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Amandine Aftalion, Francis Nier

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Abstract

Following the recent experimental realization of synthetic gauge magnetic forces, Jean Dalibard adressed the question whether the adiabatic ansatz could be mathematically justified for a model of an atom in 2 internal states, shun by a quasi resonant laser beam. In this paper, we derive rigorously the asymptotic model guessed by the physicists, and show that this asymptotic analysis contains the information about the presence of vortices. Surprisingly the main difficulties do not come from the nonlinear part but from the linear Hamiltonian. More precisely, the analysis of the nonlinear minimization problem and its asymptotic reduction to simpler ones, relies on an accurate partition of low and high frequencies (or momenta). This requires to reconsider carefully previous mathematical works about the adiabatic limit. Although the estimates are not sharp, this asymptotic analysis provides a good insight about the validity of the asymptotic picture, with respect to the size of the many parameters initially put in the complete model.

1 Introduction

A lot of interest, both in the mathematical and physical community, has been devoted in the past 10 years to the study of the rotation of a Bose Einstein condensate: experiments [MCWD1, MCWD2], theoretical works (see [Coo, Fet] for reviews), mathematical contributions [Aft, LiSe]. In [MCWD1, MCWD2], a rotating laser beam is superimposed on the magnetic trap holding the atoms in order to spin up the condensate by creating a harmonic anisotropic rotating potential. Recently, a new experimental device has emerged which consists in realizing artificial or synthetic gauge magnetic forces and leads to the formation of vortex lattices at rest in the lab frame [LCGPS]. A colloquium [DGJO] has analyzed in detail the artificial gauge fields and their manifestations. In order to

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understand the main ingredients of the physics of geometrical gauge fields, [DGJO] (see also [GCYRD]) introduced a toy model: a single quantum particle state with a 2 level internal structure. A key issue is to determine whether one internal state can be followed adiabatically. A question raised by Jean Dalibard is to analyze in particular whether vortex formation may break down the adiabatic process. In [DGJO], some conditions are provided, that we want to analyze from a mathematical point of view.

We are interested in the minimization of the energy

$$\begin{aligned}\mathcal{E}_\kappa(\phi) &= \int_{\mathbb{R}^2} |\nabla\phi|^2 + V_\kappa(x,y)|\phi|^2 \\ &\quad + \Omega_{\kappa,\ell_\kappa}(x)\langle\phi\,,\begin{pmatrix}\cos(\theta_{\ell_\kappa}(x)) & e^{i\varphi_k(y)}\sin(\theta_{\ell_\kappa}(x))\\ e^{-i\varphi_k(y)}\sin(\theta_{\ell_\kappa}(x)) & -\cos(\theta_{\ell_\kappa}(x))\end{pmatrix}\phi\rangle_{\mathbb{C}^2} + \frac{G}{2}|\phi|^4\,dxdy,\end{aligned}$$

where $\phi = \begin{pmatrix}\phi_1(x,y)\\ \phi_2(x,y)\end{pmatrix} \in \mathbb{C}^2$ and $|\phi(x,y)|^2 = |\phi_1(x,y)|^2 + |\phi_2(x,y)|^2$. Here ϕ_1 and ϕ_2 are the internal degree of freedom of a particle: ground and excited state of the atom. It is assumed that the atom is shun by a quasi resonnant laser beam. The functions $\Omega_{\kappa,\ell_\kappa}(x)$, $\varphi_k(y)$ and $\theta_{\ell_\kappa}(x)$ are given as in [DGJO] by

$$\begin{aligned}&\Omega_{\kappa,\ell_\kappa}(x) = \kappa\Omega(\frac{x}{\ell_\kappa}) \text{ with } \Omega(x) = \sqrt{1+x^2}\,,\\ &\theta_{\ell_\kappa}(x) = \underline{\theta}(\frac{x}{\ell_\kappa}) \text{ with } \cos(\underline{\theta}(x)) = \frac{x}{\sqrt{x^2+1}} \quad , \quad \sin(\underline{\theta}(x)) = \frac{1}{\sqrt{x^2+1}}\\ &\varphi_k(y) = \underline{\varphi}(ky) \text{ with } \underline{\varphi}(y) = y\,.\end{aligned}$$

where $\underline{\varphi}$ is the phase of the propagating laser beam while $\underline{\theta}$ is the mixing angle. We define

$$M(x,y) = \Omega(x)\begin{pmatrix}\cos(\underline{\theta}(x)) & e^{i\underline{\varphi}(y)}\sin(\underline{\theta}(x))\\ e^{-i\underline{\varphi}(y)}\sin(\underline{\theta}(x)) & -\cos(\underline{\theta}(x))\end{pmatrix}. \tag{1.1}$$

The matrix M models the coupling between the atom and the laser. The 2×2 matrix $M(\frac{x}{\sqrt{\ell_\kappa k}}, \sqrt{\ell_\kappa k}y)$ can be diagonalized in the bases (ψ_+,ψ_-) respectively associated with the eigenvalues $\pm\Omega(x)$,

$$\psi_+ = \begin{pmatrix}C\\ Se^{-i\varphi}\end{pmatrix}\ ,\ \psi_- = \begin{pmatrix}Se^{i\varphi}\\ -C\end{pmatrix} \tag{1.2}$$

with $C = \cos(\frac{\underline{\theta}(\frac{x}{\sqrt{\ell_\kappa k}})}{2})$, $S = \sin(\frac{\underline{\theta}(\frac{x}{\sqrt{\ell_\kappa k}})}{2})$ and $\varphi = \sqrt{\ell_\kappa k}y$. When the particle follows adiabatically one of the eigenstates, for instance ψ_-, this corresponds to set formally $\varphi = u(x,y)\psi_-$, where $u(x,y)\in\mathbb{C}$. Then u minimizes a GP type energy functional with a modified trapping potential called the geometrical gauge potential. The scalar potential $V_\kappa(x,y) = V_\kappa(x,y)\,\mathrm{Id}_{\mathbb{C}^2}$ will be adjusted in order to produce a harmonic potential after the addition of the geometrical gauge potential from the adiabatic theory. We want to justify the adiabatic approximation for states close to ψ_- and analyze the error term between the initial and effective Hamiltonians.

After a rescaling, the parameter occuring in experiments have the following orders of magnitude

$$\kappa \sim 10^6 \quad , \quad G \sim 600 \quad , \quad \ell_\kappa \sim 25 \quad , \quad k \sim 50\,,$$

but other values of the parameters can be discussed. Conditions on the strength and spatial extent of the artificial potential have to be prescribed to induce large circulation. Two cases, $\ell_\kappa k \geq 1$ and $\ell_\kappa k \leq 1$, can be distinguished and the problem has to be rewritten in two different ways in order to apply semiclassical techniques. In fact, we will focus on the case $\ell_\kappa k \geq 1$, corresponding to the previous numerical values. The complete analysis is carried out in the asymptotic regime $\ell_\kappa k \to +\infty$ but some partial results are also valid for $\ell_\kappa k \leq 1$ or $\ell_\kappa k \to 0$.
A change of scale $\phi(x,y) = \sqrt{\frac{k}{\ell_\kappa}}\psi(\sqrt{\frac{k}{\ell_\kappa}}x, \sqrt{\frac{k}{\ell_\kappa}}y)$ yields a new expression for the energy:

$$\int_{\mathbb{R}^2} \frac{k}{\ell_\kappa}(|\partial_x\psi|^2 + |\partial_y\psi|^2) + V_\kappa(\sqrt{\frac{\ell_\kappa}{k}}x, \sqrt{\frac{\ell_\kappa}{k}}y)|\psi|^2$$
$$+ \kappa\Omega(\frac{x}{\sqrt{k\ell_\kappa}})\langle\psi\,, \begin{pmatrix} \cos(\underline{\theta}(\frac{x}{\sqrt{k\ell_\kappa}})) & e^{i\underline{\varphi}(\sqrt{k\ell_\kappa}y)}\sin(\underline{\theta}(\frac{x}{\sqrt{k\ell_\kappa}})) \\ e^{-i\underline{\varphi}(\sqrt{k\ell_\kappa}y)}\sin(\underline{\theta}(\frac{x}{\sqrt{k\ell_\kappa}})) & -\cos(\underline{\theta}(\frac{x}{\sqrt{k\ell_\kappa}})) \end{pmatrix}\psi\rangle_{\mathbb{C}^2}$$
$$+ \frac{Gk}{2\ell_\kappa}|\psi|^4\, dxdy$$

According to the two cases $\ell_\kappa k \geq 1$ or $\ell_\kappa k \leq 1$, we define a small parameter ε that allows to rescale the energy. In fact, we define rather the parameter $\varepsilon^{2+2\delta}$, where δ can be taken as a first step equal to $5/2$. The exponent $\delta > 0$ is a technical trick which provides the right quantitative estimates for the adiabatic approximation with a quadratic kinetic enery term. The suitable choice of this new parameter δ is dicussed further in this introduction, in Subsections 1.1 and 1.2.

if $\ell_\kappa k \geq 1$**:** then

$$\varepsilon^{2+2\delta} = \frac{k^2}{\kappa}\,,\ \delta > 0 \quad , \quad G_\varepsilon = \frac{Gk}{\kappa\ell_\kappa} = \frac{G}{k\ell_\kappa}\varepsilon^{2+2\delta}\,, \tag{1.3}$$

This leads in our example to $\varepsilon^{2+2\delta} = 2.5\ 10^{-3}$.

if $\ell_\kappa k \leq 1$**:**

$$\varepsilon^{2+2\delta} = \frac{1}{\ell_\kappa^2\kappa}\,,\ \delta > 0 \quad , \quad G_\varepsilon = \frac{Gk}{\kappa\ell_\kappa} = Gk\ell_\kappa\varepsilon^{2+2\delta}\,, \tag{1.4}$$

In both cases, this leads to a linear energy

$$\mathcal{E}_{quad,\varepsilon}(\psi) = \int_{\mathbb{R}^2} \varepsilon^{2+2\delta}\tau_x\tau_y|\nabla\psi| + V_{\varepsilon,\tau}(x,y)|\psi|^2$$
$$+\langle\psi\,,\, M(\sqrt{\frac{\tau_x}{\tau_y}}x, \sqrt{\frac{\tau_y}{\tau_x}}y)\psi\rangle_{\mathbb{C}^2}\, dxdy \tag{1.5}$$
$$= \langle\psi\,, H_{Lin}\psi\rangle\,, \tag{1.6}$$

with

$$H_{Lin} = -\varepsilon^{2\delta}\tau_x\tau_y\varepsilon^2\Delta + V_{\varepsilon,\tau}(x,y) + M(\sqrt{\frac{\tau_x}{\tau_y}}x, \sqrt{\frac{\tau_y}{\tau_x}}y)\,, \tag{1.7}$$

$$V_{\varepsilon,\tau}(x,y) = \kappa^{-1}V_\kappa(\sqrt{\frac{\ell_\kappa}{k}}x, \sqrt{\frac{\ell_\kappa}{k}}y) \quad \text{(to be fixed)}\,, \tag{1.8}$$

$$\tau = (\tau_x,\tau_y) = \begin{cases} (\frac{1}{k\ell_\kappa},1) & \text{if } k\ell_\kappa \geq 1\,, \\ (1,k\ell_\kappa) & \text{if } k\ell_\kappa \leq 1\,. \end{cases} \tag{1.9}$$

With the nonlinear term, the energy equals

$$\mathcal{E}_\varepsilon(\psi) = \langle\psi\,,\, H_{Lin}\psi\rangle + \frac{G_{\varepsilon,\tau}}{2}\int |\psi|^4\, dxdy\,, \tag{1.10}$$

$$G_{\varepsilon,\tau} = G\tau_x\tau_y\varepsilon^{2+2\delta}\,. \tag{1.11}$$

In both cases, we thus have

$$\kappa^{-1}\mathcal{E}_\kappa(\phi) = \mathcal{E}_\varepsilon(\psi) \tag{1.12}$$

At least when $\tau_x = \tau_y = 1$ and $\delta = 0$, this problem looks like the standard problem of spatial adiabatic approximation studied in [Sor], [MaSo], [PST], although it requires some adaptations because the symbols are neither bounded nor elliptic.

We shall consider the asymptotic analysis as $\varepsilon \to 0$ with uniform control with respect to the parameters $G, k\ell_\kappa$ which allow to fix the range of validity of the reduced models. Then we shall consider the asymptotic behaviour of the reduced model and the whole system as $\ell_\kappa k$ is large. Specifying the right assumptions on $V_{\varepsilon,\tau}(x,y)$ or V_κ, possibly in a scale depending on (ℓ_κ, k), is also an issue.

1.1 Main result for the Gross Pitaevskii energy

We shall choose the potential $V_{\varepsilon,\tau}$ such that after addition of the adiabatic potential in the lower energy band, the effective potential is almost harmonic, namely we assume

$$V_{\varepsilon,\tau}(x,y) = \frac{\varepsilon^{2+2\delta}}{\ell_V^2}v(\sqrt{\tau_x}x, \sqrt{\tau_x}y) + \sqrt{1+\tau_x x^2} - \varepsilon^{2+2\delta}\left[\frac{\tau_x^2}{(1+\tau_x x^2)^2} + \frac{1}{1+\tau_x x^2}\right]\,, \tag{1.13}$$

with the potential v chosen such that

$$\begin{aligned} &v(x,y) = (x^2+y^2)\chi_v(x^2+y^2) + (1-\chi_v(x^2+y^2)) \\ &\chi_v \in \mathcal{C}_0^\infty([0,2)) \quad , \quad 0 \leq \chi_v \leq 1\,, \quad \chi_v \equiv 1 \text{ on } [0,1], \end{aligned} \tag{1.14}$$

and $\ell_V > 0$ parametrizes the shape of the quadratic potential around the origin.
With these assumptions on the potential $V_{\varepsilon,\tau}$, if one chooses $\psi = u(x,y)\psi_-$, where ψ_- is

the eigenfunction corresponding to $-\Omega(x\sqrt{\tau_x/\tau_y})$ of $\Omega(x\sqrt{\tau_x/\tau_y})M(x\sqrt{\tau_x/\tau_y}, y\sqrt{\tau_y/\tau_x})$, then the linear part H_{lin} of $\mathcal{E}_\varepsilon$ is formally replaced by the scalar $\varepsilon^{2+2\delta}\tau_x\hat{H}_-$ where

$$\hat{H}_- = -\partial_x^2 - \left(\partial_y - i\frac{x}{2\sqrt{1+\tau_x x^2}}\right)^2 + \frac{1}{\ell_V^2\tau_x}v(\sqrt{\tau_x}x, \sqrt{\tau_x}y)\,. \tag{1.15}$$

In the limit $\tau_x \to 0$, a natural (ε, τ)-independent scalar reduced model emerges:

$$\mathcal{E}_H(u) = \langle u\,, \left[-\partial_x^2 - (\partial_y - \frac{ix}{2})^2 + \frac{x^2+y^2}{\ell_V^2}\right] u\rangle + \frac{G}{2}\int_{\mathbb{R}^2} |u|^4\,, \quad u(x,y) \in \mathbb{C}\,.$$

Because κ is large, or ε is small, it is natural to expect that the ground state of $\mathcal{E}_\varepsilon$ is close, up to a unitary transform, to a vector $u(x,y)\psi_-$. This is the aim of the adiabatic theory and leads to a scalar problem. In order to get good bounds on the energy, we need to study the limit τ_x small at the same time.

In all our work $\ell_V > 0$ and $G > 0$ are assumed to be fixed, while the asymptotic behaviour is studied as $\varepsilon \to 0$ and $\tau_x \to 0$.
The quadratic part of the above energy is associated with the Hamiltonian

$$H_{\ell_V} = -\partial_x^2 - (\partial_y - \frac{ix}{2})^2 + \frac{x^2+y^2}{\ell_V^2}\,, \quad (0 < \ell_V < +\infty)\,, \tag{1.16}$$

with the domain

$$\mathcal{H}_2 = \left\{ u \in L^2(\mathbb{R}^2)\,, \sum_{|\alpha|+|\beta|\leq 2} \|q^\alpha D_q^\beta u\|_{L^2} < +\infty \right\}\,, \quad q = (x,y) \in \mathbb{R}^2\,, \tag{1.17}$$

endowed with the norm $\|f\|_{\mathcal{H}_2}^2 = \sum_{|\alpha|+|\beta|\leq 2} \|q^\alpha D_q^\beta u\|_{L^2}^2$ and the corresponding distance $d_{\mathcal{H}_2}$. It is not difficult (see Section 4.2) to check that the minimization of $\mathcal{E}_H(\varphi)$ under the constraint $\|\varphi\|_{L^2} = 1$, admits solutions and that the set, Argmin $\mathcal{E}_H$, of ground states for $\mathcal{E}_H$ is a bounded set of $\mathcal{H}_2$.

Definition 1.1. *For a functional $\mathcal{E}$ defined on a Hilbert space $\mathcal{H}$ (with $+\infty$ as a possible value), we set*

$$\mathcal{E}_{min} = \inf_{\|u\|=1} \mathcal{E}(u) \quad \textit{and} \quad \text{Argmin } \mathcal{E} = \{u \in \mathcal{H},\ \mathcal{E}(u) = E_{min} \textit{ and } \|u\| = 1\}\,.$$

Theorem 1.2. *Fix the constant ℓ_V, G and δ and assume, for some $C_\delta = C(\ell_V, G, \delta) > 0$,*

$$\varepsilon^{2\delta} \leq \frac{\tau_x^{\frac{5}{3}}}{C_\delta}\,. \tag{1.18}$$

Let $\chi = (\chi_1, \chi_2)$ be a pair of cut-off functions such that $\chi_1^2 + \chi_2^2 \equiv 1$ on $\mathbb{R}^2$, $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $\chi_1 = 1$ in a neighborhood of 0.
There exist constants, $\nu_0 = \nu_{\ell_V,G} \in (0, \frac{1}{2}]$ and for any given $\delta > 0$, $\tau_\delta = \tau(\ell_V, G, \delta) > 0$, $C_{\chi,\delta} = C(\ell_V, G, \delta, \chi) > 0$, and a unitary operator $\hat{U} = \hat{U}(\varepsilon, \tau, \ell_V, G, \delta)$ which guarantee the following properties

- *The energy $\mathcal{E}_\varepsilon$ introduced in* (1.10) *admits ground states as soon as $\tau_x \leq \tau_\delta$, with*

$$|\mathcal{E}_{\varepsilon,min} - \varepsilon^{2+2\delta}\tau_x \mathcal{E}_{H,min}| \leq C_\delta \varepsilon^{2+2\delta}\tau_x^{\frac{5}{3}} .$$

- *Any $\psi \in$ Argmin $\mathcal{E}_\varepsilon$ written in the form $\psi = \hat{U}\begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$ satisfies*

$$\|a_+\|_{L^2} \leq C_\delta \varepsilon^{2+2\delta}\tau_x \quad , \quad \|a\|_{H^2} \leq \frac{C_\delta}{\tau_x} \quad , \quad \|a\|_{L^4} + \|a\|_{L^6} \leq C_\delta ,$$
$$\|\chi_2(\tau_x^{\frac{1}{9}}.)a_-\|_{L^2} \leq C_{\chi,\delta}\tau_x^{\frac{1}{3}} ,$$
$$d_{\mathcal{H}_2}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H) \leq C_{\chi,\delta}(\tau_x^{\frac{2\nu_0}{3}} + \varepsilon) ,$$

$$\|a_+\|_{L^\infty} \leq C_\delta \varepsilon^{1+\delta} \quad , \quad d_{L^\infty}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H) \leq C_{\chi,\delta}(\tau_x^{\frac{2\nu_0}{3}} + \varepsilon)^{\frac{1}{2}} .$$

All the constants can be chosen uniformly with respect to $\delta \in (0, \delta_0]$ for any fixed $\delta_0 > 0$.

Remark 1.3. *The proof is made in two steps: 1) the limit $\varepsilon \to 0$ corresponds to the adiabatic limit for the linear problem and allows to replace the linear part H_{lin}, of $\mathcal{E}_\varepsilon$, by the scalar $\varepsilon^{2+2\delta}\tau_x \hat{H}_-$ given by* (1.15)*. 2) the limit $\tau_x \to 0$ allows to reduce the asymptotic minimization problem to a simpler one where the linear Hamiltonian is exactly H_{ℓ_V} given by* (1.16)*.*
One main point in the proof is to have precise energy estimates for the limiting problem. In our case, we obtain them in the limit $\tau_x \to 0$, because explicit calculations are more easily accessible when the magnetic field is constant, that is in the case of $\mathcal{E}_{H,min}$. In theory, if one had precise energy estimates for a general τ_x for the intermediate adiabatic model with the linear part $\hat{H}_-$, complete results could be performed for general values of τ_x .

Remark 1.4. *The exponent $\nu_0 = \nu_{\ell_V,G}$ is a Lojasiewicz-Simon exponent. It is $\frac{1}{2}$ when the Lagrange multiplier associated with $u \in$ Argmin $\mathcal{E}_H$ is a simple eigenvalue of $\hat{H}_{\ell_V} + G|\psi|^2$. There are reasons to think that it is the case for generic values $(\ell_V, G) \in (0, +\infty) \times [0, +\infty)$, namely outside a subanalytic subset of dimension smaller than* 1 *(and possibly* 0*). Nevertheless for $G = 0$, there is a discrete set of values of ℓ_V, where $\nu_{\ell_V,0}$ can be made arbitrarily small.*

Consequences and applications: One issue is the possibility to observe vortices (zeroes of the wave function with circulation around them) in a system with artificial gauge as presented in [DGJO] and modelled with the energy $\mathcal{E}_\varepsilon$. One may process in the following way: once G is fixed, choose ℓ_V such that the minimizers of $\mathcal{E}_H$ have vortices (detailed conditions will be given in section 4.2). Then take $\tau_x > 0$ and $\varepsilon > 0$ small enough so that $d_{\mathcal{H}_2}(\chi(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H)$ and therefore the L^∞ distance from $\chi(\tau_x^{\frac{1}{9}}.)a_-$ to a ground state of $\mathcal{E}_H$ is small. In the above result, the constants are not explicitely controlled and

this control is worse and worse when δ increases. So it is not explicit for given numerical values of the parameters $\ell_V, G, \tau_x, \varepsilon$ or equivalently $\ell_V, G, \kappa, k, \ell_\kappa$. Nevertheless it provides a framework for numerical simulations, where the observation of vortices can be confirmed by decreasing ε and τ_x.

The definitions (1.3) of ε and (1.9) of τ_x, transform the condition (1.18) into

$$\left(\frac{k^2}{\kappa}\right)^{\frac{2\delta}{1+2\delta}} \ll \left(\frac{1}{k\ell_\kappa}\right)^{\frac{5}{3}} .$$

The larger δ, the better, but δ must not be to large because of the value of the constants C_δ, $C_{\delta,\chi}$. A value of a few units for δ does not affect them to much. Another reason for keeping δ small is that the initial small parameter is $\varepsilon^{2+2\delta}$, while the final error estimate of $d_{\mathcal{H}_2}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H)$ (or $d_{L^\infty}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H)$) contains also an $\mathcal{O}(\varepsilon)$ term. As an example for $\delta = \frac{5}{2}$, the above relation becomes

$$\frac{1}{(k\ell_\kappa)^{7/3}} \gg \frac{k^2}{\kappa} \quad , \quad \tau_x = \frac{1}{k\ell_\kappa} \quad , \quad \varepsilon = \left(\frac{k^2}{\kappa}\right)^{\frac{1}{7}} .$$

A given precision of order $\varepsilon + \tau_x^{\frac{1}{3}}$, is more easily achieved by taking k small and $\ell_\kappa = \frac{1}{\tau_x k}$ large (for example $k = 0.1$, $\ell_\kappa = 500 \times 25$ is better than the values $k = 50$, $\ell_\kappa = 25$ given in the introduction). Note also that the external potential $V_{\varepsilon,\tau}$, defined in (1.13) must be adjusted up to the order $\varepsilon^{2+2\delta} = \frac{k^2}{\kappa}$.

1.2 Gist of the analysis

Following the general idea of the founding articles [BoFo, BoOp] of Born, Fock and Oppenheimer, it is well known in the physics literature that a Hamiltonian system

$$(\varepsilon D_q)^2 + u_0(q) \begin{pmatrix} E_+(q) & 0 \\ 0 & E_-(q) \end{pmatrix} u_0(q)^*$$

is unitarily equivalent to a diagonal Hamiltonian

$$\varepsilon^2 \left[\begin{pmatrix} (D_q - A(q))^2 & 0 \\ 0 & (D_q + A(q))^2 \end{pmatrix} + |X(q)|^2 \right] + R(\varepsilon)$$

where $(D_q \mp A(q))$ are the covariant derivatives ($\mp A(q)$ is the adiabatic connection) associated with the fiber bundles $u_0(q) \begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix}$ and $u_0(q) \begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$, and $|X(q)|^2$ is the Born-Huang potential.
Since [Kat] and until recently ([NeSo, MaSo, PST, PST2]), this problem has been widely studied by mathematicians, and the remainder term is formally:

$$R(\varepsilon) = \sum_{i,j} \varepsilon^2 C_{ij}(q)(\varepsilon D_{q_i})(\varepsilon D_{q_j}) + \mathcal{O}(\varepsilon^3) ,$$

which for a typical frequency of order $\frac{1}{\varepsilon}$ has the same size as the main term. Introducing $\delta > 0$ allows to obtain at the formal level

$$\varepsilon^{2+2\delta}\left[\begin{pmatrix}(D_q - A(q))^2 & 0\\ 0 & (D_q + A(q))^2\end{pmatrix} + |X(q)|^2\right] + \sum_{i,j} \varepsilon^{2+4\delta} C_{ij}(q)(\varepsilon D_{q_i})(\varepsilon D_{q_j}) + \mathcal{O}(\varepsilon^{3+4\delta})\,,$$

where the remainder term is $\mathcal{O}(\varepsilon^{2+4\delta})$ at the typical frequency $\frac{1}{\varepsilon}$ and thus $\mathcal{O}(\varepsilon^{2\delta})$ times the size of the main term. Such an error estimate can be made in the L^2-sense when applied to some wave function ψ lying in $\{|p| \leq r\}$, with p quantized into εD_q, or more precisely fulfilling $\psi = \chi(\varepsilon D_q)\psi$ for some $\chi \in \mathcal{C}_0^\infty(\{|p| \leq 2r\})$ with

$$\|R(\varepsilon)\psi\| \leq C_\chi(\varepsilon^{2+4\delta} + C_N\varepsilon^N)\|\psi\| \leq (C'_{\chi,\delta}\varepsilon^{2\delta})\varepsilon^{2+2\delta}\|\psi\|\,.$$

where C_N essentially depends on the estimates of N-derivatives of $u_0(q)$, $E_\pm(q)$. For a fixed δ, we choose $N \geq 2 + 4\delta$.

Remark 1.5. *(on the choice of δ): another strategy could be considered in order to optimize the exponent δ, w.r.t ε: under analyticity assumptions or more generally assumptions which lead to an explicit control of C_N in terms of N, one could think of optimizing first $C_N\varepsilon^N$ w.r.t to N according to the methods of [NeSo, Sor, MaSo]. As an example with $C_N \leq N!$, this would lead to $C_N\varepsilon^N \leq Ce^{-\frac{1}{\varepsilon}}$ after choosing $N = N(\varepsilon) = \left[\frac{1}{\varepsilon}\right]$, with some $C \leq 1$. Then taking $\delta = \delta(\varepsilon) = -\frac{1}{4\varepsilon\log(\varepsilon)}$ would lead to*

$$CC_\chi(\varepsilon^{2+4\delta(\varepsilon)} + e^{-\frac{1}{\varepsilon}}) \leq 2CC_\chi e^{-\frac{1}{\varepsilon}}\|\psi\|\,.$$

When $\delta(\varepsilon) \to \infty$ and as compared with $\varepsilon^{2+2\delta(\varepsilon)}$ considered as the order 1 term, an $\mathcal{O}(\varepsilon^{2+4\delta(\varepsilon)})$ remainder term is almost of order 2.
We do not consider this optimization of δ w.r.t ε, with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = +\infty$, because our initial small parameter is $\varepsilon^{2+2\delta}$ and the final result of Theorem 1.2, (the estimate about $d_{\mathcal{H}_2}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, Argmin\ \mathcal{E}_H)$ in the nonlinear problem), contains an $\mathcal{O}(\varepsilon)$. Hence we keep $\delta > 0$ independent of ε. This is why δ is still present in the constants of Theorem 1.2.

We need to perform a frequency (or momentum) truncation. We decompose a general ψ into $\chi(\varepsilon D_q)\psi + (1-\chi(\varepsilon D_q))\psi$ and use rough estimates for the part $(1-\chi(\varepsilon D_q))\psi$ which will be compensated in the minimization problem for $\mathcal{E}_\varepsilon$ by good a priori estimates for the norm $\|(1-\chi(\varepsilon D_q))\psi\|$ of the high-frequency part. We also have to check that the unitary transform $\hat{U}$, implementing the adiabatic approximation, does not perturb much the nonlinear part of $\mathcal{E}_\varepsilon(\psi)$.

In our case, the limit $\varepsilon \to 0$ leads to the Born-Oppenheimer Hamiltonians

$$-\partial_x^2 - (\partial_y \mp \frac{ix}{2\sqrt{1+\tau_x x^2}})^2 + \frac{v(\sqrt{\tau_x}.)}{\ell_V^2\tau_x}\,,$$

which, in a second step, in the limit $\tau_x \to 0$, leads to H_{ℓ_V} (with the sign $-ix$ for the lower energy band). This means that the convergence to the Born-Oppenheimer Hamiltonian as

$\varepsilon \to 0$ has to be uniform w.r.t $\tau_x \in (0,1]$. This last point requires to reconsider carefully the work of [PST] by following the uniformity w.r.t τ of the estimates given by Weyl-Hörmander calculus ([Hor, BoLe]) for τ-dependent metric which have uniform structural constants.
This is done for the low frequency part in Section 2 while the basic tools of semiclassical calculus are reviewed and adapted in the Appendix A.
In Section 3 the error associated in the high-frequency part is considered, as well as the effect of the unitary adiabatic transformation on the nonlinear term.
Once the adiabatic approximation is well justified in this rather involved framework, the accurate analysis, as well as the comparison when τ_x is small, of the two reduced models (the one with H_{ℓ_V} and the one with $\hat{H}_-$) is carried out in Section 4. This follows the general scheme of comparison of minimization problems: 1) Write energy estimates; 2) Use bootstrap arguments and possibly Lojasiewicz-Simon inequalities in order to get comparison of minimizers in the energy space; 3) Use the Euler-Lagrange equations in order to get a better comparison in higher regularity spaces.
In Section 5, all the information of the previous sections is gathered in order to prove Theorem 1.2 : existence of a minimizer for $\mathcal{E}_\varepsilon$ in Proposition 5.3, key energy estimates in Proposition 5.4 and bounds for minimizers in Proposition 5.6.

Some comments and additional results are pointed out in Section 6, namely: 1) the question of the smallness condition of ε w.r.t τ_x appearing in Theorem 1.2; 2) the possible extension to anisotropic nonlinearities (one would have to check that the unitary transform implementing the adiabatic approximation does not perturb the nonlinear part); 3) the minimization problem for excited states, i.e. locally and approximately carried by ψ_+ instead of ψ_-; 4) the extension to the problem of the time nonlinear dynamics of adiabatically prepared states.

2 Adiabatic approximation for the linear problem

In [PST], the adiabatic approximation is completely justified for bounded symbols or when global elliptic properties of the complete matricial symbol allow to reduce to this case after spectral truncation. Unfortunately it is not the case here because the eigenvalues of the symbol of the linear part H_{Lin} are $\varepsilon^{2\delta}\tau_x\tau_y|p|_\tau^2 + V_{\varepsilon,\tau}(x,y) \pm \Omega(\sqrt{\frac{\tau_x}{\tau_y}}x)$. In [Sor], the adiabatic theory for unbounded symbols is developed after stopping the complete asymptotic expansion in an optimal way, under some analyticity assumptions, but this would be particularly tricky here with divergences occuring both in the momentum and position directions. We shall see that the sublinear divergence in position makes no difficulty after using the right Weyl-Hörmander class. The quadratic divergence in momentum, with the kinetic energy $\tau_x\tau_y|p|^2$ is solved by first considering truncated kinetic energies and using the additional scaling factor $\varepsilon^{2\delta}$, $\delta > 0$, in front of the kinetic energy term.
Our problem shows an anisotropy in the position variables (x,y). The analysis of the linear problem can be treated in $\mathbb{R}^d$. Then we split the position and momentum variables,

$q \in \mathbb{R}^d$ and $p \in \mathbb{R}^d$, into:

$$q = (q', q'') \quad , \quad p = (p', p'') \quad q', p' \in \mathbb{R}^{d'} \, , \, q'', p'' \in \mathbb{R}^{d''} \quad , \quad d' + d'' = d \, ,$$

and the pair (τ_x, τ_y) is accordingly denoted by $(\tau', \tau'') \in (0,1)^2$.

2.1 Born-Oppenheimer Hamiltonian

Consider the Hamiltonian in $\hat{H}_\varepsilon = H(q, \varepsilon D_q, \varepsilon)$ with the symbol on $\mathbb{R}^{2d}_{q,p}$

$$\begin{aligned} H(q,p,\varepsilon) &= \varepsilon^{2\delta} \tau' \tau'' |p|^2 \gamma(\tau' \tau'' |p|^2_\tau) + \mathcal{V}(q, \tau, \varepsilon) \\ &= \varepsilon^{2\delta} \tau' \tau'' |p|^2 \gamma(\tau' \tau'' |p|^2_\tau) + u_0(q,\tau,\varepsilon) \begin{pmatrix} E_+(q,\tau,\varepsilon) & 0 \\ 0 & E_-(q,\tau,\varepsilon) \end{pmatrix} u_0^*(q,\tau,\varepsilon) \end{aligned}$$

with $\delta > 0$ and $(\tau', \tau'') \in (0,1]^2$. The following properties, with the splitting of variables $q = (q', q'') \in \mathbb{R}^d$, are assumed:

$$\begin{aligned} & E_\pm \in \mathcal{S}_u \left(\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle, \frac{\frac{\tau'}{\tau''} dq'^2}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle^2} + \frac{\tau''}{\tau'} dq''^2 \right) \, , \\ & E_+(q,\tau,\varepsilon) - E_-(q, \tau\varepsilon) \geq C^{-1} \langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle \, , \\ & u_0 = (u_0^*)^{-1} \in S_u(1, \frac{\frac{\tau'}{\tau''} dq'^2}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle^2} + \frac{\tau''}{\tau'} dq''^2 ; \mathcal{M}_2(\mathbb{C})) \, , \\ & \gamma \in \mathcal{C}^\infty_0(\mathbb{R}; \mathbb{R}_+) \quad \text{with} \quad \gamma \equiv 1 \in [0, 2r_\gamma^2] \, . \end{aligned} \tag{2.1}$$

With these assumptions we are able to justify the Born-Oppenheimer adiabatic approximation for $\delta > 0$, $\tau', \tau'' \in (0,1]$. We shall work with the τ-dependent metric

$$g_\tau = \frac{\frac{\tau'}{\tau''} dq'^2}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle^2} + \frac{\tau''}{\tau'} dq''^2 + \frac{\tau' \tau'' dp^2}{\langle \sqrt{\tau' \tau''} p \rangle^2} \quad ,$$

on the phase-space $\mathbb{R}^{2d}_{q,p}$, which is checked to have uniform properties w.r.t $\tau \in (0,1]^2$ in Proposition A.8 . The exact definition of the parameter dependent Hörmander symbol classes, $\mathcal{S}_u(m, g_\tau; \mathcal{M}_2(\mathbb{C}))$, is given in Appendix A (see in particular its meaning for τ-dependent metrics in Appendix A.4).

Theorem 2.1. *There exists a unitary operator $\hat{U} = U(q, \varepsilon D_q, \tau, \varepsilon)$ with symbol*

$$\begin{aligned} & U(q,p,\tau,\varepsilon) = u_0(q,\tau,\varepsilon) + \varepsilon u_1(q,p,\tau,\varepsilon) + \varepsilon^2 u_2(q,p,\tau,\varepsilon) \, , \\ & \varepsilon^{-2\delta} u_{1,2} \in S_u(\frac{1}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle \langle \sqrt{\tau' \tau''} p \rangle^\infty_\tau}, g_\tau; \mathcal{M}_2(\mathbb{C})) \, , \end{aligned}$$

such that

$$\hat{U}^*\hat{H}\hat{U} = \begin{pmatrix} h_{BO,+}(q,\varepsilon D_q,\varepsilon) & 0 \\ 0 & h_{BO,-}(q,\varepsilon D_q,\varepsilon) \end{pmatrix} + \varepsilon^{2+4\delta}R_1(q,\varepsilon D_q,\tau,\varepsilon) + \varepsilon^{3+2\delta}R_2(q,\varepsilon D_q,\tau,\varepsilon)\,, \quad (2.2)$$

with $h_{BO\pm}(q,p,\tau,\varepsilon)$ *equal to*

$$\begin{aligned} &= \varepsilon^{2\delta}\tau'\tau''(|p\mp\varepsilon A|^2+|\varepsilon X|^2)+E_\pm \\ &= \varepsilon^{2\delta}\tau'\tau''\left[\sum_{k=1}^{d}(p_k\mp\varepsilon A_k)^2+|\varepsilon X_k|^2\right]+E_\pm\,, \end{aligned} \quad (2.3)$$

when $\sqrt{\tau'\tau''}|p| \le r_\gamma$, *and*

$$\begin{pmatrix} +A_k & X_k \\ \bar{X}_k & -A_k \end{pmatrix} = iu_0^*(\partial_{q^k}u_0)\,.$$

The remainder terms satisfy $R_1, R_2 \in S_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))$ *and* R_2 *vanishes in* $\left\{\sqrt{\tau'\tau''}|p| \le r_\gamma\right\}$ *and those estimates are uniform w.r.t* $\delta \in (0,\delta_0]$ *(and* $\tau \in (0,1]^2$, $\varepsilon \in (0,\varepsilon_0]$*).*

Remark 2.2.

