Design of LPV observers for LPV time-delay systems: an algebraic approach

Corentin Briat, Olivier Sename

<div align="center">

RESEARCH ARTICLE

Design of LPV observers for LPV Time-Delay Systems: An Algebraic Approach

</div>

<div align="center">

C. Briat[a*], O. Sename[b] and J-F. Lafay[c]

[a] ACCESS Linnaeus Centre, Division of Optimization and Systems Theory, KTH, Stockholm, Sweden;
[b] GIPSA-Lab, Grenoble-INP, Grenoble, France;
[c] IRCCyN, Nantes, France

(*May 10, 2011*)

</div>

The design of reduced-order observer for Linear Parameter Varying (LPV) time-delay systems is addressed. Necessary conditions guaranteeing critical structural properties for the observation error dynamics are first provided through nonlinear algebraic matrix equalities. An explicit parameterization of the family of observers fulfilling these necessary conditions is then derived. Finally, an approach based on Linear Matrix Inequalities (LMIs) is provided and used to select a suitable observer within this family, according to some criterion; e.g. maximization of the delay-margin or guaranteed suboptimal $\mathcal{L}_2$-gain. Examples from the literature illustrate the efficiency of the approach.

Keywords: Time-delay systems, Linear parameter varying systems, Observers, LMIs

1   Introduction

Time-delay systems are omnipresent: from engineering to economics passing through biology, ecology and social sciences. This ubiquity has made these systems more and more attractive over the last century (Kolmanovskii and Myshkis 1999, Niculescu 2001, Gu et al. 2003, Fridman 2001, Michiels and Niculescu 2007). However, most of the breakthroughs in that topic have been made within the past three decades and many problems remain open. Amongst other, the observation is a relatively few studied topic compared to stability, filtering and control. The observation of time-delay systems has been studied for instance in (Fiagbedzi and Pearson 1990, Fattouh et al. 1998, 1999b,a, Aggoune et al. 1999, Fattouh et al. 2000c,b,a,d, Sename 2001, Darouach 2001, Sename et al. 2001, Koenig et al. 2004, Darouach 2005, Koenig et al. 2006, Sename and Briat 2006, 2007, Sename 2007).

More recently, many works have been devoted to the analysis of LPV systems due to the generic framework they offer for the control of complex systems, e.g. nonlinear and Linear Time-Varying (LTV) systems (Shamma and Athans 1990, 1991, 1992, Packard 1994, Gahinet and Apkarian 1994, Apkarian and Gahinet 1995, Apkarian and Adams 1998, Wu 2001a, Scherer 2001, Iwasaki and Shibata 2001, Mehendale and Grigoriadis 2004). The emergence of the 'LPV trend' has taken the advantage of the rise of LMI-based techniques (Boyd et al. 1994) which provide a convenient framework for solving LPV-related problems.

The stability analysis and control of LPV time-delay systems have been explored in (Wu and Grigoriadis 2001, Wu 2001b, Zhang et al. 2002, Zhang and Grigoriadis 2005, Briat et al. 2007a, 2008b, 2010). However, the problem is far from being easy since this class of problems inherits of the difficulty of each subclass but, additionally, new troubles occur. As an example, many analy-

---

sis tools developed for LPV systems fail when applied to LPV time-delay systems (e.g. projection lemma, dualization lemma). Conversely, many results obtained for Linear Time Invariant (LTI) time-delay systems cannot be applied to LPV ones due to their time-varying nature, e.g. frequency domain methods or eigenvalue based analysis techniques. Finally, amongst the remaining possible approaches, Lyapunov-Krasovskii-Functionals-based techniques are certainly the most useful.

The problem of observation of LPV time-delay systems has been relatively few studied in the literature, several works are devoted to the filtering of LPV time-delay systems (Mohammadpour and Grigoriadis 2006a,b, 2007, Briat et al. 2009a). In (Briat et al. 2007b) is developed a simple method for observers design based on a 'free-weighting matrices' approach, extended to the LPV case. Compared to the authors previous work, the provided approach is much more efficient since it involves fewer decision variables and convex constraints (LMIs).

We propose in this paper to attack the problem using an algebraic approach, extended from (Darouach 2001) to the LPV case. Since the system is not time-invariant anymore, the problem is slightly more difficult than in the LTI case due to the parameter independence of some of the observer matrices. Structural necessary conditions reflecting the constraints on the desired structure for the dynamic model of the observation error are obtained. From these conditions, implicitly defining a family of observers, an explicit parameterization of all these observers is obtained. Finally, LMI conditions (Briat et al. 2008b, 2010) are then derived in order to choose a suboptimal observer which guarantees given objectives, such as a maximal delay-margin or a minimal suboptimal $\mathcal{L}_2$-gain of the transfer mapping the exogenous inputs to the observation error. Similarly, as in (Briat et al. 2009b,a, 2010), a 'slack-variable' approach is considered in order to make the design problem easier than the original LMI obtained directly from the Lyapunov-Krasovskii Theorem. The obtained conditions are both more computationally attractive and less conservative than the ones in (Briat et al. 2007b).

The paper is structured as follows: in Section 2 definitions and preliminary results are stated. Section 3 is devoted to the derivation of the main results of the paper. Finally, examples are considered in Section 4.

The notations are quite standard, for symmetric matrices A, B: $A \prec B$ stands for $A - B$ negative definite, $\mathbb{S}^n_{++}$ is the cone of symmetric positive definite matrices of dimension n. For a general matrix A, A^+ stands for its Moore-Penrose pseudoinverse, and for a square matrix A we define $A^{\mathcal{S}} = A + A^T$.

