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Mariem Sahnoun, Vincent Andrieu, Madiha Nadri

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RESEARCH ARTICLE

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Given a global nonlinear state feedback which stabilizes globally an equilibrium, the aim of this paper is to modify the local behavior of the trajectories in order to get local optimality with respect to a given quadratic cost. A sufficient condition is given in terms of Linear Matrix Inequalities (LMI) to design a locally optimal and globally stabilizing control law. This approach is illustrated on an academic inverted pendulum model in order to stabilize its upper equilibrium point. An extension of the main result is then given to address the problematic cases.

Moreover, the cases in which the previous LMI condition failed to be satisfied is addressed and a new sufficient condition is then given (which is not anymore linear).

Keywords: Nonlinear controllers, Lyapunov stabilization, Optimal control, LMI.

1 Introduction

The design of global asymptotic stabilizers for systems described by nonlinear differential equations has received many attention from the control community over the past three decades. Depending on the structure of the model, some techniques are now available to design a control law which globally stabilizes an equilibrium. For instance the backstepping (see Krstic et al. (1995) and references therein), the forwarding (see Mazenc and Praly (1996), Jankovic et al. (1996)), and some other approaches (see Kokotović and Arcak (2001)) have been widely studied.

Despite the fact that the stabilization of an equilibrium can be achieved, it is difficult to guarantee a certain performance for the closed loop system. On another hand, when the first order approximations of a nonlinear model is considered, performances issue can be handled by employing linear optimal control designs (for instance LQ or robust controllers). Moreover, with this optimal linear controller, stabilization of an equilibrium point can also be obtained but only locally. This leads to the idea of designing a new controller which unites a local linear (optimal) controller and a global one.

This uniting controller problem has already been addressed in the literature in Prieur (2001), Teel et al. (1997), Teel and Kapoor (1997), Efimov (2006) by employing some hybrid (and discontinuous) feedbacks. In the present paper, a sufficient condition is given for designing a continuous controller which unites a linear static local stabilizer and a nonlinear global one. The theory behind these developments is inspired from recent results in Andrieu and Prieur (2010) in which a continuous uniting of two control Lyapunov functions has allowed to continuously unite a local stabilizer and a non-local one (see also recent results in Clarke (2010)).

In that paper, based on the results of Andrieu and Prieur (2010), the continuous uniting control problem is investigated and some of these results are extended to the particular case in which the local controller is linear and the non-local one is global. More precisely, given a global nonlinear control which stabilizes globally asymptotically an equilibrium, the first result of the paper gives a sufficient condition to blend this controller with a local optimal controller. This sufficient condition is given in terms of Linear Matrix Inequality (LMI). This approach is then exploited to modify the local behavior of a controller which

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has been developed in Mazenc and Praly (1996) to asymptotically stabilize an inverted pendulum to its upper position.

Motivated by the fact that in some cases, the sufficient condition doesn't apply for some static linear controllers, a more general sufficient condition is given (see section 5). However, this one is not anymore in terms of linear matrix inequalities. With simple relaxation procedures, it is shown that this strategy addresses successfully the uniting problem for a large number of local stabilizers. Indeed, it is shown that statistically all local controllers can be merged with the global one on this specific inverted pendulum example.

The paper is organized as follows. In section 2, the problem under consideration is formalized. Moreover in the same section a first result which gives a sufficient condition in terms of LMI to solve the problem mentioned above is formulated in Theorem 2.3. The proof of this Theorem is given in Section 3. Section 4 is devoted to illustrate the proposed approach on an inverted pendulum system. Some further developments and a more general sufficient condition is given in section 5. Finally section 6 contains the conclusion.

Notations:

- The transpose of a matrix P is denoted P'.
- For arbitrary square matrices (P,Q) we write $P \geq Q$ if $P-Q \geq 0$; i.e., $P-Q$ is a positive semi-definite matrix. Similarly we define $P > Q$ if $P-Q > 0$; i.e., $P-Q$ is positive definite.
- $C^k(E;F)$: We denote $C^k(E;F)$ or simply C^k when this is no ambiguity on the sets, the set of functions from E to F which is of class C^k.
- Given a function V in $C^2(\mathbb{R}^n;\mathbb{R})$, $H(V)(x)$ denotes the Hessian matrix evaluated at x in $\mathbb{R}^n$, i.e. $(H(V)(x))_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}(x)$.

2 Problem statement and main result

2.1 *Problem formulation*

Throughout this paper, the following controlled nonlinear system, affine in the input is considered:

$$\dot{x} = f(x) + g(x)u \quad , \qquad x(0) = x_0 \; , \tag{1}$$

where x in $\mathbb{R}^n$ is the state vector, u in $\mathbb{R}^p$ is the control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function such that $f(0) = 0$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is a C^1 function.

The functions f being smooth, we can introduce the two matrices (F,G) in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p}$ with $F = \frac{\partial f}{\partial x}(0)$ and $G = g(0)$ describing the first order approximation of system (1).

All along this paper, it is assumed that the system (1) satisfies the following two assumptions:

Assumption 2.1 **Global Stabilization:** There exists a positive definite, proper and C^2 function $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ and a locally Lipschitz function $\phi_\infty : \mathbb{R}^n \to \mathbb{R}^p$ such that:

$$\frac{\partial V_\infty}{\partial x}(x)\left[f(x) + g(x)\phi_\infty(x)\right] < 0 \;, \forall\, x \neq 0 \;. \tag{2}$$

Assumption 2.2 **First order Controllability:** The pair of matrices (F,G) is controllable.

Under assumptions 2.1 and 2.2, the problem under consideration is a **stabilization with prescribed local behavior problem.** It can be formulated as follows:

Under Assumptions 2.1 and 2.2, given a linear (possibly optimal) local controller $u = K_0 x$ such that the *matrix $F+GK_0$ is Hurwitz, find a continuous control law $\phi_p:\mathbb{R}^n\to\mathbb{R}^p$ such that the origin of the system*

$$\dot{x}=f(x)+g(x)\phi_p(x)\ ,$$

is globally asymptotically stable and such that :

$$\frac{\partial\phi_p}{\partial x}(0)=K_0\ . \tag{3}$$

When the two functions (f,g) are such that the system is in backstepping form (see Krstic et al. (1995)), this problem has been solved in Pan et al. (2001). However, when no structure restriction are imposed on the couple (f,g) and based on the theory developed in Andrieu and Prieur (2010), a sufficient condition can be given in terms of linear matrix inequalities (LMI) which allows to solve the previous problem.