- *The remainder term is really negligible only for* $\delta > 0$. *This is explained in Subsection 2.4.*
- *The quantization of* $\varepsilon^{2\delta}\tau'\tau''(|p\mp\varepsilon A|^2+|\varepsilon X|^2)+E_\pm$ *is nothing but*

$$\varepsilon^{2+2\delta}\tau'\tau''\left[\sum_{k=1}^{d}-(\partial_{q^k}\mp iA_k)^2+|X_k|^2\right]+E_\pm\,.$$

2.2 Second order computations for space adiabatic approximate projections of the reduced Hamiltonian

We shall consider the matricial symbol, on $\mathbb{R}^{2d}_{q,p}$,

$$H(q,p,\tau,\varepsilon) = f_\varepsilon(p,\tau) + E_+(q,\tau,\varepsilon)\Pi_0(q,\tau,\varepsilon) + E_-(q,\tau,\varepsilon)(1-\Pi_0(q,\tau,\varepsilon))$$

where $\Pi_0(q,\tau,\varepsilon) = \Pi_0(q,\tau,\varepsilon)^* = \Pi_0(q,\tau,\varepsilon)^2 \in \mathcal{M}_2(\mathbb{C})$, $\gamma, E_\pm$ real-valued, with $\delta \ge 0$ and the following properties:

$$E_\pm \in \mathcal{S}_u(\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle, g_{q\tau}) \quad , \quad \Pi_0 \in S_u(1,g_{q,\tau};\mathcal{M}_2(\mathbb{C}))\,, \quad (2.4)$$

$$\text{with} \qquad g_{q,\tau} = \frac{\frac{\tau'}{\tau''}dq'^2}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle^2} + \frac{\tau''}{\tau'}dq''^2,$$

$$E_+(q,\tau,\varepsilon) - E_-(q,\tau,\varepsilon) \ge C^{-1}\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\,, \quad (2.5)$$

$$f_\varepsilon(p,\tau) = \varepsilon^{2\delta}f_1(\sqrt{\tau'\tau''}p) \quad , \quad f_1 \in \mathcal{C}_0^\infty(\mathbb{R}^d;\mathbb{R}) \quad , \quad \delta \ge 0\,.$$

For conciseness, the arguments p, q and the parameters τ, ε, will often be omitted in $\partial_p^\alpha f_\varepsilon(p)$ and $\partial_q^\alpha E_\pm(q)$ or $\partial_q^\alpha \Pi_0$.
Remember that the metric

$$g_\tau = g_{q,\tau} + \frac{\tau'\tau'' dp^2}{\langle\sqrt{\tau'\tau''}p\rangle^2} = \frac{\frac{\tau'}{\tau''}dq'^2}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle^2} + \frac{\tau''}{\tau'}dq''^2 + \frac{\tau'\tau''dp^2}{\langle\sqrt{\tau'\tau''}p\rangle^2}$$

has the gain function

$$\lambda(q,p) = \min\left(\frac{\langle\sqrt{\tau'\tau''}p\rangle\langle\sqrt{\frac{\tau'}{\tau''}}q\rangle}{\tau'}, \frac{\langle\sqrt{\tau'\tau''}p\rangle}{\tau''}\right) \geq \langle\sqrt{\tau'\tau''}p\rangle\,.$$

The ε-quantized version of the symbol $A_\varepsilon(q,p)$ will be denoted

$$\hat{A} = A(q, \varepsilon D_q)\,.$$

Note that the symbols $E_+ - E_-$ is elliptic in its class $S_u(\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle, g_\tau)$, and

$$\varepsilon^{-2\delta} f_\varepsilon \in S_u(\frac{1}{\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau)\,.$$

Our aim is to compute accurately the adiabatic projection $\Pi^{(n)}(q,p) = \Pi_0(q,\varepsilon) + \varepsilon\Pi_1(q,p,\varepsilon) + \cdots + \varepsilon^n\Pi_n(q,p,\varepsilon)$ such that

$$\hat{\Pi}^{(n)} \circ \hat{\Pi}^{(n)} = \hat{\Pi}^{(n)} + \mathcal{O}(\varepsilon^{n+1}) \quad , \quad \left[\hat{H}, \hat{\Pi}^{(n)}\right] = \mathcal{O}(\varepsilon^{n+1})\,.$$

The general theory presented in [PST], tells us that the asymptotic expansion can be pushed up to $n = \infty$, but we will do here accurate calculations up to $n = 2$ (with additional information for $n = 3$) and then discuss the influence of the factor $\varepsilon^{2\delta}$. Those are feasible and rather easy because the kinetic energy term and the two-level potential are simple. This allows to reconsider accurately the arguments sketched in [PST] for the Born-Oppenheimer case with all the technical new specificities of our example.
Like in [PST], [Sor], [MaSo], the calculations are first done at the symbolic level and we write

$$(A(\varepsilon) = \mathcal{O}_S(\varepsilon^\nu)) \Leftrightarrow \left(\varepsilon^{-\nu}A \in S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))\right)\,.$$

For a matricial symbol $A(q,p)$ (possibly depending on (ε,τ)), it is convenient to introduce the diagonal and off-diagonal parts

$$A^D(q,p) := \Pi_0(q)A(q,p)\Pi_0(q) + (1-\Pi_0(q))A(q,p)(1-\Pi_0(q)) \tag{2.6}$$
$$A^{OD}(q,p) := \Pi_0(q)A(q,p)(1-\Pi_0(q)) + (1-\Pi_0(q))A(q,p)\Pi_0(q)\,, \tag{2.7}$$

where we recall $\Pi_0(q) = \Pi_0(q,\tau,\varepsilon)$. Note the equalities

$$(AB)^D = A^D B^D + A^{OD}B^{OD} \quad , \quad (AB)^{OD} = A^D B^{OD} + A^{OD}B^D\,. \tag{2.8}$$

The "Pauli matrix"

$$\sigma_3(q,\tau,\varepsilon) = 2\Pi_0(q,\tau,\varepsilon) - \mathrm{Id}_{\mathbb{C}^2}\,, \tag{2.9}$$

will also be used, with the relations

$$\sigma_3^2(q) = \mathrm{Id}_{\mathbb{C}^2}\,, \quad \sigma_3(q)A^D(q,p)\sigma_3(q) = A^D(q,p)\,, \quad \sigma_3(q)A^{OD}(q,p)\sigma_3(q) = -A^{OD}(q,p)\,,$$
$$\sigma_3(q)A^{OD}(q,p) = \Pi_0(q)A(q,p)(1-\Pi_0(q)) - (1-\Pi_0(q))A(q,p)\Pi_0(q)\,.$$

We are looking for

$$\Pi^{(n)}(q,p,\tau,\varepsilon) = \sum_{j=0}^{n}\varepsilon^j\Pi_j(q,p,\tau,\varepsilon) \quad \in S_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))\,,$$
$$\text{with} \quad \Pi_j \in S_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))\,,$$

and such that

$$\Pi^{(n)}\sharp^\varepsilon\Pi^{(n)} - \Pi^{(n)} = \mathcal{O}_S(\varepsilon^{n+1})\,, \tag{2.10}$$
$$\Pi^{(n)*} = \Pi^{(n)}\,, \tag{2.11}$$
$$H\sharp^\varepsilon\Pi^{(n)} - \Pi^{(n)}\sharp H = \mathcal{O}_S(\varepsilon^{n+1})\,. \tag{2.12}$$

Like in [MaSo], [PST], this system is solved by induction starting from $\Pi^{(0)}(q,p,\tau,\varepsilon) = \Pi_0(q,\tau,\varepsilon)$ with

$$G_{n+1} := \varepsilon^{-(n+1)}\left[\Pi^{(n)}\sharp^\varepsilon\Pi^{(n)} - \Pi^{(n)}\right] \mod \mathcal{O}_S(\varepsilon)\,, \tag{2.13}$$
$$\Pi^D_{n+1} := -\sigma_3 G^D_{n+1}\,, \tag{2.14}$$
$$F_{n+1} := \varepsilon^{-(n+1)}\left[H\sharp^\varepsilon(\Pi^{(n)} + \varepsilon^{n+1}\Pi^D_{n+1}) - (\Pi^{(n)} + \varepsilon^{n+1}\Pi^D_{n+1})\sharp^\varepsilon H\right] \mod \mathcal{O}_S(\varepsilon)$$
$$= \varepsilon^{-(n+1)}\left[H\sharp^\varepsilon\Pi^{(n)} - \Pi^{(n)}\sharp^\varepsilon H\right] \mod \mathcal{O}_S(\varepsilon)\,, \tag{2.15}$$
$$\Pi^{OD}_{n+1} := -\frac{1}{E_+(q)-E_-(q)}\sigma_3 F^{OD}_{n+1} = \frac{1}{E_+(q)-E_-(q)}F^{OD}_{n+1}\sigma_3\,. \tag{2.16}$$

Remember that the general theory says that the principal symbol of F_{n+1} is off-diagonal, $F_{n+1} = F^{OD}_{n+1} \mod \mathcal{O}_S(\varepsilon)$, and F_{n+1} can be chosen so that

$$F_{n+1} = F^{OD}_{n+1}\,. \tag{2.17}$$

Below are the computations up to $n = 2$ in our specific case. In these computations, we shall use Einstein's summation rule $s_k t^k = \sum_k s_k t^k$ with the coordinates (p_k, q^k) or (p^k, q^k) with $p^k = p^\ell\delta_{\ell,k} = p_k$ like in the examples

$$|p|^2 = p^k p_k = p_k p_\ell \delta^{\ell,k} = p^k p^\ell \delta_{k,\ell} \quad , \quad (\partial_p f_\varepsilon).\partial_q = (\partial_{p^k} f_\varepsilon)\delta^{k,\ell}\frac{\partial}{\partial_{q^\ell}} = (\partial_{p_k} f_\varepsilon)\partial_{q^k}\,.$$

$\mathbf{n = 0}$: Start with $\Pi^{(0)} = \Pi_0(q,\tau,\varepsilon)$ and notice $\partial_p\Pi_0 \equiv 0$ and $\partial_q\Pi_0 \equiv (\partial_q\Pi_0)^{OD}$.
$\mathbf{n = 1}$: Take

$$G_1 = \Pi_0\circ\Pi_0 - \Pi_0 = 0\,,$$
$$\text{and} \quad \Pi^D_1 = 0\,.$$

Next compute

$$\begin{aligned}\varepsilon^{-1}\left[H\sharp^{\varepsilon}\Pi_0(q,\varepsilon)-\Pi_0(q,\varepsilon)\sharp^{\varepsilon}H\right] &= \varepsilon^{-1}\left[f_{\varepsilon}\sharp^{\varepsilon}\Pi_0(q,\varepsilon)-\Pi_0(q,\varepsilon)\sharp^{\varepsilon}f_{\varepsilon}\right]\\ &= -i\partial_{p_k}f_{\varepsilon}\partial_{q^k}\Pi_0 \mod \mathcal{O}_S(\varepsilon)\,,\end{aligned}$$

and take

$$F_1=-i\partial_{p_k}f_{\varepsilon}\partial_{q^k}\Pi_0=-i\partial_{p_k}f_{\varepsilon}(\partial_{q^k}\Pi_0)^{OD}\,,$$

and $$\Pi^{(1)}=\Pi_0+\frac{\varepsilon i\partial_{p_k}f_{\varepsilon}}{E_+-E_-}\sigma_3(\partial_{q^k}\Pi_0)^{OD}\,.$$

with $$\varepsilon^{-2\delta}\Pi_1=\frac{i\partial_{p_k}f_1}{E_+-E_-}\sigma_3(\partial_{q^k}\Pi_0)^{OD}\quad\in S_u\left(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^{\infty}},g_{\tau};\mathcal{M}_2(\mathbb{C})\right)\,.$$

n = 2: Consider now

$$\begin{aligned}\varepsilon^2G_2 &= \Pi^{(1)}\sharp^{\varepsilon}\Pi^{(1)}-\Pi^{(1)} \mod \mathcal{O}_S(\varepsilon^3)\\ &= \Pi_0\sharp^{\varepsilon}\Pi_0-\Pi_0+\varepsilon(\Pi_0\sharp^{\varepsilon}\Pi_1+\Pi_1\sharp^{\varepsilon}\Pi_0)-\Pi_1+\varepsilon^2\Pi_1\sharp^{\varepsilon}\Pi_1 \mod \mathcal{O}_S(\varepsilon^3)\,.\end{aligned}$$

According to (A.3), with $\Pi_0\sharp\Pi_0=\Pi_0$ and with

$$\Pi_0\Pi_1+\Pi_1\Pi_0=\Pi_0^D\Pi_1^{OD}+\Pi_1^{OD}\Pi_0^D=\Pi_1\,,$$

we can take

$$G_2=\frac{1}{2i}\left[-\partial_{q^k}\Pi_0\partial_{p_k}\Pi_1+\partial_{p_k}\Pi_1\partial_{q^k}\Pi_0\right]+\Pi_1^2\,.$$

The first term $-\partial_{q^k}\Pi_0\partial_{p_k}\Pi_1$, with Einstein's summation rule, equals

$$\begin{aligned}-\partial_{q^k}\Pi_0\partial_{p_k}\Pi_1 &= -\frac{i\partial^2_{p_kp_\ell}f_{\varepsilon}}{(E_+-E_-)}(\partial_{q^k}\Pi_0)^{OD}\sigma_3(\partial_{q^\ell}\Pi_0)^{OD}\\ &= \frac{i\partial^2_{p_kp_\ell}f_{\varepsilon}}{(E_+-E_-)}\sigma_3(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,,\end{aligned}$$

while the second term $+(\partial_{p_k}\Pi_1)(\partial_{q^k}\Pi_0)$ gives the same result.
The third term Π_1^2 is given by

$$\Pi_1^2=-\frac{(\partial_{p_k}f_{\varepsilon})(\partial_{p_\ell}f_{\varepsilon})}{(E_+-E_-)^2}\sigma_3(\partial_{q^k}\Pi_0)^{OD}\sigma_3(\partial_{q^\ell}\Pi_0)^{OD}=\frac{(\partial_{p_k}f_{\varepsilon})(\partial_{p_\ell}f_{\varepsilon})}{(E_+-E_-)^2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,.$$

Hence the diagonal second order correction is given by

$$\begin{aligned}&(E_+-E_-)^2\Pi_2^D=-(E_+-E_-)^2\sigma_3G_2\\ &=-\left[(\partial_{p_k}f_{\varepsilon})(\partial_{p_\ell}f_{\varepsilon})+(E_+-E_-)(\partial^2_{p_kp_\ell}f_{\varepsilon})\sigma_3\right](\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\end{aligned}$$

and satisfies

$$\varepsilon^{-2\delta}\Pi_2^D\in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^{\infty}},g_{\tau};\mathcal{M}_2(\mathbb{C}))\,.$$

Consider now Π_2^{OD}: By referring to (2.17), F_2 can be chosen as the off-diagonal part of $\varepsilon^{-2}\left[H\sharp^\varepsilon\Pi^{(1)}-\Pi^{(1)}\sharp^\varepsilon H\right]$ which, according to the previous steps and (A.3), equals

$$-\frac{1}{8}\left[\partial^2_{p_k,p_\ell}H\partial^2_{q^k,q^\ell}\Pi_0-\partial^2_{q^k,q^\ell}\Pi_0\partial^2_{p_k,p_\ell}H\right]$$
$$+\frac{1}{2i}\left[\partial_{p_k}H\partial_{q^k}\Pi_1-\partial_{q^k}H\partial_{p_k}\Pi_1-\partial_{p_k}\Pi_1\partial_{q^k}H+\partial_{q^k}\Pi_1\partial_{p_k}H\right]\quad \mathrm{mod}\ \mathcal{O}_S(\varepsilon)\,. \quad (2.18)$$

Since $\partial^2_{p_k,p_\ell}H=(\partial^2_{p_k,p_\ell}f_\varepsilon)$ as a scalar symbol commutes with $\partial^2_{q^kq^\ell}\Pi_0$, the first term of (2.18) vanishes.
Similarly the factor $\partial_{q^k}\Pi_1$ appearing in the second term of (2.18) contains three terms

$$\partial_{q^k}\Pi_1=-i\frac{\partial_{q^k}(E_+-E_-)}{(E_+-E_-)^2}\sigma_3(\partial_{p_\ell}f_\varepsilon)(\partial_{q^\ell}\Pi_0)+\frac{i(\partial_{p_\ell}f_\varepsilon)}{(E_+-E_-)}\sigma_3(\partial_{q^\ell}\Pi_0)+\frac{i(\partial_{p_\ell}f_\varepsilon)}{(E_+-E_-)}\sigma_3(\partial^2_{q^kq^\ell}\Pi_0)\,,$$

where the second one is diagonal. Since $\partial_{p_k}H=\partial_{p_k}f_\varepsilon$ is diagonal, we get

$$\frac{1}{2i}\left[\partial_{p_k}H\partial_{q^k}\Pi_1+\partial_{q^k}\Pi_1\partial_{p_k}H\right]^{OD}=(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)\sigma_3\left(\partial_{q^k}\left[(E_+-E_-)^{-1}\partial_{q^\ell}\Pi_0\right]\right)^{OD}\,.$$

In the quantity $-\partial_{q^k}H\partial_{p_k}\Pi_1-\partial_{p_k}\Pi_1\partial_{q^k}H$, the derivatives

$$\partial_{p_k}\Pi_1=\frac{i(\partial^2_{p_kp_\ell}f_\varepsilon)}{(E_+-E_-)}\sigma_3(\partial_{q^\ell}\Pi_0)^{OD}$$

are off-diagonal factors, while

$$\partial_{q^k}H=(\partial_{q^k}E_+)\Pi_0+(\partial_{q^k}E_-)(1-\Pi_0)+(E_+-E_-)(\partial_{q^k}\Pi_0)^{OD}\,.$$

Since

$$(\partial_{q^k}E_+)\Pi_0\sigma_3(\partial_{q^\ell}\Pi_0)+\sigma_3(\partial_{q^\ell}\Pi_0)(\partial_{q^k}E_+)\Pi_0=(\partial_{q^k}E_+)\sigma_3(\partial_{q^\ell}\Pi_0)$$
$$\text{and}\qquad (\partial_{q^k}E_-)(1-\Pi_0)\sigma_3(\partial_{q^\ell}\Pi_0)+\sigma_3(\partial_{q^\ell}\Pi_0)(\partial_{q^k}E_-)(1-\Pi_0)=(\partial_{q^k}E_-)\sigma_3(\partial_{q^\ell}\Pi_0)\,,$$

we get

$$-\frac{1}{2i}\left[\partial_{q^k}H\partial_{p_k}\Pi_1+\partial_{p_k}\Pi_1\partial_{q^k}H\right]^{OD}=-\frac{\partial^2_{p_kp_\ell}f_\varepsilon}{2(E_+-E_-)}(\partial_{q^k}(E_++E_-))\sigma_3(\partial_{q^\ell}\Pi_0)\,.$$

This leads to

$$F_2=(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)\sigma_3\left(\partial_{q^k}\left[(E_+-E_-)^{-1}\partial_{q^\ell}\Pi_0\right]\right)^{OD}$$
$$-\frac{\partial^2_{p_kp_\ell}f_\varepsilon}{(E_+-E_-)}(\partial_{q^k}(E_++E_-))\sigma_3(\partial_{q^\ell}\Pi_0)\,.$$

and

$$\Pi_2^{OD} = -\frac{(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{(E_+ - E_-)} \left(\partial_{q^k} \left[(E_+ - E_-)^{-1} \partial_{q^\ell} \Pi_0\right]\right)^{OD} + \frac{\partial^2_{p_k p_\ell} f_\varepsilon}{2(E_+ - E_-)^2} (\partial_{q^k}(E_+ + E_-))(\partial_{q^\ell} \Pi_0)\,,$$

with

$$\varepsilon^{-2\delta} \Pi_2^{OD} \in S_u(\frac{1}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle \langle \sqrt{\tau'\tau''} p \rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))\,.$$

We have almost proved the

Proposition 2.3. *The pseudodifferential operator* $\hat{\Pi}(\varepsilon) = \hat{\Pi}(q, \varepsilon D_q, \varepsilon)$ *given by*

$$\Pi(q,p,\varepsilon) = \Pi_0(q,\varepsilon) + \varepsilon \Pi_1(q,p,\varepsilon) + \varepsilon^2 \Pi_2(q,p,\varepsilon)$$

with

$$\Pi_1(q,p,\varepsilon) = \Pi_1(q,p,\varepsilon)^{OD} = \frac{i \partial_{p_k} f_\varepsilon}{(E_+ - E_-)} \sigma_3 (\partial_{q^k} \Pi_0)\,,$$

$$\Pi_2(q,p,\varepsilon) = \Pi_2(q,p,\varepsilon)^D + \Pi_2(q,p,\varepsilon)^{OD}\,,$$

$$\Pi_2(q,p,\varepsilon)^D = -\frac{1}{(E_+ - E_-)^2} \left((\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon) + (E_+ - E)(\partial^2_{p_k p_\ell} f_\varepsilon)\right) (\partial_{q^k} \Pi_0)(\partial_{q^\ell} \Pi_0)\,,$$

$$\Pi_2(q,p,\varepsilon)^{OD} = -\frac{(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{(E_+ - E_-)} \left(\partial_{q^k} \left[(E_+ - E_-)^{-1} \partial_{q^\ell} \Pi_0\right]\right)^{OD} + 2\frac{\partial^2_{p_k p_\ell} f_\varepsilon}{(E_+ - E_-)^3} (\partial_{q^k}(E_+ + E_-))(\partial_{q^\ell} \Pi_0)\,,$$

and

$$\varepsilon^{-2\delta} \Pi_1, \varepsilon^{-2\delta} \Pi_2 \in S_u(\frac{1}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle \langle \sqrt{\tau'\tau''} p \rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))\,,$$

satisfies

$$\hat{\Pi} \circ \hat{\Pi} = \hat{\Pi} + \mathcal{O}(\varepsilon^{3+2\delta}) \quad , \quad \hat{\Pi}^* = \hat{\Pi} \quad , \quad \left[\hat{H}, \hat{\Pi}\right] = \mathcal{O}(\varepsilon^{3+2\delta})\,,$$

in $\mathcal{L}(L^2(\mathbb{R}^d; \mathbb{C}^2))$*. Moreover the estimates of the remainder terms in* $\mathcal{L}(L^2(\mathbb{R}^d; \mathbb{C}^2))$*, do not depend on the parameter* $\tau = (\tau', \tau'') \in (0,1]^2$ *and* $\delta > 0$*, as soon as* $\varepsilon \in (0, \varepsilon_0)$*.*

Proof: The above construction gives immediately

$$\hat{\Pi} \circ \hat{\Pi} = \hat{\Pi} + \mathcal{O}(\varepsilon^3) \quad , \quad \hat{\Pi}^* = \hat{\Pi} \quad , \quad \left[\hat{H}, \hat{\Pi}\right] = \mathcal{O}(\varepsilon^3)\,,$$

The first improved estimates come from the fact that

$$\hat{\Pi} \circ \hat{\Pi} - \hat{\Pi}$$

contains only terms which are Moyal products with a Π_1 or a Π_2 factor, with cancellations up to the ε^2 coefficient. Both of them have seminorms of order $\varepsilon^{2\delta}$.
For the last one, this is a similar argument after decomposing

$$\left[\hat{H}, \hat{\Pi}\right] = \left[\varepsilon^{2\delta} f_1(\sqrt{\tau'\tau''}\varepsilon D_q), \hat{\Pi}_0\right] + \left[\hat{H}, \varepsilon \hat{\Pi}_1 + \varepsilon^2 \hat{\Pi}_2\right]\,.$$

□

The above result can be improved after considering what happens at step $n = 3$ when $f_{\varepsilon,\tau}(p) = \varepsilon^{2\delta} f_1(\sqrt{\tau'\tau''}p)$ is at most quadratic w.r.t p in some region. Before this, let us examine the remainders of order 3 in ε.
$\mathbf{n = 3}$: The remainder term

$$\begin{aligned}\varepsilon^3 G_3 &= \Pi^{(2)} \sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)} \\ &= \varepsilon\left(\Pi_0 \sharp^\varepsilon \Pi_1 + \Pi_1 \sharp^\varepsilon \Pi_0 - \Pi_1\right) + \varepsilon^2\left(\Pi_1 \sharp^\varepsilon \Pi_1 + \Pi_0 \sharp^\varepsilon \Pi_2 + \Pi_2 \sharp^\varepsilon \Pi_0 - \Pi_2\right) \\ &\quad + \varepsilon^3(\Pi_1 \sharp^\varepsilon \Pi_2 + \Pi_2 \sharp^\varepsilon \Pi_1) + \varepsilon^4 \Pi_2 \sharp^\varepsilon \Pi_2 \,.\end{aligned}$$

Using the construction of Π_1 and Π_2 and the fact that $\varepsilon^{-2\delta}\Pi_1$ and $\varepsilon^{-2\delta}\Pi_2$ belong to $S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle}, g_\tau; \mathcal{M}_2(\mathbb{C}))$, the expansion of the Moyal product (A.2)-(A.3) tells us

$$\Pi^{(2)} \sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)} = \varepsilon^3\left[AG(\partial_q^2\Pi_0, \partial_p^2\Pi_1, \varepsilon) + BG(\partial_q\Pi_0, \partial_p\Pi_2, \varepsilon)\right] + \varepsilon^{3+4\delta} R_G \qquad (2.19)$$

where $R_G \in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$ and $AG(.,\varepsilon)$, $BG(.,\varepsilon)$ have an asymptotic expansion in terms of ε, of which all the terms are bilinear differential expressions of their arguments. For the commutator with $\hat{H}$, write

$$\begin{aligned}\left[H\sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)}\sharp^\varepsilon H\right] &= \left[f_\varepsilon(p)\sharp^\varepsilon \Pi_0(q) - \Pi_0(q)\sharp^\varepsilon f_\varepsilon(p)\right] \\ &\quad + \left[f_\varepsilon(p)\sharp^\varepsilon(\varepsilon\Pi_1 + \varepsilon^2\Pi_2) - (\varepsilon\Pi_1 + \varepsilon^2\Pi_2)\sharp^\varepsilon f_\varepsilon(p)\right] \\ &\quad + \left[(E_+\Pi_0 + E_-(1-\Pi_0))\sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)}\sharp^\varepsilon(E_+\Pi_0 + E_-(1-\Pi_0))\right]\end{aligned}$$

After eliminating all the terms which are cancelled while constructing Π_1 and Π_2, the contributions of all three terms of the right-hand side can be analyzed. The contribution of the third term is similar to what we got for G_3:

$$\varepsilon^3\left[AH(\partial_q^2(E_\pm\Pi_0), \partial_p^2\Pi_1, \varepsilon) + BH(\partial_q(E_\pm\Pi_0), \partial_p\Pi_2, \varepsilon)\right] + \varepsilon^{3+4\delta} R_{H,1}$$

with $R_{H,1} \in S_u(\frac{1}{\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$. With the uniform estimate of $\varepsilon^{-2\delta}\Pi_1$, $\varepsilon^{-2\delta}\Pi_2$ and $\varepsilon^{-2\delta} f_\varepsilon$, the contribution of the second term is estimated as $\varepsilon^{3+4\delta} R_{H,2}$ with $R_{H,2} \in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$. The first term is of the form $\varepsilon^3 CH(\partial_p^3 f_\varepsilon, \partial_q^3\Pi_0, \varepsilon)$ and we get

$$\begin{aligned}\left[H\sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)}\sharp^\varepsilon H\right] = \varepsilon^3\big[&CH(\partial_p^3 f_\varepsilon, \partial_q^3\Pi_0, \varepsilon) + \\ &AH(\partial_q^2(E_\pm\Pi_0), \partial_p^2\Pi_1, \varepsilon) + BH(\partial_q(E_\pm\Pi_0), \partial_p\Pi_2, \varepsilon)\big] + \varepsilon^{3+4\delta} R_H \qquad (2.20)\end{aligned}$$

where the expansions of AH, BH and CH w.r.t to ε have terms which are bilinear differential expression of their arguments, and $R_H \in S_u(\frac{1}{\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$.

Proposition 2.4. *With $f_\varepsilon(p) = \varepsilon^{2\delta} f_1(\sqrt{\tau'\tau''}p)$, assume that the third differential $\partial_p^3 f_1$ vanishes in $\{|p| \leq r\}$ and fix $r' \in (0,r)$. Then the remainders of Proposition 2.3 equal*

$$\hat{\Pi} \circ \hat{\Pi} - \hat{\Pi} = \varepsilon^{3+2\delta} \hat{R}_{G,1} + \varepsilon^{3+4\delta} \hat{R}_{G,2} \tag{2.21}$$

$$\left[\hat{H}, \hat{\Pi}\right] = \varepsilon^{3+2\delta} \hat{R}_{H,1} + \varepsilon^{3+4\delta} \hat{R}_{H,2} \tag{2.22}$$

with $R_{G,1or2} \in OpS_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$, $R_{H,1or2} \in OpS_u(\frac{1}{\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$ and $R_{GorH,1} \equiv 0$ in $\left\{|\sqrt{\tau'\tau''}p| < r'\right\}$. Those estimates are uniform for $\tau \in (0,1]^2$, $\delta \in [0, \delta_0]$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof: After noticing that the symbol Π_1 is a linear expression in $\partial_p f_\varepsilon$ while the symbol Π_2 is the sum of a linear expression of $\partial_p^2 f_\varepsilon$ and quadratic expression in $\partial_p f_\varepsilon$, the identities (2.19) and (2.20) imply

$$\begin{aligned} \Pi^{(2)} \sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)} &= \varepsilon^{3+2\delta+N} R_N + \varepsilon^{3+4\delta} R_G \,, \\ \left[H \sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)} \sharp^\varepsilon H\right] &= \varepsilon^{3+2\delta+N} R_N + \varepsilon^{3+4\delta} R_H \,, \end{aligned}$$

for an arbitrary large $N \in \mathbb{N}$, in $\left\{\sqrt{\tau'\tau''}|p| < 2r\right\}$. Choose[1] $N \geq 2\delta$ and take a cut-off function $\chi \in \mathcal{C}_0^\infty(\{|p| < r\})$ such that $\chi \equiv 1$ in a neigborhood $\{|p| \leq r'\}$. Writing for the symbol $S = \Pi^{(2)} \sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)}$ or $S = \left[H \sharp^\varepsilon \Pi^{(2)} - \Pi^{(2)} \sharp^\varepsilon H\right]$,

$$S = S \times \chi(\sqrt{\tau'\tau''}p) + S \times (1 - \chi(\sqrt{\tau'\tau''}p)) \,,$$

yields the result. □

2.3 Unitaries and effective Hamiltonian

We strengthen a little bit the assumptions (2.4)-(2.5), with the condition

$$\Pi_0(q,\tau,\varepsilon) = u_0(q,\tau,\varepsilon) P_+ u_0(q,\tau,\varepsilon)^* \tag{2.23}$$

$$\text{with} \qquad P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \,,\; u_0 = (u_0^*)^{-1} \in S(1, g_{q,\tau}; \mathcal{M}_2(\mathbb{C})) \,,$$

fulfilled in our example. The operator $\hat{u}_0$ is nothing but the local unitary transformation $u_0(q,\tau)$, on $L^2(\mathbb{R}^d; \mathbb{C}^2)$.
With the approximate projection $\hat{\Pi} = \hat{\Pi}(q, \varepsilon D_q, \tau, \varepsilon)$ given in Proposition 2.3, Proposition A.4 tells us that a true orthogonal projection $\hat{P}$ can be associated when ε_0 is chosen small enough, by taking

$$\hat{P} = \frac{1}{2i\pi} \int_{|z-1|=1/2} (z - \hat{\Pi})^{-1} \, dz \,, \tag{2.24}$$

[1] Here the estimates become δ-dependent, because a large δ requires a large N. It is uniformly controlled when $\delta \leq \delta_0$.

with

$$\hat{P} = P(q, \varepsilon D_q, \tau, \varepsilon) \quad , \quad P \in S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))\,, \tag{2.25}$$

$$P(q,p,\tau,\varepsilon) - \Pi(q,p,\tau,\varepsilon) = \varepsilon^{3+2\delta} R_1(q,p,\tau,\varepsilon) + \varepsilon^{3+4\delta} R_2(q,p,\tau,\varepsilon)\,, \tag{2.26}$$

$$\text{with} \quad R_1, R_2 \in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))\,,$$

$$\hat{P}\circ\hat{P} = \hat{P} = \hat{P}^*\,,$$

$$\left[\hat{H}, \hat{P}\right] = \varepsilon^{3+2\delta}\hat{C}_1(\tau,\varepsilon) + \varepsilon^{3+4\delta}\hat{C}_2(\tau,\varepsilon)\,, \tag{2.27}$$

$$\text{with} \quad C_1, C_2 \in S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))\,,$$

$$\text{and} \quad \left\|\hat{P} - \hat{u}_0 P_+ \hat{u}_0^*\right\|_{\mathcal{L}(L^2)} \leq C\varepsilon\,. \tag{2.28}$$

For a general $f_1 \in \mathcal{C}_0^\infty$, R_2 and C_2 are included in the main remainder term. When f_1 is quadratic in $\{|p| < r\}$, then one can assume that R_1 and C_1 vanishes in $\left\{\sqrt{\tau'\tau''}|p| < r'\right\}$ for $r' < r$, according to Proposition 2.4 and Proposition A.4
Instead of constructing unitaries between $\hat{P}(\tau,\varepsilon)$ and P_+ by the induction presented in [PST] and similar to (2.13), (2.14),(2.15),(2.16), we use like in [MaSo] Nagy's formula ([NeSo], [MaSo], [PST])

$$\begin{aligned} & P_2 = W P_1 W^* \quad , \quad W^*W = WW^* = 1 \\ \text{with} \quad & W = (1-(P_2-P_1)^2)^{-1/2}\left[P_2P_1 + (1-P_2)(1-P_1)\right]\,, \\ \text{when} \quad & P_j = P_j^2 = P_j^* \quad \text{for } j = 1,2\,, \quad \text{and } \|P_2 - P_1\|_{\mathcal{L}} < 1\,, \end{aligned} \tag{2.29}$$

easier to handle for direct second order computations in our case.