2   Definitions and Preliminary Results

The objective of the paper is to determine whether there exists an r-order LPV observer of the form:

$$
\begin{aligned}
\dot{\xi}(t) &= M_0(\rho)\xi(t) + M_h(\rho)\xi(t - h(t)) + S(\rho)u(t) + N_0(\rho)y(t) + N_h(\rho)y(t - h(t)) \\
\hat{z}(t) &= \xi(t) + Hy(t)
\end{aligned}
\tag{1}
$$

for the class of LPV time-delay systems defined by

$$
\begin{aligned}
\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) + E(\rho)w(t) \\
y(t) &= Cx(t), \qquad C \text{ full row rank} \\
z(t) &= Tx(t), \qquad T \text{ full row rank} \\
x(\theta) &= \phi(\theta), \ \theta \in [-h_m, 0]
\end{aligned}
\tag{2}
$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $\xi \in \mathbb{R}^r$, $u \in \mathbb{R}^q$, $z \in \mathbb{R}^r$, $y \in \mathbb{R}^s$ and $\phi \in C([-h_m, 0], \mathbb{R}^n)$ are the system state, the exogenous input, the observer state, the known system control input, the

signal to estimate, the measured output and the functional initial condition. In order to obtain an observer which depends on the parameter values $\rho(t)$ only (and not on their derivative $\dot{\rho}(t)$), the matrices C, H and T are set as parameter independent. Note that C can always be made parameter independent through an appropriate filtering of the measured output $y(t)$.

The parameter vector $\rho(\cdot)$ is assumed to belong to the set:

$$\mathscr{P} := \{\rho : \mathbb{R}_+ \to U_\rho \subset \mathbb{R}^p, \dot{\rho} \in co\{U_\nu\}\} \tag{3}$$

where U_ρ is a compact and connected set of $\mathbb{R}^p$, U_ν is the set of vertices of the parameter derivative space and $co\{\cdot\}$ is the convex-hull operator. The time-varying delay $h(t)$ is assumed to belong to the set

$$\mathscr{H} := \left\{h : \mathbb{R}_+ \to [0, h_m], \ \dot{h}(t) \le \mu < 1\right\} \tag{4}$$

where $0 < h_m < +\infty$.

With the above ingredients in mind, the main problem of the paper can be stated as follows:

Problem 2.1 Find an LPV time-delay observer of the form (1) for system (2) which

1) asymptotically stabilizes the observation error $e(t) = z(t) - \hat{z}(t)$, i.e. $||e(t)|| \to 0$ as $t \to +\infty$ when $w \equiv 0$ and $||\phi|| \ne 0$; and
2) provides a guaranteed (suboptimal) $\mathcal{L}_2$ gain from the exogenous input w to the estimation error e, i.e. $||e||_{\mathcal{L}_2} \le \gamma ||w||_{\mathcal{L}_2}$, $\forall w \in \mathcal{L}_2$ and for some $\gamma > 0$ when $||\phi|| = 0$.

The observation error defined above is governed by the following dynamical equation:

$$\begin{aligned}
\dot{e}(t) = {}& M_0(\rho)e(t) + M_h(\rho)e(t - h(t)) + (TA(\rho) - M_0(\rho)F - N_0(\rho)C - HCA(\rho))x(t) \\
& + (TA_h(\rho) - M_h(\rho)F - N_h(\rho)C - HCA_h(\rho))x(t - h(t)) + (FB(\rho) - S(\rho))u(t) \\
& + FE(\rho)w(t)
\end{aligned} \tag{5}$$

where $F = T - HC$. In order to make the observer error independent of the system state and the control input, the following nonlinear algebraic conditions must be fulfilled:

$$TA(\rho) - M_0(\rho)F - N_0(\rho)C - HCA(\rho) = 0, \tag{6}$$

$$TA_h(\rho) - M_h(\rho)F - N_h(\rho)C - HCA_h(\rho) = 0, \tag{7}$$

$$S(\rho) - FB(\rho) = 0. \tag{8}$$

If one the above conditions is not satisfied, say the first one, then the observation error cannot converge to 0 when the current state $x(t)$ is different from 0. Indeed, in such a case, the state will act as an exogenous input on the observation error dynamical model (5). This suggests that when the matrices are uncertain, it is unlikely possible to observe exactly the state of the system due to the practical impossibility of fulfilling the above equalities. This also suggests that when the system matrices are exactly known, it is theoretically possible to observe unstable systems[1]. This is basically not true when the system matrices are partially unknown. Note that the condition for the cancelation of the control input can always be satisfied since $S(\rho)$ is only involved in (8). So, assuming all the above conditions hold, the resulting dynamical model for the observation error writes

$$\dot{e}(t) = M_0(\rho)e(t) + M_h(\rho)e(t - h(t)) + (T - HC)E(\rho)w(t). \tag{9}$$

---

[1]This is however hardly possible in practice due to approximation/numerical errors.

This leads to the following proposition:

Proposition 2.2:   *There exists an r-order LPV observer of the form (1) for system (2) ensuring an $\mathcal{L}_2$ gain for the transfer $w \rightarrow e$ lower than γ if and only if*

1) *the observation error $e(t)$ is asymptotically stable. i.e. $e(t) \rightarrow 0$ as $t \rightarrow +\infty$ when $w \equiv 0$ and $||\phi|| \neq 0$;*

2) *The nonlinear algebraic equalities*
   2a) $TA(\rho) - M_0(\rho)(T - HC) - N_0(\rho)C - HCA(\rho) = 0$
   2b) $TA_h(\rho) - M_h(\rho)(T - HC) - N_h(\rho)C - HCA_h(\rho) = 0$
   2c) $S(\rho) = (T - HC)B(\rho)$
   hold for all $\rho \in U_\rho$;

3) $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ *for all $w \in \mathcal{L}_2$ and $||\phi|| = 0$.*

Hence, the above result incorporates all the conditions an observer must fulfill to solve Problem 2.1. Indeed, the stability and performance constraints are given by 1) and 3) respectively while the constraints 2) are structural constraints imposed on the observer to decouple the observation error from the known inputs and the system state.

3   Observer design

We state in this section the conditions for the existence of an observer (1) for system (2).