Theorem 2.3 ***LMI sufficient condition:*** *Assume the system (1) is such that there exist two functions V_∞,ϕ_∞ satisfying assumption 2.1 and such that the pair of matrices (F,G) satisfies assumption 2.2. Given P_0 a positive definite symmetric matrix in $\mathbb{R}^{n\times n}$ and K_0 a matrix in $\mathbb{R}^{n\times p}$ such that:*

$$P_0(F+GK_0)+(F+GK_0)^TP_0<0\ , \tag{4}$$

if there exists a matrix K_m in $\mathbb{R}^{n\times p}$ satisfying the following matrix inequalities

$$\begin{cases} P_0(F+GK_m)+(F+GK_m)^TP_0<0 \\ P_\infty(F+GK_m)+(F+GK_m)^TP_\infty<0 \end{cases}, \tag{5}$$

where, $P_\infty=H(V_\infty)(0)$ then, there exists a proper, positive definite and C^2 function $V_p:\mathbb{R}^n\to\mathbb{R}_+$ and a locally Lipschitz function $\phi_p:\mathbb{R}^n\to\mathbb{R}^p$ such that:

$$\frac{\partial V_p}{\partial x}(x)\Big[f(x)+g(x)\phi_p(x)\Big]<0\ ,\forall\, x\neq 0\ . \tag{6}$$

and there exists a positive real number r_∞ such that $\phi_p(x)=K_0x$ and $V_p(x)=x'P_0x$ for all x verifying $V_\infty(x)<r_\infty$.

It can be checked that Theorem 2.3 gives a sufficient condition to solve the *stabilization with prescribed local behavior problem*. Indeed, with Equation (4), the matrix K_0 is such that $F+GK_0$ is Hurwitz and moreover the function ϕ_p satisfies (3) (since the function V_∞ is positive definite, it yields $\phi_p(x)=K_0x$ in a neighborhood of the origin). The proof of Theorem 2.3 is given in section 3.

Since Theorem 2.3 gives a sufficient condition in terms of linear matrix inequalities, it allows to employ the efficient LMI solvers to check wether or not this LMI condition is satisfied. These tools are used in section 4 to employ Theorem 2.3 and to modify the local behavior of a global controller on a inverted pendulum. However, as shown in Remark **1** of Section 4.4, for some linear local controllers, this sufficient condition doesn't hold. In section 5, an extension of Theorem 2.3 is given which allows to overcome this difficulty.

2.2 *Discussion*

It can be noticed that Assumption 2.1 is a strong assumption. However, depending on the structure of the functions f and g some tools are now available allowing the design of the globally stabilizing controller ϕ_∞ and its associated Lyapunov function (backstepping, forwarding, feedback linearization, passivation,

...). Note that in Mazenc and Praly (1996), employing forwarding techniques a global controller for the model of an inverted pendulum is given. This one is studied in section 4.

Considering Assumption 2.2, a local controller ensuring local asymptotic stabilization of the origin of system (1) can be designed. Among the controls which provide asymptotic stabilization of the origin, the problem of guaranteeing a certain performance can be addressed.

One interesting aspect of this uniting methodology is the one regarding the H_∞ robust control design. Indeed, assume that the nonlinear system given in equation (1) is affected by some external disturbances as:

$$\dot{x} = f(x) + g(x)u + h(x)d \,, \tag{7}$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is a locally Lipschitz function and d in $C_0(\mathbb{R};\mathbb{R}^m)$ is an unknown external disturbance. In this case, following the H_∞ design methodology (see Başar and Bernhard (1995)) the control law must satisfy two distinct objectives:

i) The first is to guarantee the asymptotic stability of the origin when the disturbance vanishes.
ii) The second is to guarantee a given attenuation level of a quadratic functional of the state and control in the $\mathcal{L}^2$ framework. More precisely, given a positive definite matrix Q in $\mathbb{R}^{n\times n}$, a positive semi-definite matrix R in $\mathbb{R}^{p\times p}$ and a positive real number γ (the attenuation level) we want to find a stabilizing control feedback law $u = \phi_0(x)$ such that the following inequality is satisfied for all t in $\mathbb{R}$:

$$\int_0^t x_{u,w}(s)'Qx_{u,w}(s) + u(s)'Ru(s)ds \le \gamma^2 \int_0^t |d(s)|^2 ds \,, \tag{8}$$

where $x_{u,w}(\cdot)$ denotes the solution of system (7) initialized to the origin.

Solving this problem relies on the construction of a solution to a nonlinear Hamilton Jacobi Bellman equality[1] which can be difficult (or impossible) to solve. However, if we focuss on the linear approximation of system (7), then this problem can be solved locally. The first order approximation of system (7) is a linear system defined as,

$$\dot{x} = Fx + Gu + Hd \tag{9}$$

with $H = h(0)$. In this case, the Hamilton Jacobi Bellman equality is an algebraic equation defined as:

$$P_0F + F'P_0 + \frac{1}{\gamma^2}P_0HH'P_0 - P_0GR^{-1}G'P_0 + Q = 0 \,, \tag{10}$$

where the solution P_0 is a definite positive matrix in $\mathbb{R}^{n\times n}$, and a robust linear control for system (9) solving the disturbance attenuation as defined by inequality (8) is given as

$$u = \phi_0(x) = -R^{-1}G'P_0x \,. \tag{11}$$

[1]Following the nonlinear robust control design methodology, a way to solve this problem is to find a positive definite and proper smooth function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the Hamilton Jacobi Bellman equation

$$\frac{\partial V_0}{\partial x}(x)f(x) + \frac{1}{4\gamma^2}\frac{\partial V_0}{\partial x}(x)h(x)h(x)'\left(\frac{\partial V_0}{\partial x}(x)\right)' - \frac{1}{4}\frac{\partial V_0}{\partial x}(x)g(x)R^{-1}g(x)'\left(\frac{\partial V_0}{\partial x}(x)\right)' + x'Qx = 0$$

In this case, the solution to the control problem is simply

$$\phi_0(x) = -\frac{1}{2}R^{-1}g(x)'\left(\frac{\partial V_0}{\partial x}(x)\right)'$$

However, the computation of the solution to the Hamilton Jacobi Bellman equality is difficult in practice when dealing with nonlinear systems.

In section 4, this type of local controller is united with a global controller obtained by forwarding for the model of an inverted pendulum.

As seen from the LMI sufficient condition (i.e. inequalities (5)), we are interested in finding a common controller for two different lyapunov functions. This is in some aspect a dual problem from a usual problem in robust control design in which a unique lyapunov function is associated to different controllers (see Boyd et al. (1994) for further details).

Note also that given K_0 a locally stabilizing (possibly optimal) controller, many matrices P_0 are solution to the Lyapunov inequality (4). Among the solutions to this Lyapunov inequality, we need to find one such that inequalities (5) are satisfied.

Finally, it has to be noticed that inequalities (5) implies that locally V_∞ is a strict control Lyapunov function. This implies that this approach may fail when considering globally stabilizing controller which associated Lyapunov functions are not strict. This is for instance the case with most of the global controller obtained using some passivation arguments.

3 Proof of Theorem 2.3

The proof of Theorem 2.3 is based on the tools developed in Andrieu and Prieur (2010). Consequently, in a first step we review the result obtained in that paper.