Proposition 2.5. *With the definitions* (2.23) *and* (2.24) *after Proposition 2.3, there exists a unitary operator* $\hat{U}$ *on* $L^2(\mathbb{R};\mathbb{C}^2)$ *such that*

$$\hat{P} = \hat{U} P_+ \hat{U}^*\,,$$

$$\hat{U} = U(q, \varepsilon D_q, \tau, \varepsilon) \quad , \quad U \in S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))\,,$$

$$\textit{with} \quad U(q,p,\tau,\varepsilon) = u_0(q,\tau,\varepsilon) + \varepsilon u_1(q,p,\tau,\varepsilon) + \varepsilon^2 u_2(q,p,\tau,\varepsilon)\,,$$

$$u_1(q,p,\varepsilon) = -\frac{i\partial_{p_k} f_\varepsilon}{(E_+ - E_-)}(\partial_{q^k}\Pi_0)u_0\,,$$

$$\textit{and} \quad \varepsilon^{-2\delta}u_1, \varepsilon^{-2\delta}u_2 \in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))\,.$$

Moreover, when f_1 *is a quadratic function in* $\{|p| < r\}$ *and* r' *is fixed in* $(0,r)$*, the term* u_2 *can be decomposed into*

$$u_2(q,p,\tau,\varepsilon) = \varepsilon^{2\delta} v_2(q,p,\tau,\varepsilon) + \varepsilon^{4\delta}\widetilde{v_2}(q,p,\tau,\varepsilon)$$

where v_2 *does not depend on* p *in* $\left\{\sqrt{\tau'\tau''}|p| < r'\right\}$ *and* $v_2, \widetilde{v}_2 \in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$ *.*

Proof: The notation $\hat{R}_3$ will denote a generic remainder term of the form $\hat{R} = R(q, \varepsilon D_q, \tau, \varepsilon)$ with $R \in S_u(\frac{1}{\langle\sqrt{\frac{\tau'}{\tau''}}q'\rangle\langle\sqrt{\tau'\tau''}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$. The notation $\underline{\hat{R}}$ is used for a symbol $\underline{R}$, like R but which vanishes around $\left\{\sqrt{\tau'\tau''}|p| < r'\right\}$. We apply Nagy's formula (2.29) with $P_1 = \hat{u}_0 P_+ \hat{u}_0^* = \Pi_0(q, \tau, \varepsilon)$ and $P_2 = \hat{P}(\tau, \varepsilon)$ with

$$\hat{P} = \Pi_0(q, \tau, \varepsilon) + \varepsilon\Pi_1(q, \varepsilon D_q, \tau, \varepsilon) + \varepsilon^2\Pi_2(q, \varepsilon D_q, \tau, \varepsilon) + \varepsilon^{3+2\delta}\underline{\hat{R}} + \varepsilon^{3+4\delta}\hat{R}\,.$$

In the expression of $\hat{W}(\varepsilon)$ given by (2.29), the first factor is nothing but

$$(1 - (\hat{P} - \Pi_0(q, \tau, \varepsilon))^2)^{-1/2} = 1 + \frac{\varepsilon^2}{2}(\Pi_1)^2(q, \varepsilon D_q, \tau, \varepsilon) + \varepsilon^{3+4\delta}\hat{R}\,,$$

owing to $P - \Pi_0 = \mathcal{O}_S(\varepsilon^{2\delta})$. In the factor $[P_2P_1 + (1 - P_2)(1 - P_1)]$, the first term equals

$$\begin{aligned}\hat{P} \circ \Pi_0(q, \varepsilon) = \Pi_0(q, \varepsilon) + \varepsilon\Pi_1(q, \tau, \varepsilon D_q, \tau, \varepsilon) \circ \Pi_0(q, \tau, \varepsilon)\\ + \varepsilon^2\Pi_2(q, \varepsilon D_q, \tau, \varepsilon) \circ \Pi_0(q, \tau, \varepsilon) + \varepsilon^{3+2\delta}\underline{\hat{R}} + \varepsilon^{3+4\delta}\hat{R}\,,\end{aligned}$$

while the second term is

$$\begin{aligned}(1 - \hat{P}) \circ (1 - \Pi_0(q, \tau, \varepsilon)) = (1 - \Pi_0(q, \tau, \varepsilon)) - \varepsilon\Pi_1(q, \varepsilon D_q, \tau, \varepsilon) \circ (1 - \Pi_0(q, \tau, \varepsilon))\\ - \varepsilon^2\Pi_2(q, \varepsilon D_q, \tau, \varepsilon) \circ (1 - \Pi_0(q, \tau\varepsilon)) + \varepsilon^{3+2\delta}\underline{\hat{R}} + \varepsilon^{3+4\delta}\hat{R}\,.\end{aligned}$$

Hence we get

$$\begin{aligned}\hat{W} = [P_2P_1 + (1 - P_2)(1 - P_1)] = 1 + \varepsilon\Pi_1(q, \varepsilon D_q, \tau, \varepsilon) \circ \sigma_3(q, \tau)\\ + \varepsilon^2\Pi_2(q, \varepsilon D_q, \tau, \varepsilon) \circ \sigma_3(q, \tau) + \varepsilon^{3+2\delta}\underline{\hat{R}} + \varepsilon^{3+4\delta}\hat{R}\,.\end{aligned}$$

The operator $\hat{U}(\varepsilon)$ is given by $\hat{U}(\varepsilon) = \hat{W}(\varepsilon) \circ \hat{u}_0$. The semiclassical calculus recalled in (A.2)-(A.3) yields the result. In the decomposition of u_2, when f_1 is quadratic around 0, the terms which are linear in $\varepsilon^{2\delta}$ come with the second derivative of f_ε, which does not depend on p. □

Proposition 2.6. *Introduce the notation for $k \in \{1, \ldots, d\}$*

$$u_0^*(\partial_{q^k} u_0) = -i \begin{pmatrix} A_k & X_k \\ \overline{X_k} & -A_k \end{pmatrix}.$$

If $\hat{U}$ is the unitary operator introduced in Proposition 2.5, the conjugated Hamiltonian $\hat{U}(\varepsilon)^*\hat{H}(\varepsilon)\hat{U}(\varepsilon)$ equals

$$\hat{U}^*\hat{H}\hat{U} = \begin{pmatrix} \hat{h}_+ & 0 \\ 0 & \hat{h}_- \end{pmatrix} + \varepsilon^{3+2\delta}\hat{R}_1 + \varepsilon^{3+4\delta}\hat{R}_2 \tag{2.30}$$

where the remainder terms $\hat{R}_{1,2} = R_{1,2}(q, \varepsilon D_q, \tau, \varepsilon)$ *satisfy* $R_{1,2} \in S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))$ *and additionally* $R_1 \equiv 0$ *in* $\left\{\sqrt{\tau'\tau''}|p| < r'\right\}$ *when* f_1 *is quadratic in* $\{|p| < r\}$*, with* $r' < r$ *. The symbol* h_+ *and* h_- *are given by*

$$\begin{aligned} h_+ &= f_\varepsilon(p) + E_+(q) - \varepsilon(\partial_p f_\varepsilon).A + \frac{\varepsilon^2}{2}(\partial^2_{p_k p_\ell} f_\varepsilon) A_k A_\ell \\ &\quad + \frac{\varepsilon^2(\partial^2_{p_k p_\ell} f_\varepsilon)}{2} X_k \overline{X}_\ell + \frac{\varepsilon^2(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{E_+ - E_-} X_k \overline{X}_\ell \,, \qquad (2.31) \\ h_- &= f_\varepsilon(p) + E_-(q) + \varepsilon(\partial_p f_\varepsilon).A + \frac{\varepsilon^2}{2}(\partial^2_{p_k p_\ell} f_\varepsilon) A_k A_\ell \\ &\quad + \frac{\varepsilon^2(\partial^2_{p_k p_\ell} f_\varepsilon)}{2} \overline{X_k} X_\ell - \frac{\varepsilon^2(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{E_+ - E_-} \overline{X_k} X_\ell \,. \qquad (2.32) \end{aligned}$$

Proof: From the semiclassical calculus, we already know that $\hat{U}^* \hat{H} \hat{U}$ is a semiclassical operator with a symbol in $\mathcal{S}_u(\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle, g_\tau; \mathcal{M}_2(\mathbb{C}))$. Its off-diagonal part equals

$$\begin{aligned} (1 - P_+)\hat{U}^* \hat{H} \hat{U} P_+ \;+\; & P_+ \hat{U}^* \hat{H} \hat{U} (1 - P_+) \\ & = \hat{U} \left[\hat{P}, \left[\hat{P}, \hat{H}\right]\right] \hat{U} \,. \end{aligned}$$

The almost diagonal form (2.30) of $\hat{U}^* \hat{H} \hat{U}$ is then a consequence of (2.27) .
For the second result, it is necessary to compute the diagonal part of the symbol $U^* \sharp^\varepsilon H \sharp^\varepsilon U$ up to $\mathcal{O}(\varepsilon^{3+2\delta})$ in $S_u(1, g_\tau; \mathcal{M}_2)$. Let us compute the diagonal part of

$$\mathcal{B} := (u_0^* + \varepsilon u_1^* + \varepsilon^2 u_2^*) \sharp^\varepsilon H \sharp^\varepsilon (u_0 + \varepsilon u_1 + \varepsilon^2 u_2) \,,$$

or equivalently $(u_0 \mathcal{B} u_0^*)^D$ with our notations.
Since $u_0 = u_0(q, \tau, \varepsilon)$ and $H = \varepsilon^{2\delta} f_1(\sqrt{\tau'\tau''} p) + \mathcal{V}(q, \tau, \varepsilon)$, the first Moyal product equals according to (A.2)-(A.3),

$$\begin{aligned} (u_0^* + \varepsilon u_1^* + \varepsilon^2 u_2^*) \sharp^\varepsilon H &= u_0^* H + \varepsilon u_1^* H + \varepsilon^2 u_2^* H \\ &\quad + \frac{\varepsilon}{2i} \{u_0^*, H\} + \frac{\varepsilon^2}{2i} \{u_1^*, H\} - \frac{\varepsilon^2}{8} (\partial^2_{q^k q^\ell} u_0^*)(\partial^2_{p_k p_\ell} H) + \varepsilon^{3+2\delta} R \\ &= u_0^* H + \varepsilon \left[u_1^* H - \frac{\partial_{p_k} f_\varepsilon}{2i} \partial_{q^k} u_0^* \right] + \varepsilon^2 \left[u_2^* H + \frac{1}{2i} \{u_1^*, H\} \right. \\ &\quad \left. - \frac{1}{8} (\partial^2_{q^k q^\ell} u_0^*)(\partial^2_{p_k p_\ell} f_\varepsilon) \right] + \varepsilon^{3+2\delta} \underline{R} + \varepsilon^{3+4\delta} R \,, \end{aligned}$$

where R and $\underline{R}$ denote generic element of $S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))$, with the additional property that $\underline{R}$ vanishes in $\left\{\sqrt{\tau'\tau''}|p| < r'\right\}$ when f_1 is quadratic in $\{|p| \leq r\}$ with $r' < r$. The reason for the possible decomposition of the remainder, comes again from the fact that at the third order the remainder term proportional to $\varepsilon^{3+2\delta}$ arise with the third derivative

of f_ε, the second derivative w.r.t p of u_1 and the first derivative w.r.t p of u_2.
In the same way, the complete expression of $\mathcal{B}$ is given by

$$\begin{aligned}\mathcal{B}(\varepsilon) = u_0^* H u_0 + \varepsilon \left[u_0^* H u_1 + u_1^* H u_0 - \frac{(\partial_{p_k} f_\varepsilon)}{2i} (\partial_{q^k} u_0^*) u_0 + \frac{(\partial_{p_k} f_\varepsilon)}{2i} u_0^* (\partial_{q^k} u_0) \right] \\ + \varepsilon^2 \left[u_2^* H u_0 + u_0^* H u_2 + \frac{1}{2i} \{u_0^* H, u_1\} + \frac{1}{2i} \{u_1^* H, u_0\} + \frac{1}{4} \left\{ (\partial_{p_k} f_\varepsilon)(\partial_{q^k} u_0^*), u_0 \right\} \right. \\ + u_1^* H u_1 - \frac{(\partial_{p_k} f_\varepsilon)}{2i} (\partial_{q^k} u_0^*) u_1 + \frac{1}{2i} \{u_1^*, H\} u_0 \\ \left. - \frac{1}{8} (\partial^2_{q^k q^\ell} u_0^*)(\partial^2_{p_k p_\ell} f_\varepsilon)) u_0 - \frac{1}{8} u_0^* (\partial^2_{p_k p_\ell} f_\varepsilon)(\partial^2_{q^k q^\ell} u_0) \right] + \varepsilon^{3+2\delta} \underline{R} + \varepsilon^{3+4\delta} R \,.\end{aligned}$$

By recalling that $(u_1 u_0^*)^D = 0$ by Proposition 2.5, we get

$$(u_0 \mathcal{B}(\varepsilon) u_0^*)^D = H + \varepsilon \frac{i}{2} (\partial_{p_k} f_\varepsilon) \left[u_0 (\partial_{q^k} u_0^*) - (\partial_{q^k} u_0) u_0^* \right]^D + \varepsilon^2 \mathcal{B}_2^D + \varepsilon^{3+2\delta} \underline{R} + \varepsilon^{3+4\delta} R \,.$$

where $\mathcal{B}_2^D$ is made of several terms to be analyzed. We need the relations

$$\begin{aligned} \partial_{q^j} u_0^* &= -u_0^* (\partial_{q^j} u_0) u_0^* \,, & (2.33) \\ \partial^2_{q^j q^{j'}} u_0^* &= u_0^* (\partial_{q^{j'}} u_0) u_0^* (\partial_{q^j} u_0) u_0^* + u_0^* (\partial_{q^j} u_0) u_0^* (\partial_{q^{j'}} u_0) u_0^* - u_0^* (\partial^2_{q^j q^{j'}} u_0) u_0^* \\ &= -(\partial_{q^{j'}} u_0^*)(\partial_{q^j} u_0) u_0^* - (\partial_{q^j} u_0^*)(\partial_{q^{j'}} u_0) u_0^* - u_0^* (\partial^2_{q^j q^{j'}} u_0) u_0^* \,, & (2.34) \\ \text{and} \qquad [u_0 (\partial_q u_0^*)]^{OD} &= -(\partial_q \Pi_0) \sigma_3 \,, & (2.35) \end{aligned}$$

coming from $u_0^* u_0 = 1$ and the differentiation of $u_0^* \Pi_0 u_0 = P_+$.
For example, the first one simplifies the $\mathcal{O}(\varepsilon)$-term into

$$(u_0 \mathcal{B}(\varepsilon) u_0^*)^D = H - \varepsilon i (\partial_{p_k} f_\varepsilon) \left[(\partial_{q^k} u_0) u_0^* \right]^D + \varepsilon^2 \mathcal{B}_2^D + \varepsilon^{3+2\delta} \underline{R} + \varepsilon^{3+4\delta} R \,.$$

Many cancelations appear after assembling all the terms in $\mathcal{B}_2^D$. We need accurate expressions for all of them:

- By using again $(u_1 u_0^*)^D = 0$, the term $[u_0 u_2^* H + H u_2 u_0^* + u_0 u_1^* H u_1 u_0^*]^D$ equals

  $$\begin{aligned}&(f_\varepsilon + E_+) \Pi_0 (u_0 u_2^* + u_2 u_0^*) \Pi_0 + (f_\varepsilon + E_-)(1 - \Pi_0)(u_0 u_2^* + u_2 u_0^*)(1 - \Pi_0) \\ &+ (f_\varepsilon + E_-) \Pi_0 (u_0 u_1^* u_1 u_0^*) \Pi_0 + (f_\varepsilon + E_+)(1 - \Pi_0)(u_0 u_1^* u_1 u_0^*)(1 - \Pi_0)\end{aligned}$$

  In the relation

  $$u_0^* u_2 + u_2^* u_0 + u_1^* u_1 + \frac{1}{2i} \{u_0^*, u_1\} + \frac{1}{2i} \{u_1^*, u_0\} = \varepsilon^{1+2\delta} \underline{R_1} + \varepsilon^{1+4\delta} R_1$$

  the remainder terms satisfy $R_1, \underline{R}_1 \in S_u(\frac{1}{\langle \sqrt{\frac{\tau'}{\tau''}} q' \rangle \langle \sqrt{\tau' \tau''} p \rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$, with the same convention as for $R, \underline{R}$. This identity is obtained by writing that the $\mathcal{O}(\varepsilon^2)$ remainder of $U^* \sharp^\varepsilon U - 1$ vanishes and by noticing that the remainder $\underline{R_1} + \varepsilon^{2\delta} R_1$ involves

second derivatives of u_1 w.r.t p and first derivatives of u_2 w.r.t p. We obtain

$$[u_0u_2^*H + Hu_2u_0^* + u_0u_1^*Hu_1u_0^*]^D = (E_- - E_+)(u_0u_1^*u_1u_0^*)^D\sigma_3 \\ - \frac{1}{2i}H\left[u_0\left\{u_0^*, u_1\right\}u_0^* + u_0\left\{u_1^*, u_0\right\}u_0^*\right]^D + \varepsilon^{1+2\delta}\underline{\tilde{R}} + \varepsilon^{1+4\delta}\tilde{R}\,.$$

with $\tilde{R}, \underline{\tilde{R}} \in S_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))$, again with the same convention.
Again with $(\partial_p u_1)u_0^* = (\partial_p u_1 u_0^*)^{OD} = \frac{(\partial^2_{p,p_k} f)}{i(E_+ - E_-)}\partial_{q^k}\Pi_0$ and (2.35), the last factor is

$$\begin{aligned}\left[u_0\left\{u_0^*, u_1\right\}u_0^* + u_0\left\{u_1^*, u_0\right\}u_0^*\right]^D &= -(u_0\partial_{q^\ell}u_0^*)^{OD}\frac{(\partial^2_{p_\ell p_k} f_\varepsilon)}{i(E_+ - E_-)}(\partial_{q^k}\Pi_0) \\ &\quad - \frac{(\partial^2_{p_\ell p_k} f_\varepsilon)}{i(E_+ - E_-)}(\partial_{q^k}\Pi_0)((\partial_{q^\ell}u_0)u_0^*)^{OD} \\ &= -2\frac{(\partial^2_{p_\ell p_k} f_\varepsilon)}{i(E_+ - E_-)}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3\,. \qquad (2.36)\end{aligned}$$

With $u_1u_0^* = \frac{(\partial_{p_k} f_\varepsilon)}{i(E_+ - E_-)}(\partial_{q^k}\Pi_0)$, we have proved

$$[u_0u_2^*H + Hu_2u_0^* + u_0u_1^*Hu_1u_0^*]^D = -\frac{(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{(E_+ - E_-)}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3 \\ - H\frac{(\partial^2_{p_\ell p_k} f_\varepsilon)}{E_+ - E_-}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3 + \varepsilon^{1+2\delta}\underline{\tilde{R}_1} + \varepsilon^{1+4\delta}\tilde{R}_1\,. \quad (2.37)$$

- The term $\left[\frac{1}{2i}u_0\left\{u_0^* f_\varepsilon, u_1\right\}u_0^* + \frac{1}{2i}u_0\left\{u_1^* f_\varepsilon, u_0\right\}u_0^* - \frac{(\partial_{p_k} f_\varepsilon)}{2i}u_0(\partial_{q^k}u_0^*)u_1u_0^*\right]^D$ equals

$$\frac{1}{2i}f_\varepsilon\left[u_0\left\{u_0^*, u_1\right\}u_0^* + u_0\left\{u_1^*, u_0\right\}u_0^*\right]^D \\ + \frac{(\partial_{p_k} f_\varepsilon)}{2i}\left[(\partial_{q^k}u_1)u_0^* + u_0u_1^*(\partial_{q^k}u_0)u_0^* - u_0(\partial_{q^k}u_0^*)u_1u_0^*\right]^D\,.$$

The diagonal part of $(\partial_{q^k}u_1)u_0^*$ is

$$\left[(\partial_{q^k}u_1)u_0^*\right]^D = \frac{\partial_{p_\ell} f_\varepsilon}{i(E_+ - E_-)}(\partial^2_{q^k q^\ell}\Pi_0)^D\,.$$

But differentiating the relation $(\partial_{q^k}\Pi_0)\Pi_0 + \Pi_0(\partial_{q^\ell}\Pi_0) = \partial_{q^k}\Pi_0$ w.r.t q^ℓ leads to

$$(\partial^2_{q^k q^\ell}\Pi_0)^D = -(\partial_{q^k}\Pi_0\partial_{q^\ell}\Pi_0 + \partial_{q^\ell}\Pi_0\partial_{q^k}\Pi_0)\sigma_3\,. \qquad (2.38)$$

With (2.35) and $(u_1u_0^*) = (u_1u_0^*)^{OD} = \frac{\partial_{p_\ell} f_\varepsilon}{i(E_+ - E-)}\partial_{q^\ell}\Pi_0$, we obtain

$$\begin{aligned}\left[u_0u_1^*(\partial_{q^k}u_0)u_0^* - u_0(\partial_{q^k}u_0^*)u_1u_0^*\right]^D &= \left[-u_0u_1^*\sigma_3(\partial_{q^k}\Pi_0) + (\partial_{q^k}\Pi_0)\sigma_3u_1u_0^*\right] \\ &= -\frac{(\partial_{p_\ell} f_\varepsilon)}{i(E_+ - E_-)}\left[(\partial_{q^\ell}\Pi_0)(\partial_{q^k}\Pi_0) + (\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\right]\sigma_3\,.\end{aligned}$$

We have found

$$\left[\frac{1}{2i}u_0\left\{u_0^* f_\varepsilon, u_1\right\}u_0^* + \frac{1}{2i}u_0\left\{u_1^* f_\varepsilon, u_0\right\}u_0^* - \frac{(\partial_{p_k} f_\varepsilon)}{2i}u_0(\partial_{q^k}u_0^*)u_1 u_0^*\right]^D$$
$$= \frac{f_\varepsilon(\partial^2_{p_k p_\ell} f_\varepsilon)}{E_+ - E_-}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3 + 2\frac{(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{E_+ - E_-}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3\,. \quad (2.39)$$

- The diagonal part of $\frac{1}{2i}u_0\left[\{u_0^*\mathcal{V}, u_1\} + \{u_1^*\mathcal{V}, u_0\}\right]u_0^*$ equals

$$\frac{1}{2i}\left[-u_0(\partial_{q^k}u_0^*)\mathcal{V}(\partial_{p_k}u_1)u_0^* - (\partial_{q^k}\mathcal{V})(\partial_{p_k}u_1)u_0^* + u_0(\partial_{p_k}u_1^*)\mathcal{V}(\partial_{q^k}u_0)u_0^*\right]^D\,.$$

By using (2.35) with

$$(\partial_{p_k}u_1)u_0^* = [(\partial_{p_k}u_1)u_0^*]^{OD} = \frac{\partial^2_{p_k p_\ell} f_\varepsilon}{i(E_+ - E_-)}(\partial_{q^\ell}\Pi_0)\,,$$
$$(\partial_q\mathcal{V})^{OD} = (E_+ - E_-)(\partial_q\Pi_0)\,,$$
$$\text{and}\qquad E_-\Pi_0 + E_+(1-\Pi_0) = \mathcal{V} + (E_- - E_+)\sigma_3\,,$$

it becomes

$$\frac{\partial^2_{p_k p_\ell} f_\varepsilon}{2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0) + \frac{1}{2i}\mathcal{V}\left[u_0\left\{u_0^*, u_1\right\}u_0^* + u_0\left\{u_1^*, u_0\right\}u_0^*\right]^D$$
$$+ \frac{1}{2i}(E_- - E_+)\left[u_0\left\{u_0^*, u_1\right\}u_0^* + u_0\left\{u_1^*, u_0\right\}u_0^*\right]^D\sigma_3\,.$$

The relation (2.36) yields

$$\frac{1}{2i}\left[u_0\left\{u_0^*\mathcal{V}, u_1\right\}u_0^* + u_0\left\{u_1^*\mathcal{V}, u_0\right\}u_0^*\right] = -\frac{\partial^2_{p_k p_\ell} f_\varepsilon}{2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)$$
$$+ \mathcal{V}\frac{\partial^2_{p_k p_\ell} f_\varepsilon}{E_+ - E_-}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,. \quad (2.40)$$

- The term $\frac{1}{2i}\left[u_0\left\{u_1^*, H\right\}\right]^D$ is the sum of two terms

$$\frac{1}{2i}\left[u_0\left\{u_1^*, f_\varepsilon\right\}\right]^D + \frac{1}{2i}\left[u_0\left\{u_1^*, \mathcal{V}\right\}\right]^D\,.$$

Since $u_0\partial_{p_\ell}u_1^* = i\frac{\partial^2_{p_k p_\ell} f_\varepsilon}{(E_+ - E_-)}(\partial_{q^k}\Pi_0)$ is off-diagonal while

$$(\partial_{q^\ell}\mathcal{V})^{OD} = (E_+ - E_-)(\partial_{q^\ell}\Pi_0)\,,$$

the second term equals

$$\frac{\partial^2_{p_k p_\ell} f_\varepsilon}{2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,.$$

The first term a priori contains more terms because u_0^* has to be differentiated:

$$\frac{1}{2}\Big[-(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)\partial_{q^\ell}\left(\frac{1}{E_+-E_-}\right)(\partial_{q^k}\Pi_0)-\frac{(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)}{E_+-E_-}(\partial^2_{q^\ell q^k}\Pi_0)$$
$$-u_0(\partial_{q^\ell}u_0^*)\frac{(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)}{E_+-E_-}(\partial_{q^k}\Pi_0)\Big]^D .$$

The first part is off-diagonal and vanishes after taking the diagonal part. By using again (2.35) and (2.38) and $\partial_q\Pi_0=(\partial_q\Pi_0)^{OD}$, we obtain

$$\frac{(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)}{E_+-E_-}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3+\frac{(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)}{E_+-E_-}(\partial_{q^k}\Pi_0)\sigma_3(\partial_{q^\ell}\Pi_0)=0\,.$$

Hence we have proved

$$\frac{1}{2i}\left[u_0\left\{u_1^*,H\right\}\right]^D=\frac{\partial^2_{p_kp_\ell}f_\varepsilon}{2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,. \tag{2.41}$$

- The last term is

$$T=\frac{1}{8}\left[2u_0\left\{(\partial_{p_k}f_\varepsilon)(\partial_{q^k}u_0^*),u_0\right\}u_0^*-u_0(\partial^2_{q^kq^\ell}u_0^*)(\partial^2_{p_kp_\ell}f_\varepsilon)-(\partial^2_{p_kp_\ell}f_\varepsilon)(\partial^2_{q^kq^\ell}u_0)u_0^*\right]^D .$$

Forgetting the $\frac{1}{8}$ factor, the first part equals $2(\partial^2_{p_kp_\ell}f_\varepsilon)u_0(\partial_{q^k}u_0^*)(\partial_{q^\ell}u_0)u_0^*$. By using (2.34) in the second part gives

$$8T=4(\partial_{p_kp_\ell}f_\varepsilon)\left[u_0(\partial_{q^k}u_0^*)(\partial_{q^\ell}u_0)u_0^*\right]^D .$$

Owing to (2.8) and (2.35), this gives

$$2T=(\partial^2_{p_kp_\ell}f_\varepsilon)[u_0(\partial_{q^k}u_0^*)]^D[(\partial_{q^\ell}u_0)u_0^*]^D+(\partial^2_{p_kp_\ell}f_\varepsilon)(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,.$$

With $u_0(\partial_{q^k}u_0^*)=-(\partial_{q^k}u_0)u_0^*$, the last term equals

$$T=-\frac{(\partial^2_{p_kp_\ell}f_\varepsilon)}{2}[(\partial_{q^k}u_0)u_0^*]^D[(\partial_{q^\ell}u_0)u_0^*]^D+\frac{(\partial^2_{p_kp_\ell}f_\varepsilon)}{2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\,. \tag{2.42}$$

By summing (2.37),(2.39),(2.40),(2.41),(2.42), we obtain

$$B_2^D=-\frac{(\partial^2_{p_kp_\ell}f_\varepsilon)}{2}[(\partial_{q^k}u_0)u_0^*]^D[(\partial_{q^\ell}u_0)u_0^*]^D+\frac{(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)}{E_+-E_-}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3$$
$$+\frac{\partial^2_{p_kp_\ell}f_\varepsilon}{2}(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)+\varepsilon^{1+2\delta}\underline{\tilde R_1}+\varepsilon^{1+4\delta}\tilde R_1\,.$$

Hence the diagonal symbol that we seek, is

$$\begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} = \begin{pmatrix} f_\varepsilon(p,\tau)+E_+(q,\tau) & 0 \\ 0 & f_\varepsilon(p,\tau)+E_-(q,\tau) \end{pmatrix} - i\varepsilon(\partial_{p_k} f_\varepsilon) u_0^* \left[(\partial_{q^k} u_0)u_0^*\right]^D u_0$$
$$- \varepsilon^2 \frac{\partial^2_{p_k p_\ell} f_\varepsilon}{2} u_0^* \left[(\partial_{q^k} u_0)u_0^*\right]^D \left[(\partial_{q^k} u_0)u_0^*\right]^D u_0 + \varepsilon^2 \frac{(\partial_{p_k} f_\varepsilon)(\partial_{p_\ell} f_\varepsilon)}{E_+ - E_-} u_0^*(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0)\sigma_3 u_0$$
$$+ \varepsilon^2 \frac{\partial^2_{p_k p_\ell} f_\varepsilon}{2} u_0^*(\partial_{q^k}\Pi_0)(\partial_{q^\ell}\Pi_0) u_0 \,.$$

The symbol $u_0^* \left[(\partial_{q^k} u_0)u_0^*\right]^D u_0$ equals

$$u_0^*\Pi_0(\partial_{q^k} u_0)u_0^*\Pi_0 u_0 + u_0^*(1-\Pi_0)(\partial_{q^k} u_0)u_0^*(1-\Pi_0)u_0$$
$$= P_+ u_0^*(\partial_{q^k} u_0)P_+ + (1-P_+)u_0^*(\partial_{q^k} u_0)(1-P_+) = -i \begin{pmatrix} A_k & 0 \\ 0 & -A_k \end{pmatrix} .$$

For the last term we deduce from $\Pi_0 = u_0 P_+ u_0^*$ and $(\partial_q u_0)u_0^* + u_0(\partial_q u_0) = 0$,

$$\begin{aligned} &P_+ \left[u_0^*(\partial_{q^k}\Pi_0)u_0\right]\left[u_0^*(\partial_{q^\ell}\Pi_0)u_0\right] P_+ \\ &\quad = P_+ \left[u_0^*(\partial_{q^k} u_0)P_+ + P_+(\partial_{q^k} u_0^*)u_0\right]\left[u_0^*(\partial_{q^\ell} u_0)P_+ + P_+(\partial_{q^\ell} u_0^*)u_0\right] P_+ \\ &\quad = P_+ u_0^*(\partial_{q^k} u_0)P_+ u_0^*(\partial_{q^\ell} u_0)P_+ + P_+ u_0^*(\partial_{q^k} u_0)P_+(\partial_{q^\ell} u_0^*)u_0 P_+ \\ &\qquad + P_+(\partial_{q^k} u_0^*)u_0 u_0^*(\partial_{q^\ell} u_0)P_+ + P_+(\partial_{q^k} u_0^*)u_0 P_+(\partial_{q^\ell} u_0^*)u_0 P_+ \\ &\quad = -P_+ u_0^*(\partial_{q^k} u_0)(1-P_+)u_0^*(\partial_{q^\ell} u_0)P_+ = X_k \overline{X}_\ell \,. \end{aligned}$$

Taking the bracket with $(1-P_+)$ is even simpler and gives

$$(1-P_+)\left[u_0^*(\partial_{q^k}\Pi_0)u_0\right]\left[u_0^*(\partial_{q^\ell}\Pi_0)u_0\right](1-P_+)$$
$$= -(1-P_+)u_0^*(\partial_{q^k} u_0)P_+ u_0^*(\partial_{q^\ell} u_0)(1-P_+) = \overline{X_k} X_\ell \,.$$

This ends the proof. □

2.4 Discussion about the adiabatic approximation of the Born-Oppenheimer Hamiltonian

The Theorem 2.1 is a direct application of Proposition 2.6 by taking

$$f_\varepsilon(p) = \varepsilon^{2\delta}\tau'\tau''|p|^2\gamma(\tau'\tau''|p|^2) \,.$$

The operators

$$\begin{aligned} h_{\pm,BO}(q,\varepsilon D_q,\varepsilon) &= \varepsilon^{2\delta}\tau'\tau'' \left[\sum_{k=1}^d (p_k \mp \varepsilon A_k)^2\right]^{Weyl} + \varepsilon^{2+2\delta}\tau'\tau'' \sum_{k=1}^d |X_k|^2 \\ &= \varepsilon^{2+2\delta}\tau'\tau'' \left[-\Delta + |A|^2 \mp 2\left[(\frac{1}{i}\nabla).A + A.(\frac{1}{i}\nabla)\right] + |X|^2\right] \\ &= \varepsilon^{2+2\delta}\tau'\tau'' \left[|\frac{1}{i}\nabla \mp A|^2 + |X|^2\right] , \end{aligned}$$

is nothing but the usual adiabatic effective Hamiltonians which can be found in the physics litterature, including the Born-Huang potential $|X|^2 = \sum_{k=1}^{d} |X_k|^2$. We refer to [PST] and [PST2] for a discussion of the various presentations of the calculations and additional references.
Even in the region $\{|p| \leq r_\gamma\}$ with p quantized into εD_q, this approximation makes sense, only for $\delta > 0$, because of the additional term

$$\mp 2 \frac{\varepsilon^{2+4\delta} \tau' \tau''}{E_+ - E_-} \bar{X}_k X_\ell (\varepsilon \partial_{q_k})(\varepsilon \partial_{q_\ell})$$

coming from the last terms of (2.31) and (2.32), that we have included in the remainder. It is not surprising (see [Sor], [MaSo]) that the degree of the differential operators increases with the degree in ε in the adiabatic expansion of Schrödinger type Hamiltonians. The argument of physicists says that this effective Hamiltonian is used for relatively small frequencies (or momentum) so that $X_k X_\ell p^k p^\ell$ is negligible w.r.t $|p - \varepsilon A|^2 + \varepsilon^2 |X|^2$. The introduction of the additional factor $\varepsilon^{2\delta}$ with $\delta > 0$ provides a mathematically accurate and rather flexible implementation of this approximation.

3 Adaptation of the adiabatic asymptotics to the full nonlinear minimization problem

In this section, we adapt our rather general adiabatic result to our nonlinear problem. In a first step, we give an explicit form of Theorem 2.1 in our specific framework. Those results are effective when applied with wave functions localized in the frequency variable, $\psi = \chi(\sqrt{\tau_x \tau_y}\varepsilon D_q)\psi$ for some compactly supported χ. It could suffice if we considered minimizing the energy among such well prepared quantum states. We can do better by using a partition of unity in the frequency variable, which will be combined, in the end, with the a priori estimates coming from the complete and reduced minimization problems. Finally an estimate of the effect of the unitary transform $\hat{U}$ on the nonlinear term is provided.

3.1 Adiabatic approximation for the explicit Schrödinger Hamiltonian

Let us specify the result of Theorem 2.1 by going back to the coordinates $q = (q', q'') = (x, y)$ and $\tau = (\tau', \tau'') = (\tau_x, \tau_y)$. Provided that $V_{\varepsilon,\tau}(x, y)$ fulfills the proper assumptions, the unitary transform introduced in Theorem 2.1 transforms H_{Lin} given by (1.7) into the Born-Oppenheimer Hamiltonian with a good accuracy in the low frequency region, that is when applied to wave functions ψ such that $\psi = \chi(\sqrt{\tau_x \tau_y}\varepsilon D_q)\psi$ for some compactly supported χ.

The operator H_{Lin} is the ε-quantization of the symbol

$$\varepsilon^{2\delta}\tau_x\tau_y|p|^2 + u_0(q,\tau)\begin{pmatrix} E_+(q,\tau,\varepsilon) & 0 \\ 0 & E_-(q,\tau,\varepsilon)\end{pmatrix} u_0(q,\tau)^*\,,$$

$$\text{with} \qquad E_\pm(q,\tau,\varepsilon) = V_{\varepsilon,\tau}(x,y) \pm \Omega(\sqrt{\frac{\tau_x}{\tau_y}}x) = V_{\varepsilon,\tau}(x,y) \pm \sqrt{1+\frac{\tau_x}{\tau_y}x^2}\,.$$

The operator $u_0(q,\tau)$ equals

$$u_0(q,\tau) = \begin{pmatrix} C & Se^{i\varphi} \\ Se^{-i\varphi} & -C\end{pmatrix} = u_0(q,\tau)^*$$

$$\text{with} \qquad C = \cos(\frac{\theta}{2})\,, \quad S = \sin(\frac{\theta}{2})\,.$$

$$\text{and} \qquad \cot(\theta) = \sqrt{\frac{\tau_x}{\tau_y}}x \quad , \quad \varphi = \sqrt{\frac{\tau_y}{\tau_x}}y\,.$$

Proposition 3.1. *Assume $\delta \in (0,\delta_0]$, $(\tau_x,\tau_y) \in (0,1]^2$ and $V_{\varepsilon,\tau} \in S_u(\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle, \frac{\frac{\tau_x}{\tau_y}dx^2}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle^2} + \frac{\tau_y}{\tau_x}dy^2)$ then the matricial potential*

$$\mathcal{V}(q,\tau,\varepsilon) = V_{\varepsilon,\tau}(x,y) + \Omega(\sqrt{\frac{\tau_x}{\tau_y}}x)\begin{pmatrix} \cos(\theta) & e^{i\varphi}\sin(\theta) \\ e^{-i\varphi}\sin(\theta) & -\cos(\theta)\end{pmatrix}$$

fulfills the assumption of Theorem 2.1.
Choose a cut-off γ, $\gamma(|p|^2) \equiv 1$ in a neighborhood of $\{|p| \le r_\gamma\}$, as in Theorem 2.1 and consider a cut-off function $\chi \in \mathcal{C}_0^\infty((-r_\gamma^2, r_\gamma^2))$. When $\hat{U} = U(q,\varepsilon D_q,\tau,\varepsilon) \in OpS_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))$ is given in Theorem 2.1, the identities

$$\hat{U}^* H_{Lin}\hat{U}\chi(\tau_x\tau_y|\varepsilon D_q|^2) = \varepsilon^{2+2\delta}\tau_x\tau_y\begin{pmatrix} e^{+i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}}\hat{H}_+e^{-i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}} & 0 \\ 0 & e^{-i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}}\hat{H}_-e^{i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}}\end{pmatrix}\chi(\tau_x\tau_y|\varepsilon D_q|^2)$$
$$+\varepsilon^{2+4\delta}R_1(\tau,\varepsilon)\,, \tag{3.1}$$

$$H_{Lin}\chi(\tau_x\tau_y|\varepsilon D_q|^2) = \varepsilon^{2+2\delta}\tau_x\tau_y\hat{U}\begin{pmatrix} e^{+i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}}\hat{H}_+e^{-i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}} & 0 \\ 0 & e^{-i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}}\hat{H}_-e^{i\sqrt{\frac{\tau_y}{\tau_x}}\frac{y}{2}}\end{pmatrix}\hat{U}^*\chi(\tau_x\tau_y|\varepsilon D_q|^2)$$
$$+\varepsilon^{2+4\delta}R_2(\tau,\varepsilon)\,, \tag{3.2}$$

hold with

$$\hat{H}_+ = -\partial_x^2 - \left(\partial_y + i\frac{x}{2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\right)^2 + \frac{1}{\varepsilon^{+2+2\delta}\tau_x\tau_y}\left[V_{\varepsilon,\tau}(x,y) + \sqrt{1+\frac{\tau_x}{\tau_y}x^2}\right] + W_\tau(x,y)\,,$$

$$\hat{H}_- = -\partial_x^2 - \left(\partial_y - i\frac{x}{2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\right)^2 + \frac{1}{\varepsilon^{2+2\delta}\tau_x\tau_y}\left[V_{\varepsilon,\tau}(x,y) - \sqrt{1+\frac{\tau_x}{\tau_y}x^2}\right] + W_\tau(x,y)\,,$$

$$W_\tau(x,y) = \frac{\tau_x}{\tau_y(1+\frac{\tau_x}{\tau_y}x^2)^2} + \frac{\tau_y}{\tau_x(1+\frac{\tau_x}{\tau_y}x^2)}\,,$$

and the estimates

$$\|R_1(\tau,\varepsilon)\|_{\mathcal{L}(L^2)} + \|R_2(\tau,\varepsilon)\|_{\mathcal{L}(L^2)} \leq C$$

which are uniform w.r.t $\delta \in (0,\delta_0]$, $\tau \in (0,1]^2$ *and* $\varepsilon \in (0,\varepsilon_0]$.