Theorem 3.1:   *The conditions 2) of Proposition 2.2 can be satisfied if and only if one of the following equivalent statements hold:*

1) *The constant matrix $H \in \mathbb{R}^{r \times s}$ is such that the equality*

$$[\varphi(\rho) - H\psi(\rho)][I - \phi^+\phi] = 0 \tag{10}$$

holds for all $\rho \in U_\rho$ where

$$\phi = \begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \end{bmatrix}, \quad \psi(\rho) = \begin{bmatrix} CA(\rho) & CA_h(\rho) \end{bmatrix} \text{ and } \varphi(\rho) = \begin{bmatrix} TA(\rho) & TA_h(\rho) \end{bmatrix}.$$

2) *The constant matrix $H \in \mathbb{R}^{r \times s}$ is such that the equality*

$$\text{rank} \begin{bmatrix} \phi \\ \varphi(\rho) - H\psi(\rho) \end{bmatrix} = \text{rank}[\phi] \tag{11}$$

holds for all $\rho \in U_\rho$ with the matrices defined above.

Proof : Since 2c) in Proposition 2.2 can always be satisfied, the remaining problem is the existence of solutions to 2a) and 2b). First, consider the linear algebra problem

$$\mathcal{O}(\rho)\phi = \varphi(\rho) - H\psi(\rho) \tag{12}$$

where the parameter-independent matrix H is placed on the right-hand side and the remaining parameter-dependent unknown matrices $\mathcal{O}(\rho) = \begin{bmatrix} M_0(\rho) & M_h(\rho) & K_0(\rho) & K_h(\rho) \end{bmatrix}$, $K_0(\rho) = N_0(\rho) - M_0(\rho)H$ and $K_h(\rho) = N_h(\rho) - M_h(\rho)H$ are parameterized in terms of this matrix H. According to (Skelton et al. 1997), there exist solutions to such an equation if and only if (10) holds for some $H \in \mathbb{R}^{r \times s}$ and for all $\rho \in U_\rho$. The rank condition (11) is obtained using standard linear

algebra by considering the equality:

$$\begin{bmatrix} \mathcal{O}(\rho) & H \end{bmatrix} \begin{bmatrix} \phi \\ \psi(\rho) \end{bmatrix} = \varphi(\rho). \tag{13}$$

□

An immediate conclusion of the above result concerns the case when the matrix ϕ is full column rank. In such a case, the equality $I - \phi^+\phi = 0$ holds and hence statement 1) of Theorem 3.1 holds for any $H \in \mathbb{R}^{r \times s}$. Since $\phi \in \mathbb{R}^{2(r+s) \times 2n}$, then the matrix can be full column rank only when $r + s \geq n$ (i.e. $\dim(z) + \dim(y) \geq \dim(x)$). For instance, if a full-order observer is sought, then $r = n$, $T = I_n$ and ϕ is automatically full-column rank. When the matrix ϕ is not full-column rank, i.e. $I - \phi^+\phi \neq 0$, then the matrix $H \in \mathbb{R}^{r \times s}$ must be chosen such that one of the statements of Theorem 3.1 is satisfied.

Conditions 2) of Proposition 2.2 implicitly define a matching family of observers for system (2). Assuming one of the conditions of Theorem 3.1 is satisfied, it is possible to explicitly parameterize this family, as stated in the following lemma:

Lemma 3.2:   *When the conditions of Theorem 3.1 are fulfilled, the observer matrices are given by the expressions $M_0(\rho) = \Theta(\rho) - L(\rho)\Xi$, $M_h(\rho) = \Upsilon - L(\rho)\Omega$ where $L(\rho)$ is an arbitrary matrix with appropriate dimensions and*

$$\begin{aligned}
\Theta(\rho) &= TA(\rho)U - HCA(\rho)U - [\varphi(\rho) - H\psi(\rho)]\phi^+\Delta_0 CU \\
\Xi &= -(I - \phi\phi^+)\Delta_0 CU \\
\Upsilon(\rho) &= TA_h(\rho)U - HCA_h(\rho)U - [\varphi(\rho) - H\psi(\rho)]\phi^+\Delta_h CU \\
\Omega &= -(I - \phi\phi^+)\Delta_h CU \\
S(\rho) &= FB(\rho) \\
N_0(\rho) &= \mathcal{O}_s(\rho)\Delta_0 + M_0(\rho)H \\
N_h(\rho) &= \mathcal{O}_s(\rho)\Delta_h + M_h(\rho)H \\
F &= T - HC \\
\mathcal{O}_s(\rho) &= [\varphi(\rho) - H\psi(\rho)]\phi^+ - L(\rho)(I - \phi\phi^+)
\end{aligned} \tag{14}$$

with

$$\Delta_0 = \begin{bmatrix} 0 \\ 0 \\ I_s \\ 0 \end{bmatrix} \text{ and } \Delta_h = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_s \end{bmatrix}.$$

Proof : Assuming a solution to the equation (13) exists, then the general solution is expressed as (Skelton et al. 1997):

$$\mathcal{O}_s(\rho) = [\varphi(\rho) - H\psi(\rho)]\phi^+ - L(\rho)(I - \phi\phi^+) \tag{15}$$

where $L(\rho)$ is an arbitrary matrix function. Then, rewrite (13) as

$$M_0(\rho)T = TA(\rho) - \begin{bmatrix} K_0(\rho) & H \end{bmatrix} \begin{bmatrix} C \\ CA(\rho) \end{bmatrix}, \tag{16}$$

$$M_h(\rho)T = TA_h(\rho) - \begin{bmatrix} K_h(\rho) & H \end{bmatrix} \begin{bmatrix} C \\ CA_h(\rho) \end{bmatrix}. \tag{17}$$

Using the fact that T is a full row rank matrix, it is thus possible to build a matrix U such that

6

$TU = I_r$. Right multiplying (16) and (16) by U yields

$$M_0(\rho) = TA(\rho)U - \begin{bmatrix} K_0(\rho) & H \end{bmatrix} \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U,$$
$$M_h(\rho) = TA_h(\rho)U - \begin{bmatrix} K_h(\rho) & H \end{bmatrix} \begin{bmatrix} C \\ CA_h(\rho) \end{bmatrix} U. \tag{18}$$

We the above defined Δ_0 and Δ_h, it comes $K_0(\rho) = \mathcal{O}_s(\rho)\Delta_0$ and $K_h(\rho) = \mathcal{O}_s(\rho)\Delta_h$. Finally, the algebraic relations of Lemma 3.2 are obtained after substitution of $K_0(\rho)$ and $K_h(\rho)$ into the expressions (18). $\qquad\square$

After substitution of the explicit expressions of the matrices $M_0(\rho), M_h(\rho)$ and H into the dynamical model (9), we get

$$\dot{e}(t) = (\Theta(\rho) - L(\rho)\Xi)e(t) + (\Upsilon(\rho) - L(\rho)\Omega)e(t - h(t)) + (T - HC)E(\rho)w(t). \tag{19}$$

Obviously, the obtained dynamical model for the observation error is parameterized by the free matrix function $L(\rho)$ and, possibly, the constant matrix H if not already been chosen to have Theorem 3.1 satisfied. So, the remaining problem consists of suitable choice for $L(\rho)$ and H which leads to stability and performance for (19).

