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# ON POLYNOMIAL STABILITY FOR SKEW-EVOLUTION SEMIFLOWS ON BANACH SPACES

CODRUȚA STOICA

ABSTRACT. Many of the processes that arise in engineering, physics or economics can be described by mathematical models that imply nonlinear evolution equations. Of great interest is, as we emphasize in this paper, to study the solutions of differential equations using an original concept, the skew-evolution semiflows, which generalize the classic notions of evolution operators and skew-product semiflows. The techniques from the domain of nonautonomous equations in infinite dimensions with unbounded coefficients are extended for the study of the above categories.

The main concern of this paper is to give definitions, examples, connections and characterizations for various concepts for the asymptotic properties of stability of solutions for evolution equations in a nonuniform setting.

## 1. INTRODUCTION

In recent years, there were defined and developed concepts of the control theory, as stability, controllability and observability, based on the fact that the dynamical systems which describe processes from engineering, physics or economics are extremely complex and the identification of the proper mathematical model is difficult.

The theory of asymptotic properties for evolution equations has witnessed lately an explosive development and is an important way towards interesting applications. The aim of our paper is to develop a framework which enables us to obtain characterizations in a unitary approach in a nonuniform setting for the asymptotic stability on Banach spaces. The notion of skew-evolution semiflow considered in this paper seem to be more appropriate for the study of the asymptotic behavior of the solutions of evolution equations in the nonuniform case. They depend on three variables, contrary to a skew-product semiflow or an evolution operator, for which they are generalizations and which depend only on two. Hence, the study of asymptotic behaviors for skew-evolution semiflows in the nonuniform setting arises as natural, relative to the third variable.

The definitions of various types of stability are illustrated by examples and the connections between them are emphasized. Our aim is also to give some integral characterizations for them. We use a concept of nonuniform exponential stability, given and studied in by L. Barreira and C. Valls in

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[2], which we call "*Barreira-Valls exponential stability*". The polynomial stability was studied in [3].

Some generalizations for the results obtained for skew-evolution semiflows in the uniform setting in [7], [8] and [9] are proved in the nonuniform case.

## 2. SKEW-EVOLUTION SEMIFLOWS

This section gives the notion of skew-evolution semiflow on a Banach space, defined by means of an evolution semiflow and of an evolution cocycle.

Let  $(X, d)$  be a metric space,  $V$  a Banach space and  $V^*$  its topological dual. Let  $\mathcal{B}(V)$  be the space of all  $V$ -valued bounded operators defined on  $V$ . The norm of vectors on  $V$  and on  $V^*$  and of operators on  $\mathcal{B}(V)$  is denoted by  $\|\cdot\|$ .  $I$  is the identity operator. Let us denote  $Y = X \times V$  and  $T = \{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\}$ .

**Definition 2.1.** A mapping  $\varphi : T \times X \rightarrow X$  is said to be evolution semiflow on  $X$  if following properties are satisfied:

- (es<sub>1</sub>)  $\varphi(t, t, x) = x, \forall (t, x) \in \mathbb{R}_+ \times X$ ;
- (es<sub>2</sub>)  $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X$ .

**Definition 2.2.** A mapping  $\Phi : T \times X \rightarrow \mathcal{B}(V)$  is called evolution cocycle over an evolution semiflow  $\varphi$  if it satisfies following properties:

- (ec<sub>1</sub>)  $\Phi(t, t, x) = I, \forall t \geq 0, \forall x \in X$ ;
- (ec<sub>2</sub>)  $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X$ .

Let  $\Phi$  be an evolution cocycle over an evolution semiflow  $\varphi$ . The mapping

$$(2.1) \quad C : T \times Y \rightarrow Y, C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$$

is called *skew-evolution semiflow* on  $Y$ .

