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# POLYGONS OF THE LORENTZIAN PLANE AND SPHERICAL POLYHEDRA

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ABSTRACT. It is known that the space of convex polygons in the Euclidean plane with fixed normals, up to homotheties and translations, endowed with the area form, is isometric to a hyperbolic polyhedron. In this note we show a class of convex polygons in the Lorentzian plane such that their moduli space, if the normals are fixed and endowed with a suitable area, is isometric to a spherical polyhedron. These polygons have an infinite number of vertices, are space-like, contained in the future cone of the origin, and setwise invariant under the action of a linear isometry.

## 1. INTRODUCTION

It was proved in [Bavard and Ghys, 1992] that the space of convex polygons in the Euclidean plane, up to homotheties and translations, with fixed outward normals to the edges, endowed with the area form, is isometric to a hyperbolic polyhedron. It was an application of [Thurston, 1998] to the case of polygons. It was noted in [Fillastre, 2011a] that the construction of [Bavard and Ghys, 1992] can be seen as an application of the mixed-volume theory. (See [Fillastre, 2011a] for other related references.) Here we adapt the mixed-volume theory to a particular class of convex polygons in the Lorentzian plane (volume will be called area in this dimension). Despite the fact that the definition of the polygons in the present case could seem more intricate than in the Euclidean case, the theory appears to be simpler. The generalization to all dimension and to objects more general than polyhedral ones is the subject of [Fillastre, 2011b].

## 2. EQUIVARIANT CONVEX POLYGONS IN THE LORENTZIAN PLANE

The *Lorentzian plane* is  $\mathbb{R}^2$  endowed with the bilinear form  $\langle x, y \rangle_1 = x_1 y_1 - x_2 y_2$ . A vector  $x$  is called *space-like* (resp. *time-like*) if  $\langle x, x \rangle_1$  is positive (resp. negative). In this paper, a *future vector* is a time-like vector whose first coordinate is positive. The *angle* between two non-collinear future vectors  $x$  and  $y$  is the unique  $\varphi > 0$  such that [Ratcliffe, 2006, (3.1.7)]

$$(1) \quad \cosh \varphi = - \frac{\langle x, y \rangle_1}{\sqrt{\langle x, x \rangle_1 \langle y, y \rangle_1}}.$$

The set of future vectors

$$\mathcal{F} := \{x \in \mathbb{R}^2 \mid \langle x, x \rangle_1 < 0, x_1 > 0\}$$

is a convex cone. If  $a$  is a vector,

$$a^\perp := \{x \in \mathbb{R}^2 \mid \langle x, a \rangle_1 = \langle a, a \rangle_1\}$$

is the affine hyperplane over the vector hyperplane orthogonal for  $\langle \cdot, \cdot \rangle_1$  to  $a$ , and passing through  $a$ .

In  $\mathcal{F}$ , the *hyperbolic line*

$$\mathbb{H} := \{x \in \mathbb{R}^2 \mid \langle x, x \rangle_1 = -1, x_1 > 0\}$$

is the set of unitary future vectors. It will play the role of the unit circle in the Euclidean plane.  $\mathbb{H}$  is globally invariant under the action of the linear isometries of the Lorentzian plane called *hyperbolic translations*:

$$H_t := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, t \in \mathbb{R}.$$

**Definition 2.1.** Let  $(\eta_1, \dots, \eta_n)$ ,  $n \geq 1$ , be pairwise non-collinear future vectors in the Lorentzian plane, and let  $t > 0$ . A  $t$ -convex polygon is the intersection of the half-planes bounded by the lines

$$(H_t^k \eta_i)^\perp, \forall k \in \mathbb{Z}, \forall i = 1, \dots, n.$$

The half-planes are chosen such that the vectors  $\eta_i$  are inward pointing.

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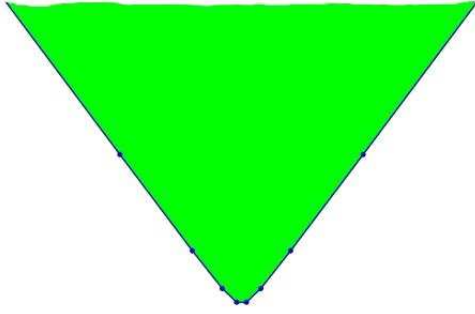


FIGURE 1. A piece of the  $\sinh^{-1}(1)$ -convex polygon made from the vector  $(0, 1)$ . All the edges have the same length for the induced metric.