Proof: Following the approach of Section 2, the Hamiltonian H_{Lin} is decomposed into

$$H_{Lin} = H(q,\varepsilon D_q,\tau,\varepsilon) + R_\gamma(\varepsilon D_q,\tau,\varepsilon)$$

$$\text{with} \quad H(q,p,\varepsilon) = \varepsilon^{2\delta}\tau_x\tau_y|p|_\tau^2\gamma(\tau_x\tau_y|p|^2) + u_0(q,\tau)\begin{pmatrix} E_+(q,\tau,\varepsilon) & 0 \\ 0 & E_-(q,\tau,\varepsilon)\end{pmatrix} u_0(q,\tau)^*,$$

$$\text{and} \quad R_\gamma(p,\tau,\varepsilon) = \varepsilon^{2\delta}\tau_x\tau_y|p|^2(1-\gamma(\tau_x\tau_y|p|^2)).$$

The Hamiltonian $H(q,\varepsilon D_q,\tau,\varepsilon)$ fulfills the assumptions of Theorem 2.1, with the metric $g_\tau = \frac{\frac{\tau_x}{\tau_y}dx^2}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle^2} + \frac{\tau_y}{\tau_x}dy^2 + \frac{\tau_x\tau_y dp^2}{\langle\sqrt{\tau_x\tau_y}p\rangle_\tau^2}$, because we assumed $V_{\varepsilon,\tau} \in S_u(\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle, \frac{\frac{\tau_x}{\tau_y}dx^2}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle^2} + \frac{\tau_y}{\tau_x}dy^2)$ and the gap $E_+(q,\tau,\varepsilon) - E_-(q,\tau,\varepsilon)$ equals $2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}$. Actually the estimates

$$|\partial_x^\alpha\partial_y^\beta u_0(x,y)| \leq C_{\alpha,\beta}\left\langle\sqrt{\frac{\tau_x}{\tau_y}}x\right\rangle^{-|\alpha|}\left(\frac{\tau_x}{\tau_y}\right)^{\frac{|\alpha|-|\beta|}{2}}$$

are due to $\partial_x\theta = -\frac{\sqrt{\tau_x}}{\sqrt{\tau_y}(1+\frac{\tau_x}{\tau_y}x^2)}$ and $\partial_y e^{i\varphi} = i\sqrt{\frac{\tau_y}{\tau_x}}e^{i\varphi}$. Moreover the explicit computation with $\alpha+\beta=1$ leads to

$$\begin{pmatrix} A_x & X_x \\ \overline{X_x} & -A_x\end{pmatrix} = iu_0^*(\partial_x u_0) = \frac{i\partial_x\theta}{2}\begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0\end{pmatrix} \tag{3.3}$$

$$\text{and} \quad \begin{pmatrix} A_y & X_y \\ \overline{X_y} & -A_y\end{pmatrix} = iu_0^*(\partial_y u_0) = \sqrt{\frac{\tau_y}{\tau_x}}S\begin{pmatrix} S & -Ce^{i\varphi} \\ -Ce^{-i\varphi} & -S\end{pmatrix}. \tag{3.4}$$

Hence the effective Hamiltonians $h_{B0,\pm}(q,\varepsilon D_q,\varepsilon)$ restricted to the region $\{|p|_\tau \leq r_\gamma\}$ are given by the symbols

$$h_{BO,\pm} = \varepsilon^{2\delta}\tau_x\tau_y\left[p_x^2 + (p_y \mp \varepsilon\sqrt{\frac{\tau_y}{\tau_x}}\sin^2(\frac{\theta}{2}))^2 + \frac{1}{4}(|\partial_x\theta|^2 + \frac{\tau_y}{\tau_x}\sin^2(\theta))\right] + V_{\varepsilon,\tau}(x,y) \pm \sqrt{1+\frac{\tau_x}{\tau_y}x^2}.$$

With $\cos(\theta) = \frac{\sqrt{\tau_x}x}{\sqrt{\tau_y}\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}$ and $\sin(\theta) = \frac{1}{\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}$, the Schrödinger Hamiltonian corresponding to the RHS is $\varepsilon^{2+2\delta}\tau_x\tau_y e^{\pm i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_\pm e^{\mp i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}$ with

$$\hat{H}_\pm = -\partial_x^2 - \left(\partial_y \pm i\frac{x}{2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\right)^2 + \frac{\tau_x}{\tau_y(1+\frac{\tau_x}{\tau_y}x^2)^2} + \frac{\tau_y}{\tau_x(1+\frac{\tau_x}{\tau_y}x^2)} + \frac{1}{\tau_x\tau_y\varepsilon^{2+2\delta}}\left[V_{\varepsilon,\tau}(x,y) \pm \sqrt{1+\frac{\tau_x}{\tau_y}x^2}\right].$$

The remainder term in Theorem 2.1 is

$$\varepsilon^{2+4\delta}R_1(q,\varepsilon D_q,\tau,\varepsilon)+\varepsilon^{3+2\delta})R_2(q,\varepsilon D_q,\tau,\varepsilon)$$

with $R_{1,2}\in S_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))$ and where R_2 vanishes in a neighborhood of $\left\{\sqrt{\tau_x\tau_y}|p|\leq r_\gamma\right\}$. The first term provides the expected $\mathcal{O}(\varepsilon^{2+4\delta})$ estimate in $L^2(\mathbb{R}^2;\mathbb{C}^2)$.
It remains to check the effect of truncations. All the factors, including the left terms

$$R_\gamma(p,\tau,\varepsilon)\,,\quad \varepsilon^{2\delta}\tau_x\tau_y\left[p_x^2+(p_y\mp\varepsilon\sin^2(\frac{\theta}{2}))^2\right](1-\gamma(\tau_x\tau_y|p|^2))$$

belong to $OpS_u(\langle\sqrt{\tau_x\tau_y}p\rangle^2,g_\tau;\mathcal{M}_2(\mathbb{C}))$. For any $a,b\in OpS_u(\langle\sqrt{\tau_x\tau_y}p\rangle^2,g_\tau;\mathcal{M}_2(\mathbb{C}))$, where b vanishes in a neighborhood of $\left\{\sqrt{\tau_x\tau_y}|p|\leq r_\gamma\right\}$, and two cut-off functions $\chi_1,\chi_2\in\mathcal{C}_0^\infty((-r_\gamma^2,r_\gamma^2))$ such that $\chi_1\prec\chi_2$ (see Definition A.2), the pseudo-differential calculus says

$$\begin{aligned}&(1-\chi_2(\tau_x\tau_y|p|^2))\sharp^\varepsilon a\sharp^\varepsilon\chi_1(\tau_x\tau_y|p|^2)\in\mathcal{N}_{u,g_\tau}\,,\\&b\sharp^\varepsilon\chi_1(|p|_\tau^2)\in\mathcal{N}_{u,g_\tau}\,,\end{aligned}$$

with uniform estimates of all the seminorms w.r.t $\tau\in(0,1]^2$ and $\delta\in(0,\delta_0]$. Applying this with $\chi_1=\chi$ and various χ_2 such that $\chi_1\prec\chi_2\prec\gamma$, implies that the remainder terms due to truncations are $\mathcal{O}(\varepsilon^N)$ elements of $\mathcal{L}(L^2)$ for any $N\in\mathbb{N}$, uniformly w.r.t $\tau\in(0,1]^2$ and $\delta\in(0,\delta_0]$. Fixing $N\geq 2+4\delta_0$ ends the proof of (3.1).
For (3.2) use (3.1) with a cut-off function χ_1 such that $\chi\prec\chi_1$ and conjugate with $\hat{U}$:

$$\begin{aligned}H_{Lin}\hat{U}\chi_1(\tau_x\tau_y|\varepsilon D_q|^2)\hat{U}^*=&\\\varepsilon^{2+2\delta}\tau_x\tau_y\hat{U}\begin{pmatrix}e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_+e^{-i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}&0\\0&e^{-i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_-e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\end{pmatrix}&\chi_1(\tau_x\tau_y|\varepsilon D_q|^2)\hat{U}^*\\&+\varepsilon^{2+4\delta}R_1'(\varepsilon)\,.\end{aligned}$$

Right-composing with $\chi(\tau_x\tau_y|\varepsilon D_q|^2)$ and noticing that

$$\chi_1(\tau_x\tau_y|\varepsilon D_q|^2)\hat{U}^*\chi(\tau_x\tau_y|\varepsilon D_q|^2)-\hat{U}^*\chi(\tau_x\tau_y|\varepsilon D_q|^2)=R(q,\varepsilon D_q,\tau,\varepsilon)\,,$$

with $R\in\mathcal{N}_{u,g_\tau}$ lead to (3.2) like above. □

3.2 Linear energy estimates for non truncated states

Proposition 3.2. *Assume $\delta\in(0,\delta_0]$, $\tau=(\tau_x,\tau_y)\in(0,1]^2$ and that $V_{\varepsilon,\tau}$ belongs to the parametric symbol class $S_u(\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle,\frac{\frac{\tau_x}{\tau_y}dx^2}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle^2}+\frac{\tau_y}{\tau_x}dy^2)$. Set $\hat{\chi}=\chi(\tau_x\tau_y|\varepsilon D_q|^2)$ for $\chi\in\mathcal{C}_0^\infty((-r_\gamma^2,r_\gamma^2))$. When $\hat{U}=U(q,\varepsilon D_q,\tau,\varepsilon)\in OpS_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))$ is the unitary operator*

given in Theorem 2.1 and parametrized by a truncation in $\{\sqrt{\tau_x\tau_y}|p| < r_\gamma\}$*, then for any* $\chi \in \mathcal{C}_0^\infty((-r_\gamma^2, r_\gamma^2))$ *the estimates*

$$\left|\left\langle \psi\,,\, \left[\hat{U}^* H_{Lin}\hat{U} - \varepsilon^{2+2\delta}\tau_x\tau_y H_{BO}\right]\psi\right\rangle\right| \tag{3.5}$$
$$\leq C\varepsilon^{1+2\delta}\left[\varepsilon^{1+2\delta}\|\psi\|_{L^2}^2 + \|(1-\hat{\chi})\psi\|_{L^2}^2 + \|(1-\hat{\chi})\psi\|_{L^2} \times \|\sqrt{\tau_x\tau_y}|\varepsilon D_q|(1-\hat{\chi})\psi\|_{L^2}\right],$$
$$\left|\left\langle \psi\,,\, \left[H_{Lin} - \varepsilon^{2+2\delta}\tau_x\tau_y\hat{U} H_{BO}\hat{U}^*\right]\psi\right\rangle\right| \tag{3.6}$$
$$\leq C\varepsilon^{1+2\delta}\left[\varepsilon^{1+2\delta}\|\psi\|_{L^2}^2 + \|(1-\hat{\chi})\psi\|_{L^2}^2 + \|(1-\hat{\chi})\psi\|_{L^2} \times \|\sqrt{\tau_x\tau_y}|\varepsilon D_q|(1-\hat{\chi})\psi\|_{L^2}\right],$$

hold uniformly w.r.t $\tau \in (0,1]^2$ *and* $\delta \in (0, \delta_0]$*, for all* $\psi \in L^2(\mathbb{R}^2;\mathbb{C}^2)$*, with*

$$H_{BO} = \begin{pmatrix} e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_+ e^{-i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y} & 0 \\ 0 & e^{-i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_- e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y} \end{pmatrix}$$

$$\hat{H}_+ = -\partial_x^2 - \left(\partial_y + i\frac{x}{2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\right)^2 + \varepsilon^{-2-2\delta}\left[V_{\varepsilon,\tau}(x,y) + \sqrt{1+x^2}\right] + W_\tau(x,y)\,,\,,$$

$$\hat{H}_- = -\partial_x^2 - \left(\partial_y - i\frac{x}{2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\right)^2 + \varepsilon^{-2-2\delta}\left[V_{\varepsilon,\tau}(x,y) - \sqrt{1+x^2}\right] + W_\tau(x,y)$$

$$W_\tau(x,y) = \frac{\tau_x}{\tau_y(1+\frac{\tau_x}{\tau_y}x^2)^2} + \frac{\tau_y}{\tau_x(1+\frac{\tau_x}{\tau_y}x^2)}\,.$$

Proof: Set

$$\mathcal{D} = \hat{U}^* H_{Lin}\hat{U} - \varepsilon^{2+2\delta}\tau_x\tau_y \begin{pmatrix} e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_+ e^{-i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y} & 0 \\ 0 & e^{-i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\hat{H}_- e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y} \end{pmatrix}$$

and bound the terms $\langle\hat{\chi}\psi\,,\,\mathcal{D}\psi\rangle$ and $\langle(1-\hat{\chi}\psi\,,\,\mathcal{D}\hat{\chi}\psi\rangle$ by $C\varepsilon^{2+4\delta}\|\psi\|_{L^2}^2$ with the help of Proposition 3.1. The remaining term is

$$\langle(1-\hat{\chi})\psi\,,\,\mathcal{D}(1-\hat{\chi}(\tau_x\tau_y|\varepsilon D_q|^2))\psi\rangle\,.$$

The operator $\mathcal{D}$ can be decomposed according to $\mathcal{D} = \varepsilon^{2+2\delta}\tau_x\tau_y\mathcal{D}_{kin} + \mathcal{D}_{pot}$ with

$$\mathcal{D}_{kin} = \hat{U}^*|D_q|^2\hat{U} - \begin{pmatrix} |D_q - A|^2 + |X|^2 & 0 \\ 0 & |D_q + A|^2 + |X|_\tau^2 \end{pmatrix}, \tag{3.7}$$

$$\mathcal{D}_{pot} = \hat{U}^*\hat{U}_0 \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}\hat{U}_0^*\hat{U} - V_{\varepsilon,\tau}(x,y) - \sqrt{1+\frac{\tau_x}{\tau_y}x^2}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.8}$$

The normalization in (3.8) allows to use directly the semiclassical calculus if one remembers that $\hat{U}_0^*\hat{U} = \mathrm{Id} + \varepsilon\hat{R}$ and $\hat{U}^*\hat{U}_0 = \mathrm{Id} + \varepsilon\hat{R}'$ with $\varepsilon^{-2\delta}R, \varepsilon^{-2\delta}R' \in S_u(\frac{1}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle}, g_\tau; \mathcal{M}_2(\mathbb{C}))$ and $E_\pm(q,\tau,\varepsilon) = V_{\varepsilon,\tau}(x,y) \pm \sqrt{1+\frac{\tau_x}{\tau_y}x^2}$. We obtain

$$|\langle(1-\hat{\chi})\psi\,,\,\mathcal{D}_{pot}(1-\hat{\chi}))\psi\rangle| \leq C\varepsilon^{1+2\delta}\|(1-\hat{\chi})\psi\|_{L^2}^2\,,$$

which corresponds to the second term of our right-hand sides.
The kinetic energy term (3.7) is decomposed into $\varepsilon^{2+2\delta}\tau_x\tau_y\mathcal{D}_{kin} = \varepsilon^{2+2\delta}\tau_x\tau_y\mathcal{D}^0_{kin} + \mathcal{D}^1_{kin}$ with

$$\mathcal{D}^1_{kin} = \varepsilon^{2\delta}\hat{U}^*\tau_x\tau_y|\varepsilon D_q|^2\hat{U} - \varepsilon^{2\delta}\hat{U}_0^*\tau_x\tau_y|\varepsilon D_q|^2\hat{U}_0\,, \tag{3.9}$$

$$\mathcal{D}^0_{kin} = \hat{U}_0^*|D_q|^2\hat{U}_0 - \begin{pmatrix} |D_q - A|^2 + |X|^2 & 0 \\ 0 & |D_q + A|^2 + |X|^2_\tau \end{pmatrix}. \tag{3.10}$$

Writing (3.9) in the form

$$\mathcal{D}^1_{kin} = \varepsilon^{2\delta}(\hat{U}^* - \hat{U}_0^*)\tau_x\tau_y|\varepsilon D_q|^2\hat{U} + \varepsilon^{2\delta}\hat{U}_0^*\tau_x\tau_y|\varepsilon D_q|^2(\hat{U} - \hat{U}_0)\,,$$

while $\tau_x\tau_y|\varepsilon D_q|^2 \in OpS_u(\langle\sqrt{\tau_x\tau_y}p\rangle, g_\tau; \mathcal{M}_2(\mathbb{C}))$, $\hat{U}, \hat{U}_0 \in OpS_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))$ and $\varepsilon^{-1-2\delta}(\hat{U} - \hat{U}_0) \in OpS_u(\frac{1}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle\langle\sqrt{\tau_x\tau_y}p\rangle^\infty_\tau}, g_\tau; \mathcal{M}_2(\mathbb{C}))$, leads to

$$\left|\left\langle(1-\hat{\chi})\psi\,,\, \mathcal{D}^1_{kin}(1-\hat{\chi})\psi\right\rangle\right| \leq C\varepsilon^{1+4\delta}\|(1-\hat{\chi})\psi\|^2_{L^2}\,.$$

which is even smaller than the $\mathcal{D}_{pot}$ upper bound.
In (3.10), the first term can be computed via

$$\langle\Phi\,,\, U_0^*|D_q|^2_\tau U_0(q)\Psi\rangle = \sum_{q_j\in\{x,y\}}\langle D_{q_j}(u_0(q,\tau)\Phi)\,,\, D_{q_j}(u_0(q,\tau)\Psi)\rangle$$

with $D_{q_j}(u_0 f) = u_0(D_{q_j} - iu_0^*\partial_{q_j}u_0)f$ and equals

$$\begin{aligned}\hat{U}_0^*|D_q|^2\hat{U}_0 &= |D_q - iu_0^*\partial_q u_0|^2 \\ &= \sum_{q_j\in\{x,y\}}\Big[D^2_{q_j} - (iu_0^*(q)\partial_{q_j}u_0(q))^2)D_{q_j} - D_{q_j}(iu_0^*(q)\partial_{q_j}u_0(q))^2) \\ &\qquad\qquad +(iu_0^*(q)\partial_{q_j}u_0(q))^2\Big]\,.\end{aligned}$$

Meanwhile expanding the entries of the second term in (3.10) gives

$$|D_q \mp A(q)|^2 = \sum_{q_j\in\{x,y\}}\left[D^2_{q_j} \mp A_j(q)D_{q_j} \mp D_{q_j}A_j(q) + A_j(q)^2\right]\,.$$

By using the expresions (3.3) and (3.4) for A, X and $iu_0^*\partial_q u_0$, we obtain

$$\mathcal{D}^0_{kin} = \frac{1}{2}\begin{pmatrix} 0 & R_- \\ R_+ & 0\end{pmatrix}$$

$$\text{with}\qquad R_\pm = \pm i(D_x(\partial_x\theta) + (\partial_x\theta)D_x)e^{\mp i\varphi} - \sqrt{\frac{\tau_y}{\tau_x}}\sin(\theta)(D_y e^{\mp i\varphi} + e^{\mp i\varphi}D_y)$$

$$\text{and}\qquad (\partial_x\theta) = -\frac{\sqrt{\tau_x}}{\sqrt{\tau_y}(1+\frac{\tau_x}{\tau_y}x^2)}\quad,\quad \sin(\theta) = \frac{1}{\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\,.$$

Therefore, we obtain

$$\left|\left\langle(1-\hat\chi)\psi\,,\,\mathcal{D}^0_{kin}(1-\hat\chi)\psi\right\rangle\right| \leq 4\max(\sqrt{\frac{\tau_x}{\tau_y}},\sqrt{\frac{\tau_y}{\tau_x}})\||D_q|(1-\hat\chi)\psi\|_{L^2}\|(1-\hat\chi)\psi\|_{L^2}\,.$$

and, owing to $\tau_x,\tau_y\in(0,1]$,

$$\left|\left\langle(1-\hat\chi)\psi\,,\,\varepsilon^{2+2\delta}\tau_x\tau_y\mathcal{D}^0_{kin}(1-\hat\chi)\psi\right\rangle\right| \leq 4\varepsilon^{1+2\delta}\|\sqrt{\tau_x\tau_y}|\varepsilon D_q|(1-\hat\chi)\psi\|_{L^2}\|(1-\hat\chi)\psi\|_{L^2}\,.$$

This ends the proof of (3.5).
For (3.6) it suffices to replace ψ in (3.5) by $\hat U^*\psi$ with a cut-off function χ_1 such that $\chi\prec\chi_1$ and then to use $(1-\chi_1(|\varepsilon D_q|^2_\tau))\hat U^*\chi(|\varepsilon D_q|^2_\tau)\in Op\mathcal{N}_{u,g_\tau}$, with uniform seminorm estimates w.r.t $\tau\in(0,1]^2$ and $\delta\in(0,\delta_0]$. The L^2-norm of the corresponding additional error term is $\mathcal{O}(\varepsilon^N)$, for any N, and one fixes $N\geq 2+4\delta_0$. □

3.3 Control of the nonlinear term

In this subsection, we estimate the effect of the operator $\hat U=U(q,\varepsilon D_q,\tau,\varepsilon)$ belonging to $OpS_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))$ on the nonlinear term $\int_{\mathbb{R}^2}|\psi|^4\,dxdy$.

Proposition 3.3. *Let $\hat U$ be the unitary operator introduced in Theorem 2.1. The inequalities*

$$\int|\psi(x,y)|^4\,dxdy\geq(1-C\varepsilon^{1+2\delta})\int|(\hat U\psi)(x,y)|^4\,dxdy\,, \tag{3.11}$$

and

$$\int|\psi(x,y)|^4\,dxdy\geq(1-C\varepsilon^{1+2\delta})\int|(\hat U^*\psi)(x,y)|^4\,dxdy \tag{3.12}$$

hold for any $\psi\in L^4(\mathbb{R}^2;\mathbb{C}^2)$.

Proof: For ψ_1 and ψ_2 belonging to $L^4(\mathbb{R}^2;\mathbb{C}^2)$, the local relations

$$\begin{aligned}
|\psi_1|^4(q) &= \left(|\psi_2(q)|^2+2\,\mathbb{R}\mathrm{e}\langle\psi_2(q)\,,\,(\psi_1-\psi_2)(q)\rangle+|\psi_1(q)|^2\right)^2\\
&= |\psi_2(q)|^4+2|\psi_2|^2|\psi_1-\psi_2|^2+4\left(\mathbb{R}\mathrm{e}\langle\psi_2\,,\,\psi_1-\psi_2\rangle+\frac{1}{2}|\psi_1-\psi_2|^2\right)^2\\
&\qquad +4|\psi_2|^2\,\mathbb{R}\mathrm{e}\langle\psi_2\,,\,(\psi_1-\psi_2)\rangle\\
&\geq |\psi_2(q)|^4-4|\psi_2(q)|^3|\psi_1(q)-\psi_2(q)|\,,
\end{aligned}$$

is integrated w.r.t $q=(x,y)\in\mathbb{R}^2$, with Hölder inequality, into

$$\|\psi_1\|^4_{L^4}=\int|\psi_1|^4\,dxdy\geq\|\psi_2\|^4_{L^4}-4\|\psi_2\|^3_{L^4}\|\psi_1-\psi_2\|_{L^4}\,.$$

With $\psi_1=\hat U_0\psi=u_0(q)\psi$, $|\psi(q)|^2=|\psi_1(q)|^2$ for all $q\in\mathbb{R}^2$, and $\psi_2=\hat U\psi=\psi_1+(\hat U-\hat U_0)\psi$, we obtain

$$\|\psi\|^4_{L^4}=\|\psi_1\|^4_{L^4}\geq\|\hat U\psi\|^4_{L^4}-4\|\hat U\psi\|^3_{L^4}\|(\hat U-\hat U_0)\psi\|_{L^4}\,.$$

The operator $\hat{U} - \hat{U}_0$ equals $\varepsilon^{1+2\delta} r(q, \varepsilon D_q, \tau, \varepsilon)$ with $r \in S_u(\frac{1}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle\langle\sqrt{\tau_x\tau_y}p\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$, where we recall $g_\tau = \frac{\frac{\tau_x}{\tau_y}dx^2}{\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle^2} + \frac{\tau_y}{\tau_x}dy^2 + \frac{\tau_x\tau_y dp^2}{\langle\sqrt{\tau_x\tau_y}p\rangle^2}$. After introducing the isometric transform on $L^4(\mathbb{R}^2;\mathbb{C}^2)$

$$(T_{\varepsilon,\tau}\varphi)(x,y) = \varepsilon\sqrt{\tau_x\tau_y}\varphi(\varepsilon\sqrt{\tau_x\tau_y}x, \varepsilon\sqrt{\tau_x\tau_y}y)\,,$$

the difference $\hat{U} - \hat{U}_0$ becomes

$$\hat{U} - \hat{U}_0 = \varepsilon^{1+\delta} T^{-1}_{\varepsilon,\tau} r_1(\varepsilon\tau_x x, \varepsilon\tau_y y, D_x, D_y, \varepsilon, \tau) T_{\varepsilon,\tau}$$

with r_1 uniformly bounded in $S(1, \frac{d(\tau_x x)^2}{\langle\tau_x x\rangle^2} + d(\tau_y y)^2 + \frac{dp^2}{\langle p\rangle^2}; \mathcal{M}_2(\mathbb{C}))$. A fortiori, the symbol $r_1(\varepsilon\tau_x x, \varepsilon\tau_y y, \xi, \eta; \varepsilon, \tau)$ is uniformly bounded in $S(1, dq^2 + \frac{dp^2}{\langle p\rangle^2})$ and the Lemma 3.4 below provides the uniform bound

$$\|\hat{U} - \hat{U}_0\|_{\mathcal{L}(L^4)} \leq C_0\varepsilon^{1+\delta}\,.$$

We have proved

$$\|\psi\|^4_{L^4} \geq \|\hat{U}\psi\|^4_{L^4} - C\varepsilon^{1+2\delta}\|\hat{U}\psi\|^3_{L^4}\|\psi\|_{L^4}\,.$$

which implies (3.11). The second inequality (3.12) is proved similarly with $\hat{U}^* = \hat{U}_0^* + (\hat{U}^* - \hat{U}_0^*)$. □

The result below is a particular case of the general L^p bound, $1 < p < \infty$, for pseudodifferential operator in $OpS(1, dq^2 + \frac{dp^2}{\langle p\rangle^2}; \mathcal{L}(\mathcal{H}_1;\mathcal{H}_2))$, $\mathcal{H}_i$ Hilbert spaces, stated in [Tay]-Proposition 5.7 and relying on Calderon-Zygmund analysis of singular integral operators.

Lemma 3.4. *For any $p \in (1, +\infty)$, there exists a seminorm $\mathbf{n}$ on $S(1, dq^2 + \frac{dp^2}{\langle p\rangle^2}; \mathcal{M}_2(\mathbb{C}))$ such that*

$$\forall a \in S(1, dq^2 + \frac{dp^2}{\langle p\rangle^2}; \mathcal{M}_2(\mathbb{C})), \quad \|a(q, D_q)\|_{\mathcal{L}(L^p)} \leq \mathbf{n}(a)\,.$$

Proof: The Proposition 5.7 of [Tay] says that for any $a \in S(1, dq^2 + \frac{dp^2}{\langle p\rangle}^2; \mathcal{M}_2(\mathbb{C}))$, the operator $a(q, D_q)$ is bounded on $L^p(\mathbb{R}^2;\mathbb{C}^2)$. It is not difficult to follow the control of the constants in the previous pages of [Tay] in order to check that $\|a(q, D_q)\|_{\mathcal{L}(L^p)}$ is estimated by a seminorm of a. More efficiently, one can refer to the uniform boundedness principle, because $S(1, dq^2 + \frac{dp^2}{\langle p\rangle^2}; \mathcal{M}_2(\mathbb{C})) \ni a \to a(q, D_q) \in \mathcal{L}(L^p(\mathbb{R}^2;\mathbb{C}^2))$ is a linear operator from a Fréchet space to a Banach space. □

4 Reduced minimization problems

In this section, we assume that potential $V_{\varepsilon,\tau}$ satisfies (1.13)-(1.14). After the first paragraph of this section and in the rest of the paper, we focus on the case $\tau_y = 1$, $\tau_x \to 0$ (and $\varepsilon \to 0$). Two reduced problems have to be considered: 1) the one obtained as $\varepsilon \to 0$ and τ_x is fixed; 2) the one derived from the previous one as $\tau_x \to 0$ and which is

parameterized only by (G, ℓ_V). The linear part of this latter reduced problem is a purely quadratic Schrödinger Hamiltonian (with a constant magnetic field), from which many a priori information can be obtained. This section is divided into three parts. First we specify the potential $V_{\varepsilon,\tau}$ and check our main assumptions for the general theory. Then we review some properties of the reduced Gross-Pitaevskii problem parametrized by (G, ℓ_V). Finally we make the comparison with the reduced Gross-Pitaevskii problem parametrized by (G, ℓ_V, τ_x) as $\tau_x \to 0$ and deduce properties which will be necessary for the study of the complete minimization problem.

4.1 Reduced minimization problems

Lemma 4.1. *The potential $V_{\varepsilon,\tau}$ defined by (1.13)-(1.14) belongs to $S_u\left(\langle\sqrt{\frac{\tau_x}{\tau_y}}x\rangle, g_{q,\tau}\right)$ with the metric $g_{q,\tau} = \frac{\tau_x dx^2}{\tau_y\langle 1+\frac{\tau_x}{\tau_y}x^2\rangle} + \frac{\tau_y}{\tau_x}dy^2$.*

Proof: After the change of variable $(x', y') = (\sqrt{\frac{\tau_x}{\tau_y}}x, \sqrt{\frac{\tau_y}{\tau_x}}y)$, it is equivalent to check

$$\frac{\varepsilon^{2+2\delta}}{\ell_V^2}v(\tau_y x, \tau_x y) + \sqrt{1+x^2} - \varepsilon^{2+2\delta}\left[\frac{\tau_x^2}{(1+x^2)^2} + \frac{\tau_y^2}{1+x^2}\right] \in S_u(\langle x\rangle, \frac{dx^2}{\langle x\rangle^2} + dy^2)\,.$$

It is done if $v(\tau_y x, \tau_x y) \in S_u(\langle x\rangle, \frac{dx^2}{\langle x\rangle^2} + dy^2)$. We know $v \in S(1, \frac{dx^2+dy^2}{1+x^2+y^2})$. Hence for all $(\alpha, \beta) \in \mathbb{N}^2$ there exists $C_{\alpha,\beta} > 0$ such that

$$\begin{aligned}\forall \tau \in (0,1]^2, \forall x, y \in \mathbb{R}^2, \quad |\partial_x^\alpha \partial_y^\beta \left(v(\tau_y x, \tau_x y)\right)| &\leq C_{\alpha,\beta}\frac{\tau_x^\beta \tau_y^\alpha}{(1+\tau_y^2 x^2 + \tau_x^2 y^2)^{-\frac{\alpha+\beta}{2}}} \\ &\leq C_{\alpha,\beta}\frac{1}{(\frac{1}{\tau_y^2}+x^2)^{\alpha/2}} \leq C_{\alpha,\beta}\langle x\rangle^{1-|\alpha|}\,,\end{aligned}$$

which is what we seek. □

If the error terms of Proposition 3.2 and Proposition 3.3 are assumed to be negligible, the energy $\mathcal{E}_\varepsilon(\psi)$ of a state $\psi = \hat{U}^* \begin{pmatrix} 0 \\ a_- \end{pmatrix}$ is close to

$$\begin{aligned} &\varepsilon^{2+2\delta}\tau_x\tau_y\langle a_-\,, \hat{H}_- a_-\rangle + \frac{G_{\varepsilon,\tau}}{2}\int |a_-|^4\, dxdy = \varepsilon^{2+2\delta}\tau_x\tau_y\mathcal{E}_\tau(a_-)\,, \\ \text{with} \quad &\mathcal{E}_\tau(a_-) = \langle a_-\,, \hat{H}_- a_-\rangle + \frac{G}{2}\int |a_-|^4\, dxdy\,, \qquad (4.1) \\ &\hat{H}_- = -\partial_x^2 - \left(\partial_y - i\frac{x}{2\sqrt{1+\frac{\tau_x}{\tau_y}x^2}}\right)^2 + \frac{1}{\ell_V^2\tau_x\tau_y}v(\sqrt{\tau_x\tau_y}x, \sqrt{\tau_x\tau_y}y)\,. \end{aligned}$$

with the potential v chosen from (1.14).
When $\frac{\tau_x}{\tau_y}$ and $\tau_x\tau_y$ are small, in particular in the regime $\tau_x \ll 1$ and $\tau_y = 1$ that we shall consider, this energy is well approximated by

$$\mathcal{E}_H(\varphi) = \langle \varphi\,,\, \left[-\partial_x^2 - (\partial_y - \frac{ix}{2})^2 + \frac{x^2+y^2}{\ell_V^2}\right]\varphi\rangle + \frac{G}{2}\int |\varphi|^4\,, \tag{4.2}$$

as this will be checked and specified in the next paragraph. Although more general asymptotics could be considered, we concentrate from now on the regime $\tau_x \ll 1$, $\tau_y = 1$. The parameters ℓ_V and G are assumed to be fixed as $\tau_x \to 0$.

In order to prove that the ground states of $\mathcal{E}_H$ and $\mathcal{E}_\varepsilon$ are close, we need to have good estimates on the energy $\mathcal{E}_H$.

4.2 Properties of the harmonic approximation

Let us recall that the energy functional $\mathcal{E}_H$ does not any more depend on τ_x and is parameterized only by (G, ℓ_V). Let us start with its properties. We introduce the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are given by

$$\mathcal{H}_s = \left\{ u \in L^2(\mathbb{R}^2) \sum_{|\alpha|+|\beta|\le s} \|q^\alpha D_q^\beta u\|_{L^2} < +\infty \right\}, \quad s = 1, 2, (q = (x, y)) \tag{4.3}$$

endowed with the norm $\|u\|^2_{\mathcal{H}_s} = \sum_{|\alpha|+|\beta|\le s} \|q^\alpha D_q^\beta u\|^2_{L^2}$. For a compact set K of $\mathcal{H}_s$ and for $u \in \mathcal{H}_s$, the distance $d_s(u, K)$ follows the usual definition $\min_{v\in K} \|u - v\|_{\mathcal{H}_s}$. The self-ajoint operator associated with the linear part of $\mathcal{E}_H$ is denoted by

$$H_{\ell_V} = -\partial_x^2 - (\partial_y - \frac{ix}{2})^2 + \frac{x^2+y^2}{\ell_V^2}\,, \quad (0 < \ell_V < +\infty)\,.$$

Its domain is $\mathcal{H}_2$ while its form domain is $\mathcal{H}_1$. Note also the compact embeddings $\mathcal{H}_2 \subset\subset \mathcal{H}_1 \subset\subset L^2 \cap L^4$. Following the general scheme presented in [HiPr, Sjo], its spectrum equals

$$\sigma(H_{\ell_V}) = \left\{(1+2n_+)r_+ + (1+2n_-)r_-\,, \quad (n_+, n_-) \in \mathbb{N}^2\right\}$$

with $r_\pm = \frac{1}{2\sqrt{2}}\sqrt{1 + \frac{8}{\ell_V^2} \pm \sqrt{1 + \frac{4}{\ell_V^2}}}$.