**Example 2.1.** Let us denote  $\mathcal{C} = \mathcal{C}(\mathbb{R}, \mathbb{R})$  the set of all continuous functions  $x : \mathbb{R} \rightarrow \mathbb{R}$ , endowed with the topology of uniform convergence on compact subsets of  $\mathbb{R}$ . The set  $\mathcal{C}$  is metrizable with respect to the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}, \text{ where } d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

We consider for every  $n \in \mathbb{N}^*$  a decreasing function

$$x_n : \mathbb{R}_+ \rightarrow \left( \frac{1}{2n+1}, \frac{1}{2n} \right), \text{ with the property } \lim_{t \rightarrow \infty} x_n(t) = \frac{1}{2n+1}.$$

We denote

$$x_n^s(t) = x_n(t+s), \forall t, s \geq 0.$$

Let  $X$  be the closure in  $\mathcal{C}$  of the set  $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$ . The mapping

$$\varphi : T \times X \rightarrow X, \varphi(t, s, x) = x_{t-s}, \text{ where } x_{t-s}(\tau) = x(t-s+\tau), \forall \tau \geq 0,$$

is an evolution semiflow on  $X$ . Let us consider the Banach space  $V = \mathbb{R}^p$ ,  $p \geq 1$ , with the norm  $\|(v_1, \dots, v_p)\| = |v_1| + \dots + |v_p|$ . Then the mapping

$$\Phi : T \times X \rightarrow \mathcal{B}(V), \Phi(t, s, x)v = \left( e^{\alpha_1 \int_s^t x(\tau-s)d\tau} v_1, \dots, e^{\alpha_p \int_s^t x(\tau-s)d\tau} v_p \right),$$

where  $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$  is fixed, is an evolution cocycle over the evolution semiflow  $\varphi$  and  $C = (\varphi, \Phi)$  is a skew-evolution semiflow on  $Y$ .

**Remark 2.1.** The concept of skew-evolution semiflow is a natural generalization of the notion of skew-product semiflow, considered by M. Megan, A.L. Sasu and B. Sasu[5].

Following example emphasizes the fact that the skew-evolution semiflows generalize the concept of skew-product semiflows, which appear as operatorial solutions of nonlinear equations.

**Example 2.2.** Let  $X$  be the metric space defined as in Example 2.1. The mapping  $\varphi_0 : \mathbb{R}_+ \times X \rightarrow X$ ,  $\varphi_0(t, x) = x_t$ , where  $x_t(\tau) = x(t + \tau)$ ,  $\forall \tau \geq 0$ , is a semiflow on  $X$ . Let us consider for every  $x \in X$  the parabolic system with Neumann's boundary conditions:

$$(2.2) \quad \begin{cases} \frac{\partial v}{\partial t}(t, y) = x(t) \frac{\partial^2 v}{\partial y^2}(t, y), & t > 0, y \in (0, 1) \\ v(0, y) = v_0(y), & y \in (0, 1) \\ \frac{\partial v}{\partial y}(t, 0) = \frac{\partial v}{\partial y}(t, 1) = 0, & t > 0. \end{cases}$$

Let  $V = \mathcal{L}^2(0, 1)$  be a separable Hilbert space with the orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$ ,  $e_0 = 1$ ,  $e_n(y) = \sqrt{2} \cos n\pi y$ , where  $y \in (0, 1)$ ,  $n \in \mathbb{N}$ .

We denote  $D(A) = \{v \in \mathcal{L}^2(0, 1), v(0) = v(1) = 0\}$  and we define the operator

$$A : D(A) \subset V \rightarrow V, Av = \frac{d^2 v}{dy^2},$$

which generates a  $C_0$ -semigroup  $S$ , defined by

$$S(t)v = \sum_{n=0}^{\infty} e^{-n^2 \pi^2 t} \langle v, e_n \rangle e_n,$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $V$ . We define for every  $x \in X$ ,  $A(x) : D(A) \subset V \rightarrow V$ ,  $A(x) = x(0)A$ , which allows us to rewrite system (2.2) in  $V$  as

$$(2.3) \quad \begin{cases} \dot{v}(t) = A(\varphi_0(t, x))v(t), & t > 0 \\ v(0) = v_0. \end{cases}$$