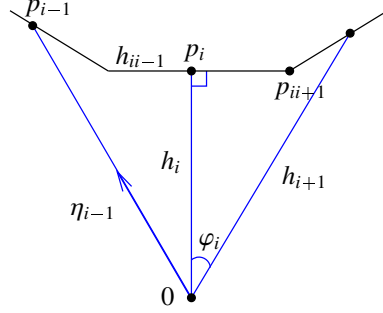


FIGURE 2. Notations for a  $t$ -convex polygon.

Let us consider a  $t$ -convex polygon made from a single future vector  $\eta$  (see Figure 1). Without loss of generality, consider that  $\eta \in \mathbb{H}$ . Let  $u, v$  be two different vectors in the orbit of  $\eta$ , and let  $x$  be the intersection between  $u^\perp$  and  $v^\perp$ . As  $\langle x, u \rangle_1 = \langle x, v \rangle_1 = -1$ ,  $x$  is orthogonal to  $u - v$ , which is a space-like vector (compute its norm with the help of (1)). Hence  $x$  is time-like, and as  $u^\perp$  and  $v^\perp$  never meet the past cone,  $x$  is future. It is easy to deduce that the  $t$ -convex polygon is contained in  $\mathcal{F}$ . As any  $t$ -convex polygon is the intersection of a finite number of  $t$ -convex polygons made of a single orbit, it is contained in  $\mathcal{F}$ .

Let  $P$  be a  $t$ -convex polygon and let  $\eta_i$  be the inward unit normal of the edge  $e_i$  of  $P$ . The edge at the left (resp. right) of  $e_i$  is denoted by  $e_{i-1}$  (resp.  $e_{i+1}$ ). Let  $p_i$  be the foot of the perpendicular from the origin to the line containing  $e_i$ . The positive number  $h_i = \sqrt{-\langle p_i, p_i \rangle_1}$  is a *support number* of  $P$ . In particular,  $p_i = h_i \eta_i$ . Let  $p_{ii+1}$  be the vertex between  $e_i$  and  $e_{i+1}$ . We denote by  $h_{ii+1}$  the signed distance from  $p_i$  to  $p_{ii+1}$ : it is non negative if  $p_i$  is in the same side of  $e_{i+1}$  than  $P$ . The angle between  $\eta_i$  and  $\eta_{i+1}$  is denoted by  $\varphi_i$ . See Figure 2.

**Lemma 2.2.** *With the notations introduced above,*

$$(2) \quad h_{ii+1} = \frac{h_i \cosh \varphi_i - h_{i+1}}{\sinh \varphi_i}, h_{ii-1} = \frac{h_i \cosh \varphi_{i-1} - h_{i-1}}{\sinh \varphi_{i-1}}.$$

*Proof.* By definition,  $h_{ii+1}$  is non negative when  $\langle p_i - p_{ii+1}, \eta_{i+1} \rangle_1 \leq 0$ , i.e.  $-(h_{i+1} - h_i \cosh \varphi_i) \geq 0$ . Hence

$$h_{ii+1} = -\frac{h_{i+1} - h_i \cosh \varphi_i}{|h_{i+1} - h_i \cosh \varphi_i|} \sqrt{\langle p_{ii+1} - p_i, p_{ii+1} - p_i \rangle_1}.$$

Up to an orientation and time orientation preserving linear isometry, one can take  $\eta_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . In particular  $p_i = \begin{pmatrix} 0 \\ h_i \end{pmatrix}$  and  $(p_{ii+1})_2 = h_i$  hence  $\langle p_{ii+1} - p_i, p_{ii+1} - p_i \rangle_1 = (p_{ii+1})_1^2$ . We also have  $\eta_{i+1} = \begin{pmatrix} \sinh \varphi_i \\ \cosh \varphi_i \end{pmatrix}$ , and as  $\langle p_{ii+1}, \eta_{i+1} \rangle_1 = -h_{i+1}$  we get  $(p_{ii+1})_1 = \frac{-h_{i+1} + h_i \cosh \varphi_i}{\sinh \varphi_i}$ . The proof for  $h_{ii-1}$  is similar, considering  $\eta_{i-1} = \begin{pmatrix} -\sinh \varphi_i \\ \cosh \varphi_i \end{pmatrix}$ .  $\square$