3.1 *Continuously uniting local and non-local controller*

In Andrieu and Prieur (2010), a sufficient condition is given to allow the construction of a continuous control law which unites a local and a non-local one and preserves the global stability of the closed loop systems. This approach is based on the uniting of two Control Lyapunov Functions. One of the result obtained in that paper can be summarized as follows:

Theorem 3.1 ***Given in Andrieu and Prieur (2010) :*** *Let $\phi_0 : \mathbb{R}^n \to \mathbb{R}^p$ and $\phi_\infty : \mathbb{R}^n \to \mathbb{R}^p$ be two locally Lipschitz functions, $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ be two C^1 positive definite and proper functions, R_0 and r_∞ be two positive real numbers such that the following holds.*

i) Local stabilizability: For all x in $\{x : 0 < V_0(x) \leq R_0\}$,

$$\frac{\partial V_0}{\partial x}(x) f(x) + \frac{\partial V_0}{\partial x}(x) g(x) \phi_0(x) < 0 ; \tag{12}$$

ii) Non-local stabilizability: For all x in $\{x : V_\infty(x) \geq r_\infty\}$

$$\frac{\partial V_\infty}{\partial x}(x) f(x) + \frac{\partial V_\infty}{\partial x}(x) g(x) \phi_\infty(x) < 0 ; \tag{13}$$

iii) Covering assumption:

$$\{x : V_\infty(x) > r_\infty\} \cup \{x : V_0(x) < R_0\} = \mathbb{R}^n ; \tag{14}$$

iv) Uniting CLF assumption: For all x in $\{x : V_\infty(x) > r_\infty, V_0(x) < R_0\}$ there exists u_x in $\mathbb{R}^p$ such that:

$$\frac{\partial V_0}{\partial x}(x) f(x) + \frac{\partial V_0}{\partial x}(x) g(x) u_x < 0 ; \frac{\partial V_\infty}{\partial x}(x) f(x) + \frac{\partial V_\infty}{\partial x}(x) g(x) u_x < 0 . \tag{15}$$

Then, there exists a locally Lipschitz function $\phi_p : \mathbb{R}^n \to \mathbb{R}^p$ and a positive definite and proper C^1 function $V_p : \mathbb{R}^n \to \mathbb{R}_+$ which solves the uniting controller problem, i.e. such that

i) *Local property:* $\phi_p(x) = \phi_0(x)$ *and* $V_p(x) = V_0(x)$ *for all* x *such that* $V_\infty(x) \leq r_\infty$ *;*
ii) *Non-local property:* $\phi_p(x) = \phi_\infty(x)$ *and* $V_p(x) = V_\infty(x)$ *for all* x *such that* $V_0(x) \geq R_0$ *;*
iii) *Global stabilizability:*

$$\frac{\partial V_p}{\partial x}(x)\left[f(x)+g(x)\phi_p(x)\right] < 0 \ , \forall\, x \neq 0 \ . \tag{16}$$

This result is not presented in this way in Andrieu and Prieur (2010) but can be easily obtained from (Andrieu and Prieur 2010, Theorem 3.1) and (Andrieu and Prieur 2010, Proposition 2.2).

The idea of the proof in Andrieu and Prieur (2010) is to design a controller which is a continuous path going from $\phi_0(x)$ for x small toward $\phi_\infty(x)$ for larger values of the state. The global asymptotic stability of the origin is ensured by adding a sufficiently large term which depends on the *uniting control Lyapunov function* constructed from V_0 and V_∞. More precisely, the function $\phi_p : \mathbb{R}^n \to \mathbb{R}^p$ obtained from Theorem 3.1 and which is a solution to the uniting controller problem is defined as

$$\phi_p(x) \ = \ \mathcal{H}(x) \ - \ k\,c(x)\left(\frac{\partial V_p}{\partial x}(x) g(x)\right)' \ , \forall x \in \mathbb{R}^n \ , \tag{17}$$

where $V_p : \mathbb{R}^n \to \mathbb{R}_+$ is the united control Lyapunov function constructed employing the result in (Andrieu and Prieur 2010, Theorem 2.1). To be precise, this function unites the local and nonlocal control Lyapunov functions V_0 and V_∞ and is given for all x in $\mathbb{R}^n$ by

$$V_p(x) \ = \ R_0\left[\varphi_0(V_0(x))+\varphi_\infty(V_\infty(x))\right]V_\infty(x) + \ r_\infty\left[1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))\right]V_0(x) \ , \tag{18}$$

where $\varphi_0 : \mathbb{R}_+ \to [0,1]$ and $\varphi_\infty : \mathbb{R}_+ \to [0,1]$ are two continuously differentiable non-decreasing functions satisfying[1]:

$$\varphi_0(s)\begin{cases} = \ 0 & \forall s \leq r_0 \\ > \ 0 & \forall r_0 < s < R_0 \\ = \ \frac{1}{2} & \forall s \geq R_0 \end{cases} , \ \varphi_\infty(s)\begin{cases} = \ 0 & \forall s \leq r_\infty \\ > \ 0 & \forall r_\infty < s < R_\infty \\ = \ \frac{1}{2} & \forall s \geq R_\infty \end{cases} , \tag{21}$$

and where $r_0 = \max_{\{x : V_\infty(x) \leq r_\infty\}} V_0(x)$, and $R_\infty = \min_{\{x : V_0(x) \geq R_0\}} V_\infty(x)$. In (17) the function $\mathcal{H}$ continuously interpolates the two controllers ϕ_0 and ϕ_∞ and is given as

$$\mathcal{H}(x) \ = \ \upsilon(x)\,\phi_0(x) \ + \ [1-\upsilon(x)]\,\phi_\infty(x)$$

where υ is any continuous function[2] such that

$$\upsilon(x) \ = \ \begin{cases} 1 \ \text{ if } V_\infty(x) \ \leq \ r_\infty \ , \\ 0 \ \text{ if } V_0(x) \ \geq \ R_0 \ . \end{cases}$$

---

[1]For instance, φ_0 and φ_∞ can be defined as:

$$\varphi_0(s) \ = \ \frac{3}{2}\left(\frac{s-r_0}{R_0-r_0}\right)^2 - \left(\frac{s-r_0}{R_0-r_0}\right)^3 \ , \ s \in [r_0, R_0] \ , \tag{19}$$

$$\varphi_\infty(s) \ = \ \frac{3}{2}\left(\frac{s-r_\infty}{R_\infty-r_\infty}\right)^2 - \left(\frac{s-r_\infty}{R_\infty-r_\infty}\right)^3 \ , \ s \in [r_\infty, R_\infty] \ . \tag{20}$$

[2]For instance, giving φ_0 and φ_∞ defined in (21), a possible choice is: $\upsilon(x) = 1 - \varphi_0(V_0(x)) - \varphi_\infty(V_\infty(x))$. (22)

Also, in (17) the function c is any continuous function such that[3]

$$c(x)\begin{cases} = 0 \ \text{ if } V_0(x) \ \geq \ R_0 \ \text{ or } V_\infty(x) \ \leq \ r_\infty \ , \\ > 0 \ \text{ if } V_0(x) \ < \ R_0 \ \text{ and } V_\infty(x) \ > \ r_\infty \ , \end{cases} \tag{24}$$

and k is a positive real number sufficiently large to ensure that V_m is a Lyapunov function of the closed-loop system. The existence of k is obtained employing compactness arguments (see analogous arguments in (Andrieu et al. 2008, Lemma 2.13)).