The mapping  $\Phi_0 : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$ ,  $\Phi_0(t, x)v = S\left(\int_0^t x(s)ds\right)v$  is a cocycle over the semiflow  $\varphi_0$  and  $C_0 = (\varphi_0, \Phi_0)$  is a linear skew-product semiflow strongly continuous on  $Y$ . Also, for all  $v_0 \in D(A)$ , we have obtained that  $v(t) = \Phi(t, x)v_0$ ,  $t \geq 0$ , is a strongly solution of system (2.3). As  $C_0 = (\varphi_0, \Phi_0)$  is a skew-product semiflow on  $Y$ , then the mapping  $C : T \times Y \rightarrow Y$ ,  $C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$ , where

$$\varphi(t, s, x) = \varphi_0(t - s, x) \text{ and } \Phi(t, s, x) = \Phi_0(t - s, x), \quad \forall (t, s, x) \in T \times X$$

is a skew-evolution semiflow on  $Y$ . Hence, the skew-evolution semiflows generalize the notion of skew-evolution semiflows.

More directly, if  $\Phi(t, s, x)$  is the solution of the Cauchy problem

$$\begin{cases} v'(t) = A(\varphi(t, s, x))v(t), & t > s \\ v(s) = x, \end{cases}$$

then  $C = (\varphi, \Phi)$  is a linear skew-evolution semiflow.

Next example shows that the skew-evolution semiflows generalize the notion of evolution operators.

**Example 2.3.** For  $X = \mathbb{R}_+$ , the mapping  $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $\varphi(t, s, x) = x$  is an evolution semiflow. For every evolution cocycle  $\Phi$  over  $\varphi$ , we obtain that the mapping  $E_\Phi : T \rightarrow \mathcal{B}(V)$ ,  $E_\Phi(t, s) = \Phi(t, s, 0)$  is an evolution operator on  $V$ .

**Example 2.4.** If  $C = (\varphi, \Phi)$  denotes a skew-evolution semiflow and  $\alpha \in \mathbb{R}$  a parameter, then  $C_\alpha = (\varphi, \Phi_\alpha)$ , where

$$(2.4) \quad \Phi_\alpha : T \times X \rightarrow \mathcal{B}(V), \quad \Phi_\alpha(t, t_0, x) = e^{\alpha(t-t_0)}\Phi(t, t_0, x),$$

is also a skew-evolution semiflow, called the  $\alpha$ -shifted skew-evolution semiflow.

### 3. EXPONENTIAL STABILITY

In this section we consider several concepts of exponential stability for skew-evolution semiflows. For each, we will consider an equivalent relation of definition. Some connections between these concepts are established. We will emphasize that they are not equivalent.

The nonuniform exponential stability is given by

**Definition 3.1.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is exponentially stable (e.s.) if there exist a mapping  $N : \mathbb{R}_+ \rightarrow [1, \infty)$  and a constant  $\alpha > 0$  such that, for all  $(t, s) \in T$ , following relation takes place:

$$(3.1) \quad \|\Phi(t, t_0, x)v\| \leq N(s)e^{-\alpha t} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

A concept of nonuniform exponential stability, which we will name "Barreira-Valls exponential stability", is given by L. Barreira and C. Valls in [2] for evolution equations.

**Definition 3.2.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is Barreira-Valls exponentially stable (BV.e.s.) if there exist some constants  $N \geq 1$ ,  $\alpha > 0$  and  $\beta$  such that, for all  $(t, s), (s, t_0) \in T$ , the relation holds:

$$(3.2) \quad \|\Phi(t, t_0, x)v\| \leq Ne^{-\alpha t} e^{\beta s} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

The asymptotic property of nonuniform stability is considered in

**Definition 3.3.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is stable (s.) if there exists a mapping  $N : \mathbb{R}_+ \rightarrow [1, \infty)$  such that, for all  $(t, s), (s, t_0) \in T$ , the relation is true:

$$(3.3) \quad \|\Phi(t, t_0, x)v\| \leq N(s) \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

Let us remind the property of exponential growth for skew-evolution semiflows, given by

**Definition 3.4.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  has exponential growth (e.g.) if there exist two mappings  $M, \omega : \mathbb{R}_+ \rightarrow [1, \infty)$ ,  $\omega$  nondecreasing, such that, for all  $(t, s), (s, t_0) \in T$ , we have:

$$(3.4) \quad \|\Phi(t, t_0, x)v\| \leq M(s)e^{\omega(s)(t-s)} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

**Remark 3.1.** The relations concerning the previously defined asymptotic properties for skew-evolution semiflows are given by

$$(3.5) \quad (BV.e.s.) \implies (e.s.) \implies (s.) \implies (e.g.)$$

The reciprocal statements are not true, as shown in what follows.