Proposition 4.2. *The functional $\mathcal{E}_H$ admits minima on $\{u \in \mathcal{H}_1, \|u\|_{L^2} = 1\}$, with a minimum value $\mathcal{E}_{H,min}$ satisfying*

$$\mathcal{E}_{H,min} \ge r_+ + r_- \ge \frac{\sqrt{2}}{2}\,.$$

The set of minimizers Argmin $\mathcal{E}_H$ *is a bounded subset of $\mathcal{H}_2$ and therefore a compact subset of $\{u \in \mathcal{H}_1, \|u\|_{L^2} = 1\}$. Morever for any $\varphi \in$ Argmin $\mathcal{E}_H$, φ is an eigenvector of*

$H_0 + G|\varphi|^2$. *Finally there exist two constants* $C = C_{\ell_V,G} > 0$ *and* $\nu = \nu_{\ell_V,G} \in (0, 1/2]$ *such that*

$$\forall u \in \mathcal{H}_1\,, (\|u\|_{L^2} = 1, \mathcal{E}_H(u) \leq \mathcal{E}_{H,min} + 1) \Rightarrow (d_{\mathcal{H}_1}(u, \text{Argmin } \mathcal{E}_H) \leq C(\mathcal{E}_H(u) - \mathcal{E}_{H,min})^\nu)\ . \tag{4.4}$$

Proof: On $\mathcal{H}_1$ and $\mathcal{H}_2$, the scalar products

$$\begin{aligned}\langle u\,,\, v\rangle_{1,\ell_V} &= \langle u\,,\, H_{\ell_V} v\rangle_{L^2} \\ \langle u\,,\, v\rangle_{2,\ell_V} &= \langle H_{\ell_V} u\,,\, H_{\ell_V} v\rangle_{L^2}\end{aligned}$$

provide norms $\|u\|_{k,\ell_V}$, $k = 1, 2$, respectively equivalent to $\|u\|_{\mathcal{H}_k}$. In this proof, all the "uniform" estimates are actually parameterized by (G, ℓ_V) . The nonlinearity $\frac{G}{2}\int |u|^4(x)\, dx$ as well as the constraint $\|u\|_{L^2} = 1$ are continuous functions on $L^2 \cap L^4$ while the quadratic part of $\mathcal{E}_H(u)$ is simply $\|u\|^2_{1,\ell_V}$ with $\mathcal{E}_H(u) \geq \|u\|^2_{1,\ell_V} \geq r_+ + r_- \geq \frac{\sqrt{2}}{2}$. The compact embedding $\mathcal{H}_1 \subset\subset L^2 \cap L^4$ thus implies that the infimum $\inf_{u\in\mathcal{H}_1\,,\,\|u\|_{L^2}=1} \mathcal{E}_H(u)$ is achieved. A minimizer $\varphi \in \text{Argmin } \mathcal{E}_H$ solves in a distributional sense the Euler-Lagrange equation

$$H_{\ell_V}\varphi + G|\varphi|^2 u = \lambda_\varphi \varphi$$

where λ_φ is the Lagrange multiplier associated with the constraint $\|\varphi\|_{L^2} = 1$. By taking the scalar product with φ, one obtains the bounds for λ_φ:

$$\mathcal{E}_{H,min} \leq \lambda_\varphi \leq 2\mathcal{E}_{H,min}\,.$$

Since $\mathcal{H}_1$ is also (compactly) embedded in $L^6(\mathbb{R}^2)$, the equation

$$H_{\ell_V}\varphi = -G|\varphi|^2\varphi + \lambda_\varphi\varphi$$

ensures that $\|\varphi\|_{2,\ell_V}$ is uniformly bounded on Argmin $\mathcal{E}_H$. Therefore Argmin $\mathcal{E}_H$ is a bounded susbset of $\mathcal{H}_2$ and a compact subset of $\mathcal{H}_1$. In a $\mathcal{H}_1$-neighborhood of $\varphi \in$ Argmin $\mathcal{E}_H$ ($\|\varphi\|_{L^2} = 1$), the L^2-sphere $\{u \in \mathcal{H}_1,\, \|u\|_{L^2} = 1\}$ can be parametrized by

$$u = (1 - \|v\|^2_{L^2})\varphi + v\,, \quad \langle\varphi\,,\, v\rangle_{L^2} = 0\,.$$

Notice also that the potential $G|\varphi|^2$ is a relatively compact perturbation of H_{ℓ_V}, so that $H_{\ell_V} + G|\varphi|^2$ is a self-adjoint operator in $L^2(\mathbb{R}^2)$ with domain $\mathcal{H}_2$ and with a compact resolvent. With $\langle\varphi\,,\, v\rangle_{1,\ell_V} = -G\int_{\mathbb{R}^2} |\varphi|^2\overline{\varphi}v\, dx$ for φ is an eigenvector of $H_{\ell_V} + G|\varphi|^2$ and $v \perp \varphi$, the energy $\mathcal{E}_H(u)$ becomes

$$\begin{aligned}\mathcal{E}_H((1 - \|v\|^2_{L^2})\varphi + v) &= \|v\|^2_{1,\ell_V} + (1 - \|v\|_{L^2})\langle\varphi\,,\,\varphi\rangle_{1,\ell_V} \\ &\quad -2G(1 - \|v\|^2_{L^2})\,\mathbb{R}\text{e}\int_{\mathbb{R}^2} |\varphi|^2\overline{\varphi}v\, dx + \frac{G}{2}\int_{\mathbb{R}^2} |(1 - \|\varphi\|^2_{L^2})\varphi + v|^4\, dx \\ &= \|v\|^2_{1,\ell_V} + F_\varphi(v)\,,\end{aligned}$$

where v lies in the closed subset $\mathcal{H}_{1,\varphi} = \{v \in \mathcal{H}_1, \langle\varphi\,, v\rangle_{L^2} = 0\}$ of $\mathcal{H}_1$ and $F_\varphi(v)$ is the composition of the compact embedding $\mathcal{H}_1 \to L^2 \cap L^4$ with a real analytic, real-valued,

functional on $L^2 \cap L^4$. Hence on $\mathcal{H}_{1,\varphi}$ endowed with the scalar product $\langle\ ,\ \rangle_{1,\ell_V}$, the Hessian of $\mathcal{E}_H((1-\|v\|_{L^2}^2)\varphi + v)$ equals $\mathrm{Id} + D^2F_{\ell_V}(0)$, with $D^2F_{\ell_V}(0)$ compact (and self-adjoint). We can apply the Lojasiewicz-Simon inequality which says that there exist two constants $C_\varphi > 0$, $\nu_\varphi \in (0, 1/2]$, such that

$$\|v\|_{1,\ell_V} \leq C_\varphi \left(\mathcal{E}_H((1-\|v\|_{L^2}^2)\varphi + v) - \mathcal{E}_{H,min}\right)^{\nu_\varphi} .$$

Since the set Argmin $\mathcal{E}_H$ is a compact subset of $\mathcal{H}_1$, it can be covered by a finite number of neighborhoods of $\varphi_i \in$ Argmin $\mathcal{E}_H$, $1 \leq i \leq N$, where a Lojasiewicz-Simon inequality holds. Take

$$\nu_{\ell_V,G} = \min_{1\leq i\leq N} \nu_{\varphi_i} \quad \text{and} \quad C_{\ell_V,G} = 2 \max_{1\leq i\leq N} C_{\varphi_i} .$$

□

Remark 4.3. *The Lojasiewicz inequality is a classical result of real algebraic geometry (see a.e. [Loj, BCR]) proved by Lojasiewicz after Tarski-Seidenberg Theorem. It is usually written as* $|\nabla f(x)| \leq C f(x)^\nu$ *with* $\nu \in (0, 1]$ *for a real analytic function of* x *lying around* x_0 *with* $f(x_0) = 0$. *The variational form is a variant of it. It was extended to the infinite dimensional case with applications to PDE's by L. Simon in [Sim]. We refer the reader also to [Chi, HaJe, Hua] and [BDLM] for recent texts and references concerned with the infinite dimensional case or the extension with o-minimal structures.*

The nonlinear Euler-Lagrange equation is usually studied after linearization via the Liapunov-Schmidt process. Here using some coordinate representation of the constraint submanifold, especially when it is a sphere for a simple norm, allows to use directly the standard result for the minimization of real analytic functionals.

When the minimization problem is non degenerate at every $\varphi \in$ Argmin $\mathcal{E}_H$, *i.e. in the present case when the kernel of* $\mathrm{Id} + D^2F_\varphi(0)$ *is rectricted to* $\{0\}$, *the compact set* Argmin $\mathcal{E}_H$ *is made of a finite number of point. When* ℓ_V *is fixed so that* r_+ *and* r_- *are rationnally independent, the spectrum of* H_{ℓ_V} *is made of simple eigenvalues and when* G *is small enough,* $G < G_{\ell_V}$, *the non degeneracy assumption is satisfied via a perturbation argument from the case* $G = 0$. *For large* G, *we can only say that the set of* $(G, \ell_V) \in (0, +\infty)^2$ *such that all the minima are non degenerate,* $\nu_{\ell_V,G} = 1/2$, *is a subanalytic subset of* $\mathbb{R}^2$. *In our case with a linear part* H_{ℓ_V} *which is a complex operator with no rotational symetry, no standard methods like in [AJR] allow to reduce the minimization problem to some radial nonlinear ODE.*

From the information given by the Lowest-Landau-Level reduction, when G *and* ℓ_V *are large, the supposed hexagonal symmetry, after removing some trivial rotational invariance, of the problem (see [ABN, Nie]) suggests that there are presumably several minimizers.*

A change of variable $\varphi(x, y)e^{-ixy/4} = \alpha u(\alpha x, \alpha y)$ with $\alpha^2 = 1/(\ell_V\sqrt{G})$ leads to

$$\mathcal{E}_H(\varphi) = \frac{1}{\ell_V\sqrt{G}}\tilde{\mathcal{E}}_H(u) = \frac{1}{\ell_V\sqrt{G}} \int |(\nabla - \frac{i}{2}\ell_V\sqrt{G}e_z \times r)u|^2 + G(r^2|u|^2 + \frac{1}{2}|u|^4). \quad (4.5)$$

This implies that $\ell_V\sqrt{G}$ is equivalent to a rotation value. We have the following results from the literature

- when $\ell_V\sqrt{G}$ is small and G is large, the minimizer is unique up to rotation and vortex-free [AJR]: namely $u(x,y) = f(r)e^{ic}$ for some real number c, where f does not vanish. If $\ell_V\sqrt{G} = 0$, this is an adaptation of a result of [BrOs]. When $\ell_V\sqrt{G}$ is non zero, this requires refined estimates for the jacobian.

- when $\ell_V\sqrt{G}$ is large, then vortices are expected in the system and this can be analyzed in details in the LLL regime (lowest Landau level) if additionally $\sqrt{G}/\ell_V$ is small [AB, ABN]. More precisely, if $\sqrt{G}/\ell_V$ is small, then

$$\inf \tilde{\mathcal{E}}_H - \frac{1}{2} - \inf E_{LLL} = o\left(\frac{\sqrt{G}}{\ell_V}\right) \tag{4.6}$$

  where

$$E_{LLL}(u) = \int G(r^2|u|^2 + \frac{1}{2}|u|^4) \tag{4.7}$$

  for functions u such that $u(x,y)e^{\ell_V\sqrt{G}r^2/4}$ is a holomorphic function of $x+iy$. This space is called the LLL. If u is a ground state of $\tilde{\mathcal{E}}_H$ and w its projection onto the LLL, then $|u-w|$ tends to 0 in H^1 and $C^{0,\alpha}$ as $\sqrt{G}/\ell_V$ tends to 0. If additionally, $\ell_V\sqrt{G}$ is large, then one can estimate $\inf E_{LLL}$ [ABN] thanks to test functions with vortices and $\inf E_{LLL} = O\left(\frac{\sqrt{G}}{\ell_V}\right)$.

- if $\ell_V\sqrt{G}$ is large, and $\sqrt{G}/\ell_V$ is large, then this is a Thomas Fermi regime where the energy can be estimated as well [Aft] and is of order $\sqrt{G}/\ell_V$.

We complete the previous result with another comparison statement which will be useful in the sequel.

Proposition 4.4. *There exists $C = C_{\ell_V,G} > 0$ such that when $u \in \mathcal{H}_1$ satisfy $\mathcal{E}_H(u) \leq \mathcal{E}_{H,min} + 1$, $\|u\|_{L^2} = 1$, and solves*

$$H_{\ell_V,G}u + G|u|^2u = \lambda_u u + r$$

with $\lambda_u \in \mathbb{R}$ and $r \in L^2$, then

- *$u \in \mathcal{H}_2$;*

- *there exists $u_0 \in$ Argmin $\mathcal{E}_H$, with Lagrange multiplier λ_{u_0}, such that*

$$|\lambda_u - \lambda_{u_0}| + \|u - u_0\|_{\mathcal{H}_2} \leq C\left(\|r\|_{L^2} + (\mathcal{E}_H(u) - \mathcal{E}_{H,min})^\nu\right) ,$$

  where $\nu = \nu_{\ell_V,G} \in (0, \frac{1}{2}]$ is the exponent given in Proposition 4.2.

Proof: Since Argmin $\mathcal{E}_H$ is compact, Proposition 4.2 already provides $u_0 \in$ Argmin $\mathcal{E}_H$ such that

$$\|u - u_0\|_{\mathcal{H}_1} \leq C(\mathcal{E}_H(u) - \mathcal{E}_{H,min})^\nu .$$

Taking the difference of the equation for u and the Euler-Lagrange equation for u_0, we obtain

$$H_{\ell_V,G}(u-u_0) = (\lambda_u - \lambda_{u_0})\, u_0 + \lambda_u(u-u_0) + G(|u_0|^2u_0 - |u|^2u) + r\,.$$

Taking the scalar product with u_0, with

$$\||u_0|^2u_0 - |u|^2u\|_{L^2} \leq C\left(\mathcal{E}_H(u_0) + \mathcal{E}_H(u)\right)\|u-u_0\|_{\mathcal{H}_1} \leq C'\left(\mathcal{E}_H(u) - \mathcal{E}_{H,min}\right)^\nu$$

implies

$$|\lambda_u - \lambda_{u_0}| \leq C''\left(\|r\|_{L^2} + \left(\mathcal{E}_H(u) - \mathcal{E}_{H,min}\right)^\nu\right)\,.$$

Using the ellipticity of $H_{\ell_V,G}$ and the equivalence of the norms $\|\varphi\|_{\mathcal{H}_2}$ and $\|H_{\ell_V,G}\varphi\|_{L^2}$ ends the proof. □

4.3 Comparison of the two reduced minimization problems

In the regime $\tau_y = 1$ and $\tau_x \to 0$, while $\ell_V > 0$ and $G > 0$ are fixed, we compare the two minimization problems for the energies $\mathcal{E}_\tau$ and $\mathcal{E}_H$ defined in (4.1)-(4.2). We start with the next Lemma which is a simple application of the so called IMS localization formula (see a.e. [CFKS]). We shall use the functional spaces $\mathcal{H}_s$ defined by (4.3) associated with $\mathcal{E}_H$ as well as the standard Sobolev spaces $H^s(\mathbb{R}^2)$ associated with $\mathcal{E}_\tau$, with $s = 1, 2$ and $\mathcal{H}_s \subset H^s(\mathbb{R}^2)$.

Lemma 4.5. *Let $\chi_1, \chi_2 \in \mathcal{C}_b^\infty(\mathbb{R}^2)$ satisfy $\chi_1^2 + \chi_2^2 = 1$, $\operatorname{supp}\chi_1 \subset \{x^2+y^2<1\}$ and take $\alpha \in (0, \frac{1}{2}]$. Then the following identities*

$$\forall u \in \mathcal{H}_1\,, \quad \mathcal{E}_H(u) = \mathcal{E}_H(\chi_1(\tau_x^\alpha .)u) + \mathcal{E}_H(\chi_2(\tau_x^\alpha .)u) - \tau_x^{2\alpha}\sum_{j=1}^2 \int_{\mathbb{R}^2} |(\nabla\chi_j)(\tau_x^\alpha .)|^2|u|^2 + G\int_{\mathbb{R}^2}(\chi_1^2\chi_2^2)(\tau_x^\alpha .)|u|^4\,, \quad (4.8)$$

with the same formula for $\mathcal{E}_\tau(u)$ when $u \in H^1(\mathbb{R}^2)$, and

$$\forall u \in H^1(\mathbb{R}^2)\,, \quad \mathcal{E}_\tau(u) = \mathcal{E}_H(\chi_1(\tau_x^\alpha .)u) + \mathcal{E}_\tau(\chi_2(\tau_x^\alpha .)u) - \tau_x^{2\alpha}\sum_{j=1}^2 \int_{\mathbb{R}^2} |(\nabla\chi_j)(\tau_x^\alpha .)|^2|u|^2 + G\int_{\mathbb{R}^2}(\chi_1^2\chi_2^2)(\tau_x^\alpha .)|u|^4 + R(u)\,, \quad (4.9)$$

hold with

$$|R(u)| \leq \frac{1}{4}\left(\mathcal{E}_\tau(\chi_1(\tau_x^\alpha .)u)^{1/2} + \mathcal{E}_H(\chi_1(\tau_x^\alpha .)u)^{1/2}\right)\|u\|_{L^2}\tau_x^{1-3\alpha}\,.$$

Proof: The first identity is a direct application of the IMS localization formula (see a.e. [CFKS]) which comes from the identity

$$P\chi^2 P - \chi P^2 \chi = [P,\chi]^2 - \frac{1}{2}\left[\chi^2, P\right] P - \frac{1}{2} P\left[P, \chi^2\right]$$

when P is a differential operator of order ≤ 1 and χ is a $\mathcal{C}^\infty$ function. Simply combine it with the identity

$$|u|^4 = |\chi_1(\tau_x^\alpha .)u|^4 + |\chi_2(\tau_x^\alpha .)u|^4 + 2\chi_1^2\chi_2^2(\tau_x^\alpha .)|u|^4 .$$

Using the same argument for $\mathcal{E}_\tau$ provides the same identity after replacing $\mathcal{E}_H$ with $\mathcal{E}_\tau$, and it suffices to compare $\mathcal{E}_\tau(\chi_1(\tau_x^\alpha .)u)$ with $\mathcal{E}_H(\chi_1(\tau_x^\alpha .)u)$. The definition (1.14) of the potential v and the condition $\alpha \leq \frac{1}{2}$ imply

$$v(\tau_x^{1/2}.) = \tau_x(x^2+y^2) \quad \text{on } \operatorname{supp}\chi_1(\tau_x^\alpha .) .$$

Therefore, we obtain, by setting $u_\tau = \chi_1(\tau_x^\alpha .)u$,

$$\begin{aligned}
&|\mathcal{E}_\tau(\chi_1(\tau_x^\alpha .)u) - \mathcal{E}_H(\chi_1(\tau_x^\alpha .)u)| = \left| \int_{\mathbb{R}^2} |(\partial_y - i\frac{x}{2\sqrt{1+\tau_x x^2}})u_\tau|^2 - |(\partial_y - i\frac{x}{2})u_\tau|^2 \right| \\
&\leq \left(\|(\partial_y - i\frac{x}{2\sqrt{1+\tau_x x^2}})u_\tau\|_{L^2} + \|(\partial_y - i\frac{x}{2})u_\tau\|_{L^2} \right) \|\frac{\tau_x x^3/2}{1+\sqrt{1+\tau_x x^2}}\chi_1(\tau_x^\alpha .)u\|_{L^2} \\
&\leq \frac{1}{4}\left(\mathcal{E}_\tau(\chi_1(\tau_x^\alpha .)u)^{1/2} + \mathcal{E}_H(\chi_1(\tau_x^\alpha .)u)^{1/2}\right) \|u\|_{L^2}\tau_x^{1-3\alpha} .
\end{aligned}$$

□

Proposition 4.6. *For any given $(\ell_V, G) \in (0,+\infty)^2$, there exists $\tau_{\ell_V,G} > 0$ such that the following properties hold when $\tau_x \leq \tau_{\ell_V,G}$.*

- *The minimization problem*

$$\inf_{u\in H^1(\mathbb{R}^2)\,,\,\|u\|_{L^2}=1} \mathcal{E}_\tau(u)$$

  admits a solution $u \in H^1(\mathbb{R}^2)$.

- *A solution $u \in H^1(\mathbb{R}^2)$ to the above minimization problem, solves an Euler-Lagrange equation*

$$\left[-\partial_x^2 - (\partial_y - \frac{i}{2}\frac{x}{\sqrt{1+\tau_x x^2}})^2 + \frac{v(\tau_x^{1/2}.)}{\ell_V^2\tau_x} + G|u|^2\right] u = \lambda_u u$$

  with $0 \leq \lambda_u \leq 2\mathcal{E}_{\tau,min}$ and belongs to $H^2(\mathbb{R}^2)$.

- *Moreover the minimum value* $\mathcal{E}_{\tau,min} = \min_{u\in H^1(\mathbb{R}^2)\,,\,\|u\|_{L^2}=1} \mathcal{E}_\tau(u)$ *satisfies the estimate*

$$|\mathcal{E}_{\tau,min} - \mathcal{E}_{H,min}| \leq C_{\ell_V,G}\tau_x^{2/3}\,.$$

- *For* $u \in \text{Argmin}\ \mathcal{E}_\tau$ *and any pairs* $\chi = (\chi_1,\chi_2)$ *in* $\mathcal{C}_b^\infty(\mathbb{R}^2)^2$ *such that* $\chi_1^2+\chi_2^2 = 1$ *with* $\text{supp}\,\chi_1 \subset \{x^2+y^2<1\}$ *and* $\chi_1 \equiv 1$ *in* $\{x^2+y^2 \leq 1/2\}$*, the functions* $\chi_j(\tau_x^{1/9}.)u$*,* $j=1,2$*, satisfy*

$$\|\chi_2(\tau_x^{1/9}.)u\|_{L^2}^2 \leq C_{\chi,\ell_V,G}\tau_x^{2/3} \tag{4.10}$$
$$\mathcal{E}_\tau(\chi_1(\tau_x^{1/9}.)u) \leq \mathcal{E}_{\tau,min} + C_{\chi,\ell_V,G}\tau_x^{2/3} \tag{4.11}$$
$$d_{\mathcal{H}_2}(\chi_1(\tau_x^{1/9}.)u, \text{Argmin}\ \mathcal{E}_H) \leq C_{\chi,\ell_V,G}\tau_x^{2\nu_{\ell_V,G}/3} \quad , \quad \nu_{\ell_V,G} \in (0,\frac{1}{2}]\,. \tag{4.12}$$

A constant $C_{a,b,c}$ *is a constant which is (or can be) fixed once* (a,b,c) *are given.*

Proof: Fix ℓ_V and G. We drop the indices ℓ_V, G in the constants. The exponent α will be fixed to the value $\frac{1}{9}$ within the proof.
First step, upper bound for $\inf\{\mathcal{E}_\tau(u)\,,\ u\in H^1(\mathbb{R}^2)\,,\,\|u\|_{L^2}=1\}$: Let $\chi = (\chi_1,\chi_2)$ and $\tilde\chi = (\tilde\chi_1,\tilde\chi_2)$ be two pairs as in our statement such that $\tilde\chi_1 \prec \chi_1$ according to Definition A.2. Take $u_0 \in \text{Argmin}\ \mathcal{E}_H \subset \mathcal{H}_1$. According to Proposition 4.2, it belongs to a bounded set to $\mathcal{H}_2$ so that $\||q|^2u_0\|_{L^2}$, with $q=(x,y)$, is uniformly bounded. Hence, $0 \notin \text{supp}\,\nabla\tilde\chi_j \cup \text{supp}\,\tilde\chi_1\tilde\chi_2$ implies

$$\int_{\mathbb{R}^2} |\nabla\tilde\chi_j(\tau_x^\alpha.)|^2|u_0|^2 = \mathcal{O}(\tau_x^{4\alpha}) \quad \text{while} \quad \int_{\mathbb{R}^2} \tilde\chi_1^2\tilde\chi_2^2(\tau_x^\alpha.)|u_0|^4 \geq 0\,.$$

Lemma 4.5 above with the pair $\tilde\chi$ and $\alpha \in (0,\frac{1}{2}]$ gives:

$$\mathcal{E}_{H,min} = \mathcal{E}_H(u_0) \geq \mathcal{E}_H(\tilde\chi_1(\tau_x^\alpha.)u_0) + \mathcal{E}_H(\tilde\chi_2(\tau_x^\alpha.)u_0) - C\tau_x^{6\alpha}\,.$$

On $\text{supp}\,\tilde\chi_2(\tau_x^\alpha.)$, the potential $\frac{v(\tau_x^{1/2}.)}{\ell_v^2\tau_x}$ is bounded from below by $\frac{1}{C'\tau_x^{2\alpha}}$. Thus we get

$$\mathcal{E}_{H,min}\left(\|\tilde\chi_1u_0\|_{L^2}^2 + \|\tilde\chi_2u_0\|_{L^2}^2\right) \geq \mathcal{E}_{H,min}\|\tilde\chi_1u_0\|_{L^2}^2 + \frac{1}{C'\tau_x^{2\alpha}}\|\tilde\chi_2u_0\|_{L^2}^2 - C\tau_x^{6\alpha}\,,$$

and finally

$$\|\tilde\chi_2u_0\|_{L^2}^2 \leq C''\tau_x^{8\alpha}\,, \quad \|\tilde\chi_1u_0\|_{L^2}^2 = 1+\mathcal{O}(\tau_x^{8\alpha})\,,$$

as soon as $\tau_x < (C'\mathcal{E}_{H,min})^{-1/2\alpha}$.

The function $u_1 = \|\tilde\chi_1u_0\|_{L^2}^{-1}\tilde\chi_1u_0$ is normalized with

$$\mathcal{E}_{H,min} \leq \mathcal{E}_H(u_1) \leq \mathcal{E}_{H,min} + \mathcal{O}(\tau_x^{6\alpha})\,,$$

and $\chi_1(\tau_x^\alpha .)u_1 = u_1$, $\chi_2(\tau_x^\alpha .)u_1 = 0$. Applying the second formula of Lemma 4.5 with, now, the pair χ, leads to

$$\mathcal{E}_\tau(u_1) = \mathcal{E}_H(u_1) + R(u_1) = \mathcal{E}_{H,min} + \mathcal{O}(\tau_x^{6\alpha}) + R(u_1)$$
$$\text{with} \qquad R(u_1) \leq \frac{1}{4}(\mathcal{E}_\tau(u_1)^{1/2} + (\mathcal{E}_{H,min} + \mathcal{O}(\tau_x^{6\alpha}))^{1/2})\tau_x^{1-3\alpha} .$$

With the estimate $\frac{\sqrt{2}}{2} \leq \mathcal{E}_{H,\min} \leq C$, we deduce

$$\mathcal{E}_\tau(u_1) = \mathcal{E}_{H,min} + \mathcal{O}(\tau_x^{6\alpha} + \tau_x^{1-3\alpha}) .$$

It's time to fix α to the value $\frac{1}{9}$ so that $\tau_x^{6\alpha} = \tau_x^{1-3\alpha} = \tau_x^{2/3}$ and

$$\inf_{u\in H^1(\mathbb{R}^2)} \mathcal{E}_\tau(u) \leq \mathcal{E}_\tau(u_1) \leq \mathcal{E}_{H,min} + \kappa\tau_x^{2/3} .$$

Second step - Existence of a minimizer: Once the function $u_1 \in H^1(\mathbb{R}^2)$ has been constructed as above, consider $\tau_x < \tau_0$ with $\mathcal{E}_{H,min} + \kappa\tau_0^{2/3} \leq \frac{1}{\ell_V^2\tau_0}$. The functional

$$\mathcal{E}_\tau(u) - \frac{1}{\ell_V^2\tau_x}\|u\|_{L^2}^2 ,$$

is the sum of a convex strongly continuous functional (and therefore weakly continuous) on $H^1(\mathbb{R}^2)$ and a negative functional

$$\langle u, \frac{1}{\ell_V^2\tau_x}\left[v(\tau_x^{1/2}.) - 1\right]_- u\rangle .$$

Due to the compact support of $v-1$, it is also continuous w.r.t the weak topology on $H^1(\mathbb{R}^2)$. Out a minimizing sequence $(u_n)_{n\in\mathbb{N}^*}$, extract a weakly converging subsequence in $H^1(\mathbb{R}^2)$. The weak limit, u_∞, satisfies

$$\mathcal{E}_\tau(u_\infty) - \frac{1}{\ell_V^2\tau_x}\|u_\infty\|_{L^2}^2 = \lim_{k\to\infty} \mathcal{E}_\tau(u_{n_k}) - \frac{1}{\ell_V^2\tau_x}\|u_{n_k}\|_{L^2}^2 .$$

with $\|u_\infty\|_{L^2} \leq 1$. The same convergence holds also for the energy $\mathcal{E}_\tau(u) - \frac{1}{2\ell_V^2\tau_x}\|u\|_{L^2}^2$, so that actually $\|u_\infty\|_{L^2} = \lim_{k\to\infty}\|u_{n_k}\|_{L^2} = 1$ and u_∞ realizes the minimum of $\mathcal{E}_\tau(u)$ under the constraint $\|u\|_{L^2} = 1$.
The Euler-Lagrange equation can thus be written, with the stated straightforward consequences.
Third step - a priori estimate for minimizers of $\mathcal{E}_\tau$: Let $u \in H^1(\mathbb{R}^2)$ satisfy $\|u\|_{L^2} = 1$ and $\mathcal{E}_\tau(u) = \mathcal{E}_{\tau,min} \leq \mathcal{E}_H + \kappa\tau_x^{2/3}$. Take two pairs $\tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2)$ and $\chi' = (\chi_1', \chi_2')$, like in our statement, and such that $\chi_1' \prec \tilde{\chi}_1$. The identities (4.9) for $\mathcal{E}_\tau$ and (4.9) in Lemma 4.5 provide

$$\mathcal{E}_{\tau,min} \geq \mathcal{E}_\tau(\tilde{\chi}_1(\tau_x^{1/9}.)u) - C_{\tilde{\chi}}\tau_x^{2/9}$$
$$\mathcal{E}_{\tau,min} \geq \mathcal{E}_H(\tilde{\chi}_1(\tau_x^{1/9}.)u) - C_{\tilde{\chi}}\tau_x^{2/9} - \frac{1}{4}\left[\mathcal{E}_\tau(\tilde{\chi}_1(\tau_x^{1/9}.)u)^{1/2} + \mathcal{E}_H(\tilde{\chi}_1(\tau_x^{1/9}.)u)^{1/2}\right]\tau_x^{2/3} .$$

The first line says

$$\mathcal{E}_\tau(\tilde\chi_1(\tau_x^{1/9}.)u) \le \mathcal{E}_{\tau,min} + C_{\tilde\chi}\tau_x^{2/9} \le \mathcal{E}_{H,min} + \kappa\tau_x^{2/3} + C_{\tilde\chi}\tau_x^{2/9} \le C'_{\tilde\chi}\,,$$

which combined with the second line provides the uniform estimate

$$\mathcal{E}_\tau(\tilde\chi_1(\tau_x^{1/9}.)u) \le C''_{\tilde\chi}\,.$$

Therefore $\tilde\chi_1(\tau_x^{1/9}.)u$ is uniformly bounded in $\mathcal{H}_1$ with respect to τ_x. Consider now the Euler-Lagrange equation

$$\left[-\partial_x^2 - (\partial_y - \frac{i}{2}\frac{x}{\sqrt{1+\tau_x x^2}})^2 + \frac{v(\tau_x^{1/2}.)}{\ell_V^2\tau_x} + G|u|^2\right] u = \lambda_u u$$

and write its local version for $u'_1 = \chi'_1(\tau_x^{2/9}.)u$ in the form

$$\left[-\partial_x^2 - (\partial_y - \frac{ix}{2})^2 + \frac{x^2+y^2}{\ell_V^2}\right] u'_1 = \lambda_u u'_1 - G|\tilde u_1|^2 u'_1 + f_u^1 + f_u^2 \qquad (4.13)$$

$$\text{with} \qquad f_u^1 = -ix(\frac{1}{\sqrt{1+\tau_x x^2}} - 1)\chi'(\tau_x^{1/9}.)\partial_y\tilde u_1 - \frac{x^2}{4}\frac{\tau_x x^2}{1+\tau_x x^2}u'_1$$

$$\text{and} \qquad f_u^2 = -2\tau_x^{1/9}(\nabla\chi'_1)(\tau_x^{1/9}.).\nabla\tilde u_1 - \tau_x^{2/9}(\Delta\chi'_1)(\tau_x^{1/9}.)\tilde u_1\,,$$

after setting $\tilde u_1 = \tilde\chi_1(\tau_x^{2/9}.)u$. Both functions, $\tilde u_1$ and therefore $u'_1 = \chi'_1(\tau_x^{2/9}.)\tilde u_1$ are uniformly estimated in $\mathcal{H}_1$ and therefore in $H^1(\mathbb{R}^2)$. From the imbedding $H^1(\mathbb{R}^2) \subset L^6(\mathbb{R}^2)$, the term $G|\tilde u_1|^2u'_1$ is uniformly bounded in $L^2(\mathbb{R}^2)$. For the term f_u^1, the support condition $\operatorname{supp}\chi'_1(\tau_x^{1/9}.) \subset \left\{|x| \le \tau_x^{-1/9}\right\}$ imply

$$\|f_u^1\|_{L^2} \le C_{\chi'}(\tau_x^{1-1/3} + \tau_x^{1-4/9}) \le C''_{\chi'}\,,$$

while the estimate

$$\|f_u^2\| \le C_{\chi'}\tau_x^{1/9} \le C''_{\chi'}$$

is straightforward. Hence the right-hand side of (4.13) is uniformly bounded in $L^2(\mathbb{R}^2)$ and we have proved

$$\|\chi'_1(\tau_x^{1/9}.)u\|_{\mathcal{H}_2} \le C^3_{\chi'} \qquad (4.14)$$

for any good pair of cut-offs $\chi' = (\chi'_1, \chi'_2)$.

Fourth step- accurate comparison of minimal energies: We already know $\mathcal{E}_{\tau,min} \le \mathcal{E}_{H,min} + \kappa\tau_x^{2/3}$ and we want to check the reverse inequality. Consider a minimizer u of $\mathcal{E}_\tau$ and take two pairs of cut-off $\chi = (\chi_1, \chi_2)$ and $\chi' = (\chi'_1, \chi'_2)$, such that $\chi_1 \prec \chi'_1$. The identity (4.8) for $\mathcal{E}_\tau$ and (4.9) of Lemma 4.5 used with χ imply

$$\mathcal{E}_{\tau,min} \ge \mathcal{E}_\tau(\chi_1(\tau_x^{1/9}.)u) + \mathcal{E}_\tau(\chi_2(\tau_x^{1/9}.)u) - \tau_x^{2/9}\sum_{j=1}^2\int_{\mathbb{R}^2}|(\nabla\chi_j)(\tau_x^{1/9}.)|^2|u|^2 \qquad (4.15)$$

$$\mathcal{E}_{\tau,min} \ge \mathcal{E}_H(\chi_1(\tau_x^{1/9}.)u) - \tau_x^{2/9}\sum_{j=1}^2\int_{\mathbb{R}^2}|(\nabla\chi_j)(\tau_x^{1/9}.)|^2|u|^2 + R(u)\,. \qquad (4.16)$$

After setting $u_1' = \chi_1'(\tau_x^{1/9}.)u$, we get the bound

$$\int_{\mathbb{R}^2} |(\nabla\chi_j)(\tau_x^{1/9}.)|^2|u|^2 = \int_{\mathbb{R}^2} \frac{|(\nabla\chi_j)(\tau_x^{1/9}.)|^2}{|q|^4}||q|^2u_1'|^2 \leq C_\chi\tau_x^{4/9}\,,$$

while we already know from the third step the bounds $\mathcal{E}_\bullet(\chi_1(\tau_x^{1/9}.)u) \leq C_\chi$ when $\bullet$ stands for τ or H, which implies $|R(u)| \leq C_\chi\tau_x^{2/3}$. From (4.15), we deduce, as we did in the first step with the energy $\mathcal{E}_H$,

$$\|\chi_2(\tau_x^{1/9}.)u\|_{L^2}^2 \leq C_\chi\tau_x^{2/3} \quad , \quad \|\chi_1(\tau_x^{1/9}.)u\|_{L^2} = 1 + \mathcal{O}(\tau_x^{2/3})\,,$$

while the second line implies

$$\mathcal{E}_{\tau,min} \geq \left(1 - C_\chi\tau_x^{2/3}\right)\mathcal{E}_{H,min} - C_\chi'\tau_x^{2/3} \geq \mathcal{E}_{H,min} - C_\chi''\tau_x^{2/3}\,.$$

Fifth step- accurate comparison of minimizers: The function $u_1 = \|\chi_1(\tau_x^{1/9}.)u\|_{L^2}^{-1}\chi_1(\tau_x^{1/9}.)u$, satisfies

$$\mathcal{E}_H(u_1) \leq \mathcal{E}_{H,min} + C_\chi\tau_x^{2/3}$$

while $\chi_1(\tau_x^{1/9}.)u$ solves the equation (4.13) for some pair $\tilde\chi = (\tilde\chi_1, \tilde\chi_2)$ such that $\chi_1 \prec \tilde\chi_1$ after replacing (χ', u_1') with $(\chi, \chi_1(\tau_x^{1/9}.)u)$. After normalization by setting $\tilde u_1 = \|\chi(\tau_x^{1/9}.)u\|_{L^2}^{-1}\chi_1(\tau_x^{1/9}.)u$ it becomes

$$\begin{aligned}
& H_{\ell_V,G}u_1 + G|u_1|^2u_1 = \lambda_u u_1 + f_u^1 + f_u^2 + f_u^3 \qquad (4.17)\\
& u_1 = \|\chi(\tau_x^{1/9}.)u\|_{L^2}^{-1}\chi_1(\tau_x^{1/9}.)u \quad , \quad \tilde u_1 = \|\chi(\tau_x^{1/9}.)u\|_{L^2}^{-1}\tilde\chi_1(\tau_x^{1/9}.)u\,,\\
\text{with} \quad & f_u^1 = -ix(\frac{1}{\sqrt{1+\tau_x x^2}} - 1)\chi(\tau_x^{1/9}.)\partial_y\tilde u_1 - \frac{x^2}{4}\frac{\tau_x x^2}{1+\tau_x x^2}u_1\\
& f_u^2 = -2\tau_x^{1/9}(\nabla\chi_1)(\tau_x^{1/9}.).\nabla\tilde u_1 - \tau_x^{2/9}(\Delta\chi_1)(\tau_x^{1/9}.)\tilde u_1\,,\\
\text{and} \quad & f_u^3 = G\|\chi_1(\tau_x^{1/9}.)u\|_{L^2}^2(|\tilde u_1|^2 - |u_1|^2)u_1 + G(1 - \|\chi_1(\tau_x^{1/9}.)u\|_{L^2}^2)|u_1|^2u_1
\end{aligned}$$

The estimate (4.14), for any new good pair $\chi' = (\chi_1', \chi_2')$ such that $\chi_1 \prec \tilde\chi_1 \prec \chi_1'$, implies that the terms f_u^1 and f_u^2 of the right-hand side of (4.17) have an L^2-norm of order $\tau_x^{2/3}$. For the third term the estimate (4.14) also implies that

$$(|\tilde u_1|^2 - |u_1|^2)u_1 = \left(1 - \chi_1^2(\tau_x^{1/9}.)\right)|\tilde u_1|^2u_1\,,$$

has an L^2-norm of order $\mathcal{O}(\tau_x^{2/3})$ (use the L^∞ bound for $|\tilde u_1|^2$ with $\||q|^2u_1\|_{L^2} \leq C_\chi$). We conclude by applying Proposition 4.4. □

We end this section with a comparison property similar to Proposition 4.4.