Remark 1: It seems important to note that, under some circumstances, it can be possible to remove completely the influence of the exogenous inputs independently of $L(\rho)$ by choosing appropriately the matrix $H \in \mathbb{R}^{r \times s}$. Indeed, if the matrix H can be chosen such that both $(T - HC)E(\rho) = 0$ and Theorem 3.1 are satisfied, then the observation error evolution is totally decoupled from the exogenous input. In such a case, it is clear that the $\mathcal{L}_2$ gain from w to e is 0 and $L(\rho)$ is the only remaining degree of freedom and must be chosen according to some stability constraints only, e.g. pole placement, maximal delay margin.

The following theorem provides a way of choosing the matrices $L(\rho)$ and H such that both the state observation error is asymptotically stable for all $(\rho, h) \in \mathscr{P} \times \mathscr{H}$ and the $\mathcal{L}_2$ gain from the exogenous input to the observation error is lower than $\gamma > 0$.

Theorem 3.3: *There exists a parameter dependent observer (1) such that Problem 2.1 is solved if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^r$, a matrix function $\bar{L} : U_\rho \to \mathbb{R}^{r \times (2r+2s)}$, constant matrices $Q, R \in \mathbb{S}_{++}^r$, $X \in \mathbb{R}^{r \times r}$, $\bar{H} \in \mathbb{R}^{r \times r}$ and a positive scalar γ such that the LMI:*

$$\begin{bmatrix} -X^{\mathcal{S}} & \Sigma_{12}(\rho) & \Sigma_{13}(\rho) & \Sigma_{14}(\rho) & 0 & X^T & h_m R \\ \star & \Sigma_{22} & R & 0 & I_r & 0 & 0 \\ \star & \star & \Sigma_{33} & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_m R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{20}$$

holds for all $\rho \in U_\rho$, $\nu \in U_\nu$ where

$$
\begin{aligned}
\Sigma_{12}(\rho) &= X^T \Theta_0(\rho) - \bar{H}\Theta_H(\rho) - \bar{L}(\rho)\Xi \\
\Sigma_{13}(\rho) &= X^T \Upsilon_0(\rho) - \bar{H}\Upsilon_H(\rho) - \bar{L}(\rho)\Omega \\
\Sigma_{14}(\rho) &= (X^T T - \bar{H}C)E(\rho) \\
\Sigma_{22}(\rho) &= \frac{\partial P(\rho)}{\partial \rho}\nu - P(\rho) + Q - R \\
\Sigma_{33}(\rho) &= -(1-\mu)Q - R \\
\Theta_0(\rho) &= TA(\rho)U - \varphi(\rho)\phi^+\Delta_0 CU \\
\Theta_H(\rho) &= CA(\rho)U - \psi(\rho)\phi^+\Delta_0 CU \\
\Upsilon_0(\rho) &= TA_h(\rho)U - \varphi(\rho)\phi^+\Delta_H CU \\
\Upsilon_H(\rho) &= CA_h(\rho)U - \psi(\rho)\phi^+\Delta_H CU
\end{aligned}
\tag{21}
$$

In such a case, the observer ensures $||e||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ and is obtained using Lemma 3.2 with $L(\rho) = X^{-T}\bar{L}(\rho)$ and $H = X^{-T}\bar{H}$.

Proof : The proof is given in Appendix A. □

Remark 2 : If H has already been chosen to satisfy Theorem 3.1, then it is sufficient to change $\bar{H}$ by $X^T H$ in the above result. In such a case, the matrix X can now be made parameter dependent, i.e. $X : U_\rho \to \mathbb{R}^{r \times r}$ and the LMI is solved for $P(\rho), X(\rho), Q, R, \bar{L}(\rho)$ and γ.

Such a parameter dependent LMI can be solved by different techniques including for instance gridding (Apkarian and Adams 1998) or any other technique (Scherer 2006).

4   Examples

4.1   *Example 1: Full-order observer for stable systems*

Let us consider the following system

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1+0.2\rho(t) \\ -2 & -3+0.1\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho(t) & 0.1 \\ -0.2+0.1\rho(t) & -0.3 \end{bmatrix} x_h(t) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) + \begin{bmatrix} 1+\rho \\ 2+\rho \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \\
z(t) &= x(t) \\
\rho(t) &\in [-1, 1] \\
\dot{\rho}(t) &\in [-1, 1]
\end{aligned}
\tag{22}
$$

slightly modified from (Mohammadpour and Grigoriadis 2007).