The following example presents a skew-evolution semiflow which is exponentially stable but not Barreira-Valls exponentially stable.

**Example 3.1.** Let  $X = \mathbb{R}_+$ . The mapping  $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $\varphi(t, s, x) = x$  is an evolution semiflow on  $\mathbb{R}_+$ .

Let us consider a continuous function  $u : \mathbb{R}_+ \rightarrow [1, \infty)$  with

$$u(n) = e^{n \cdot 2^{2n}} \quad \text{and} \quad u\left(n + \frac{1}{2^{2n}}\right) = e^4.$$

We define

$$\Phi_u(t, s, x)v = \frac{u(s)e^s}{u(t)e^t}v, \quad \text{where } (t, s) \in T, (x, v) \in Y.$$

As following relation

$$\|\Phi_u(t, s, x)v\| \leq u(s)e^s e^{-t} \|v\|$$

holds for all  $(t, s, x, v) \in T \times Y$ , it results that the skew-evolution semiflow  $C_u = (\varphi, \Phi_u)$  is exponentially stable.

Let us now suppose that the skew-evolution semiflow  $C_u = (\varphi, \Phi_u)$  is Barreira-Valls exponentially stable. Then, according to Definition 3.2, there exist  $N \geq 1$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $t_1 > 0$  such that

$$\frac{u(s)e^s}{u(t)e^t} \leq N e^{-\alpha t} e^{\beta s}, \quad \forall t \geq s \geq t_1.$$

For  $t = n + \frac{1}{2^{2n}}$  and  $s = n$  it follows that

$$e^{n(2^{2n}+1)} \leq N e^{n+\frac{1}{2^{2n}}+4} e^{-\alpha\left(n+\frac{1}{2^{2n}}\right)} e^{\beta n},$$

which is equivalent with

$$e^{n(2^{2n}-\beta)} \leq N e^{\frac{1}{2^{2n}}+4-\alpha\left(n+\frac{1}{2^{2n}}\right)}.$$

For  $n \rightarrow \infty$ , a contradiction is obtained, which proves that  $C_u$  is not Barreira-Valls exponentially stable.

There exist skew-evolution semiflows that are stable but not exponentially stable, as results from the following

**Example 3.2.** Let us consider  $X = \mathbb{R}_+$ ,  $V = \mathbb{R}$  and

$$u : \mathbb{R}_+ \rightarrow [1, \infty) \quad \text{with the property} \quad \lim_{t \rightarrow \infty} \frac{u(t)}{e^t} = 0.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v$$

is an evolution cocycle. As  $|\Phi_u(t, s, x)v| \leq u(s)|v|$ ,  $\forall (t, s, x, v) \in T \times Y$ , it follows that  $C_u = (\varphi, \Phi_u)$  is a stable skew-evolution semiflow, for every evolution semiflow  $\varphi$ .

On the other hand, if we suppose that  $C_u$  is exponentially stable, according to Definition 3.1, there exist a mapping  $N : \mathbb{R}_+ \rightarrow [1, \infty)$  and a constant  $\alpha > 0$  such that, for all  $(t, s), (s, t_0) \in T$ , we have

$$\|\Phi_u(t, t_0, x)v\| \leq N(s)e^{-\alpha t} \|\Phi_u(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

It follows that

$$\frac{u(s)}{N(s)} \leq \frac{u(t)}{e^{\alpha t}}.$$

For  $t \rightarrow \infty$  we obtain a contradiction, and, hence,  $C_u$  is not exponentially stable.

Following example gives a skew-evolution semiflow that has exponential growth but is not stable.