3.2 *Proof of Theorem 2.3*

The idea of the proof is to show that with matrix inequalities (5) the four points of Theorem 3.1 are satisfied and consequently the controller (17) is a solution to the stabilization with prescribed local behavior problem.

Proof of Theorem 2.3: Consider $V_0(x) = x^T P_0 x$. Along the trajectories of System (1) with $u = K_0 x$, the function V_0 satisfies:

$$\dot{\overbrace{V_0(x)}} = x^T S_0 x + \triangle_0(x) \ ,$$

where S_0 is matrix in $\mathbb{R}^{n\times n}$ defined as

$$S_0 = P_0(F + GK_0) + (F + GK_0)^T P_0 \ ,$$

and where $\triangle_0 : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function defined as,

$$\triangle_0(x) = 2x^T P_0[(f(x) - Fx) + (g(x) - G)K_0 x] \ .$$

It can be checked that inequality (4) implies that S_0 is a symmetric negative definite matrix. Moreover, the function $\triangle_0$ satisfies,

$$\triangle_0(0) = 0 \ , \ \frac{\partial \triangle_0}{\partial x}(0) = 0 \ , H(\triangle_0)(0) = 0 \ .$$

Hence, it yields:

$$\triangle_0(x) = \circ(|x|^2) \ .$$

Consequently, $\dot{\overbrace{V_0(x)}} < 0$ along the trajectories of the System (1) with $u = K_0 x$ for all sufficiently small x. Hence Item 1 of Theorem 3.1 is satisfied with R_0 small enough.

On another hand, with Assumption 2.1, Item 2 of Theorem 3.1 is trivially satisfied for all $r_\infty > 0$. The functions V_0 and V_∞ being proper and definite positive, Item 3 is satisfied provided r_∞ is selected sufficiently small.

Now, along the trajectories of System (1) with $u = K_m x$, it yields,

$$\dot{\overbrace{V_\infty(x)}} = 2x^T P_\infty(f(x) + g(x)K_m x) + \left(\frac{\partial V_\infty}{\partial x}(x) - 2x^T P_\infty\right)(f(x) + g(x)K_m x) \ .$$

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[3]For instance, a possible choice is $c(x) = \max\{0, (R_0 - V_0(x))(V_\infty(x) - r_\infty)\}$ (23)

Which can be rewritten,

$$\overset{\cdot}{\overbrace{V_\infty(x)}} = x^T S_\infty x + \triangle_\infty(x) \ ,$$

where S_∞ is matrix in $\mathbb{R}^{n\times n}$ defined as

$$S_\infty = P_\infty(F+GK_m) + (F+GK_m)^T P_\infty \ ,$$

and where $\triangle_\infty : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function defined as,

$$\triangle_\infty(x) = 2x^T P_\infty[f(x) - Fx + (g(x)-G)K_m x] + \left(\frac{\partial V_\infty}{\partial x}(x) - 2x^T P_\infty\right)(f(x)+g(x)K_m x) \ .$$

Note that with (5), S_∞ is a symmetric negative definite matrix. Moreover since $\triangle_\infty$ satisfies

$$\triangle_\infty(0) = 0 \ , \frac{\partial \triangle_\infty}{\partial x}(0) = 0, H(\triangle_\infty)(0) = 0 \ ,$$

it yields

$$\triangle_\infty(x) = \circ(|x|^2) \ .$$

Consequently, along the trajectories of System (1) with $u = K_m x$ for all x small

$$\overset{\cdot}{\overbrace{V_\infty(x)}} < 0 \ . \tag{25}$$

It can be checked that the same conclusion holds with the function V_0. In other word, along the trajectories of System (1) with $u = K_m x$ for all x small

$$\overset{\cdot}{\overbrace{V_0(x)}} < 0 \ . \tag{26}$$

Inequalities (25) and (26) implies that the control law $u = K_m x$ makes strictly negative the time derivative of the two functions V_0 and V_∞ for x small enough. Hence, Item 4 of Theorem 3.1 is satisfied provided R_0 and r_∞ are small enough.

With Theorem 3.1, it yields that there exists a continuous function ϕ_p (for instance the one defined in (17)) which makes the origin of the system $\dot{x} = f(x) + g(x)\phi_p(x)$ globally asymptotically stable with associated Lyapunov function V_p defined in (18) and for all x such that $V_\infty(x) < r_\infty$ then $\phi(x) = K_0 x$ and $V_p(x) = x' P_0 x$. This ends the proof of Theorem 2.3.

4 Application to the inverted pendulum

The inverted pendulum is a classical example in control theory. The goal is to apply control torque to stabilize the inverted pendulum and raise it to its upper equilibrium position while the displacement of the carriage is brought to zero.

In our context, the control law has to ensure the overall stability of the system and a local disturbance attenuation level for a given quadratic cost with respect to some external disturbances on the model.

4.1 *Dynamical model*

Consider the inverted pendulum constituted of a movable carriage in translation on a horizontal axis. The pendulum, while being fixed on the carriage is free to rotate. We consider the rigid rod of negligible mass, and we define M the mass of the carriage (in gramme), m the mass of the pendulum (in gramme), l the length of the rod (in meter), χ the position of the carriage from origin (in meter), θ the angle between the pendulum and the vertical (in rad).

Using the equation of Euler-Lagrange, we get the following model given in terms of differential equations:

$$\begin{cases} \ddot{\chi}\cos(\theta)+l\ddot{\theta}-g\sin(\theta)=0\ , \\ (M+m)\,\ddot{\chi}+ml\cos(\theta)\ddot{\theta}-ml\sin(\theta)\dot{\theta}^2=F+d\ , \end{cases} \tag{27}$$

where F is the horizontal acceleration acting on the cart and is an unknown disturbance (which can be related to friction). This model can be rewritten in state space form as

$$\begin{cases} \ddot{\chi}=\dfrac{u+d+ml\dot{\theta}^2\sin(\theta)-mg\sin(\theta)\cos(\theta)}{M+m\sin(\theta)^2}\ , \\ \ddot{\theta}=\dfrac{g}{l}\sin(\theta)-\dfrac{1}{l}\cos(\theta)\left[\dfrac{u+d+ml\dot{\theta}^2\sin(\theta)-mg\sin(\theta)\cos(\theta)}{M+m\sin(\theta)^2}\right]\ , \end{cases} \tag{28}$$

where the state $x=(\chi,\dot{\chi},\theta,\dot{\theta})$ is in $\mathbb{R}\times\mathbb{R}\times\left(-\frac{\Pi}{2},\frac{\Pi}{2}\right)\times\mathbb{R}$ and the control input $u=F$ is in $\mathbb{R}$.

The physical data taken for our experiments are the following.

$$l=0.26m\ ,\ M=600g\ ,\ m=100g\ ,\ g=9.81\ .$$

4.2 *Globally stabilizing control law using forwarding*

We are interested in this paragraph in the control law given by Mazenc and Praly using the technique of forwarding or adding integrators (see Mazenc and Praly (1996)). In this subsection we don't consider the external disturbances.