Proposition 4.7. *Let ℓ_V, G be fixed positive numbers and take $u \in H^1(\mathbb{R}^2)$ such that*

$$\mathcal{E}_\tau(u) \leq \mathcal{E}_{\tau,min} + C_{\ell_{V,G}}\tau_x^{2/3}$$

and which solves

$$-\partial_x^2 u - (\partial_y - i\frac{x}{\sqrt{1+\tau_x x^2}})^2 + \frac{v(\tau_x^{1/2}.)}{\ell_V^2 \tau_x} u + G|u|^2 u = \lambda_u u + r_u$$

with $\lambda_u \in \mathbb{R}$ *and* $\|r_u\|_{L^2} \leq 1$ *. Then for any pair* $\chi = (\chi_1, \chi_2) \in \mathcal{C}_b^\infty(\mathbb{R}^2)$ *so that* $\chi_1^2 + \chi_2^2 = 1$ *with* $\operatorname{supp} \chi_1 \subset \{x^2 + y^2 < 1\}$ *and* $\chi_1 \equiv 1$ *in* $\{x^2 + y^2 \leq 1/2\}$ *, there exists* $\tau_{\chi,\ell_V,G}$ *and* C_{χ,ℓ_V} *such that*

$$\|\chi_2(\tau_x^{1/9}.)u\|_{L^2}^2 \leq C_{\chi,\ell_V,G}\tau_x^{2/3} \tag{4.18}$$

$$|\mathcal{E}_\tau(\chi_1(\tau_x^{1/9}.)u) - \mathcal{E}_{H,min}| \leq C_{\chi,\ell_V,G}\tau_x^{2/3} \tag{4.19}$$

$$d_{\mathcal{H}_2}(\chi_1(\tau_x^{1/9}.)u, \text{Argmin } \mathcal{E}_H) \leq C_{\chi,\ell_V,G}(\tau_x^{2\nu_{\ell_{V,G}}/3} + \|r_u\|_{L^2}) , \tag{4.20}$$

when $\tau_x < \tau_{\chi,\ell_V,G}$ *and where* $\nu_{\ell_V,G} \in (0, \frac{1}{2}]$ *is the exponent given in Proposition 4.2.*

Proof: The analysis follows essentially the same line as the study of the minimizers of $\mathcal{E}_\tau$ in the proof of Proposition 4.6. By taking two pairs $\chi' = (\chi_1', \chi_2')$ and $\tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2)$ such that $\chi_1' \prec \tilde{\chi_1}$, we obtain successively like in the Third Step in the proof of Proposition 4.6 :

- $\mathcal{E}_\tau(\tilde{\chi}_1(\tau_x^{1/9}.)u) + \mathcal{E}_\tau(\tilde{\chi}_2(\tau_x^{1/9}.)u) \leq C_{\tilde{\chi}}$;
- $\left[-\partial_x^2 - (\partial_y - \frac{ix}{2})^2 + \frac{x^2+y^2}{\ell_V^2}\right] u_1' = \lambda_u u_1' - G|\tilde{u}_1|^2 u_1' + f_u^1 + f_u^2 + r_u$ where u_1',$\tilde{u}_1$, $f_u^{1,2}$ have the same expressions as in (4.13);
- $\|\chi_1'(\tau_x^{1/9}.)u\|_{\mathcal{H}_2} \leq C_{\chi'}$ owing to $\|r_u\|_{L^2} \leq 1$.

From the last estimate, the refined comparaison of energies like in the Fourth Step gives for a pair $\chi = (\chi_1, \chi_2)$ such that $\chi_1 \prec \chi_1'$:

$$\|\chi_2(\tau_x^{1/9}.)u\|_{L^2}^2 \leq C_\chi \tau_x^{2/3} \quad \mathcal{E}_H(u_1) \leq \mathcal{E}_{H,min} + C_\chi \tau_x^{2/3} ,$$

with $u_1 = \|\chi_1(\tau_x^{1/9}.)u\|^{-1}\chi_1(\tau_x^{1/9}.)u$ and $C_\chi \tau_x^{2/3} \leq 1$ for $\tau_x \leq \tau_\chi$. The equation (4.17) is replaced by

$$H_{\ell_V,G}u_1 + G|u_1|^2 u_1 = \lambda_u u_1 + f_u^1 + f_u^2 + f_u^3 + \|\chi_1(\tau_x^{1/9}.)u\|^{-1} r_u$$

without changing the expressions of $f_u^{1,2,3}$. Again the estimate $\|\chi_1'(\tau_x^{1/9}.)u\|_{\mathcal{H}_2} \leq C_{\chi'}$ is used with various cut-offs χ_1', in order to get $\|f_u^1 + f_u^2 + f_u^3\|_{L^2} = \mathcal{O}(\tau_x^{2/3})$. We conclude with the help of Proposition 4.4 applied to u_1 . □

5 Analysis of the complete minimization problem

We consider the complete minimization problem for the energy

$$\mathcal{E}_\varepsilon(\psi) = \langle \psi, H_{Lin}\psi \rangle + \frac{G_{\varepsilon,\tau}}{2} \int |\psi^4| = \varepsilon^{2+2\delta}\tau_x \left[\langle \psi, \varepsilon^{-2-2\delta}\tau_x^{-1} H_{Lin}\psi \rangle + \frac{G}{2} \int |\psi^4| \right]$$

and compare its solutions to the minimization of the reduced energies $\mathcal{E}_\tau$ and $\mathcal{E}_H$, introduced in the previous sections. We work with $\tau_y = 1$, $\tau_x \to 0$, $\varepsilon \to 0$, while ℓ_V, G and $\delta \in (0, \delta_0]$ are fixed. The analysis follows the same lines as the proof of Proposition 4.6.

5.1 Upper bound for $\inf\{\mathcal{E}_\varepsilon(\psi), \|\psi\|_{L^2}=1\}$

The potential V_ε is chosen according to (1.13)-(1.14) while ℓ_V and G are fixed. The parameter τ_x is assumed to be smaller than $\tau_{\ell_V,G}$ so that the minimal energy $\mathcal{E}_{\tau,min}(\tau_x)$ of $\mathcal{E}_\tau$ is achieved (see Proposition 4.6) and $|\mathcal{E}_{\tau,min}(\tau_x) - \mathcal{E}_{H,min}| \leq C_{\ell_V,G}\tau_x^{2/3}$. Moreover Proposition 4.6 also says that by truncating an element of Argmin $\mathcal{E}_\tau$, one can find $a_- \in \mathcal{H}_2$ such that

$$\|a_-\|_{L^2} = 1\,, \quad |\mathcal{E}_\tau(a_-) - \mathcal{E}_{H,min}| \leq C_{\ell_V,G}\tau_x^{2/3} \quad \text{and} \quad \|a_-\|_{\mathcal{H}_2} \leq C_{\ell_V,G}\,. \tag{5.1}$$

Proposition 5.1. *Under the above assumptions, take* $\psi = \hat{U}\begin{pmatrix} 0 \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$ *where* a_- *satisfies* (5.1) *and* $\hat{U} = U(q, \varepsilon D_q, \tau, \varepsilon)$ *is the unitary operator introduced in Theorem 2.1. The estimate*

$$\left|\mathcal{E}_\varepsilon(\psi) - \varepsilon^{2+2\delta}\tau_x\mathcal{E}_\tau(a_-)\right| \leq C_{\ell_V,G}\varepsilon^{2+4\delta}\,. \tag{5.2}$$

hold uniformly w.r.t $\tau_x \in (0, \tau_{\ell_V,G}]$ *and* $\delta \in (0, \delta_0]$.

Proof: Let us compare first the linear part by estimating

$$\left|\langle\psi\,, H_{Lin}\psi\rangle - \varepsilon^{2+2\delta}\tau_x\langle a_-\,, \left[-\partial_x^2 - (\partial_y - i\frac{x}{2\sqrt{1+\tau_x x^2}})^2 + \frac{v(\sqrt{\tau_x}.)}{\ell_V^2\tau_x}\right]a_-\rangle\right| \\ = \left|\langle\psi\,, (H_{Lin} - \varepsilon^{2+2\delta}\tau_x U^* H_{BO} U)\psi\rangle\right|$$

By Proposition 3.2, it suffices to estimate

$$\|(1-\hat{\chi})\psi\|_{L^2} \quad \text{and} \quad \|\sqrt{\tau_x}|\varepsilon D_q|(1-\hat{\chi})\psi\|_{L^2}$$

for some given cut-off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\hat{\chi} = \chi(\tau_x\varepsilon^2 D_q^2)$. Notice

$$e^{i\frac{y}{2\sqrt{\tau_x}}}(\sqrt{\tau_x}\varepsilon D_q)e^{-i\frac{y}{2\sqrt{\tau_x}}} = \begin{pmatrix} \sqrt{\tau_x}\varepsilon D_x \\ \sqrt{\tau_x}\varepsilon D_y - \frac{\varepsilon}{2} \end{pmatrix}$$

Hence by using the functional calculus of $\sqrt{\varepsilon}D_q$, we can say that there exists a cut-off $\chi' \in \mathcal{C}_0^\infty(-r_\gamma^2, r_\gamma^2)$, with $\chi' \equiv 1$ around 0 and $\chi' \prec \chi$ such that

$$e^{i\frac{y}{2\sqrt{\tau_x}}}(1-\hat{\chi})^2 e^{-i\frac{y}{2\sqrt{\tau_x}}} \leq (1-\hat{\chi'})^2\,, \\ \text{and} \quad e^{i\frac{y}{2\sqrt{\tau_x}}}(\tau_x\varepsilon^2|D_q|^2)(1-\hat{\chi})^2 e^{-i\frac{y}{2\sqrt{\tau_x}}} \leq 2(\tau_x\varepsilon^2|D_q|^2)(1-\hat{\chi'})^2 + 2(1-\hat{\chi'})^2$$

as soon as $\varepsilon \leq \varepsilon_0 \leq 1$, for a convenient choice of ε_0 and r_γ. By using $\|a\|_{H^2(\mathbb{R}^2)} \leq C_{\ell_V,G}$, we deduce

$$\begin{aligned} \|(1-\hat{\chi})\psi\|_{L^2}^2 \leq \|(1-\hat{\chi'})a_-\|_{L^2}^2 &\leq C_{\ell_V,G}\tau_x^2\varepsilon^4 \quad \text{and} \\ \|(\sqrt{\tau_x}\varepsilon|D_q|)(1-\hat{\chi})\psi\|_{L^2}^2 &\leq 2\|(\sqrt{\tau_x}\varepsilon|D_q|)(1-\hat{\chi'})a_-\|_{L^2}^2 + 2\|(1-\hat{\chi'})a_-\|_{L^2}^2 \\ &\leq C_{\ell_V,G}\tau_x\varepsilon^2\,. \end{aligned}$$

By Proposition 3.2, we obtain

$$\left|\langle\psi\,,(H_{Lin}-\varepsilon^{2+2\delta}\tau_x U^*H_{BO}U)\psi\rangle\right| \\ \leq C_{\ell_V,G}\left[\varepsilon^{2+4\delta}+\varepsilon^{5+2\delta}\tau_x^2+\varepsilon^{4+2\delta}\tau_x^{3/2}\right]\leq C'_{\ell_V,G}\varepsilon^{2+4\delta}\,. \quad (5.3)$$

For the nonlinear part of the energy, Proposition 3.3 gives

$$(1-C\varepsilon^{1+2\delta})\int_{\mathbb{R}^2}|a_-|^4\leq\int_{\mathbb{R}^2}|\psi|^4\leq(1+C\varepsilon^{1+2\delta})\int_{\mathbb{R}^2}|a_-|^4$$

and the bound, $\int_{\mathbb{R}^2}|e^{-i\frac{y}{2\sqrt{\tau_x}}}a_-|^4=\int_{\mathbb{R}^2}|a_-|^4\leq C_{\ell_V,G}$, leads to

$$\left|\frac{G\tau_x\varepsilon^{2+2\delta}}{2}\int_{\mathbb{R}^2}|\psi|^4-\varepsilon^{2+2\delta}\frac{G\tau_x}{2}\int_{\mathbb{R}^2}|a_-|^4\right|\leq C_{\ell_V,G}\varepsilon^{2+2\delta}\tau_x\times\varepsilon^{1+2\delta}\,,$$

which is smaller than the error term for the linear part. This ends the proof of (5.2). □

Remark 5.2.

- *The energy $\mathcal{E}_\tau(a_-)=\mathcal{E}_{H,min}+\mathcal{O}(\tau_x^{2/3})$. Therefore the error given by* (5.2) *is relevant, as compared with the energy scale of $\varepsilon^{2+2\delta}\tau_x\mathcal{E}_{\tau,min}$, when*

$$\varepsilon^{2\delta}\leq c_{\ell_V,G}\tau_x\,, \quad (5.4)$$

with $c_{\ell_V,G}$ small enough, and accurate when

$$\varepsilon^{2\delta}\leq C_{\ell_V,G}\tau_x^{5/3}\,. \quad (5.5)$$

Remember that the constants $c_{\ell_V,G}$ and $C_{\ell_V,G}$ depend also on δ_0, when $\delta\in(0,\delta_0]$.

- *It is interesting to notice that the worst term in the right-hand side of* (5.3) *comes from the error of order $\mathcal{O}(\varepsilon^{2+2\delta})$ in the Born-Oppenheimer approximation. There seems to be no way to get an additional factor τ_x^ν with $\nu>0$ because the initial problem is rapidly oscillatory in the y-variable in a τ_x-dependent scale. This can be seen on the gain associated with the metric g_τ, for $\tau_y=1$ and $\tau_x>0$, which is simply $\langle\sqrt{\tau}_x p\rangle$, of order 1 in the (very) low frequencies.*

5.2 Existence of a minimizer for $\mathcal{E}_\varepsilon$

With the choice (1.13)-(1.14) of the potential $V_{\varepsilon,\tau}$, the linear Hamiltonian H_{Lin} can be written

$$H_{Lin}=-\varepsilon^{2+2\delta}\tau_x\Delta+u_0(q,\tau)\begin{pmatrix}2\sqrt{1+\tau_x x^2}&0\\0&0\end{pmatrix}u_0(q,\tau)^* \\ +\varepsilon^{2+2\delta}\frac{v(\tau_x^{1/2}.)}{\ell_V^2}-\varepsilon^{2+2\delta}W_\tau(x,y)\,,$$

$$\text{with}\qquad W_\tau(x,y)=\frac{\tau_x^2}{(1+\tau_x x^2)^2}+\frac{1}{1+\tau_x x^2}\,.$$

For $t \leq \frac{\varepsilon^{2+2\delta}}{2\ell_V^2}$, the negative part $\left(\varepsilon^{2+2\delta}\frac{v(\sqrt{\tau_x}.)}{\ell_V^2} - t\right)_-$ is compactly supported. Set

$$H_{Lin,t,+} = -\varepsilon^{2+2\delta}\tau_x\Delta + u_0(q,\tau)\begin{pmatrix} 2\sqrt{1+\tau_x x^2} & 0 \\ 0 & 0 \end{pmatrix} u_0(q,\tau)^* + \left(\varepsilon^{2+2\delta}\frac{v(\sqrt{\tau_x}.)}{\ell_V^2} - t\right)_+ ,$$

so that

$$\mathcal{E}_\varepsilon(\psi) - t\|\psi\|^2 = \langle\psi\,,\, H_{Lin,t,+}\psi\rangle + \frac{G_{\varepsilon,\tau}}{2}\int|\psi|^4 + \langle\psi\,,\,\varepsilon^{2+2\delta}\left[\left(\frac{v(\sqrt{\tau_x}.)}{\ell_V^2} - \frac{t}{\varepsilon^{2+2\delta}}\right)_- - W_\tau\right]\psi\rangle\,. \tag{5.6}$$

Proposition 5.3. *Assume $\varepsilon \leq \varepsilon_{\ell_V,G}$ and $\tau_x \leq \tau_{\ell_V,G}$ with $\tau_{\ell_V,G}$ small enough. Then the infimum*

$$\inf\{\mathcal{E}_\varepsilon(u), \quad u \in H^1(\mathbb{R}^2), \|u\|_{L^2} = 1\}$$

is achieved. Any element ψ of Argmin $\mathcal{E}_\varepsilon$ *solves an Euler-Lagrange equation*

$$H_{Lin}\psi + G_{\varepsilon,\tau}|\psi|^2\psi = \lambda_\psi\psi$$

with the estimates

$$|\mathcal{E}_\varepsilon(\psi)| + |\lambda_\psi| \leq C_{\ell_V,G}\varepsilon^{2+2\delta} \quad , \quad \|(1+\tau_x|D_q|^2)^{1/2}\psi\|_{L^2} \leq C_{\ell_V,G}\,,$$
$$\mathcal{E}_\varepsilon(\psi) = \mathcal{E}_{\varepsilon,min} \leq \varepsilon^{2+2\delta}\left[\mathcal{E}_{\tau,min} + C_{\ell_V,G}\varepsilon^{2\delta})\right] \leq \varepsilon^{2+2\delta}\left[\tau_x\mathcal{E}_{H,min} + C'_{\ell_V,G}(\tau_x^{5/3} + \varepsilon^{2\delta})\right]\,.$$

Proof: From Proposition 5.1, we know that

$$\inf_{\|u\|_{L^2}=1}\mathcal{E}_\varepsilon(u) \leq \varepsilon^{2+2\delta}\tau_x\mathcal{E}_{\tau,min} + C_{\ell_V,G}\varepsilon^{2+4\delta} =: \frac{t}{2}$$

For $\varepsilon \leq \varepsilon_{\ell_V,G}$ and $\tau_x \leq \tau_{\ell_V,G}$, t smaller than $\frac{\varepsilon^{2+2\delta}}{2\ell_V^2}$. Consider the decomposition (5.6) for the energy $\mathcal{E}_\varepsilon(u) - t\|u\|_{L^2}^2$. By the same argument (convexity of the positive part and compacity of the negative part) as we used for $\mathcal{E}_\tau$ in the proof of Proposition 4.6 (second step), a weak limit of an extracted sequence of minimizers in $H^1(\mathbb{R}^2)$ is a minimum for $\mathcal{E}_\varepsilon$ on $\{\|u\|_{L^2} = 1\}$.
A element ψ of Argmin $\mathcal{E}_\varepsilon$ satisfies

$$\begin{aligned}
\varepsilon^{2+2\delta}\tau_x\left[\langle\psi\,,\,-\Delta\psi\rangle + \frac{G}{2}\int|\psi|^4\right] &\leq \langle\psi\,,\,H_{Lin,t,+}\psi\rangle + \frac{G_{\varepsilon,\tau}}{2}\int|\psi|^4 \\
&\leq \mathcal{E}_{\varepsilon,min} - \langle\psi\,,\,\varepsilon^{2+2\delta}\left[\left(\frac{v(\sqrt{\tau_x}.)}{\ell_V^2} - \frac{t}{\varepsilon^{2+2\delta}}\right)_- - W_\tau\right]\psi\rangle \\
&\leq \frac{t}{2} + t + (1+\tau_x^2)\varepsilon^{2+2\delta} \leq C'_{\ell_V,G}\varepsilon^{2+2\delta}\,,
\end{aligned}$$

by recalling $v \geq 0$ for the last line. This implies

$$\|\psi\|_{H^1}^2 \leq C'_{\ell_V,G}\tau_x^{-1} \quad , \quad \|\psi\|_{L^4}^4 \leq \frac{2C'_{\ell_V,G}}{G}\tau_x^{-1}\,,$$

and by interpolation with $\|u\|_{L^6} \leq C\|\nabla u\|_{L^2}^{2/3}\|u\|_{L^2}^{1/3}$, $\|\psi\|_{L^6} = \mathcal{O}(\tau_x^{-1/3})$. The first inequality with $\|\psi\|_{L^2} = 1$, gives $\|(1+\tau|D_q|^2)^{1/2}\psi\|_{L^2} = \mathcal{O}(1)$.
The Euler-Lagrange equation

$$H_{Lin}\psi + G_{\varepsilon,\tau}|\psi|^2\psi = \lambda\psi$$

implies

$$\begin{aligned} |\lambda| &\leq 2\left[\langle\psi\,,\,H_{Lin,t,+}\psi\rangle + \frac{G_{\varepsilon,\tau}}{2}\int|\psi|^4\right] - \langle\psi\,,\,\varepsilon^{2+2\delta}\left[\left(\frac{v(\sqrt{\tau_x}.)}{\ell_V^2} - \frac{t}{\varepsilon^{2+2\delta}}\right)_- - W_\tau\right]\psi\rangle \\ &\leq C''_{\ell_V,G}\varepsilon^{2+2\delta}\,. \end{aligned}$$

Similarly, the lower bound $\mathcal{E}_\varepsilon(\psi) \geq -2\varepsilon^{2+2\delta}$ is due to $W_\tau \geq -2$. The upper bound of $\mathcal{E}_\varepsilon(\psi) = \mathcal{E}_{\varepsilon,min}$ comes from Proposition 5.1 and (5.1). □

5.3 Comparison of minimal energies between $\mathcal{E}_\varepsilon$ and $\mathcal{E}_\tau$

In this subsection, we specify a priori estimates for the minimizers of $\mathcal{E}_\varepsilon$ and compare the energies $\mathcal{E}_{\varepsilon,min}$ and $\mathcal{E}_{\tau,min}$ without imposing relations between $\varepsilon^{2\delta}$ and τ_x. This is not necessary at this level, if one uses carefully bootstrap arguments.

Proposition 5.4. *Let $V_{\varepsilon,\tau}$ be given by* (1.13)-(1.14) *and assume $\tau_x \leq \tau_{\ell_V,G}$ and $\varepsilon \leq \varepsilon_{\ell_V,G}$ so that $\mathcal{E}_\varepsilon$ admits a ground state according to Proposition 5.3. The operator $\hat{U}$ is the unitary transform provided by Theorem 2.1 and an element $\psi \in$ Argmin $\mathcal{E}_\varepsilon$ is written $\hat{U}\begin{pmatrix} e^{+i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$. Then, the estimates*

$$\|(1+\tau_x|D_q|^2)^{1/2}a\|_{L^2} \leq C_{\ell_V,G}\,, \tag{5.7}$$

$$\|a_+\|_{L^2}^2 + G_{\varepsilon,\tau}\int|a|^4 \leq C_{\ell_V,G}\varepsilon^{2+4\delta} + \mathcal{E}_{\varepsilon,min}\,, \tag{5.8}$$

$$|\mathcal{E}_\varepsilon(\psi) - \varepsilon^{2+2\delta}\tau_x\mathcal{E}_\tau(a_-)| \leq C_{\ell_V,G}\varepsilon^{2+4\delta}\,, \tag{5.9}$$

$$|\mathcal{E}_{\varepsilon,min} - \varepsilon^{2+2\delta}\tau_x\mathcal{E}_{\tau,min}| \leq C_{\ell_V,G}\varepsilon^{2+4\delta}\,. \tag{5.10}$$

hold with right-hand sides which can be replaced by $C_{\ell_V,G}\varepsilon^{2+2\delta}\tau_x$ when $\varepsilon^{2\delta} \leq c_{\ell_V,G}\tau_x$.

Proof: For $\tilde{a} = \begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$ and $a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$, the norms $\|(1+\tau|D_q|^2)^{1/2}a\|_{L^2}$ and $\|(1+\tau_x|D_q|^2)^{1/2}\tilde{a}\|_{L^2}$ are uniformly equivalent because

$$e^{\pm i\frac{y}{2\sqrt{\tau_x}}}(\sqrt{\tau_x}D_y)e^{\mp i\frac{y}{2\sqrt{\tau_x}}} = (\sqrt{\tau_x}D_y) \mp \frac{1}{2}\,.$$

Hence it suffices to estimate $\sqrt{\tau_x}\varepsilon D_q\hat{U}^*\psi$. Remember that $\hat{U} = U(q,\varepsilon D_q,\tau,\varepsilon)$ with $U \in S_u(1,g_\tau;\mathcal{M}_2(\mathbb{C}))$, while $\sqrt{\tau_x}p \in S_u(\langle\sqrt{\tau_x}p\rangle, g_\tau;\mathbb{R}^2)$. With $\|(1+\tau|D_q|^2)^{1/2}\psi\|_{L^2} \leq C_{\ell_V,G}$ in Proposition 5.3, simply compute:

$$\sqrt{\tau_x}\varepsilon D_q\hat{U}^*\psi = \varepsilon\hat{U}^*\sqrt{\tau_x}D_q\psi + \left[\sqrt{\tau_x}\varepsilon D_q, \hat{U}^*\right]\psi = \mathcal{O}(\varepsilon) \quad \text{in } L^2(\mathbb{R}^2;\mathbb{C}^2)\,,$$

and divide by ε for (5.7).
In order to compare the energies, consider first the linear part by writing

$$\left|\langle\psi\,,H_{Lin}\psi\rangle-\varepsilon^{2+2\delta}\tau_x\left[\langle a_+\,,\,\hat{H}_+a_+\rangle+\langle a_-\,,\,\hat{H}_-a_-\rangle\right]\right|=\left|\langle\tilde{a}\,,(\hat{U}^*H_{Lin}\hat{U}-\varepsilon^{2+2\delta}H_{BO})\tilde{a}\rangle\right|\,,$$

with

$$\begin{aligned}\tilde{a}&=\begin{pmatrix}e^{i\frac{y}{2\sqrt{\tau_x}}}a_+\\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_-\end{pmatrix}\,,\\ H_{BO}&=\begin{pmatrix}e^{i\frac{y}{2\sqrt{\tau_x}}}\hat{H}_+e^{-i\frac{y}{2\sqrt{\tau_x}}} & 0\\ 0 & e^{-i\frac{y}{2\sqrt{\tau_x}}}\hat{H}_-e^{i\frac{y}{2\sqrt{\tau_x}}}\end{pmatrix}\,,\\ \hat{H}_\pm&=-\partial_x^2-(\partial_y\pm i\frac{x}{2\sqrt{1+\tau_x x^2}})^2+\frac{v(\sqrt{\tau_x}.)}{\ell_V^2\tau_x}+\frac{(1\pm1)}{\varepsilon^{2+2\delta}\tau_x}(1+\tau_x x^2)^{1/2}\,.\end{aligned}$$

Here it is convenient to write

$$\varepsilon^{2+2\delta}\tau_xH_{BO}=\varepsilon^{2\delta}\hat{B}+\begin{pmatrix}2(1+\tau_xx^2)^{1/2}\\ 0\end{pmatrix}$$

$$\text{with}\qquad\hat{B}=\begin{pmatrix}\hat{B}_+ & 0\\ 0 & \hat{B}_-\end{pmatrix}\quad,\quad\hat{B}_\pm=B_\pm(q,\varepsilon D_q,\tau,\varepsilon)\tag{5.11}$$

$$\text{and}\qquad B_\pm(q,p,\tau,\varepsilon)=\tau_xp_x^2+(\sqrt{\tau_x}p_y\pm\frac{\varepsilon}{2}(\frac{\sqrt{\tau_x}x}{\sqrt{1+\tau_xx^2}}-\sqrt{\tau_x}))^2+\frac{\varepsilon^2v(\sqrt{\tau_x}.)}{\ell_V^2}\tag{5.12}$$

With the notation $\hat{\chi}=\chi(\tau_x\varepsilon^2|D_q|^2)$ of Proposition 3.2, Lemma 5.5 below says in particular

$$\begin{aligned}&\|(1-\hat{\chi})\psi\|_{L^2}^2\le C_{\ell_V,G}\left[\langle\tilde{a},\hat{B}\tilde{a}\rangle+\varepsilon^{1+2\delta}\|\tilde{a}\|_{L^2}^2\right]\,,\\ &\|\sqrt{\tau_x}\varepsilon|D_q|(1-\hat{\chi})\psi\|_{L^2}^2\le C_{\ell_V,G}\left[\langle\tilde{a},\hat{B}\tilde{a}\rangle+\varepsilon^{1+2\delta}\|\tilde{a}\|_{L^2}^2\right]\,.\end{aligned}$$

Proposition 3.2 yields

$$\left|\langle\tilde{a}\,,\left(\hat{U}^*H_{Lin}\hat{U}-\varepsilon^{2\delta}\hat{B}-\begin{pmatrix}2(1+\tau_xx^2)^{1/2}\\ 0\end{pmatrix}\right)\tilde{a}\rangle\right|\le C_{\ell_V,G}\left[\varepsilon^{2+4\delta}+\varepsilon^{1+2\delta}\langle\tilde{a}\,,\,\hat{B}\tilde{a}\rangle\right]$$

For the nonlinear part of the energy, Proposition 3.3 gives

$$(1-C\varepsilon^{1+2\delta})\int_{\mathbb{R}^2}|\psi|^4\le\int_{\mathbb{R}^2}|\tilde{a}|^4=\int_{\mathbb{R}^2}|a|^4\le(1+C\varepsilon^{1+2\delta})\int_{\mathbb{R}^2}|\psi|^4$$

with $\tilde{a}=\begin{pmatrix}e^{i\frac{y}{2\sqrt{\tau_x}}}a_+\\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_-\end{pmatrix}$ and $a=\begin{pmatrix}a_+\\ a_-\end{pmatrix}$. The bound $\int_{\mathbb{R}^2}|\psi|^4\le C_{\ell_V,G}\tau_x^{-1}$ coming from $\mathcal{E}_\varepsilon(\psi)=\mathcal{O}(\varepsilon^{2+2\delta})$ with $\varepsilon^{2+2\delta}W_\tau=\mathcal{O}(\varepsilon^{2+2\delta})$, leads to

$$\left|\frac{G\tau_x\varepsilon^{2+2\delta}}{2}\int_{\mathbb{R}^2}|\psi|^4-\frac{G\tau_x\varepsilon^{2+2\delta}}{2}\int_{\mathbb{R}^2}|\tilde{a}|^4\right|\le C_{\ell_V,G}\varepsilon^{2+2\delta}\tau_x\times\varepsilon^{1+2\delta}\tau_x^{-1}=C_{\ell_V,G}\varepsilon^{3+4\delta}\,,$$

which is smaller than the error term for the linear part. We have proved

$$\left|\mathcal{E}_\varepsilon(\psi) - \varepsilon^{2\delta}\langle\tilde{a}\,,\,\hat{B}\tilde{a}\rangle - \langle\tilde{a}_+\,,\,2(1+\tau_x x^2)^{1/2}\tilde{a}_+\rangle - \frac{G_{\varepsilon,\tau}}{2}\int|a|^4\right| \leq C_{\ell_V,G}\left[\varepsilon^{2+4\delta} + \varepsilon\times\varepsilon^{2\delta}\langle\tilde{a}\,,\,\hat{B}\tilde{a}\rangle\right].$$

With $\mathcal{E}_\varepsilon(\psi) = \mathcal{E}_{\varepsilon,min}$ $(= \mathcal{O}(\varepsilon^{2+2\delta}))$, this gives

$$(1-C\varepsilon)\varepsilon^{2\delta}\langle\tilde{a}\,,\,\hat{B}a\rangle + 2\langle\tilde{a}_+\,,\,(1+\tau_x x^2)^{1/2}\tilde{a}_+\rangle + \frac{G_{\varepsilon,\tau}}{2}\int|a|^4 \leq C\varepsilon^{2+4\delta} + \mathcal{E}_{\varepsilon,min}\,.$$

where all the terms of the left-hand side are now non negative. We deduce (5.8) and by bootstrapping

$$\varepsilon^{2+2\delta}\tau_x\langle a_-\,,\,\hat{H}_-a_-\rangle + \frac{G_{\varepsilon,\tau}}{2}\int|a|^4 \leq \varepsilon^{2\delta}\langle\tilde{a}\,,\,\hat{B}\tilde{a}\rangle + \frac{G_{\varepsilon,\tau}}{2}\int|\tilde{a}|^4 \leq \mathcal{E}_{\varepsilon,min} + C_{\ell_V,G}\varepsilon^{2+4\delta}\,.$$

Additionally, $\int|a|^4 = \int\left(|a_-|^2+|a_+|^2\right)^2 \geq \int|a_-|^4$ also gives

$$0 \leq \|a_-\|_{L^2}^4\mathcal{E}_{\tau,min} \leq \mathcal{E}_\tau(a_-) \leq C_{\ell_V,G}\frac{\varepsilon^{2\delta}}{\tau_x} + \frac{\mathcal{E}_{\varepsilon,min}}{\varepsilon^{2+2\delta}\tau_x} \leq \mathcal{E}_{\tau,min} + C'_{\ell_V,G}\frac{\varepsilon^{2\delta}}{\tau_x}\,.$$

With $\|a_-\|_{L^2}^2 = 1-\|a_+\|^2 = 1+\mathcal{O}(\varepsilon^{2+2\delta}) = 1+\mathcal{O}(\frac{\varepsilon^{2\delta}}{\tau_x})$ and $\mathcal{E}_{\tau,min} = E_{H,min} + \mathcal{O}(\tau_x^{2/3})$, this finally leads to

$$\left|\frac{\mathcal{E}_{\varepsilon,min}}{\varepsilon^{2+2\delta}\tau_x} - \mathcal{E}_\tau(a_-)\right| \leq C''_{\ell_V,G}\frac{\varepsilon^{2\delta}}{\tau_x}\,.$$

□

Lemma 5.5. *Assume $B(q,p,\tau,\varepsilon) = \tau_x|p|^2 + \varepsilon r$ with $r \in S_u(\langle\sqrt{\tau_x}p\rangle, g_\tau; \mathcal{M}_2(\mathbb{C}))$, take any $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ and set $\hat{B} = B(q,\varepsilon D_q,\tau,\varepsilon)$ and $\hat{\chi} = \chi(\tau_x\varepsilon|D_q|^2)$. Then there exists $\varepsilon_{B,\chi} > 0$ and $C_{B,\chi} > 0$ such that the estimates*

$$\text{and} \qquad \|(1-\hat{\chi})u\|_{L^2}^2 + \|\sqrt{\tau_x}|\varepsilon D_q|(1-\hat{\chi})u\|_{L^2}^2 \leq C_{B,\chi}\left[\langle u\,,\,\hat{B}u\rangle + \varepsilon^{1+2\delta}\|u\|_{L^2}^2\right],$$

$$\text{and} \qquad \|(1-\hat{\chi})u\|_{L^2} + \|\sqrt{\tau_x}|\varepsilon D_q|(1-\hat{\chi})u\|_{L^2} \leq C_{B,\chi}\left[\|\hat{B}u\|_{L^2} + \varepsilon^{1+2\delta}\|u\|_{L^2}\right],$$

hold uniformly w.r.t $\delta \in (0,\delta_0]$ and $\varepsilon \in (0,\varepsilon_{B,\chi})$.