**Example 3.3.** We consider  $X = \mathbb{R}_+$ ,  $V = \mathbb{R}$  and

$$u : \mathbb{R}_+ \rightarrow [1, \infty) \text{ with the property } \lim_{t \rightarrow \infty} \frac{e^t}{u(t)} = \infty.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_u(t, s, x)v = \frac{u(s)e^t}{u(t)e^s}v$$

is an evolution cocycle. We have  $|\Phi(t, s, x)v| \leq u(s)e^{t-s}|v|$ ,  $\forall (t, s, x, v) \in T \times Y$ . Hence,  $C_u = (\varphi, \Phi_u)$  is a skew-evolution semiflow, over every evolution semiflow  $\varphi$ , and has exponential growth.

Let us suppose that  $C_u$  is stable. According to Definition 3.3, there exists a mapping  $N : \mathbb{R}_+ \rightarrow [1, \infty)$  such that  $u(s)e^t \leq N(s)u(t)e^s$ , for all  $(t, s) \in T$ . If  $t \rightarrow \infty$ , a contradiction is obtained. Hence,  $C_u$  is not stable.

#### 4. POLYNOMIAL STABILITY

In this section, we introduce a new concept of nonuniform stability for skew-evolution semiflows, given by the next

**Definition 4.1.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is called polynomially stable (p.s.) if there exist a mapping  $N : \mathbb{R}_+ \rightarrow [1, \infty)$  and a constant  $\gamma > 0$  such that:

$$(4.1) \quad \|\Phi(t, s, x)v\| ds \leq N(s)(t-s)^{-\gamma} \|v\|,$$

for all  $t > s \geq 0$  and all  $(x, v) \in Y$ .

**Remark 4.1.** If a skew-evolution semiflow  $C$  is exponentially stable, then it is polynomially stable.

$$(e.s.) \implies (p.s.)$$

The reciprocal statement is not true, as shown in

**Example 4.1.** Let  $X = \mathbb{R}_+$ ,  $V = \mathbb{R}$  and the mapping  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  given by  $u(t) = t + 1$ . The mapping  $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , where  $\varphi(t, s, x) = x$  is an evolution semiflow on  $\mathbb{R}_+$ . We consider

$$\Phi_u : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as we have

$$|\Phi_u(t, s, x)v| \leq \frac{s^2}{t}|v| = s\frac{s}{t}|v|, \quad \forall t \geq s \geq 1 = t_0, \quad \forall (x, v) \in Y,$$

it follows that  $C = (\varphi, \Phi)$  is a Barreira-Valls polynomially stable skew-evolution semiflow.

If we suppose that  $C$  is exponentially stable, according to Definition 3.1, there exist  $N : \mathbb{R}_+ \rightarrow [1, \infty)$  and  $\alpha > 0$  such that

$$\frac{s+1}{t+1} \leq N(s)e^{-\alpha t}, \quad \forall t \geq s \geq t_0,$$

which is equivalent with

$$\frac{e^{\alpha t}}{t+1} \leq \frac{N(t_0)}{t_0+1}, \quad \forall t \geq t_0,$$

and which, for  $t \rightarrow \infty$ , leads to a contradiction. Hence,  $C$  is not exponentially stable.

**Remark 4.2.** For  $\alpha \geq \beta$  in Definition 3.2, a Barreira-Valls exponentially stable skew-evolution semiflow  $C$  is polynomially stable.

$$(B.V.e.s.) \implies (p.s.)$$

**Example 4.2.** Let us consider  $X = \mathbb{R}_+$ ,  $V = \mathbb{R}$  and the mapping  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  given by  $u(t) = t^2 + 1$ . The mapping  $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , where  $\varphi(t, s, x) = t - s + x$  is an evolution semiflow on  $\mathbb{R}_+$ . We define

$$\Phi_u : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as the relation

$$|\Phi_u(t, s, x)v| \leq (s^2 + 1)(t - s)^{-2}|v|, \quad \forall t > s \geq 0, \quad \forall (x, v) \in Y$$

holds, it follows that  $C = (\varphi, \Phi)$  is a polynomially stable skew-evolution semiflow. On the other hand,  $C$  is not Barreira-Valls exponentially stable.

