra. That method can be useful for systems where the effective field theory is described by a similar action such as Eq. (56) which is one of the simplest form that can be written out from a unique pair of Grassmann variables. Application of this method to other extensions of the BC Hamiltonian, such as the Blume-Emery-Griffiths model [3], is also possible. Finally, a partial bosonization of the system leads to a *mixed* representation of the model not only in term of fermions but also in term of *hard core bosons*, as written explicitly in Eq. 44. This result could be useful to see the tricritical point as a special location in the phase diagram where a possible hidden symmetry between fermion and boson variables may appear.

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