Proof: For $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi_0 \geq 0$ and $\chi_0 \equiv 1$ around 0, the symbol $\chi_0(\tau_x p^2) + B$ is an elliptic symbol in $S_u(\langle\sqrt{\tau_x}p\rangle^2, g_\tau : \mathcal{M}_2(\mathbb{C}))$. For $\varepsilon > 0$ small enough according to (χ_0, B), its quantization $\hat{\chi}_0 + \hat{B}$ is invertible and its inverse belongs to $OpS_u(\langle\sqrt{\tau_x}p\rangle, g_\tau, \mathcal{M}_2(\mathbb{C}))$. Next we notice that in

$$(1-\hat{\chi}) = (1-\hat{\chi})\circ\left[\hat{\chi}_0+\hat{B}\right]^{-1}\circ\hat{B} + (1-\hat{\chi})\circ\left[\hat{\chi}_0+\hat{B}\right]^{-1}\circ\hat{\chi}_0\,,$$

the last term belongs to $Op\mathcal{N}_{u,g_\tau}$ if $\chi_0 \prec \chi$. All the estimates are consequences of

$$(1-\hat{\chi}) = (1-\hat{\chi})\circ\left[\hat{\chi}_0+\hat{B}\right]^{-1}\circ\hat{B} + \varepsilon^{1+2\delta}\rho\,,$$

with $\rho \in S_u(\frac{1}{\langle p_\tau\rangle^\infty}, g_\tau; \mathcal{M}_2(\mathbb{C}))$. □

5.4 Adiabatic Euler-Lagrange equation

As suggested by Proposition 4.7, or the last steps in the proof of Proposition 4.6, an accurate comparison of minimizers requires some comparison of the Euler-Lagrange equations. We check here that the a_- component in Proposition 5.4 solves approximately the Euler-Lagrange equations for minimizers of $\mathcal{E}_\tau$. Here the bootstrap argument is made in terms of operators instead of quadratic forms. In order to get reliable results, we now assume

$$\varepsilon^{2\delta} \leq c_{\ell_V,G}\tau_x\,,$$

with $c_{\ell_V,G}$ chosen small enough.
With such an assumption we know:

- from (5.10) and $|\mathcal{E}_{\tau,min}(\tau_x) - \mathcal{E}_{H,min}| = \mathcal{O}(\tau_x^{2/3})$,

$$C^{-1}\varepsilon^{2+2\delta}\tau_x \leq \mathcal{E}_{\varepsilon,min} \leq C\varepsilon^{2+2\delta}\tau_x\,.$$

- When $\psi \in \text{Argmin } \mathcal{E}_{\varepsilon,min}$ is written $\psi = \hat{U}\begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-\frac{y}{2\sqrt{\tau_x}}}a_+ \end{pmatrix}$ the L^4-norm of a is uniformly bounded, $\int |a|^4 \leq C_{\ell_V,G}$, according to (5.8). By applying Lemma 3.4 this also gives $\int |\psi|^4 \leq C_{\ell_V,G}$. We also have $\|a_+\|_{L^2}^2 \leq C_{\ell_V,G}\varepsilon^{2+2\delta}\tau_x$.

- The Lagrange multiplier λ_ψ, associated with $\psi \in \text{Argmin } \mathcal{E}_{\varepsilon,min}$, equals $\mathcal{E}_\varepsilon(\psi) + \frac{G_{\varepsilon,\tau}}{2}\int |\psi|^4$. Thus it is of order $\mathcal{O}(\varepsilon^{2+2\delta}\tau_x)$.

Proposition 5.6. *Under the same assumptions as in Proposition 5.4 and with the above condition $\varepsilon^{2\delta} \leq c_{\ell_V,G}\tau_x$, write a minimizer ψ of $\mathcal{E}_\varepsilon$ in the form $\psi = \hat{U}\begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$. Then the component a_- solves the equation*

$$\hat{H}_- a_- + G\int |a_-|^2 a_- = \lambda_\psi a_- + r_\varepsilon \quad \textit{with} \quad \|r_\varepsilon\|_{L^2} \leq C_{\ell_V,G}(\varepsilon + \frac{\varepsilon^{2\delta}}{\tau_x})\,,$$

while $\|a_+\|_{L^2} \leq C_{\ell_V,G}\varepsilon^{2+2\delta}\tau_x$.
The L^p-norms of a and ψ are uniformly bounded by $C_{\ell_V,G}$ for $p \in [2,6]$.
Moreover if $\hat{B}$ is the operator defined in (5.11)-(5.12), *$\tilde{a} = \begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$ satisfies*

$$\|\hat{B}\tilde{a}\|_{L^2} \leq C_{\ell_V,G}\varepsilon^2\tau_x\,. \tag{5.13}$$

Remark 5.7. *The relation of $\|\hat{B}\tilde{a}\|_{L^2}$ with $\|\tilde{a}\|_{H^2}$ is given by*

$$C^{-1}\left[\varepsilon^2\tau_x\|D_q^2\tilde{a}\| + \varepsilon^2\|\tilde{a}\|_{L^2}\right] \leq \|\hat{B}\tilde{a}\|_{L^2} + \varepsilon^2\|\tilde{a}\|_{L^2} \leq C\left[\varepsilon^2\tau_x\|D_q^2\tilde{a}\| + \varepsilon^2\|\tilde{a}\|_{L^2}\right]\,.$$

Note that the L^2 remainder terms have a factor ε^2 and not $\varepsilon^2\tau_x$.

Proof: Playing with the Euler-Lagrange equation for ψ, we shall first prove (5.13) argument as we did for $\langle \tilde{a}\,,\,\hat{B}a\rangle$ in the variational proof of Proposition 5.4 and then use it in order to estimate $\|r\|_{L^2}$. The Euler-Lagrange equation for ψ

$$H_{Lin}\psi + G_{_{\varepsilon,\tau}}|\psi|^2\psi = \lambda_\psi\psi$$

becomes

$$\hat{U}^*H_{Lin}\hat{U}\tilde{a} + G_{\varepsilon,\tau}\hat{U}^*\left(|\psi|^2\psi\right) = \lambda_\psi\tilde{a}\,.$$

Remember that Born-Oppenheimer Hamiltonian is given by

$$\begin{aligned}\varepsilon^{2+2\delta}\tau_x H_{B0} &= \varepsilon^{2+2\delta}\tau_x\begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}\hat{H}_+e^{-i\frac{y}{2\sqrt{\tau_x}}} & 0\\ 0 & e^{-i\frac{y}{2\sqrt{\tau_x}}}\hat{H}_-e^{i\frac{\sqrt{\tau_y}}{2\sqrt{\tau_x}}y}\end{pmatrix},\\ &= \varepsilon^{2\delta}\hat{B} + \begin{pmatrix}2\sqrt{1+\tau_x x^2}\\ 0\end{pmatrix},\end{aligned}$$

by using the notations of (5.11)-(5.12).

Let us consider first the linear part after decomposing $\tilde{a}$ into $\tilde{a} = \hat{\chi}\tilde{a} + (1-\hat{\chi})\tilde{a}$, where the low frequency cut-off operator $\hat{\chi} = \chi(q,\varepsilon D_q,\tau,\varepsilon)$ has been introduced in Proposition 3.2:

$$\hat{U}^*H_{Lin}\hat{U}\tilde{a} = \underbrace{\hat{U}^*H_{Lin}\hat{U}\hat{\chi}\tilde{a}}_{(I)} + \underbrace{(\hat{U}^*-\hat{U}_0^*)H_{Lin}\hat{U}(1-\hat{\chi})\tilde{a} + \hat{U}_0^*H_{Lin}(\hat{U}-\hat{U}_0)(1-\hat{\chi})\tilde{a}}_{(II)} + \underbrace{\hat{U}_0^*H_{Lin}\hat{U}_0(1-\hat{\chi})\tilde{a}}_{(III)}\,.$$

The three terms are treated by reconsidering the computations done for Proposition 3.2. By inserting a cut-off $\hat{\chi}_1\,,\ \chi \prec \chi_1\,,$ in $\chi_1\hat{U}^*H_{Lin}\hat{U}\hat{\chi}\tilde{a}$ with $(1-\hat{\chi}_1)\hat{U}^*H_{Lin}\hat{U}\hat{\chi} \in Op\mathcal{N}_{u,g_\tau}$, we get

$$(I) = \varepsilon^{2+2\delta}\tau_x H_{BO}\hat{\chi}\tilde{a} + \mathcal{O}(\varepsilon^{2+4\delta}) \quad \text{in } L^2(\mathbb{R}^2)\,.$$

With $\varepsilon^{-1-2\delta}(\hat{U}-\hat{U}_0) \in OpS_u(\frac{1}{\langle\sqrt{\tau_x}p\rangle}^\infty, g_\tau; \mathcal{M}_2(\mathbb{C}))$ and by using Lemma 5.5, the second term is estimated by

$$\|(II)\|_{L^2} \le C\varepsilon^{1+2\delta}\|(1-\hat{\chi})\tilde{a}\|_{L^2} \le C'\left[\varepsilon\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} + \varepsilon^{2+4\delta}\right].$$

We write the third term as

$$(III) = \varepsilon^{2+2\delta}\tau_x H_{BO}(1-\hat{\chi})\tilde{a} + \varepsilon^{2+2\delta}\tau_x\mathcal{D}_{kin}(1-\hat{\chi})\tilde{a} + \mathcal{D}_{pot}(1-\hat{\chi})\tilde{a}\,,$$

where $\mathcal{D}_{kin}$ and $\mathcal{D}_{pot}$ are defined by (3.7)-(3.8). Following the arguments given in the proof of Proposition (3.2) after these definitions, we get

$$\begin{aligned}\|\mathcal{D}_{pot}(1-\hat{\chi})\tilde{a}\|_{L^2} &\le C\varepsilon^{1+2\delta}\|(1-\hat{\chi})\tilde{a}\|_{L^2} \le\\ &\le C'\left[\varepsilon\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} + \varepsilon^{2+4\delta}\right],\\ \|\mathcal{D}_{kin}(1-\hat{\chi})\tilde{a}\|_{L^2} &\le C\left[\varepsilon^{1+4\delta}\|(1-\hat{\chi})\tilde{a}\|_{L^2} + \varepsilon^{1+2\delta}\|(\varepsilon\sqrt{\tau_x}|D_q|)(1-\hat{\chi})\tilde{a}\|_{L^2}\right]\\ &\le C'\left[\varepsilon\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} + \varepsilon^{2+4\delta}\right].\end{aligned}$$

Hence the Euler-Lagrange equation can be written

$$\begin{aligned} & \varepsilon^{2+2\delta}\tau_x H_{BO}\tilde{a} = \lambda_\psi \tilde{a} - \varepsilon^{2+2\delta}\tau_x G\hat{U}^*\left(|\psi|^2\psi\right) + r \\ \text{with} \quad & \|r\|_{L^2} \leq C\left[\varepsilon\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} + \varepsilon^{2+4\delta}\right]. \end{aligned}$$

From Proposition 5.3, with $\varepsilon^{2\delta} \leq c\tau_x$, we know $\|\psi\|_{L^4} \leq C$. It can be transformed into $\|\tilde{a}\|_{L^4} = \|\hat{U}^*\psi\|_{L^4} \leq C$ with

$$\||\psi|^2\psi\|_{L^2} \leq C\|\psi\|_{L^6}^3 \leq C'\|\tilde{a}\|_{L^6}^3,$$

by applying Lemma 3.4, adapted for the metric g_τ like in the proof Proposition 3.3. We start from the interpolation inequality

$$\|f\|_{L^6} \leq C\|D_q^2 f\|_{L^2}^{\frac{1}{9}}\|f\|_{L^4}^{\frac{8}{9}} \leq \frac{C}{(\tau_x\varepsilon)^{\frac{2}{9}}}\|\tau_x^2\varepsilon^2 D_q^2 f\|_{L^2}^{\frac{1}{9}}\|f\|_{L^4}^{\frac{8}{9}}.$$

By introducing the operator $\tilde{B} = B(\tau_x^{-1/2}\varepsilon q, \tau_x^{\frac{1}{2}}D_q, \tau, \varepsilon)$ with

$$\begin{aligned} B_\pm(\tau_x^{-1/2}q, \tau_x^{1/2}D_q, \tau, \varepsilon) = (\tau_x\varepsilon)^2 D_q^2 \pm \frac{\varepsilon}{2}\left(\frac{x}{\sqrt{1+x^2}} - \sqrt{\tau_x}\right)(\tau_x\varepsilon)D_y \\ + \frac{\varepsilon^2}{4}\left[\left(\frac{x}{\sqrt{1+x^2}} - \sqrt{\tau_x}\right)^2 + \frac{4v}{\ell_V^2}\right], \end{aligned}$$

it becomes

$$\|f\|_{L^6} \leq \frac{C_{\ell_V}}{(\tau_x\varepsilon)^{\frac{2}{9}}}\left[\|\tilde{B}f\|_{L^2}^{\frac{1}{9}}\|f\|_{L^4}^{\frac{8}{9}} + \varepsilon^{\frac{2}{9}}\|f\|_{L^2}^{\frac{1}{9}}\|f\|_{L^4}^{\frac{8}{9}}\right].$$

The above relation with $f = \tau_x^{-\frac{1}{2}}\tilde{a}(\tau_x^{-\frac{1}{2}}.)$ which satisfies

$$\|f\|_{L^6} = \tau_x^{-\frac{1}{3}}\|\tilde{a}\|_{L^6} \quad , \quad \|f\|_{L^4} = \tau_x^{-\frac{1}{2}}\|\tilde{a}\|_{L^4} \quad \text{and} \quad \|\tilde{B}f\|_{L^2} = \|\hat{B}\tilde{a}\|_{L^2}$$

leads to

$$\begin{aligned} \tau_x^{-\frac{1}{3}}\|\tilde{a}\|_{L^6} &\leq \frac{C_{\ell_V}}{(\tau_x\varepsilon)^{\frac{2}{9}}}\left[\|\hat{B}\tilde{a}\|_{L^2}^{\frac{1}{9}}\tau_x^{-\frac{4}{9}}\|\tilde{a}\|_{L^4}^{\frac{8}{9}} + \varepsilon^{\frac{2}{9}}\|\tilde{a}\|_{L^2}^{\frac{1}{9}}\tau_x^{-\frac{4}{9}}\|\tilde{a}\|_{L^4}^{\frac{8}{9}}\right] \\ &\leq C_{\ell_V,G}\tau_x^{-\frac{1}{3}}\left[\varepsilon^{-\frac{2}{9}}\|\hat{B}\tilde{a}\|_{L^2}^{\frac{1}{9}} + 1\right], \end{aligned}$$

and finally

$$G_{\varepsilon,\tau}\||\psi|^2\psi\|_{L^2} \leq G\varepsilon^{2+2\delta}\tau_x\|\psi\|_{L^6}^3 \leq C_{\ell_V,G}\left[\varepsilon^{4/3}\tau_x\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} + \varepsilon^{2+2\delta}\tau_x\right]$$

With $|\lambda_\psi| = \mathcal{O}(\varepsilon^{2+2\delta}\tau_x)$, we obtain

$$\|\varepsilon^{2+2\delta}\tau_x H_{BO}\tilde{a}\|_{L^2} \leq C\left[\varepsilon\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} + \varepsilon^{2+4\delta} + \varepsilon^{2+2\delta}\tau_x\right],$$

and recall

$$\varepsilon^{2+2\delta\tau_x} H_{BO} = \begin{pmatrix} \varepsilon^{2\delta}\hat{B}_+ + 2\sqrt{1+\tau_x x^2} \\ \varepsilon^{2\delta}\hat{B}_- \end{pmatrix} .$$

According to Lemma 5.8 below

$$\|\varepsilon^{2\delta}\hat{B}_+ \tilde{a_+}\|_{L^2} + \|2\sqrt{1+\tau_x x^2}\tilde{a}_+\|_{L^2} \leq C\|(\varepsilon^{2\delta}\hat{B}_+ + 2\sqrt{1+\tau_x x^2})\tilde{a}_+\|_{L^2} .$$

We deduce

$$\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2} \leq C\varepsilon^{2+2\delta}\tau_x .$$

Plugging this result into the estimate of the remainder r gives

$$\varepsilon^{2+2\delta}\tau_x H_{BO}\tilde{a} = \lambda_\tau \tilde{a} + G_{\varepsilon,\tau}\hat{U}^*(|\psi|^2\psi) + \mathcal{O}(\varepsilon^{3+2\delta}\tau_x + \varepsilon^{2+4\delta}) .$$

Consider now more carefully the nonlinear term

$$\hat{U}^*\left(|\psi|^2\psi\right) = \underbrace{\hat{U}_0^*\left[|\hat{U}_0\tilde{a}|^2(\hat{U}_0\tilde{a})\right]}_{(1)} + \underbrace{(\hat{U}^* - \hat{U}_0^*)\left[|\hat{U}_0\tilde{a}|^2(\hat{U}_0\tilde{a})\right]}_{(2)} + \underbrace{\hat{U}^*\left[|\hat{U}\tilde{a}|^2(\hat{U}\tilde{a}) - |\hat{U}_0\tilde{a}|^2(\hat{U}_0\tilde{a})\right]}_{(3)} .$$

By differentiating the relation $\int |f|^4 = \int |\hat{U}_0 f|^4$ w.r.t f, the first term equals

$$(1) = |\tilde{a}|^4\tilde{a} .$$

By semiclassical calculus in the metric g_τ, the operator $\sqrt{\tau}\varepsilon D_q\hat{U}_0$ equals

$$\sqrt{\tau_x}\varepsilon D_q\hat{U}_0 = \hat{U}_0\sqrt{\tau_x}\varepsilon D_q + \left[\sqrt{\tau_x}\varepsilon D_q, \hat{U}_0\right]\psi = \hat{U}_0\sqrt{\tau_x}\varepsilon D_q + \mathcal{O}(\varepsilon) \quad \text{in } \mathcal{L}(L^2(\mathbb{R}^2;\mathbb{C}^2)) .$$

We have already proved $\|\tilde{a}\|_{L^6} \leq C$ and Lemma 3.4 leads again to

$$\|\hat{U}\tilde{a}\|_{L^6} + \|\hat{U}_0\tilde{a}\|_{L^6} \leq C .$$

With $\varepsilon^{-1-2\delta}(\hat{U} - \hat{U}_0) \in OpS_u(1, g_\tau; \mathcal{M}_2(\mathbb{C}))$, this gives

$$\|(2)\| \leq C\varepsilon^{1+2\delta} .$$

For the third term, we use

$$\| \left[|\hat{U}\tilde{a}|^2(\hat{U}\tilde{a}) - |\hat{U}_0\tilde{a}|^2(\hat{U}_0\tilde{a})\right] \|_{L^2} \leq C_0\|(\hat{U} - \hat{U}_0)\tilde{a}\|_{L^6}\left[\|\hat{U}\tilde{a}\|_{L^6}^2 + \|\hat{U}_0\tilde{a}\|_{L^6}^2\right] \leq C\varepsilon^{1+2\delta} .$$

We have proved

$$G_{\tau,\varepsilon}\hat{U}^*(|\psi|^2\psi) - G_{\tau,\varepsilon}|\tilde{a}|^2\tilde{a} = \mathcal{O}(\varepsilon^{3+4\delta}\tau_x) ,$$

which is even better than the estimate for the linear part.
For the $\tilde{a}_-$ component, we get

$$H_{BO}\begin{pmatrix} 0 \\ \tilde{a}_- \end{pmatrix} + G(|\tilde{a}_-|^2 + |\tilde{a}_+|^2)\tilde{a}_- = \frac{\lambda_\psi}{\varepsilon^{2+2\delta}\tau_x}\tilde{a}_- + \mathcal{O}(\varepsilon + \frac{\varepsilon^{2\delta}}{\tau_x}) ,$$

and it remains to estimate the term $|\tilde{a}_+|^2\tilde{a}_-$. The first line of the system may be written

$$\varepsilon^{2\delta}\hat{B}_+\tilde{a_+} + 2\sqrt{1+\tau_x x^2}\tilde{a_+} + G_{\varepsilon,\tau}\left(|\tilde{a}_+|^2 + |\tilde{a}_-|^2\right)\tilde{a}_+ = \lambda_\psi \tilde{a}_+ + \mathcal{O}(\varepsilon^{3+2\delta}\tau_x + \varepsilon^{2+2\delta}\tau_x)\,.$$

Taking the scalar product with $\tilde{a}_+$, with $\lambda_\tau = \mathcal{O}(\varepsilon^{2+2\delta}\tau_x)$ and $\hat{B}_+ \geq 0$, gives

$$\int |\tilde{a}_+|^4 \leq C[\|\tilde{a}_+\|_{L^2}^2 + (\varepsilon + \frac{\varepsilon^{2\delta}}{\tau_x})\|a_+\|_{L^2}] \leq C'\varepsilon^{2+2\delta}\tau_x\,.$$

Then the same argument as in the estimate of $\|\tilde{a}\|_{L^6}$ gives

$$\begin{aligned}
\|\tilde{a}_+\|_{L^6} &\leq C\left[\varepsilon^{-\frac{2}{9}}\|\hat{B}_+\tilde{a}_+\|_{L^2}^{\frac{1}{9}}\|\tilde{a}_+\|_{L^4}^{\frac{8}{9}} + \|\tilde{a}_+\|_{L^2}^{\frac{1}{9}}\|\tilde{a}_+\|_{L^4}^{\frac{8}{9}}\right] \\
&\leq C'\left[(\varepsilon^{-2-2\delta}\|\varepsilon^{2\delta}\hat{B}\tilde{a}\|_{L^2})^{\frac{1}{9}} + (\varepsilon^{2+2\delta}\tau_x)^{\frac{1}{18}}\right](\varepsilon^{2+2\delta}\tau_x)^{\frac{2}{9}} \\
&\leq C''\tau_x^{\frac{1}{9}}(\varepsilon^{2+2\delta}\tau_x)^{\frac{2}{9}} \leq C''\varepsilon^{\frac{4+4\delta}{9}}\tau_x^{\frac{1}{3}}\,.
\end{aligned}$$

Again with $\|\tilde{a}\|_{L^6} \leq C$, we deduce

$$\||\tilde{a}_+|^2\tilde{a}_-\|_{L^2} \leq C_{\ell_V,G}\varepsilon^{\frac{8+8\delta}{3}}\tau_x^{\frac{2}{3}} \leq C'_{\ell_V,G}\varepsilon\,.$$

The final result is just a transcription in terms of a. □

Lemma 5.8. *Let B_+ be the symbol*

$$B_+(q,p,\tau_x,\varepsilon) = \tau_x p_x^2 + (\sqrt{\tau_x}p_y + \frac{\varepsilon}{2}(\frac{\sqrt{\tau_x}x}{\sqrt{1+\tau_x x^2}} - \sqrt{\tau_x}))^2 + \frac{\varepsilon^2 v(\sqrt{\tau_x}.)}{\ell_V^2}$$

introduced in (5.12). *By setting $\hat{B}_+ = B_+(q,\varepsilon D_q,\tau_x,\varepsilon)$, the operator $A = \varepsilon^{2\delta}\hat{B}_+ + 2\sqrt{1+\tau_x x^2}$ is self-adjoint with domain $D(A) = \{u \in L^2(\mathbb{R}^2),\, Au \in L^2(\mathbb{R}^2)\}$ as soon as $\varepsilon \leq \varepsilon_0$, with ε_0 independent of τ_x. Moreover the inequality*

$$\forall u \in D(A), \quad \|\varepsilon^{2\delta}\hat{B}_+u\|_{L^2} + \|2\sqrt{1+\tau_x}x\| \leq C\|Au\|_{L^2}$$

holds with a constant C independent of (ε,τ_x).

Proof: The operator A can be written $A = a_0(q,\varepsilon^{1+\delta}D_q,\tau_x) + \varepsilon^{1+\delta}a_1(q,\varepsilon^{1+\delta}D_q,\tau_x) + \varepsilon^{2+2\delta}a_2(q,\tau_x)$, with $a_k \in S_u(\langle\sqrt{\tau_x}p\rangle^{2-k} + \langle\sqrt{\tau_x}x\rangle^{(1-k)_+}, g_\tau)$ and

$$a_0(q,p,\tau_x) = p^2 + 2\sqrt{1+\tau_x x^2}\,.$$

Therefore the operator A is elliptic in $OpS_u(\langle\sqrt{\tau_x}p\rangle^2 + \langle\sqrt{\tau_x}x\rangle, g_\tau)$ and the result about the domain follows with

$$\|\varepsilon^{2\delta}\hat{B}_+u\|_{L^2} + \|2\sqrt{1+\tau_x}x\| \leq C(\|Au\|_{L^2} + \|u\|_{L^2})$$

for all $u \in D(A)$. We conclude with

$$2\|u\|_{L^2}^2 \leq \langle u\,,\, Au\rangle \leq \|u\|_{L^2}\|Au\|_{L^2}$$

due to $\hat{B}_+ \geq 0$. □

5.5 End of the proof of Theorem 1.2

Assume $G, \ell_V > 0$ be fixed and $\delta \in (0, \delta_0]$. Although we dropped δ_0 in our notations, all the constants in the previous inequalities depend on (G, ℓ_V, δ_0). We assume now $\tau_x \leq \tau_{\ell_V,G,\delta_0}$, $\varepsilon \leq \varepsilon_{\ell_V,G,\delta_0}$ and

$$\varepsilon^{2\delta} \leq \tau_x^{\frac{5}{3}} .$$

Proposition 5.4 says

$$|\mathcal{E}_{\varepsilon,min} - \varepsilon^{2+2\delta}\tau_x \mathcal{E}_{\tau,min}| \leq C\varepsilon^{2+4\delta} \leq C'\varepsilon^{2+2\delta}\tau_x^{\frac{5}{3}}$$

and we recall

$$|\mathcal{E}_{\tau,min} - \mathcal{E}_{H,min}| \leq C\tau_x^{\frac{2}{3}}$$

by Proposition 4.6.

When $\psi = \hat{U}\begin{pmatrix} e^{i\frac{y}{2\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{2\sqrt{\tau_x}}}a_- \end{pmatrix}$, Proposition 5.6 says

$$\|a_+\|_{L^2} \leq C\varepsilon^{2+2\delta}\tau_x \quad , \quad \|a\|_{H^2} \leq \frac{C}{\tau_x} \quad \text{and} \quad \|a\|_{L^4} + \|a\|_{L^6} \leq C ,$$

while a_- solves the approximate Euler-Lagrange equation

$$\hat{H}_- a_- + G\int |a_-|^2 a_- = \lambda_\psi a_- + r_\varepsilon \quad \text{with} \quad \|r_\varepsilon\|_{L^2} \leq C_{\ell_V,G}(\varepsilon + \frac{\varepsilon^{2\delta}}{\tau_x}) .$$

For $u = \|a_-\|_{L^2} a_-$ the energy $\mathcal{E}_\tau(u)$ satisfy

$$|\mathcal{E}_\tau(u) - \mathcal{E}_{\tau,min}| \leq C\tau_x^{\frac{2}{3}}$$

and the above equation becomes

$$\hat{H}_- u + G\int |u|^2 u = \lambda_\psi u + \mathcal{O}(\varepsilon + \frac{\varepsilon^{2\delta}}{\tau_x}) .$$

We conclude by referring to Proposition 4.7 applied to u and then renormalizing for a_-: For $\chi = (\chi_1, \chi_2)$ with $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, $\chi_1^2 + \chi_2^2 \equiv 1$, $\chi_1 = 1$ in a neighborhood of 0

$$\begin{aligned}
&\|\chi_2(\tau_x^{\frac{1}{9}}.)a_-\|_{L^2} \leq C\tau_x^{\frac{1}{3}} \\
&|\mathcal{E}_\tau(\chi_1(\tau_x^{\frac{1}{9}}a_-) - \mathcal{E}_{H,min})| \leq C\tau_x^{\frac{2}{3}} \\
&d_{\mathcal{H}_2}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H) \leq C(\tau_x^{\frac{2\nu_{\ell_V,G}}{3}} + \varepsilon) , \quad \nu_{\ell_V,G} \in (0, \frac{1}{2}] .
\end{aligned}$$

For the L^∞-estimates of a_+ and $d_{L^\infty}(\chi_1(\tau_x^{\frac{1}{9}}.)a_-, \text{Argmin } \mathcal{E}_H)$, we simply use the interpolation inequality

$$\|u\|_{L^\infty} \leq C\|u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}} \leq C\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}} ,$$

valid for any $u \in H^2(\mathbb{R}^2)$ (write $u(0) = \int_{\mathbb{R}^2} \hat{u}(\xi)\frac{d\xi}{(2\pi)^2}$, cut the integral according to $|\xi| \lessgtr R$, estimate both term by Cauchy-Schwartz with $\hat{u} = \frac{1}{|\xi|^2}(|\xi|^2\hat{u})$ when $|\xi| \geq R$, and then optimize w.r.t R).

6 Additional comments

We briefly discuss and sketch how our analysis could be adapted to other problems. No definite statement is given. Complete proofs require additional work, which may be done in the future.

6.1 About the smallness condition of ε w.r.t τ_x

In our main result, Theorem 1.2, condition (1.18) is used, namely

$$\varepsilon^{2\delta} \leq \frac{\tau_x^{\frac{5}{3}}}{C_\delta}\,.$$

One may wonder whether such a condition is necessary in order to compare the minimization problems for $\mathcal{E}_\varepsilon$ and $\varepsilon^{2+2\delta}\tau_x\mathcal{E}_H$. When comparing the minimal energies in Proposition 5.4, we found

$$|\mathcal{E}_{\varepsilon,min} - \varepsilon^{2+2\delta}\mathcal{E}_{H,min}| \leq C\varepsilon^{2+4\delta}\,,$$

while we know that $\mathcal{E}_{H,min} = \mathcal{E}_{H,min}(\ell_V, G)$ is a positive number independent of τ_x and ε. Hence it seems natural to say that $\varepsilon^{2\delta} \ll \tau_x$, at least, is required for ensuring that $\varepsilon^{2+2\delta}\tau_x\mathcal{E}_{H,min}$ is a good approximation of $\mathcal{E}_{\varepsilon,min}$. The error is made of three parts:

- the error term for the Born-Oppenheimer approximation in the low-frequency range given in Theorem 2.1;
- the error term coming from the truncated high frequency part;
- the non linear term.

The non linear term is $\frac{G\varepsilon^{2+2\delta}\tau_x}{2}\int|\psi|^4$, so that a small error in $\|\psi\|_{L^4}^4$ will give a negligible term w.r.t $\varepsilon^{2+2\delta}\tau_x\mathcal{E}_{H,min}$. The question is thus mainly about the linear problem. If one looks more carefully at the error term of Theorem 2.1, it is made of the term

$$-\varepsilon^2\frac{(\partial_{p_k}f_\varepsilon)(\partial_{p_\ell}f_\varepsilon)}{E_+ - E_-}\overline{X}_k X_\ell\,, \tag{6.1}$$

according to Proposition 2.6, and of terms coming from the third order term of Moyal products. The function f_ε is in our case $f_\varepsilon(p) = \varepsilon^{2\delta}\tau_x p^2\gamma(\tau_x p^2)$, $p = (p_x, p_y)$, $k, \ell \in \{x, y\}$, and the factors X_x and X_y computed in the proof of Proposition 3.1 are at most of order $\frac{1}{\sqrt{\tau_x}}$. Hence the quantity (6.1) is an $\mathcal{O}(\varepsilon^{2+4\delta}\tau_x)$ which is again negligible w.r.t $\varepsilon^{2+2\delta}\tau_x\mathcal{E}_{H,min}$. By considering the higher order terms in the Moyal product, the fast oscillating part of the symbol w.r.t y, at the frequency $\frac{1}{\sqrt{\tau_x}}$, deteriorates the estimates: although there are compensations with the slow variations w.r.t p_y, always multiplied by $\sqrt{\tau_x}$, only an ε^k factor without τ_x appears in the k-th order term.
Hence, computing the higher order terms, at least up to order 3, in the adiabatic approximation and then considering the question of the high-frequency truncation, is a way to understand whether the smallness of ε w.r.t τ_x is necessary.

6.2 Anisotropic nonlinearity

Our work assumes an isotropic nonlinearity. A more general nonlinear term would be

$$\frac{G\varepsilon^{2+2\delta}\tau_x}{2}\int \alpha_1|\psi_1|^4+2\alpha_{12}|\psi_1|^2|\psi_2|^2+\alpha_2|\psi_2|^4\;dxdy\,,\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

Our case is $\alpha_1=\alpha_2=\alpha_{12}=1$. Let $\psi=\hat{U}\phi$, with the unitary tranform $\hat{U}=\hat{U}_0+\varepsilon^{1+2\delta}\hat{R}$, (or conversely $\phi=\hat{U}^*\psi$). Then the same arguments as in Subsection 3.3 will lead to

$$\int \alpha_1|\psi_1|^4+2\alpha_{12}|\psi_1|^2|\psi_2|^2+\alpha_2|\psi_2|^4\;dxdy \\ =\left(\int \alpha_1|\psi_1^0|^4+2\alpha_{12}|\psi_1^0|^2|\psi_2^0|^2+\alpha_2|\psi_2^0|^4\;dxdy\right)(1+\mathcal{O}(\varepsilon^{1+2\delta}))\,,$$

after setting $\psi^0(q)=u_0(q)\phi(q)$ at every $q=(x,y)$. In our case

$$u_0(x,y)=\begin{pmatrix}C & Se^{i\varphi}\\ Se^{-i\varphi} & -C\end{pmatrix}\quad,\quad C=\cos\left(\frac{\theta}{2}\right),\,S=\sin\left(\frac{\theta}{2}\right),\quad \theta=\underline{\theta}(\sqrt{\tau_x}x)\,,\,\varphi=\frac{y}{\sqrt{\tau_x}}\,.$$

The pointwise identities

$$|\psi_1^0|^2+|\psi_2^0|^2=|\phi_1|^2+|\phi_2|^2\quad,\quad |\psi_1^0|^2-|\psi_2^0|^2=\cos(\theta)\left(|\phi_1|^2-|\phi_2|^2\right)+2\sin(\theta)\,\mathbb{R}\mathrm{e}(\overline{\phi_1e^{i\varphi}}\phi_2)$$

lead to

$$\begin{aligned}\alpha_1|\psi_1^0|^4+2\alpha_{12}|\psi_1^0|^2|\psi_2^0|^2+\alpha_2|\psi_2^0|^4 &= \frac{\alpha_1+2\alpha_{12}+\alpha_2}{4}\left(|\phi_1|^2+|\phi_2|^2\right)^2\\ &\quad+\frac{\alpha_1-\alpha_2}{2}\left(|\phi_1|^2+|\phi_2|^2\right)\left(\cos(\theta)(|\phi_1|^2-|\phi_2|^2)+2\sin(\theta)\,\mathbb{R}\mathrm{e}(\overline{\phi_1e^{i\varphi}}\phi_2)\right)\\ &\quad+\frac{\alpha_1-2\alpha_{12}+\alpha_2}{4}\left(\cos(\theta)(|\phi_1|^2-|\phi_2|^2)+2\sin(\theta)\,\mathbb{R}\mathrm{e}(\overline{\phi_1e^{i\varphi}}\phi_2)\right)^2\\ &=\left[\alpha_1\cos^4(\frac{\theta}{2})+\alpha_2\sin^4(\frac{\theta}{2})+\frac{\alpha_{12}}{2}\sin^2(\theta)\right]|\phi_1|^4\\ &\quad+\left[\alpha_1\sin^4(\frac{\theta}{2})+\alpha_2\cos^4(\frac{\theta}{2})+\frac{\alpha_{12}}{2}\sin^2(\theta)\right]|\phi_2|^4\\ &\quad+\left[\frac{(\alpha_1+\alpha_2)}{2}\sin^2(\theta)+\alpha_{12}(1+\cos^2(\theta))\right]|\phi_1|^2|\phi_2|^2\\ &\quad+\left[(\alpha_1-\alpha_2)+(\alpha_1-2\alpha_{12}+\alpha_2)\cos(\theta)\right]\sin(\theta)|\phi_1|^2\,\mathbb{R}\mathrm{e}(\overline{\phi_1e^{i\varphi}}\phi_2)\\ &\quad+\left[(\alpha_1-\alpha_2)-(\alpha_1-2\alpha_{12}+\alpha_2)\cos(\theta)\right]\sin(\theta)|\phi_2|^2\,\mathbb{R}\mathrm{e}(\overline{\phi_1e^{i\varphi}}\phi_2)\\ &\quad+(\alpha_1-2\alpha_{12}+\alpha_2)\sin^2(\theta)\,\mathbb{R}\mathrm{e}\left[(\overline{\phi_2e^{i\varphi}}\phi_2)\right]\,.\end{aligned}$$

At least three points have to be adapted from the previous analysis:

1) When we take a test function $\psi = \hat{U}\begin{pmatrix} 0 \\ e^{-i\frac{y}{\sqrt{\tau_x}}}a_- \end{pmatrix}$ the energy $\mathcal{E}_\varepsilon(\psi)$ will be close to $\varepsilon^{2+2\delta}\tau_x\mathcal{E}_\tau(a_-)$, with

$$\mathcal{E}_\tau(a_-) = \langle a_-\,,\,\hat{H}_-a_-\rangle + \frac{G}{2}\int\left[\alpha_1\cos^4(\frac{\theta}{2}) + \alpha_2\sin^4(\frac{\theta}{2}) + \frac{\alpha_{12}}{2}\sin^2(\theta)\right]|a_-|^4\;dxdy\,,$$

and $\cos(\theta) = \frac{\sqrt{\tau_x}x}{\sqrt{1+\tau_x x^2}}$. Hence before taking the limit $\tau_x \to 0$ we have a position dependent nonlinearity. This will induce another error term when comparing with the energy $\mathcal{E}_H(a_-) = \langle a_-\,,\,H_{\ell_V}a_-\rangle + \frac{G(\alpha_1+\alpha_2+2\alpha_{12})}{8}\int|a_-|^4$, in the limit $\tau_x \to 0$. Another possibility consists in considering the case when $|\alpha_2-\alpha_1| + |\alpha_{12}-\alpha_1|$ is small as $(\varepsilon,\tau_x)\to(0,0)$. The energy $\mathcal{E}_\tau(\psi)$, written as,

$$\langle a_-\,,\,\hat{H}_-a_-\rangle + \frac{G}{2}\int\left[\alpha_1 + (\alpha_2-\alpha_1)\sin^4(\frac{\theta}{2}) + \frac{(\alpha_{12}-\alpha_1)}{2}\sin^2(\theta)\right]|a_-|^4\;dxdy\,,$$

will converge to $\mathcal{E}_H(a_-) = \langle a_-\,,\,H_{\ell_V}a_-\rangle + \frac{G\alpha_1}{2}\int|a_-|^4$.