A similar concept to the nonuniform exponential growth can be considered the following nonuniform asymptotic property, given by

**Definition 4.2.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  has polynomial growth (*p.g.*) if there exist two mappings  $M, \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that:

$$(4.2) \quad \|\Phi(t, s, x)v\| \leq M(s)(t - s)^{\gamma(s)} \|v\|,$$

for all  $t > s \geq 0$  and all  $(x, v) \in Y$ .

**Remark 4.3.** If a skew-evolution semiflow  $C$  has polynomial growth, then it has exponential growth.

$$(p.g.) \implies (e.g.)$$

In order to obtain an integral characterization for the property of nonuniform polynomial stability for skew-evolution semiflows, we introduce the following concept, given by

**Definition 4.3.** A skew-evolution semiflow  $C = (\varphi, \Phi)$  is said to be  $*$ -strongly measurable ( $*$ -s.m.) if for every  $(t, t_0, x, v^*) \in T \times X \times V^*$  the mapping defined by  $s \mapsto \|\Phi(t, s, \varphi(s, t_0, x))^*v^*\|$  is measurable on  $[t_0, t]$ .

A particular class of  $*$ -strongly measurable skew-evolution semiflows is given by the next

**Definition 4.4.** *A  $*$ -strongly measurable skew-evolution semiflow  $C = (\varphi, \Phi)$  is called  $*$ -integrally stable ( $*$ -i.s.) if there exists a nondecreasing mapping  $B : \mathbb{R}_+ \rightarrow [1, \infty)$  such that:*

$$(4.3) \quad \int_s^t \|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\| d\tau \leq B(s) \|v^*\|,$$

for all  $(t, s) \in T$ , all  $x \in X$  and all  $v^* \in V^*$  with  $\|v^*\| \leq 1$ .

**Theorem 4.1.** *Let  $C = (\varphi, \Phi)$  be a  $*$ -strongly measurable skew-evolution semiflow with polynomial growth. If  $C$  is  $*$ -integrally stable, then  $C$  is stable.*

*Proof.* Let us consider the function

$$\gamma_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \gamma_1(t) = \frac{1}{1 + \gamma(t)},$$

where the mapping  $\gamma$  is given by Definition 4.2. We remark that for  $t \geq s + 1$  we have

$$\int_s^t (\tau - s)^{-\gamma(s)} d\tau = \int_0^{t-s} u^{-\gamma(s)} du \geq \int_0^1 u^{-\gamma(s)} du = \gamma_1(s).$$

Hence, it follows that

$$\begin{aligned} & \gamma_1(s) | \langle v^*, \Phi(t, s, x)v \rangle | \leq \\ & \leq \int_s^t (\tau - s)^{-\gamma(s)} \|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\| \|\Phi(\tau, s, x)v\| d\tau \leq \\ & \leq M(s) \|v\| \int_s^t \|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\| d\tau \leq M(s) B(s) \|v\| \|v^*\|, \end{aligned}$$

where the existence of function  $M$  is assured by Definition 4.2. We obtain

$$\|\Phi(t, s, x)v\| \leq M_1(s) \|v\|, \quad \forall t \geq s + 1 > s \geq 0, \quad \forall (x, v) \in Y,$$

where we have denoted

$$M_1(s) = \frac{M(s)B(s)}{\gamma(s)}, \quad s \geq 0.$$

On the other hand, for  $t \in [s, s + 1)$ , we have

$$\|\Phi(t, s, x)v\| \leq M(s)(t - s)^{\gamma(s)} \|v\| \leq M(s) \|v\|,$$

and, hence, it follows that

$$\|\Phi(t, s, x)v\| \leq [M(s) + M_1(s)] \|v\|, \quad \forall (t, s) \in T, \quad \forall (x, v) \in Y,$$

which proves that the skew-evolution semiflow  $C$  is stable.  $\square$

The main result of this section is the following

**Theorem 4.2.** *Let  $C = (\varphi, \Phi)$  be a  $*$ -strongly measurable skew-evolution semiflow with polynomial growth. If  $C$  is  $*$ -integrally stable, then  $C$  is polynomially stable.*