The matrix ϕ in Theorem 3.1 is full-column rank, hence H does not need to be chosen a priori. The structure of $L(\rho)$ is arbitrarily chosen to be $L(\rho) = L_2\rho^2 + L_1\rho + L_0$ and we also pick $P(\rho) = P_0 + P_1\rho + P_2\rho^2/2$. Using Theorem 3.3 and a gridding approach, we seek the maximal delay value for which there exists an observer and we find $h_m = 213.238$ seconds using a bisection approach. Now, letting $h_m = 0.8$, we find the minimal $\gamma = 1.39 \cdot 10^{-4}$ with the observer

$$
M_0(\rho) = \begin{bmatrix} -0.0006\rho^2 + 0.00001\rho - 5.1879 & 0 \\ 0 & -0.1\rho - 4 \end{bmatrix}, \; H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\tag{23}
$$

$$
\begin{aligned}
M_h(\rho) &= \begin{bmatrix} 0 & 0 \\ 0 & -0.4 \end{bmatrix}, & N_h(\rho) &= \begin{bmatrix} 0 \\ -0.1\rho - 0.6 \end{bmatrix}, \; N_0(\rho) = \begin{bmatrix} 0 \\ -0.1\rho - 6 \end{bmatrix}, \\
S(\rho) &= \begin{bmatrix} 0 \\ 0.000014\rho + 1 \end{bmatrix}
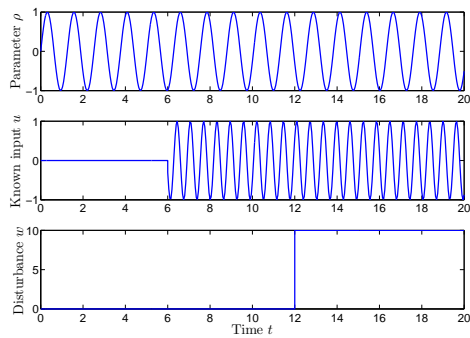\end{aligned}
\tag{24}
$$

Figure 1.   Time evolution of the parameter ρ (top), the known input u (center) and the disturbance w (bottom) for Examples 1 and 2.
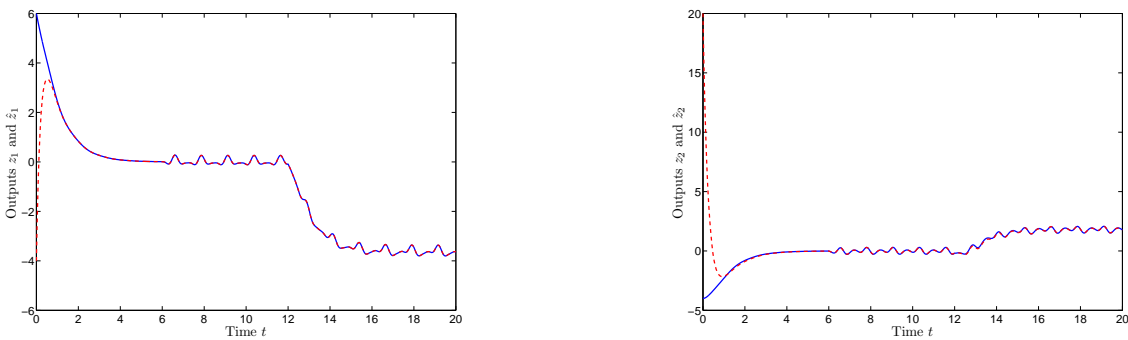

Figure 2. Actual (plain) and estimated (dashed) outputs for Example 1 - First approach (Theorem 3.1)

For simulation, the constant delay is set to $h(t) = 0.5$, $t \in \mathbb{R}_+$, the parameter is chosen as $\rho(t) = \sin(t)$, a known input signal of the form $u(t) = \sin(10t)$ is applied at $t = 6$ seconds while a step disturbance $w(t)$ of amplitude 10 is applied at $t = 12$ seconds. The state trajectories and their respective estimated values are depicted in Figure 2 where we can observe that, using the knowledge of the known input u, the state estimation is accurate even in presence of a disturbance of large amplitude, due to the high $\mathcal{L}_2$ attenuation (note that $||(T - HC)E(\rho)||_2 \simeq 10^{-6}$).

4.2   *Example 2: Reduced order observer for stable systems*

Let us consider now the same system as in Section 4.1 with the difference that

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \text{ and } z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \tag{25}$$

Hence, we would like to obtain a reduced-order controller estimating the first state from the measurement of the second state only. Since the matrix ϕ is full-column rank, then H is not restricted. Using Theorem 3.3 with $h_m = 0.8$ and a griding of the conditions, we find a minimal $\gamma = 0.2565$, showing that total decoupling is not possible in this case. The observer matrices are given by

$$\begin{array}{llll} M_0(\rho) = -2.3306, & M_h(\rho) = 0.3165\rho - 0.23306, & N_0(\rho) = 0.3165\rho + 0.2199, \\ N_h(\rho) = 0.36885\rho + 0.02199, & H \quad\;\; = -1.1653, & S(\rho) \;\; = 2.16528\rho + 3.3306. \end{array} \tag{26}$$

The simulation result is depicted in Fig. 3 where we have used the same scenario as in Example 1. In this case, however, the disturbance is responsible of an observation error, as seen in Fig. 3.
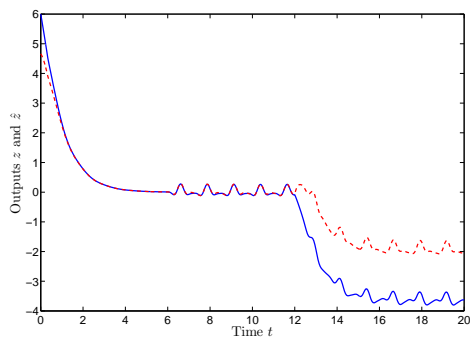
Figure 3. Actual (plain) and estimated (dashed) outputs for Example 2

4.3   *Example 3: Observation of a milling process*

Let us consider a milling process (Zhang et al. 2002) described by the model

$$\dot{x}(t) = (A_0 + \rho(t)A_1)x(t) + (A_{h0} + \rho(t)A_{h1})x(t - h) \tag{27}$$

where the delay $h = \pi/\omega$ is constant, $\rho(t) = \cos(2\omega t + \beta)$, $\beta > 0$ and $\omega > 0$ is the angular velocity. The system matrices are given by

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10 + 0.1710k) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \ A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
A_{h0} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1710k & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_{h1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{28}
$$

where $k > 0$ is the cutting stiffness constant. We assume that the displacement of the blade $x_1(t)$ and the tool $x_2(t)$ are measured while the whole state is desired to be estimated, that is

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) \text{ and } z(t) = x(t). \tag{29}$$

According to the discussion in (Zhang et al. 2002), the parameter has unbounded derivative, hence we pick $P(\rho) = P_0$ and $L(\rho) = L_0 + L_1\rho + L_2\rho^2$. For the observer design, we pick $h_m = 0.8$ (hence $h \le h_m$, $\omega \ge \pi/0.8$), $k = 0.2$ and use Theorem 3.3 where H is unconstrained since ϕ is full-column rank. For simulation, we pick $\omega = 10$ rad/s and $\beta = 7\pi/18$. The results are depicted in Fig. 4 where we can see that the observer is able to track the system state accurately.