Following Mazenc and Praly (1996), the differential equations of inverted pendulum (27) are rewritten in new coordinates with $d=0$:

$$\rho=\frac{\chi}{l}\ ,\ v=\frac{\dot{\chi}}{\sqrt{gl}}\ ,\ \theta=\theta\ ,\ \omega=\dot{\theta}\sqrt{\frac{l}{g}}\ , \tag{29}$$

and with a new control variable given by:

$$u_2=\frac{1}{g}\frac{u+ml\dot{\theta}^2\sin(\theta)-mg\sin(\theta)\cos(\theta)}{M+m\sin(\theta)^2}\ , \tag{30}$$

and, finally with a new time variable

$$t:=t\sqrt{\frac{g}{l}}\ .$$

Consequently, the following equations are obtained (Mazenc and Praly (1996)):

$$\dot{\rho}=v\ ,\ \dot{v}=u_2\ ,\ \dot{\theta}=\omega\ ,\ \dot{\omega}=\sin(\theta)-u_2\cos(\theta)\ , \tag{31}$$

with (ρ, v, ω) in $\mathbb{R}^3$ and we restrict θ to be in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. From a practical point of view, this means that the position of the mass is set above the fixation point of the bar. The forwarding approach of Mazenc and Praly consists in 3 steps :

i) stabilize the subsystem (θ, ω);
ii) stabilize the subsystem (v, θ, ω) by adding a first integration;
iii) stabilize the complete system by adding a final integration.

To obtain a non bounded domain (i.e on $\mathbb{R}^4$), the following change of coordinate is considered in Mazenc and Praly (1996):

$$\rho_1 = \rho \ , \ v_1 = v \ , \ t_1 = \tan(\theta) \ , \ r_1 = (1 + {t_1}^2)\omega \ .$$

In this case, (31) is rewritten as following:

$$\dot{\rho}_1 = v_1 \ , \ \dot{v}_1 = u_2 \ , \ \dot{t}_1 = r_1 \ , \ \dot{r}_1 = \frac{2t_1 {r_1}^2}{1+{t_1}^2} + t_1\sqrt{1+{t_1}^2} - u_2\sqrt{1+{t_1}^2} \ . \tag{32}$$

The forwarding approach consists on using a change of coordinates to each new addition of integration obtained by computing a solution to a partial differential equation and to perform a Lyapunov design at each step. With this approach the authors in Mazenc and Praly (1996) gave a control law $u_2 = \phi(\rho_1, v_1, t_1, r_1)$ which stabilizes the system globally asymptotically as following:

$$\phi(\rho_1, v_1, t_1, r_1) = \phi_1(t_1, r_1) + \phi_2(v_2, t_1, r_1) + \phi_3(\rho_3, v_2, t_1, r_1) \ , \tag{33}$$

with: $\phi_1(t_1, r_1) = 2t_1\left(1 + \frac{{r_1}^2}{(1+{t_1}^2)^{\frac{3}{2}}}\right) + r_1 \ ,$

$$\phi_2(v_2, t_1, r_1) = \frac{1}{10}v_2, \ v_2 = v_1 + 2\frac{r_1}{\sqrt{1+t_1^2}} + t_1 \ , \tag{34}$$

$$\phi_3(\rho_3, v_2, t_2, r_2) = 2[2r_1 + t_1]\sqrt{1+{t_1}^2} + 2v_2\left[10 + \frac{1}{2}|v_2|\right] + \frac{10\rho_3}{\sqrt{1+{\rho_3}^2}} \ ,$$

and $\rho_3 = \rho_1 + 2\log\left(t_1 + \sqrt{1+t_1^2}\right) + 10v_2 + v_1 + \frac{r_1}{\sqrt{1+t_1^2}}$. (35)

The associated control Lyapunov function $V : \mathbb{R}^4 \to \mathbb{R}_+$ is given as:

$$V(\rho_1, v_1, t_1, r_1) = 2{r_1}^2 + 2\left((1+{t_1}^2)^{\frac{3}{2}} - 1\right) + 2r_1t_1 + 10{v_2}^2 + \frac{1}{3}|v_2|^3 + \sqrt{1+{\rho_3}^2} - 1 \ , \tag{36}$$

where the functions v_2 and ρ_3 are given in (34) and (35).

With these data, it is shown in Mazenc and Praly (1996) that V_∞ defined in (36) satisfies along the trajectories of system (32) with $u_2 = \phi(\rho_1, v_1, t_1, r_1)$, the controller defined in (33), the following inequality

$$\dot{\overbrace{V(\rho_1, v_1, t_1, r_1)}} < 0 \ , \ \forall(\rho_1, v_1, t_1, r_1) \neq 0 \ .$$

Consequently, Assumption 2.1 is satisfied for System (32).

Going back to the coordinates of system (28), it yields a Lyapunov function V_∞ defined as,

$$V_\infty(x) = V\left(\frac{\chi}{l}, \frac{\dot{\chi}}{gl}, \tan(\theta), (1+\tan(\theta)^2)\dot{\theta}\sqrt{\frac{l}{g}}\right) \ ,$$

and a control law $u = \phi_\infty(x)$ with

$$\phi_\infty(x) = g(M + m\sin(\theta)^2)\phi\left(\frac{\chi}{l}, \frac{\dot{\chi}}{gl}, \tan(\theta), (1+\tan(\theta)^2)\dot{\theta}\sqrt{\frac{l}{g}}\right) - ml\dot{\theta}^2\sin(\theta) + mg\sin(\theta)\cos(\theta), \quad (37)$$

which satisfies Assumption 2.1 for the model (27) with the small modification that the state space is not $\mathbb{R}^4$ but $\mathbb{R} \times \mathbb{R} \times \left(-\frac{\Pi}{2}, \frac{\Pi}{2}\right) \times \mathbb{R}$.

4.3 *Locally robust stabilizing control law*

No we consider the disturbance d involved in the model (28) and we design a robust control law on the first order approximation of the model. The matrices of the first order approximation of system (28) are given as

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} + \frac{mg}{M} & 0 \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{lM} \end{bmatrix}, \; H = G\,. \quad (38)$$

Assumption 2.2 is satisfied for system (32). Consequently, a linear local stabilizing controller can be obtained. Among the possible linear local controllers which ensure local stabilization one may select one which ensures a particular attenuation level as defined in (8).

As an example, in (8) the matrix Q and the real number R are chosen as:

$$Q = \mathtt{Diag}\{0.1, 0.2, 0.3, 0.4\}\,,\; R = 5.10^{-6}. \quad (39)$$

Solving the associated Riccati equation (see equation (10)) by employing the routine (*care*) of Matlab with the attenuation level $\gamma = 0.045$, the matrix

$$P_0 \approx \begin{bmatrix} 0.20 & 0.11 & 0.26 & 0.03 \\ 0.11 & 0.19 & 0.50 & 0.05 \\ 0.26 & 0.50 & 1.98 & 0.13 \\ 0.03 & 0.05 & 0.13 & 0.01 \end{bmatrix} \quad (40)$$

is obtained. It yields that the control law (11) is given as:

$$\phi_0(x) = [141.6 \quad 289.8 \quad 1338.8 \quad 364]x \quad (41)$$

This controller guarantees an attenuation level for the linearization of the model in the sense of inequality (8). However when considering the nonlinear model (32), only local asymptotic stability can be achieved with this control law.