*Proof.* As the skew-evolution semiflow  $C = (\varphi, \Phi)$  is  $*$ -integrally stable, according to Theorem 4.1, it follows that there exists a mapping  $M_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$\begin{aligned} & | \langle v^*, \Phi(t, s, x)v \rangle | = | \langle \Phi(t, \tau, \varphi(\tau, s, x))^* v^*, \Phi(\tau, s, x)v \rangle | \leq \\ & \leq \| \Phi(\tau, s, x)v \| \| \Phi(t, \tau, \varphi(\tau, s, x))^* v^* \| \leq M_2(s) \| v \| \| \Phi(t, \tau, \varphi(\tau, s, x))^* v^* \|. \end{aligned}$$

By integrating on  $[s, t]$  we obtain for  $(x, v) \in Y$  and  $v^* \in V^*$  with  $\|v^*\| \leq 1$

$$\begin{aligned} (t-s) | \langle v^*, \Phi(t, s, x)v \rangle | & \leq M_2(s) \| v \| \int_s^t \| \Phi(t, \tau, \varphi(\tau, s, x))^* v^* \| d\tau \leq \\ & \leq M_2(s) B(s) \| v \| \| v^* \|, \end{aligned}$$

which implies

$$(t-s) \| \Phi(t, s, x)v \| \leq M_2(s) B(s) \| v \|.$$

Hence, following relation

$$\| \Phi(t, s, x)v \| \leq M_2(s) B(s) (t-s)^{-1} \| v \|$$

holds for all  $(t, s) \in T$  and all  $(x, v) \in Y$ .

Finally, it results that the skew-evolution semiflow  $C = (\varphi, \Phi)$  is polynomially stable.  $\square$

**Remark 4.4.** In [9], a variant of Theorem 4.2 for the case of uniform exponential stability is proved, as a generalization of a well known theorem of E.A. Barbashin, given in [1] for differential systems and of a result obtained in [4] by C. Buşe, M. Megan, M. Prajea and P. Preda for evolution operators. We remark that, in the nonuniform setting, the property of  $*$ -integral stability only implies the polynomial stability.

**Remark 4.5.** The reciprocal of Theorem 4.2 is not true. The skew-evolution semiflow given in Example 4.2 is polynomially stable but not  $*$ -strongly measurable. If we suppose that  $C$  is  $*$ -strongly measurable, we have

$$\int_s^t \frac{\tau^2 + 1}{t^2 + 1} d\tau = \frac{t-s}{t^2 + 1} \left( 1 + \frac{t^2 + ts + s^2}{3} \right) \leq N(s).$$

For  $t \rightarrow \infty$ , a contradiction is obtained.

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#### REFERENCES

- [1] E.A. Barbashin, Introduction dans la theorie de la stabilit e, Izd. Nauka, Moscou, 1967.
- [2] L. Barreira, C. Valls, *Stability of Nonautonomous Differential Equations*, Lecture Notes in Mathematics 1926 (2008).
- [3] L. Barreira, C. Valls, Polynomial growth rates, *Nonlinear Analysis* 71 (2009) 5208–5219.
- [4] C. Buşe, M. Megan, M. Prajea, P. Preda, *The strong variant of a Barbashin theorem on stability of solutions for nonautonomous differential equations in Banach spaces*, *Integral Equations Operator Theory* 59, No. 4 (2007) 491–500.
- [5] M. Megan, A.L. Sasu, B. Sasu, *Asymptotic Behaviour of Evolution Families*, Ed. Mirton, 2003.

- [6] M. Megan, C. Stoica, *On uniform exponential trichotomy of evolution operators in Banach spaces*, Integral Equations Operators Theory 60 No. 4 (2008) 499–506.
- [7] C. Stoica, *Uniform Asymptotic Behaviors for Skew-evolution Semiflows on Banach Spaces*, Mirton Publishing House, Timișoara, 2010.
- [8] C. Stoica, *On skew-evolution semiflows in infinite dimensional spaces*, SIAM International Conference on Emerging Topics in Dynamical Systems and Partial Differential Equations DSPDEs'10, Barcelona, Spain, June 2010.
- [9] C. Stoica, M. Megan, *On uniform exponential stability for skew-evolution semiflows on Banach spaces*, Nonlinear Analysis 72, Issues 3–4 (2010) 1305–1313.

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