4.4   *Remarks on Examples*

It is important to explain how the above matrices have been obtained. Actually, since many matrix manipulations (pseudoinverse, inverse and arithmetical operations) have been performed numerically, many errors also accumulated. Initially, the obtained matrices are rational functions of high order (approx. 10) but after a closer analysis, we can see that many poles/zeros cancelations are prevented by small numerical errors on the coefficients of the rational functions. Hence, using a least mean square approximation on each coefficient of the matrices, it is possible to simplify the overall expression of the observer. In the current example, the $\mathcal{L}_2$ norm of the error between the initial rational function and the approximant is of order 10^{-6}.
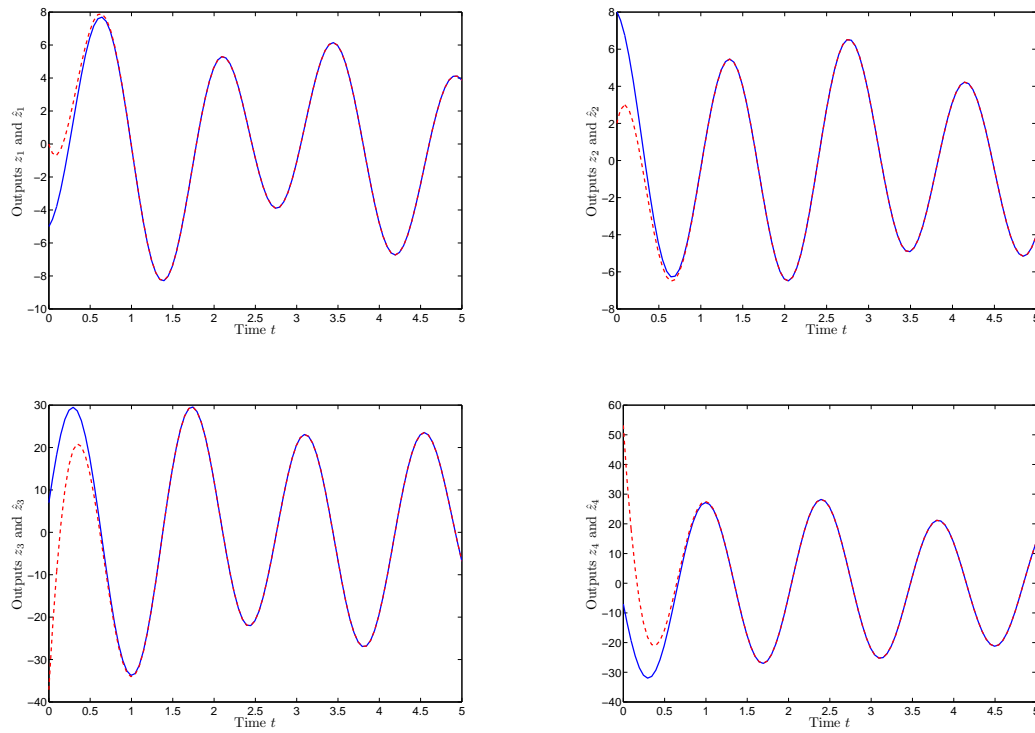
Figure 4. Actual (plain) and estimated (dashed) outputs for the milling process.

5   Conclusion

The design of a class of observers for LPV time-delay systems has been addressed. The approaches are based on the use of algebraic techniques to parameterize a family of observers which satisfy a set of structural constraints. Then, an observer which guarantees stability and performance criteria is finally determined using LMIs. The approach has been finally illustrated through examples demonstrating the relevance of the approach.

The current limitations of the approach lie in the difficulty of knowing the time-varying delay exactly. This knowledge is necessary to the implementation of the observers, extensions to the uncertain delay knowledge cases might be possible using techniques inspired from (Briat et al. 2010). Other possible extensions are possible, e.g. widening the class of systems by allowing the measured output to depend on the parameters, the exogenous inputs and the delayed state. Finally, the class of observers could also be improved by making H parameter-dependent.

Appendix A: Proof of theorem 3.3

For convenience, the following LPV time-delay system is introduced:

$$
\begin{aligned}
\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + E(\rho)w(t) \\
z(t) &= C(\rho)x(t) \\
\rho &\quad \in \mathscr{P} \\
h &\quad \in \mathscr{H}
\end{aligned}
\tag{A1}
$$

and will be used throughout the proof. Let us also consider the Lyapunov-Krasovskii functional (Briat et al. 2008a,b, 2009a,b) extended from (Han 2005, Gouaisbaut and Peaucelle 2006):

$$V(x_t) := x(t)^T P(\rho) x(t) + \int_{t-h(t)}^t x(\theta)^T Q x(\theta) d\theta + h_m \int_{-h_m}^0 \int_{t+\theta}^t \dot{x}^T(s) R \dot{x}(s) ds d\theta$$

(A2)

playing the role of the storage function in the dissipativity criterion. Define also the supply-rate $s(w,z) := \gamma w^T w - \gamma^{-1} z^T z$ which characterizes the $\mathcal{L}_2$-gain from the input w to the output z of the system (A1). Then, considering the function $H_d(t) := V(x_t) - \int_0^t s(w(\theta), z(\theta)) d\theta$, and differentiating it along the trajectories solutions of the system (A1) yields

$$\dot{H}_d \le \xi(t)^T \left(\begin{bmatrix} \beth(\rho, \dot{\rho}) & P(\rho) A_h(\rho) & P(\rho) E(\rho) \\ \star & -(1-\mu) Q & 0 \\ \star & \star & 0 \end{bmatrix} + h_m^2 \begin{bmatrix} A(\rho)^T \\ A_h(\rho)^T \\ E(\rho)^T \end{bmatrix} R \begin{bmatrix} A(\rho)^T \\ A_h(\rho)^T \\ E(\rho)^T \end{bmatrix}^T \right) \xi(t)$$
$$+ \mathcal{I} - s(w(t), z(t)),$$
$$\beth(\rho, \dot{\rho}) = A(\rho)^T P(\rho) + P(\rho) A(\rho) + \frac{\partial P(\rho)}{\partial \rho} \dot{\rho} + Q,$$
$$s(w, z) = \gamma w^T w - \gamma^{-1} z^T z,$$
$$\mathcal{I} = -h_m \int_{t-h(t)}^t \dot{x}(s)^T R \dot{x}(s) ds,$$
$$\xi(t) = \mathrm{col}[x(t), x(t-h(t)), w(t)].$$