4.4 *Synthesis of optimal control law locally and globally stable*

The aim of this subsection is to employ Theorem 2.3 to unite the local optimal controller (41) and the globally stabilizing controller (33). The matrix $P_\infty = H(V_\infty)(0)$ is given as:

$$P_\infty \approx \begin{bmatrix} 7.40 & 13.25 & 23.08 & 6.57 \\ 13.25 & 27.64 & 47.59 & 13.81 \\ 23.08 & 47.59 & 85 & 23.93 \\ 6.57 & 13.81 & 23.93 & 6.96 \end{bmatrix} \quad (42)$$

Using (42) and Theorem 2.3, the existence of a vector K_m solution of LMI (5) with F, G given in (38) has to be verified. Employing the *Yalmip* package (Löfberg (2004)) in *Matlab* in combination with the solver[1] *Sedumi* (Sturm (1999)), it is shown that a vector K_m doesn't exist with these data.

However, the following matrix $\bar{P}_0$

$$\bar{P}_0 = \begin{bmatrix} 0.64 & 0.28 & 0.62 & 0.07 \\ 0.28 & 0.62 & 0.89 & 0.16 \\ 0.62 & 0.81 & 2.94 & 0.21 \\ 0.07 & 0.16 & 0.21 & 0.04 \end{bmatrix} \tag{43}$$

is also solution to the Lyapunov inequality (26) with the same control gain K_0. In this case we get the existence of K_m which satisfies (5) and is given by $K_m = [5249 \quad 11152 \quad 19487 \quad 5671]$. From Theorem 2.3, it yields that the control law given in (17) is a global stabilizer with the prescribed local behavior.

Performances of the proposed controller are evaluated in simulation when d is modeled as a centered gaussian noise with a standard deviation equal to 4. Functions φ_0, φ_∞, υ and c are respectively defined in (19), (20), (22) and (24) with parameters $R_0 = 46.2$, $r_\infty = 0.2702$, $r_0 = 24.914$, $R_\infty = 0.5017$ and $k = 10$. The initial condition considered is:

$$\chi(0) = 10 \ , \ \dot{\chi}(0) = 0.1 \ , \ \theta(0) = 1.3 \ , \ \dot{\theta}(0) = 0.8 \ .$$

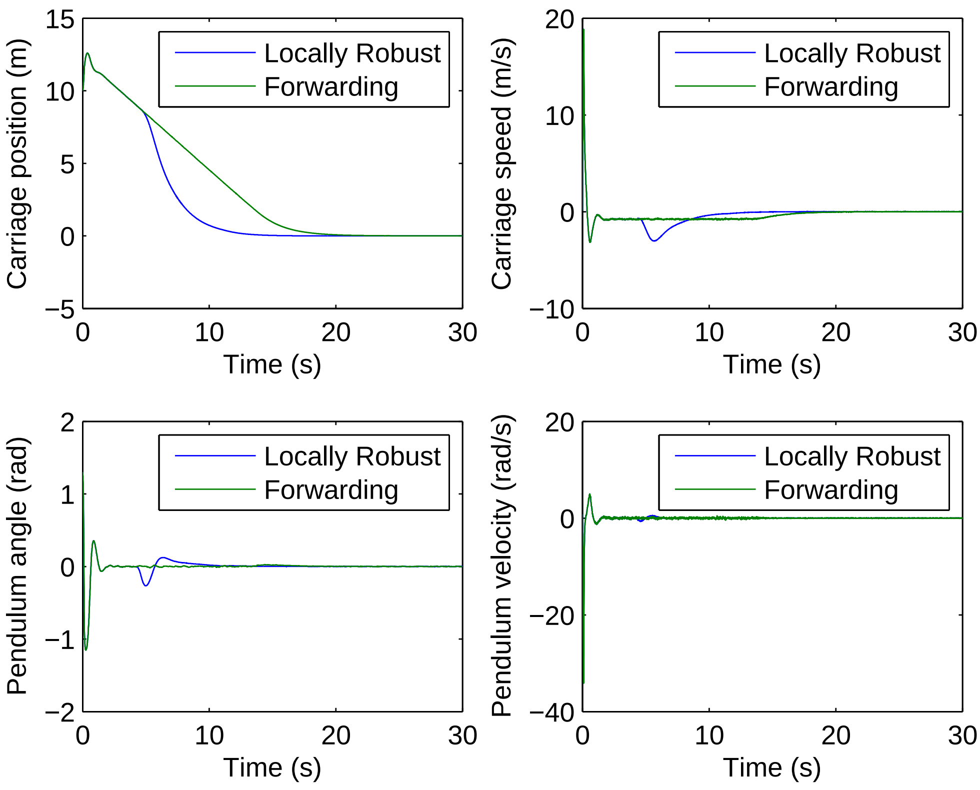

Figure 1. Evolution of the state variables when considering the forwarding controller and the locally robust one.

[1] All Matlab files can be downloaded from the website : https://sites.google.com/site/vincentandrieu/

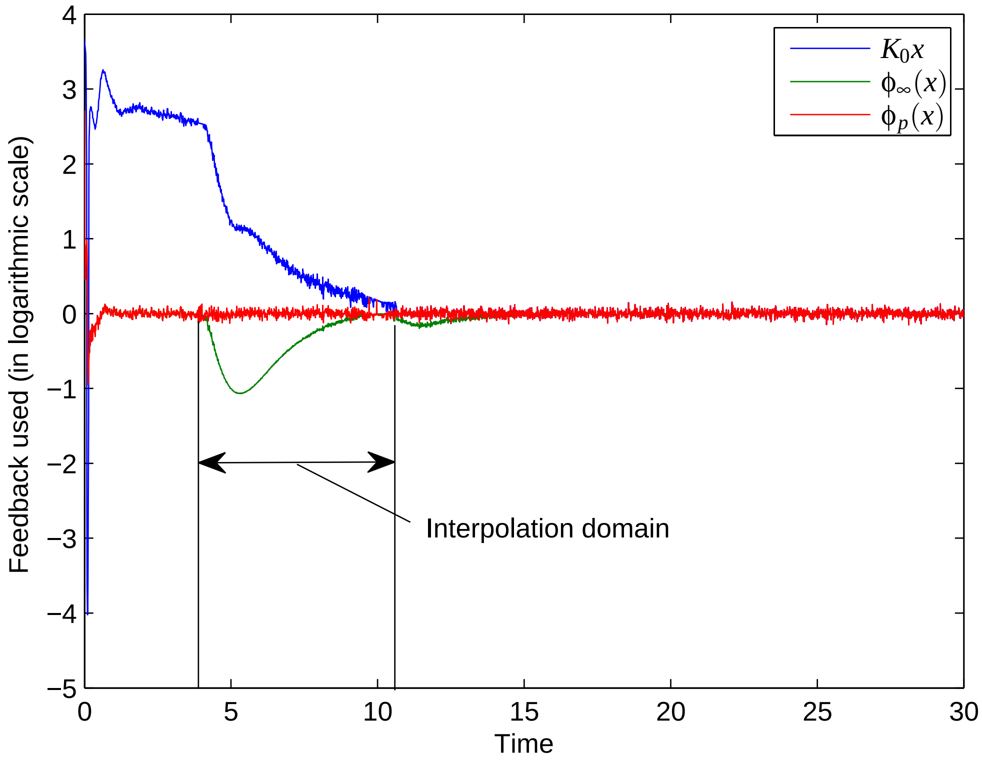

Figure 2. Comparison of the functions K_0x, $\phi_\infty(x)$ and $\phi_p(x)$ along the solution of the closed loop system.