2) The existence of a minimizer for $\mathcal{E}_\varepsilon$ and the variational argument showing that $\|a_+\|_{L^2}^2 = \mathcal{O}(\varepsilon^{2+4\delta}) + \mathcal{E}_{\varepsilon,min}$ when $\psi = \hat{U}\begin{pmatrix} e^{i\frac{y}{\sqrt{\tau_x}}}a_+ \\ e^{-i\frac{y}{\sqrt{\tau_x}}}a_- \end{pmatrix}$ is a ground state for $\mathcal{E}_\varepsilon$ will be essentially the same as in the isotropic case.

3) The analysis and the use of the Euler-Lagrange equation for ground states of $\mathcal{E}_\varepsilon$, like in Subsection 5.4, will certainly be more delicate because it will be a system, and the vanishing of the crossing terms have to be considered more carefully.

6.3 Minimization for excited states

One may consider like in [DGJO] the question of minimizing the energy, for states prepared according to ψ_+ local eigenvector of the potential. Two things have to be modified in order to adapt the previous analysis:

1) The space of states, on which the energy is minimized has to be specified. The unitary transform $\hat{U}$ introduced in Theorem 2.1 provides a simple way to formulate this minimization problem: set $\mathcal{F}_+ = \mathrm{Ran}\,\hat{U}P_+$ with $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and consider

$$\inf_{\psi\in\mathcal{F}_+\,,\,\|\psi\|=1}\mathcal{E}_\varepsilon(\psi)\,.$$

2) In order to get asymptotically as $\varepsilon\to 0$, the same scalar minimization problems with $\mathcal{E}_\tau$ and $\mathcal{E}_H$, the external potential $V_{\varepsilon,\tau}$ has to be changed. It must be now

$$V_{\varepsilon,\tau}(x,y) = \frac{\varepsilon^{2+2\delta}}{\ell_V^2}v(\sqrt{\tau_x}x,\sqrt{\tau_x}y) - \sqrt{1+\tau_x x^2} - \varepsilon^{2+2\delta}\left[\frac{\tau_x^2}{(1+\tau_x x^2)^2} + \frac{1}{1+\tau_x x^2}\right].$$

The analysis of this problem is essentially the same as for the complete minimization problem. It is even simpler because the unitary $\hat{U}$ is directly introduced. A slightly different question is about the minimization of the energy $\mathcal{E}_\varepsilon$ in the space $\mathcal{F}_+^0 = \mathrm{Ran}\,\hat{U}_0 P_+$, but the accurate comparison between $\hat{U}$ and $\hat{U}_0$ widely used through this article would lead to similar results.
Possibly this extension can even be generalized to higher rank matricial potentials with eigenvectors $\psi_1, \ldots, \psi_N$, for states modelled on any given ψ_k .

6.4 Time dynamics of adiabatically prepared states

When $\psi_0 = \hat{U}\begin{pmatrix} 0 \\ a_{-,0} \end{pmatrix}$ (resp. $\psi_0 = \hat{U}\begin{pmatrix} a_{+,0} \\ 0 \end{pmatrix}$), the question is whether the solution $\psi(t)$ to

$$i\partial_t \psi = H_{Lin}\psi + G_{\varepsilon,\tau}|\psi|^2\psi \quad , \quad \psi(t=0) = \psi_0$$

remains close to $\hat{U}\begin{pmatrix} 0 \\ a_-(t) \end{pmatrix}$ (resp. $\begin{pmatrix} a_+(t) \\ 0 \end{pmatrix}$) with

$$i\partial_t a_- = \varepsilon^{2+2\delta}\tau_x \hat{H}_- a_- + \frac{G\varepsilon^{2+2\delta}\tau_x}{2}|a_-|^2 a_- \quad , \quad a_-(t=0) = a_{-,0}\,,$$

$$\text{resp.} \qquad i\partial_t a_+ = \varepsilon^{2+2\delta}\tau_x \hat{H}_+ a_+ + \frac{G\varepsilon^{2+2\delta}\tau_x}{2}|a_+|^2 a_+ \quad , \quad a_+(t=0) = a_{+,0}\,.$$

More precisely, the question is about the range of time where this approximation is valid: what is the size of T_ε w.r.t ε such that $\sup_{t\in[-T_\varepsilon,T_\varepsilon]} \|\psi(t) - \hat{U}\begin{pmatrix} 0 \\ a_-(t) \end{pmatrix}\|$ (resp. $\sup_{t\in[-T_\varepsilon,T_\varepsilon]} \|\psi(t) - \begin{pmatrix} a_+(t) \\ 0 \end{pmatrix}\|$) remains small.

Since the approximation of $\hat{U}^* H_{Lin}\hat{U}$ by $\begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix}$ is good in the low frequency range, a natural assumption will be that the initial data are supported in the low frequency region

$$a_{0,\pm} = \chi(\varepsilon^2\tau_x D_q^2)a_{0,\pm}\,,$$

for some compactly supported χ. Then the question is whether the norm of $(1 - \tilde{\chi}(\varepsilon^2\tau_x D_q^2))\psi(t)$ remains small for $t \in [0, T_\varepsilon]$ for $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi \prec \tilde{\chi}$. Then the two last parts of Section 3, concerned with the high frequency part and the effect of $\hat{U}$ on the nonlinear term, have to be reconsidered.
Note that the adiabatically prepared state with $\psi_0 = \hat{U}\begin{pmatrix} a_{0,+} \\ 0 \end{pmatrix}$ are probably not stable for very long time, $T_\varepsilon = \mathcal{O}(e^{\frac{c}{\varepsilon}})$, because the characteristic set

$$\mathcal{C}_\lambda = \big\{(q,p) \in \mathbb{R}^4, \quad \det\big(\varepsilon^{2\delta}\tau_x p^2 + V_{\varepsilon,\tau}(q) + M(q) - \lambda\big) = 0\big\}$$

contains two components when $\lambda \geq \min E_+$, one corresponding to the higher level of $M(q)$ with $|p|^2 \leq C(\lambda)$, and another one for the lower level of $M(q)$ but with large p's.

This means that a tunnel effect will occur between the two levels, so that adiabatically prepared states, with energies close to $\lambda \geq \min E_+$, will not remain in this state for (very) large times.

A Semiclassical calculus

A.1 Short review in the scalar case

Consider a Hörmander metric, g, that is a metric on $\mathbb{R}^{2d}_{q,p}$, which satisfies the uncertainty principle, the slowness and temperance conditions (see [Hor, BoLe]) and consider g-weights (slow and tempered for g) M, M_1, M_2. The symplectic form on $\mathbb{R}^{2d}_{q,p}$ is denoted by σ :

$$\sigma(X, X') = \sum_{j=1}^{d} p^j q'_j - q_j p'^j \,, \quad X = (q, p)\,, X' = (q', p')\,.$$

The dual metric g^σ is given by $g^\sigma_X(T) = \sup_{T' \neq 0} \frac{|\sigma(T,T')|^2}{g_X(T')}$ and the gain associated with g is

$$\lambda(X) = \inf_{T \neq 0} \left(\frac{g^\sigma_X(T)}{g_X(T)} \right)^{1/2} \geq 1 \quad \text{(uncertainty)}\,. \tag{A.1}$$

In the scalar case, the space $S(M, g)$ is then the subspace of $\mathcal{C}^\infty(\mathbb{R}^{2d}_{p,q}; \mathbb{C})$ of functions such that

$$\forall N \in \mathbb{N}, \exists C_N \qquad \sup_{g_X(T_\ell)=1} |(T_1 \dots T_N a)(X)| \leq C_N M(X)\,,$$

after identifying a vector field T_ℓ with a first-order differentiation operator. For a given $N \in \mathbb{N}$, the system of seminorms $(p_{N,M,g})_{N \in \mathbb{N}}$ defined by

$$p_{N,M,g}(a) = \sup_{X \in \mathbb{R}^{2d}} \sup_{g_X(T_\ell)=1} M(X)^{-1} |(T_1, \dots, T_\ell a)(X)|$$

makes $S(M, g)$ a Fréchet space.
For $a \in \mathcal{S}'(\mathbb{R}^{2d}_{q,p})$ and $\varepsilon > 0$, the Weyl quantized operator $a^W(q, \varepsilon D_q) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is given by its kernel

$$a^W(q, \varepsilon D_q)(q, q') = \int_{\mathbb{R}^d} e^{i \frac{(q-q').p}{\varepsilon}} a(\frac{q+q'}{2}, p) \, \frac{dp}{(2\pi\varepsilon)^d}\,.$$

When no confusion is possible, we shall use the shortest notations

$$a^W(q, \varepsilon D_q) = a^W = a(q, \varepsilon D_q)\,.$$

When $a \in S(M, g)$, it sends $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^d)$) into itself and the composition $a_1 \circ a_2$ makes sense for $a_j \in S(M_j, g)$. The Moyal product $a_1 \sharp^\varepsilon a_2$ is then defined as the Weyl-

symbol of $a_1^W(q,\varepsilon D_q)\circ a_2^W(q,\varepsilon D_q)$:

$$
\begin{aligned}
a_1\sharp^\varepsilon a_2(X) &= \left(e^{\frac{i\varepsilon}{2}\sigma(D_{X_1},D_{X_2})}a_1(X_1)a_2(X_2)\right)\Big|_{X_1=X_2=X}\\
&= \sum_{j=0}^{J-1}\frac{\left(\frac{i\varepsilon}{2}\sigma(D_{X_1},D_{X_2})\right)^j}{j!}a_1(X_1)a_2(X_2)\Big|_{X_1=X_2=X}\\
&+\int_0^1\frac{(1-\theta)^{J-1}}{(J-1)!}e^{\frac{i\varepsilon}{2}\theta\sigma(D_{X_1},D_{X_2})}\left(\frac{i\varepsilon}{2}\sigma(D_{X_1},D_{X_2})\right)^J a_1(X_1)a_2(X_2)\Big|_{X_1=X_2=X}\\
&= \sum_{j=0}^{J-1}\frac{\left(\frac{i\varepsilon}{2}\sigma(D_{X_1},D_{X_2})\right)^j}{j!}a_1(X_1)a_2(X_2)\Big|_{X_1=X_2=X}+\varepsilon^J R_J(a_1,a_2,\varepsilon)(X)\,, \qquad \text{(A.2)}
\end{aligned}
$$

where $R_J(.,.,\varepsilon)$ is a uniformly continuous bilinear operator from $S(M_1,g)\times S(M_2,g)$ into $S(M_1M_2\lambda^{-J},g)$ (i.e. any seminorm of $R_J(a_1,a_2,\varepsilon)$ is uniformly controlled by some bilinear expression of a finite number of seminorms of a_1 and a_2).
The three first terms of the previous expansion are given by

$$
\begin{aligned}
a_1\sharp^\varepsilon a_2 &= a_1a_2+\frac{\varepsilon}{2i}\left[\partial_{p_k}a_1\partial_{q^k}a_2-\partial_{q^k}a_1\partial_{p_k}a_2\right]\\
&-\frac{\varepsilon^2}{8}\left[(\partial^2_{p_k,p_\ell}a_1)(\partial^2_{q^kq^\ell}a_2)+(\partial^2_{q^k,q^\ell}a_1)(\partial^2_{p_k,p_\ell}a_2)-2(\partial^2_{p_k,q^\ell}a_1)(\partial^2_{q^k,p_\ell}a_2)\right] \qquad \text{(A.3)}\\
&\quad+\varepsilon^3R_3(a_1,a_2,\varepsilon)\,,
\end{aligned}
$$

by making use of the Einstein convention $s^jt_j=\sum_j s^jt_j$.

With the small parameter $\varepsilon\in(0,\varepsilon_0)$, it is more convenient to consider the Fréchet-space $S_u(M,g)$ of bounded functions from $(0,\varepsilon_0)$ to $S(M,g)$ endowed with the seminorms:

$$P_{N,M,g}(a)=\sup_{\varepsilon\in(0,\varepsilon_0)}p_{N,M,g}(a(\varepsilon))\,.$$

The subscript $_u$ stands for uniform seminorm estimates and we shall give a variation of this definition in Subsection A.4 below. The space of ε-quantized family of symbols will be denoted by $\mathrm{Op}S_u(M,g)$:

$$(a(\varepsilon)\in S_u(M,g))\Leftrightarrow\left(a^W(q,\varepsilon D_q,\varepsilon)\in\mathrm{Op}S_u(M,g)\right)\,.$$

We recall the Beals-criterion proved in [BoCh] (see [NaNi] for the ε-dependent version) for diagonal Hörmander metrics of the form $g=\sum_{j=1}^d\frac{{dq^j}^2}{\varphi_j(X)^2}+\frac{{dp_j}^2}{\psi_j(X)^2}$.
Set $\mathfrak{D}=(D_{q^1},\dots,D_{q^d},q^1,\dots,q^d)$ and for $E\in\mathbb{N}^{2d}$ introduce the multi-commutator $\mathrm{ad}^E_{\mathfrak{D}}=\prod_{j=1}^{|E|}\mathrm{ad}^{E_j}_{\mathfrak{D}_j}$ acting on the continuous operators from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ and the weight $M_E(X)=(\varphi(X),\psi(X))^E$.
Then the Beals criterion says that an operator $A:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ belongs to $\mathrm{Op}S(1,g)$ if and only if

$$\mathrm{ad}^E_{\mathfrak{D}}A\in\mathcal{L}(L^2(\mathbb{R}^d),H^\varepsilon(M_E))$$

for all $E \in \mathbb{N}^{2d}$, when the Sobolev space $H^\varepsilon(M_E)$ is given by

$$\|u\|_{H^\varepsilon(M_E)} = \|M_E(q, \varepsilon D_q)u\|_{L^2} .$$

Moreover the family of seminorms $(Q_N)_{N\in\mathbb{N}}$, defined by,

$$Q_N(A) = \sup_{\varepsilon\in(0,\varepsilon_0)} \max_{|E|=N} \varepsilon^{-N} \| \operatorname{ad}^E_{\mathfrak{D}} A(\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d),H^\varepsilon(M_E,g))} \tag{A.4}$$

on $\mathrm{Op}S_u(M, g)$, is uniformly equivalent to the family $(P_N(a))_{N\in\mathbb{N}}$ on $S_u(M, g)$ after the identification $A(\varepsilon) = a^W(q, \varepsilon D_q, \varepsilon)$. "Uniformly" means here that the comparison of the two topologies is expressed with constants independent of $\varepsilon \in (0, \varepsilon_0)$.
Below is an example of a metric which satisfies all the assumptions and which is used in our computations

$$g = \frac{dq'^2}{(1 + |q'|^2)^{\varrho}} + dq''^2 + \frac{dp^2}{(1 + |p|^2)^{\varrho'}} , \quad m \geq 0 , \quad \varrho, \varrho' \in [0, 1] , \tag{A.5}$$

with the gain function $\lambda(q, p) = \langle p\rangle^{\varrho'}$.
It is convenient to introduce the class of negligible symbols and operators.

Definition A.1. *An element $a \in S_u(1, g)$, belongs to $\mathcal{N}_{u,g}$ if*

$$\forall N, N' \in \mathbb{N}, \exists C_{N,N'} > 0, \quad \varepsilon^{-N} a(\varepsilon) \in S_u(\lambda^{-N'}, g) .$$

Similarly, $Op\mathcal{N}_{u,g}$ denotes the ε-quantized version:

$$\left(a^W(q, \varepsilon D_q, \varepsilon) \in Op\mathcal{N}_{u,g}\right) \Leftrightarrow (a \in \mathcal{N}_{u,g}) .$$

Combined with the following relation between cut-off functions it provides easy estimates from phase-space localization.

Definition A.2. *For two cut-off functions $\chi_1, \chi_2 \in \mathcal{C}^\infty_0(\mathbb{R}^{2d})$, $0 \leq \chi_{1,2} \leq 1$, the notation $\chi_1 \prec \chi_2$ means that $\chi_2 \equiv 1$ in a neighborhood of* $\operatorname{supp}\chi_1$.

For example the pseudodifferential calculus leads to

$$(\chi_1 \prec \chi_2) \Rightarrow (\forall a \in S_u(M, g), \quad (1 - \chi_2)\sharp^\varepsilon a \sharp^\varepsilon \chi_1 \in \mathcal{N}_{u,g}) , \tag{A.6}$$

when the weight M satisfies $M \leq C\lambda^C$ for some $C > 0$.

A.2 Applications to matricial operators

Operator valued pseudodifferential calculus has been studied in [Bak]. When $\mathfrak{h}$ is a Hilbert space it suffices to tensorize the previous calculus with $\mathcal{L}(\mathfrak{h})$, which corresponds to the componentwise definition in $\mathcal{M}_n(\mathbb{C})$ when $\mathfrak{h} = \mathbb{C}^n$. The corresponding class of symbols associated with a Hörmander metric g and a g-tempered weight M is denoted by $S(M, g; \mathcal{L}(\mathfrak{h}))$

while the set of bounded families in $S(M,g;\mathcal{L}(\mathfrak{h}))$ parametrized by $\varepsilon \in (0,\varepsilon_0)$ is denoted by $S_u(M,g;\mathcal{L}(\mathfrak{h}))$.
The asymptotic expansions (A.2)-(A.3) of the Moyal product clearly holds (see [PST] for a presentation without specifying the remainder terms) if one takes care of the order of the symbols. For example, the two first terms of the expansion of a commutator $[a_1(q,\varepsilon D_q), a_2(q,\varepsilon D_Q)]$ are

$$a_1 \sharp^\varepsilon a_2 - a_2 \sharp^\varepsilon a_1 = [a_1(q,p), a_2(q,p)] + \frac{\varepsilon}{2i}(\partial_p a_1 \partial_q a_2 - \partial_q a_1 \partial_p a_2) - \frac{\varepsilon}{2i}(\partial_p a_2 \partial_q a_1 - \partial_q a_2 \partial_p a_1) + \varepsilon^2 \left[R_2(a_1,a_2,\varepsilon) - R_2(a_2,a_1,\varepsilon)\right] \quad \text{(A.7)}$$

with no simpler expression when the matricial symbols do not commmute.
When the Hörmander metric g has the form $\sum_j \frac{dq^j{}^2}{\varphi_j^2} + \frac{dp_j^2}{\psi_j^2}$ the uniform Beals criterion also holds: In the seminorms $Q(a)$ defined in (A.4) simply replace $L^2(\mathbb{R}^d)$ by the Hilbert tensor product $L^2(\mathbb{R}^d)\otimes\mathfrak{h}$ and consider the operators $\mathfrak{D}$ and $M_E(q,\varepsilon D_q)$ as the diagonal ones $\mathfrak{D}\otimes \mathrm{Id}_\mathfrak{h}$ and $M_E(,\varepsilon D_q)\otimes \mathrm{Id}_\mathfrak{h}$.
Finally the Definition A.1 of negligible symbols also makes sense for matricial symbols after replacing $\mathrm{Op}S(\lambda^{-N'},g)$ by $\mathrm{Op}S(\lambda^{-N'},g;\mathcal{L}(\mathfrak{h}))$.

We end this section with standard applications of the Beals criterion.

Proposition A.3. *Consider a diagonal Hörmander metric $g = \sum_{j=1}^d \frac{dq^j{}^2}{\varphi_j(X)^2} + \frac{dp_j^2}{\psi_j(X)^2}$ and the constant metric $g_0 = dq^2 + dp^2$. Assume that any $f \in \{\varphi_j, \psi_j,\ j\in\{1,\dots,d\}\}$ is g_0 slow and tempered weight such that $f(q,\varepsilon D_q)^s$ belongs $OpS_u(f^s,g_0)$ for any $s\in\mathbb{N}^2$.
Assume that $A \in OpS_u(1,g;\mathcal{L}(\mathfrak{h}))$ is a family of invertible operators in $\mathcal{L}(L^2(\mathbb{R}^d)\otimes\mathfrak{h})$ and such that $\|A^{-1}(\varepsilon)\|$ is uniformly bounded in w.r.t $\varepsilon\in(0,\varepsilon_0)$. Then A^{-1} belongs to $OpS_u(1,g;\mathcal{L}(\mathfrak{h}))$.*

Proof: We start from the relation

$$\mathrm{ad}_\mathfrak{D}^E A^{-1} = \sum_{|E'|=|E|, E'\in\mathbb{N}^{|E|}} c_{E,E'} A^{-1} \prod_{j=1}^{|E|} \left[\left(\mathrm{ad}_\mathfrak{D}^{E'_j} A\right) A^{-1}\right]. \quad \text{(A.8)}$$

which holds in $\mathcal{L}(L^2(\mathbb{R}^d)\otimes\mathfrak{h})$ for all $E\in\mathbb{N}^{2d}$. Hence the Beals criterion in the metric g_0 says that A^{-1} belongs to $\mathrm{Op}S_u(1,g:\mathcal{L}(\mathfrak{h}))$. In particular A^{-1} belongs to $\mathcal{L}(H^\varepsilon(M_{E"}))$ for any $E"\in\mathbb{N}^{2d}$. This allows to apply the Beals criterion in the metric g and yields the result. □

A.3 Pseudodifferential projections

An application of the Beals criterion says that a true pseudodifferential projection can be made from an approximate one at the principal symbol level. This holds for matricial symbols.

[2]This last condition is redundant after a possible modification of φ_j adn ψ_j if one refers to [BoCh], but easier to check directly in our examples than giving the general proof.

Proposition A.4. *Consider a diagonal Hörmander metric* $g = \sum_{j=1}^{d} \frac{dq^j{}^2}{\varphi_j(X)^2} + \frac{dp_j^2}{\psi_j(X)^2}$ *with the same properties as in Proposition A.3. Assume that the operator* $\hat{\Pi} \in OpS_u(1, g; \mathcal{L}(\mathfrak{h}))$ *satisfies*

$$\left(\hat{\Pi} \circ \hat{\Pi} - \hat{\Pi}\right) = \varepsilon^{\mu} \hat{R}_{\mu} + \varepsilon^{\nu} \hat{R}_{\nu} \,,$$

with $R_{\mu} \in S_u(M, g; \mathcal{L}(\mathfrak{h}))$, $R_{\nu} \in S_u(N, g; \mathcal{L}(\mathfrak{h}))$ $\nu > \mu > 0$ *and* $M, N \leq 1$. *Assume additionally that there exist* $\chi, \chi' \in S_u(1, g; \mathcal{L}(\mathfrak{h}))$, $0 \leq \chi \leq \chi' \leq 1$ *such that* $\chi \prec \chi'$ *and* $\chi' R_{\mu} = 0$.
Then for $\varepsilon_1 \leq \varepsilon_0$ *small enough, the operator*

$$\hat{P} = \frac{1}{2i\pi} \int_{|z-1|=1/2} (z - \hat{\Pi})^{-1} \, dz$$

is well defined for $\varepsilon \in (0, \varepsilon_1)$ *and satisfies*

$$\begin{aligned} & \hat{P} \circ \hat{P} = \hat{P} \quad in \quad \mathcal{L}(L^2(\mathbb{R}^d) \otimes \mathfrak{h}) \,, \\ and \quad & \varepsilon^{-\nu} \left(\hat{P} - \hat{\Pi}\right) = \varepsilon^{\mu} \hat{\Pi}_{\mu} + \varepsilon^{\nu} \hat{\Pi}_{\nu} \,, \quad OpS_u(M, g; \mathcal{M}_2(\mathbb{C})) \,, \quad (\varepsilon \leq \varepsilon_1) \,. \end{aligned}$$

with $\Pi_{\mu} \in S_u(M, g; \mathcal{L}(\mathfrak{h}))$, $\Pi_{\nu} \in S_u(N, g; \mathcal{L}(\mathfrak{h}))$ *and* $\hat{\Pi}_{\mu} \circ \hat{\chi} \in Op\mathcal{N}_{u,g}$.

Proof: The first result concerned with the definition of $\hat{P}$ is a direct application of the simple general result in Lemma A.5 applied with $T = \hat{\Pi}$ and $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathfrak{h}$. Note that our assumptions $\nu > 1$ and $M \leq 1$ ensure that $\|\hat{\Pi}^2 - \hat{\Pi}\| \leq \frac{1}{8}$ as soon as $\varepsilon \leq \varepsilon_1$ with ε_1 small enough.
Writing

$$\hat{\Pi} - \hat{P} = (\hat{\Pi}^2 - \hat{\Pi})(A_1 - A_0)$$

with

$$A_x = \frac{1}{2i\pi} \int_{|z-x|} (z - \hat{\Pi})^{-1} \, dz \,, \quad x \in \{0, 1\} \,,$$

reduces the problem to proving $A_x \in \mathrm{Op}S_u(1, g; \mathcal{L}(\mathfrak{h}))$ or $(z - \hat{\Pi})^{-1} \in \mathrm{Op}S_u(1, g; \mathcal{L}(\mathfrak{h}))$ when $\{|z - x| = 1/2\}$ with uniform bounds. But this was proved in Proposition A.3 as a consequence of the Beals criterion. This ends the proof. □

Lemma A.5. *Assume that in an Hilbert space* $\mathcal{H}$, *the operator* $T \in \mathcal{L}(\mathcal{H})$ *satisfies* $\|T^2 - T\| \leq \delta < 1/4$ *and* $\|T\| \leq C$ *then*

$$\sigma(T) \subset \{z \in \mathbb{C} \,, |z(z-1)| \leq \delta\} \subset \{z \in \mathbb{C} \,, |z| \leq c_{\delta}\} \cup \{z \in \mathbb{C} \,, |z - 1| \leq c_{\delta}\} \,, \quad c_{\delta} < \frac{1}{2} \,,$$

$$\max \left\{ \|(z - T)^{-1}\|, |z - 1| = \frac{1}{2} \right\} \leq 2 \frac{2C + 1}{1 - 4\delta} \,.$$

Moreover, the operator

$$P = \frac{1}{2i\pi} \int_{|z-1|=1/2} (z-T)^{-1}\, dz\,.$$

differs from T according to

$$T - P = (T^2 - T)(A_1 - A_0) = (A_1 - A_0)(T^2 - T)\,, \tag{A.9}$$

$$\textit{with} \qquad A_1 = \frac{1}{2i\pi} \int_{|z-1|=\frac{1}{2}} (T-z)^{-1}(1-z)^{-1}\, dz \tag{A.10}$$

$$\textit{and} \qquad A_0 = \frac{1}{2i\pi} \int_{|z|=\frac{1}{2}} (T-z)^{-1} z^{-1}\, dz\,. \tag{A.11}$$

Proof: If $z \in \sigma(T)$ then $z(z-1)$ belongs to $\sigma(T(T-1)) \subset \left\{ z \in \mathbb{C}, |z(z-1)| \leq \frac{1}{4} \right\}$ (Remember that $|z(z-1)| = \frac{1}{4}$ means $|Z - \frac{1}{4}| = \frac{1}{4}$ with $Z = (z - \frac{1}{2})^2$). Consider $z \in \mathbb{C}$ such that $|z-1| = \frac{1}{2}$, then the relation

$$(T-z)(T-(1-z)) = z(1-z) + (T^2 - T)\,,$$

with $|z(1-z)| \geq \frac{1}{4}$ and $\|T^2 - T\| \leq \delta < \frac{1}{4}$, implies

$$\|(T-z)^{-1}\| \leq \|T-(1-z)\| \|[z(1-z) + T^2 - T]^{-1}\| \leq \frac{C + \frac{1}{2}}{\frac{1}{4} - \delta} \quad \text{for } |z-1| = \frac{1}{2}\,.$$

The symmetry with respect to $z = \frac{1}{2}$ due to $(1-T)(1-T) - (1-T) = T^2 - T$ implies also

$$\|(T-z)^{-1}\| \leq \frac{C + \frac{1}{2}}{\frac{1}{4} - \delta} \quad \text{for } |z| = \frac{1}{2}\,.$$

Compute

$$\begin{aligned} T - P &= \frac{1}{2i\pi} \int_{|z-1|=\frac{1}{2}} \left[T(z-1)^{-1} - (z-T)^{-1} \right] dz \\ &= (T-1)P + (T^2 - T)A_1 \quad \text{with} \quad A_1 = \frac{1}{2i\pi} \int_{|z-1|=\frac{1}{2}} (T-z)^{-1}(1-z)^{-1}\, dz\,. \end{aligned}$$

In particular this implies to $T(1-P) = (T^2 - T)A_1$ while replacing T with $(1-T)$ and P with $1-P$ leads to

$$P - T = -T(1-P) + (T^2 - T)A_0 \quad \text{with} \quad A_0 = \frac{1}{2i\pi} \int_{|z|=\frac{1}{2}} (T-z)^{-1} z^{-1}\, dz\,.$$

Summing the two previous identities yields the result. □

A.4 Extension to parameter dependent metrics

Additionnally to the semiclassical (or adiabatic) parameter, we need other parameters $\tau = (\tau', \tau'') \in (0,1]^2$ on which the metric $g = g_\tau$ depends. In general consider $\tau \in \mathcal{T} \subset \mathbb{R}^\nu$ and a family of Hörmander metrics $(\gamma_\tau)_{\tau\in\mathcal{T}}$ defined on $\mathbb{R}^{2d}_{q,p}$.

Definition A.6. *The familly of metric $(g_\tau)_{\tau\in\mathcal{T}}$ is said admissible if the uncertainty principle* (A.1) *is satisfied and if the slowness and temperance constants C_1, C_2, N_2 involved in*

$$(slowness) \qquad \left(g_{\tau,X}(X-Y) \leq \frac{1}{C_1}\right) \Rightarrow \left(\left(\frac{g_{\tau,X}}{g_{\tau,Y}}\right)^{\pm 1} \leq C_1\right),$$

$$(temperance) \qquad \left(\frac{g_{\tau,X}}{g_Y}\right)^{\pm 1} \leq C_2(1+g^\sigma_{\tau,X}(X-Y))^{N_2},$$

can be chosen uniformly w.r.t $\tau \in \mathcal{T}$.
Accordingly a family of weights $(M_\tau)_{\tau\in\mathcal{T}}$ will be admissible if the slowness and temperance constants of M_τ w.r.t g_τ can be chosen uniform w.r.t $\tau \in \mathcal{T}$.

The important point is that all the estimates of the Weyl-Hörmander pseudodifferential calculus (see [Hor, BoLe]), including the equivalence of norms in the Beals criterion of [BoCh], occur with constants which are determined by the dimension d, the uncertainty lower bound (which is 1 here), the slowness and temperance constants. Hence all the pseudodifferential and semiclassical estimates, (operator norms or seminorms of remainder terms) are uniform w.r.t to (ε, τ) as long as the symbols, $a_k(\varepsilon,\tau)$, $k = 1, 2$, have uniformly controlled seminorm in $S(M_{\tau,k}, g_\tau)$, w.r.t $(\varepsilon,\tau) \in (0,\varepsilon_0] \times \mathcal{T}$.
The definition of symbol classes $S_u(M,g)$ with uniform control of seminorms w.r.t $\varepsilon \in (0,\varepsilon_0]$ can be extended to admissible families $(M_\tau, g_\tau)_{\tau\in\mathcal{T}}$.

Definition A.7. *For an admissible family $(M_\tau, g_\tau)_{\tau\in\mathcal{T}}$, the set of parameter dependent symbol $a(X,\varepsilon,\tau)$, $X \in \mathbb{R}^{2d}$, $(\varepsilon,\tau) \in (0,\varepsilon_0] \times \tau$ with uniform estimates*

$$\forall N \in \mathbb{N},\ \exists C_N > 0,\ \forall (\varepsilon,\tau) \in (0,\varepsilon_0] \times \mathcal{T}, \quad p_{N,M_\tau,g_\tau}(a(\varepsilon,\tau)) \leq C_N,$$

is denoted by $S_u(M_\tau, g_\tau, \mathcal{L}(\mathfrak{h}))$, $\tau \in \mathcal{T}$.
Equivalently the set of semiclassically quantized operators $a(q, \varepsilon D_q, \varepsilon, \tau)$ when the symbol a belongs to $S_u(M_\tau, g_\tau; \mathcal{L}(\mathfrak{h}))$ is denoted by $OpS_u(M_\tau, g_\tau; \mathcal{L}(\mathfrak{h}))$.
The set of negligible symbols and operators associated with $(g_\tau)_{\tau\in\mathcal{T}}$ with uniform estimates in Definition A.1 w.r.t $\tau \in \mathcal{T}$ is denoted by $\mathcal{N}_{u,g_\tau}$ and $Op\mathcal{N}_{u,g_\tau}$.

Proposition A.8. *For $\tau = (\tau', \tau'') \in (0,1]^2$ the family $(g_\tau)_{\tau\in(0,1]^2}$ defined on $\mathbb{R}^{2d} = \mathbb{R}^{2(d'+d'')}$ by*

$$g_\tau = \frac{\tau' dq'^2}{\langle\sqrt{\tau'}q'\rangle^2} + \tau'' dq''^2 + \frac{\tau'\tau'' dp^2}{\langle\sqrt{\tau'\tau''}p\rangle^2}$$

is admissible.

Proof: It is easier to consider the symplectically equivalent metric (use the transform $(q', q'', p', p'') \to (\tau'^{\frac{1}{2}} q', \tau''^{\frac{1}{2}} q'', \tau'^{-\frac{1}{2}} p', \tau''^{-\frac{1}{2}} p'')$)

$$\tilde{g}_\tau = \frac{dq'^2}{\langle q' \rangle^2} + dq'' + \frac{\tau'^2 dp'^2 + \tau''^2 dp''^2}{\langle p \rangle_\tau^2}$$

after setting $\langle p \rangle_\tau^2 = 1 + \tau'^2 p'^2 + \tau''^2 p''^2$. Firstly remember that the metric

$$g_{(1,1)} = \frac{dq'^2}{\langle q' \rangle^2} + dq'' + \frac{dp^2}{\langle p \rangle^2} .$$

is a Hörmander metric. The metric $\tilde{g}_\tau^\sigma$ is given by

$$\tilde{g}_\tau^\sigma = \frac{\langle p \rangle_\tau^2}{\tau'^2} dq'^2 + \frac{\langle p \rangle_\tau^2}{\tau''^2} dq''^2 + \langle q' \rangle^2 dp'^2 + dp''^2 .$$

Hence the uncertainty principle (A.1) is satisfied with

$$\lambda_\tau(q', q'', p', p'') = \min \left\{ \langle q' \rangle \frac{\langle p \rangle_\tau}{\tau'}, \frac{\langle p \rangle_\tau}{\tau''} \right\} \geq \langle p \rangle_\tau \geq 1 .$$

In order to check the uniform slowness and temperance of $\tilde{g}_\tau$, introduce the new variables $X_\tau = (q', q'', \tau' p', \tau'' p'')$ when $X = (q', q'', p', p'')$ with a similar definition for Y_τ and T_τ.

Slowness: Write

$$\left(\tilde{g}_{\tau,X}(X - Y) \leq \frac{1}{C_1} \right) \Leftrightarrow \left(\tilde{g}_{(1,1),X_\tau}(X_\tau - Y_\tau) \leq \frac{1}{C_1} \right) .$$

When C_1 is slowness constant of $\tilde{g}_{(1,1)}$, this implies

$$\left(\frac{\tilde{g}_{(1,1),X_\tau}}{\tilde{g}_{(1,1),Y_\tau}} \right)^{\pm 1} \leq C_1$$

which is nothing but

$$\left(\frac{\tilde{g}_{\tau,X}}{\tilde{g}_{\tau,Y}} \right)^{\pm 1} \leq C_1 .$$

Temperance: Write

$$\left(\frac{\tilde{g}_{\tau,X}(T)}{\tilde{g}_{\tau,Y}(T)} \right)^{\pm 1} = \left(\frac{\tilde{g}_{(1,1),X_\tau}(T_\tau)}{\tilde{g}_{(1,1),Y_\tau}(T_\tau)} \right)^{\pm 1} \leq C_2 (1 + \tilde{g}^\sigma_{(1,1),X_\tau}(X_\tau - Y_\tau))^{N_2} ,$$

when C_2 and N_2 are the temperance constants for $\tilde{g}_{(1,1)}$. The problem is reduced to showing

$$\forall X, T \in \mathbb{R}^{2d} , \quad \tilde{g}^\sigma_{(1,1),X_\tau}(T_\tau) \leq \tilde{g}^\sigma_{\tau,X}(T) .$$

The expression of $\tilde{g}_\tau^\sigma$ gives with $X = (q', q'', p', p'')$ and $T = (\theta', \theta'', \pi', \pi'')$

$$\tilde{g}^\sigma_{(1,1)X_\tau}(T_\tau) = \langle p \rangle_\tau^2 \theta'^2 + \langle p \rangle_\tau^2 \theta''^2 + \langle q' \rangle^2 \tau'^2 \pi'^2 + \tau''^2 \pi''^2 \leq \tilde{g}^\sigma_{\tau,X}(T)$$

owing to $\max\{\tau', \tau''\} \leq 1$.

□

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