Using the Jensen's inequality (Gu et al. 2003) on the term $\mathcal{I}$, we get

$$\mathcal{I} \le -(x(t) - x(t-h(t)))^T R (x(t) - x(t-h(t))).$$

(A3)

Then expanding the above relations, adding them up and using Schur complements, the following LMI is obtained

$$\begin{bmatrix} \beth(\rho, \dot{\rho}) - R & P(\rho) A_h(\rho) + R & P(\rho) E(\rho) & C(\rho)^T & h_m A(\rho)^T R \\ \star & -(1-\mu) Q - R & 0 & 0 & h_m A_h(\rho)^T R \\ \star & \star & -\gamma I & 0 & h_m E(\rho)^T R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

(A4)

and ensures the stability of system (A1) by virtue of the Lyapunov-Krasovskii Theorem. Due to the multiple products $P(\rho) A(\rho)$ and $RA(\rho)$ between system and Lyapunov matrices, the linearization after substitution of the matrices of (19) is not possible. Following (Briat et al. 2009a,b, 2010), the following dilated version of the above inequality is used instead:

$$\begin{bmatrix} -X(\rho)^{\mathcal{S}} & X(\rho)^T A(\rho) + P(\rho) & X(\rho)^T A_h(\rho) & X(\rho)^T E(\rho) & 0 & X^T & h_m R \\ \star & \frac{\partial P}{\partial \rho} \dot{\rho} - P(\rho) + Q - R & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -Q_\mu - R & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -hR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

(A5)

where $X(\rho) \in \mathbb{R}^{r \times r}$ is an additional free matrix. As explained in the above references, the LMI (A5) is not equivalent to (A4) but is only sufficient.

Substitute the observation error dynamic model in (A5) with $C(\rho) = I_r$, expand the expression and perform the change of variables $\bar{H} = X^T H$, $\bar{L}(\rho) = X^T L(\rho)$ (where X is set constant since H is constant) to obtain the LMI (20). Finally, since the LMI is affine in $\dot{\rho}$, then using a convexity argument, it is enough to check the feasibility of the LMIs at the vertices of $co\{U_\nu\}$, namely on the set U_ν. This completes the proof.

References

W. Aggoune, M. Boutayeb, and M. Darouach. Observers design for a class of non linear systems with time-varying delay. In *Proc. 38th IEEE Confer. on Decision & Control*, pages 2912–2913, Phoenix, Arizona, USA, 1999.

P. Apkarian and R.J. Adams. Advanced gain-scheduling techniques for uncertain systems. *IEEE Transactions on Automatic Control*, 6:21–32, 1998.

P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers. *IEEE Transactions on Automatic Control*, 5:853–864, 1995.

S. Boyd, L. El-Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. PA, SIAM, Philadelphia, 1994.

C. Briat, O. Sename, and J-F. Lafay. A LFT/$\mathcal{H}_\infty$ state-feedback design for linear parameter varying time delay systems. In *European Control Conference 2007, Kos, Greece*, 2007a.

C. Briat, O. Sename, and J-F. Lafay. Full order LPV/$\mathcal{H}_\infty$ observers for LPV time-delay systems. In *3rd IFAC Symposium on System Structure and Control 2007, Foz do Iguacu, Brazil*, 2007b.

C. Briat, O. Sename, and J-F. Lafay. A full-block $\mathcal{S}$-procedure application to delay-dependent $\mathcal{H}_\infty$ state-feedback control of uncertain time-delay systems. In *17th IFAC World Congress, Korea, Seoul*, 2008a.

C. Briat, O. Sename, and J-F. Lafay. Parameter dependent state-feedback control of LPV time delay systems with time varying delays using a projection approach. In *17th IFAC World Congress, Korea, Seoul*, 2008b.

C. Briat, O. Sename, and J-F. Lafay. $\mathcal{H}_\infty$ filtering of uncertain LPV systems with time-delays. In *10th European Control Conference, 2009, Budapest, Hungary*, 2009a.

C. Briat, O. Sename, and J-F. Lafay. Memory resilient gain-scheduled state-feedback control of time-delay systems with time-varying delays. In *ROCOND'09*, Haifa, Israel, 2009b.

C. Briat, O. Sename, and J.F. Lafay. Memory resilient gain-scheduled state-feedback control of uncertain LTI/LPV systems with time-varying delays. *Systems & Control Letters*, 59:451–459, 2010.

M. Darouach. Linear functional observers for systems with delays in state variables. *IEEE Transactions on Automatic Control*, 46(3):491–496, 2001.

M. Darouach. Reduced-order observers for linear neutral delay systems. *IEEE Transactions on Automatic Control*, 50(9):1407–1413, 2005.

A. Fattouh, O. Sename, and J.-M. Dion. $\mathcal{H}_\infty$ observer design for time-delay systems. In *Proc. 37th IEEE Confer. on Decision & Control*, pages 4545–4546, Tampa, Florida, USA, 1998.

A. Fattouh, O. Sename, and J.-M. Dion. α-decay rate observer design for linear systems with delayed state. In *Proc. 8th European Control Confer. ECC'99*, Karlsruhe, Germany, 1999a.

A. Fattouh, O. Sename, and J.-M. Dion. Robust observer design for time-delay systems: A riccati equation approach. *Kybernetika*, 35(6):753–764, 1999b.