The evolution of the state variables are depicted in Figure 1. As it can be seen the use of the locally robust controller ensures a faster convergence rate. Moreover, as seen in Figure 3 its robustness with respect to the measurement noise seems to be improved.

As can be seen in Figure 2 at the moment $t = 3.95(sec)$, the control law of the modified Forwarding leaves the usual forwarding control law up to time $t = 10.68(sec)$ where it reaches the locally optimal control law.

As seen on Figure 4, in the interpolation domain the control law increase due to the high gain parameter k.

Remark 1 : There are locally optimal or robust control laws for which the matrix inequality (5) does not have a solution. To evaluate the frequency of these problematic cases, a statistical study on the frequency of solvability of the *LMI* condition given in Theorem 5.1 is done using the data obtained from the inverted pendulum studied previously in the next Section.

4.5 *Statistical study*

In this paragraph, we make a statistical study of the solvability frequency of the sufficient condition given in terms of *LMI* in the Theorem 5.1 with the data obtained from the inverted pendulum studied previously.

To numerically estimate the frequency of the problematic cases in which there is no solution to the LMI sufficient condition (5), we develop a statistical approach. To do this, we restrict ourselves to consider the set of local optimal LQ controllers $u = -G'P_0x$ where each P_0 is solution to the algebraic Riccati

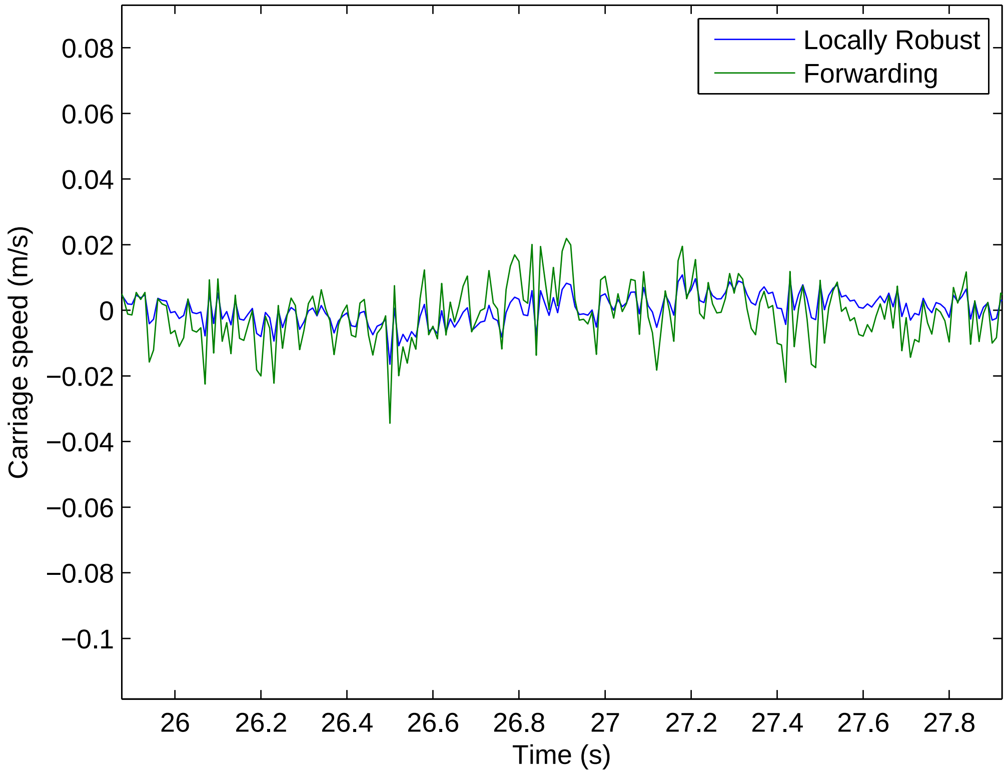

Figure 3. Comparison of the steady-state response of the carriage speed when using the forwarding or the locally robust control law.

equation parameterized by a matrix Γ in $\mathbb{R}^{4\times 4}$ and given as[1],

$$F'P_0 + P_0F - P_0GG'P_0 + \Gamma'\Gamma = 0\ .$$

To perform a statistical study, on this set of controllers, elements of Γ are given by uncorrelated uniformly distributed random variables in $[0,1]$, and we have simulated a number of draws. For each of these draws, we solve the corresponding Riccati equation and we obtain a local LQ controller K_0 and its associated Lyapunov function $x'P_0x$. With P_0 we check the corresponding *LMI* condition in (5) employing the *Yalmip* package (Löfberg (2004)) in *Matlab* in combination with the solver[1] *Sedumi* (Sturm (1999)).

We set the number of draws to 10000. For each of these draws, the matrix Γ is obtained using the routine (*rand*) of Matlab. We repeated this manipulation to test the pertinence of our approach. The values given in the following tabular are the percentage of cases for which we have obtained a solution to the LMI test (5):

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[1]Note that compare to usual *LQ* approaches, we are not losing any generality by setting $R = 1$ since we can normalize the cost
[1]All Matlab files can be downloaded from the website : https://sites.google.com/site/vincentandrieu/