A. Fattouh, O. Sename, and J.-M. Dion. An LMI approach to robust observer design for linear time-delay systems. In *Proc. 39th IEEE Confer. on Decision & Control*, Sydney, Australia, December, 12-15, 2000a.

A. Fattouh, O. Sename, and J.-M. Dion. $\mathcal{H}_\infty$ controller and observer design for linear systems with point and distributed time-delays: An LMI approach. In *2nd IFAC Workshop on Linear Time Delay Systems*, Ancône, Italy, 2000b.

A. Fattouh, O. Sename, and J.-M. Dion. Robust observer design for linear uncertain time-delay

systems: A factorization approach. In *14th Int. Symp. on Mathematical Theory of Networks and Systems*, Perpignan, France, June, 19-23, 2000c.

A. Fattouh, O. Sename, and J.-M. Dion. Robust observer design for linear uncertain time-delay systems: A factorization approach. In *14th Int. Symp. on Mathematical Theory of Networks and Systems*, Perpignan, France, June, 19-23, 2000d.

Y. A. Fiagbedzi and A. E. Pearson. A state observer for systems described by functional differential equations. *Automatica*, 26(2):321–331, 1990.

E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43(4):309–319, 2001.

P. Gahinet and P. Apkarian. A linear matrix inequality approach to $\mathcal{H}_\infty$ control. *International Journal of Robust and Nonlinear Control*, 4:421–448, 1994.

F. Gouaisbaut and D. Peaucelle. Delay dependent robust stability of time delay-systems. In 5th *IFAC Symposium on Robust Control Design*, Toulouse, France, 2006.

K. Gu, V.L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems.* Birkhäuser, 2003.

Q.L. Han. Absolute stability of time-delay systems with sector-bounded nonlinearity. *Automatica*, 41:2171–2176, 2005.

T. Iwasaki and G. Shibata. LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Transactions on Automatic Control*, 46:1195–1208, 2001.

D. Koenig, B. Marx, and O. Sename. Unknown inputs proportional integral observers for descriptors systems with multiple delays and unknown inputs. In *American Control Conference, Boston, Massachusetts, USA*, 2004.

D. Koenig, D. Jacquet, and S. Mammar. Delay dependent $\mathcal{H}_\infty$ observer for linear delay descriptor systems. In *American Control Conference 2006, Minneapolis, USA*, 2006.

V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

C. S. Mehendale and K. M. Grigoriadis. A new approach to LPV gain-scheduling design and implementation. In *43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas*, 2004.

W. Michiels and S.I. Niculescu. *Stability and stabilization of time-delay systems. An eigenvalue based approach.* SIAM Publication, Philadelphia, USA, 2007.

J. Mohammadpour and K. M. Grigoriadis. Rate-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for time varying state delayed LPV systems. In *Conference on Decision and Control, San Diego, USA*, 2006a.

J. Mohammadpour and K. M. Grigoriadis. Delay-dependent $\mathcal{H}_\infty$ filtering for a class of time-delayed LPV systems. In *American Control Conference, Minneapolis, USA*, 2006b.

J. Mohammadpour and K. M. Grigoriadis. Less conservative results of delay-depdendent $\mathcal{H}_\infty$ filtering for a class of time-delayed LPV systems. *International Journal of Control*, 80(2): 281–291, 2007.

S.-I. Niculescu. *Delay effects on stability. A robust control approach*, volume 269. Springer-Verlag: Heidelbeg, 2001.

A. Packard. Gain scheduling via Linear Fractional Transformations. *Systems & Control Letters*, 22:79–92, 1994.

C. W. Scherer. LPV control and full-block multipliers. *Automatica*, 37:361–375, 2001.

C. W. Scherer. LMI relaxations in robust control. *European Journal of Control - Special Issue on 'Linear Matrix Inequalities in Control' (preprint available at http://www.dcsc.tudelft.nl/ cscherer/papers/EJC06.pdf)*, 12:3–29, 2006.

0. Sename and C. Briat. Observer-based $\mathcal{H}_\infty$ control for time-delay systems: a new lmi solution. In *European Control Conference 2007, Kos, Greece*, 2007.

O. Sename. New trends in design of observers for time-delay systems. *Kybernetika*, 37(4): 427–458, 2001.

O. Sename. Is a mixed design of observer-controllers for time-delay systems interesting ? *Asian Journal of Control*, 9(2):180–189, 2007.

O. Sename and C. Briat. Observer-based $\mathcal{H}_\infty$ control for time-delay systems: a new LMI solution. In *6th IFAC Workshop on Time Delay Systems*, L'Aquila, Italy, 2006.

O. Sename, A. Fattouh, and J-M. Dion. Further results on unknown input observers design for time-delay systems. In *40th IEEE Conference on Decision and Control*, Orlando, Florida, USA, 2001.

J. Shamma and M. Athans. Analysis of nonlinear gain scheduled control systems. *IEEE Transactions on Automatic Control*, 35:898–907, 1990.

J. Shamma and M. Athans. Guaranteed properties of gain scheduled control of linear parameter-varying plants. *Automatica*, 27:559–564, 1991.

J. S. Shamma and M. Athans. Gain scheduling: potential hazards and possible remedies. *IEEE Contr. Syst. Magazine*, 12(3):101–107, 1992.

R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, 1997.

F. Wu. A generalized LPV system analysis and control synthesis framework. *International Journal of Control*, 74:745–759, 2001a.

F. Wu. Delay dependent induced $\mathcal{L}_2$-norm analysis and control for LPV systems with state delays. In *ASME International Mechanical Engineering Congress and Exposition*, 2001b.

F. Wu and K.M. Grigoriadis. LPV systems with parameter-varying time delays: analysis and control. *Automatica*, 37:221–229, 2001.

F. Zhang and K.M. Grigoriadis. Delay-dependent stability analysis and $\mathcal{H}_\infty$ control for state-delayed LPV system. In *Mediterranean Conference on Control and Automation*, pages 1532–1537, 2005.

X. Zhang, P. Tsiotras, and C. Knospe. Stability analysis of LPV time-delayed systems. *International Journal of Control*, 75:538–558, 2002.