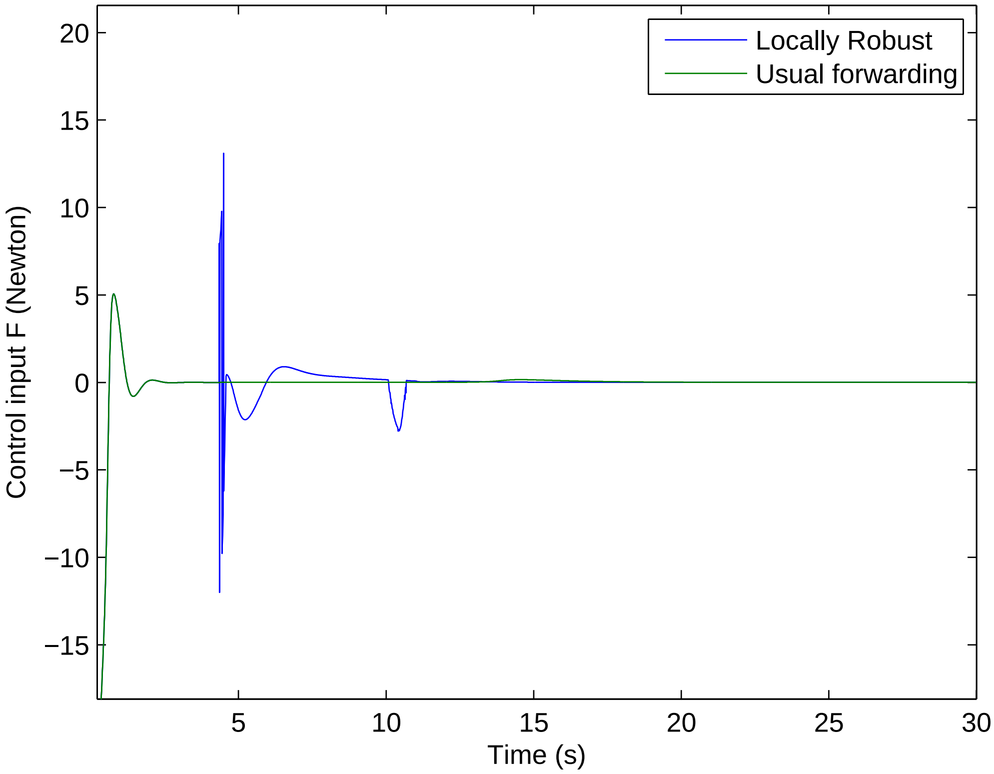

Figure 4. Comparison of the controlled input in the undisturbed case.

Test	Percentage of success
1	36,32%
2	36,14%
3	36,87%
4	35,64%
5	35,97%
6	36,20%
7	36,72%
8	36,55%
9	35,92%
10	36,10%

The mean of these percentage is 36.24% and the standard deviation is 1.13 (which is relatively low). Thus, we conclude that the frequency of 36.24% appears to be representative of the mathematical expectation of the solvability of our sufficient condition in the case of the inverted pendulum for LQ controllers.

5 Extension of theorem 2.3

To apply our approach and design a globally stabilizing control law and locally optimal, we have to solve the LMI test (5). However, as we have seen in the statistical analysis, in most cases, this is not possible. A solution to the problem where the local optimal control does not solve the LMI test is to use a transient Lyapunov function. Indeed, we have the following result.

Theorem 5.1 ***Extension*** *Under Assumptions 2.1 and 2.2, given P_0 a symmetric definite positive matrix in $\mathbb{R}^{n\times n}$ and K_0 a matrix in $\mathbb{R}^{n\times p}$ such that:*

$$P_0(F+GK_0)+(F+GK_0)^T P_0 < 0\ . \tag{44}$$

If there are two matrices $K_{m,1}$ and $K_{m,2}$ in $\mathbb{R}^{n\times p}$, P_m a definite positive matrix in $\mathbb{R}^{n\times n}$ such that the following matrix inequalities are satisfied:

$$\begin{cases} P_0(F+GK_{m,1})+(F+GK_{m,1})^T P_0 < 0 \\ P_m(F+GK_{m,1})+(F+GK_{m,1})^T P_m < 0 \\ P_m(F+GK_{m,2})+(F+GK_{m,2})^T P_m < 0 \\ P_\infty(F+GK_{m,2})+(F+GK_{m,2})^T P_\infty < 0 \end{cases}, \tag{45}$$

where the matrix $P_\infty = H(V_\infty)(0)$ then, there exists a continuous function $\phi_p : \mathbb{R}^n \to \mathbb{R}^p$ such that the origin of the system $\dot{x} = f(x)+g(x)\phi_p(x)$ is globally asymptotically stable, and there exists a positive sufficiently small real number r_∞ such that $\phi_p(x) = K_0 x$ for all x verifying $V_m(x) < r_\infty$.

Proof : The proof of Theorem 5.1 is a direct consequence of Theorem 2.3. Indeed, if the two last matrix inequalities in (45) are satisfied, we can apply Theorem 2.3 to obtain a locally Lipschitz control law ϕ_m, a proper and definite positive C^1 function $V_{m,1}$, and a positive real number $r_{\infty,m}$ sufficiently small such that

$$\frac{\partial V_{m,1}}{\partial x}(x)\left[f(x)+g(x)\phi_m(x)\right] < 0\ , \forall\, x \neq 0\ , \tag{46}$$

and such that for all x such that $V_\infty(x) < r_{\infty,m}$ then

$$\phi_m(x) = K_{m,1}x\ ,\ V_{m,1}(x) = x' P_m x\ .$$

With the two first inequalities in (45), we can use another time Theorem 2.3 to obtain a function ϕ_p, a real r_∞ small enough such that the origin of the system: $\dot{x} = f(x)+g(x)\phi_p(x)$ is globally asymptotically stable, and for all x such that $V_m(x) < r_\infty$ then $\phi_p(x) = K_0 x$. □

It has to be noticed that this result does not come in the form of a linear matrix inequality. Therefore, it is not possible to employ the usual LMI resolution tool to directly solve this sufficient condition. However, by randomly selecting the matrix P_m, inequalities (45) become linear in the unknowns $K_{m,1}$, $K_{m,2}$.

Consequently, given K_0, the local controller and its associated Lyapunov function P_0, we can employ the following algorithm:

`While the matrix inequalities (45) is not satisfied`

i) `Select randomly a positive definite matrix` Q_m `in` $\mathbb{R}^{n\times n}$.
ii) `Solve the associated Riccati equation` $F'P_0 + P_0F - P_0GG'P_0 + Q_m = 0$ `to get a` P_m `matrix in` $\mathbb{R}^{n\times n}$ `which defines a CLF.`
iii) `Check if the matrix inequalities (45) is satisfied.`

Employing this simple algorithm, we have shown numerically that with 10000 different P_0 and K_0, in all cases it was possible to find a P_m such that the matrix inequalities (45) was satisfied. Note that, with this algorithm the maximal number of transient CLF P_m that has to be tested was 22.

Consequently, it seems that with Theorem 5.1, it is possible to design a globally stabilizing controller such that its first order approximation can be solution of all possible optimal (LQ) or robust (H_∞) problem on this specific example.

6 Conclusion

A method to obtain a globally stabilizing control law with a pre-selected local behavior (robustness or optimality) is presented in that paper. This approach is based on the use of a technique recently developed in Andrieu and Prieur (2010). A sufficient condition in terms of LMI is first given. This approach is illustrated in an academic problem of stabilizing an inverted pendulum to its upper equilibrium position. We modify the local behavior of the globally stabilizing control law obtained by forwarding in Mazenc and Praly (1996) in order to get a locally robust control law. Moreover, it is shown numerically that nearly 36.24% of LQ controllers can be reproduced locally with this approach. By extending this approach and using a transitional Lyapunov function, it is shown numerically that 100% of LQ controller can be combined with the global control law. The results show the advantage of this technique to change the local behavior of a controlled nonlinear system which in practice is difficult to tune.

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