### 3. AREA OF $t$ -CONVEX POLYGONS

Let  $P$  be a convex  $t$  polygon. Choose an edge and denote its inward unit normal by  $\eta_1$ . We denote the inward unit normal of the edge on the right by  $\eta_2$ , and so on until  $\eta_{n+1} = H_t(\eta_1)$ . The edges with normals  $\eta_1, \dots, \eta_n$  are the *fundamental edges* of  $P$ . The number  $h_i(P)$  is the support number of the edge with normal  $\eta_i$ . If  $\varphi_i$  is the angle between  $\eta_i$  and  $\eta_{i+1}$  we have

$$(3) \quad \varphi_1 + \varphi_2 + \dots + \varphi_n = t.$$

The *area* of  $P$  is

$$A(P) = \frac{1}{2} \sum_{i=1}^n h_i(P) \ell_i(P)$$

where the sum is on the fundamental edges, and  $\ell_i(P) = h_{ii-1}(P) + h_{ii+1}(P)$  is the length of the  $i$ th fundamental edge (hence positive).  $A(P)$  is the area (in the sense of the Lebesgue measure) in a fundamental domain for the action of  $H_t$  of the complement of  $P$  in  $\mathcal{F}$ . Formula (2) allow us to extend formally the definition of  $A$  to any vector  $h \in \mathbb{R}^n$ :

$$A(h) = \frac{1}{2} \sum_{i=1}^n h_i \frac{h_i \cosh \varphi_{i-1} - h_{i-1}}{\sinh \varphi_{i-1}} + h_i \frac{h_i \cosh \varphi_i - h_{i+1}}{\sinh \varphi_i}.$$

In other words, we identify  $P$  with the support numbers of its fundamental edges, and the vectors  $\eta_1, \dots, \eta_n$  are in bijection with the standard basis of  $\mathbb{R}^n$ . Of course  $P$  is then uniquely determined by the vector made of its support numbers.

If  $n = 1$ , there is only one angle between the unit inward normal  $\eta$  and its image under  $H_t$ , which is equal to  $t$ , and  $A(h) = h^2 \frac{\cosh t - 1}{\sinh t}$ . If  $n \geq 2$ , we introduce the *mixed-area*

$$A(h, k) = \frac{1}{2} \sum_{i=1}^n h_i \frac{k_i \cosh \varphi_{i-1} - k_{i-1}}{\sinh \varphi_{i-1}} + h_i \frac{k_i \cosh \varphi_i - k_{i+1}}{\sinh \varphi_i},$$

which is the polarization of  $A$ . Effectively, it is clearly a bilinear form, and

$$(4) \quad A(\eta_k, \eta_j) = \begin{cases} 0 & \text{if } 2 \leq |j - k| \leq n + 1 \\ -\frac{1}{2} \frac{1}{\sinh \varphi_{k-1}} & \text{if } j = k - 1 \\ -\frac{1}{2} \frac{1}{\sinh \varphi_k} & \text{if } j = k + 1 \\ \frac{1}{2} \left( \frac{\cosh \varphi_{k-1}}{\sinh \varphi_{k-1}} + \frac{\cosh \varphi_k}{\sinh \varphi_k} \right) & \text{if } j = k \end{cases}$$

( $\eta_i$  is identified with the  $i$ th vector of the standard basis of  $\mathbb{R}^n$ ) so  $A$  is symmetric. We also obtain the following key result, which is analog to a fundamental result of Brunn–Minkowski theory for convex Euclidean polygons.

**Proposition 3.1.** *The symmetric bilinear form  $A$  is positive definite. The Cauchy–Schwarz inequality applied to  $t$ -convex polygons gives the reversed Minkowski inequality:*

$$A(P, Q)^2 \leq A(P)A(Q),$$

with equality if and only if  $P$  and  $Q$  are homothetic.

*Proof.* As  $\cosh \varphi_k > 1$ , the matrix  $(A(u_k, u_j))_{k,j}$  is strictly diagonally dominant, and symmetric with positive diagonal entries, hence positive definite, see for example [Varga, 2000, 1.22].  $\square$

#### 4. SPHERICAL ORTHOSCHEMES

For a given  $t > 0$ , we consider the set of convex  $t$ -polygons with a given set of inward normals (i.e. the set of convex  $t$ -polygons with parallel edges), up to global isometries of the Lorentzian plane. Note that this set is known if and only if one knows the angles  $\varphi_1, \varphi_2, \dots, \varphi_n$  between the inward normals of the fundamental edges. The  $t$ -convex polygons are identified with their support numbers, so we are looking at a subset of  $\mathbb{R}^n$ . It is made of polygons with fundamental edges of positive length. From (2) the length is a linear function of the support vectors, so  $\mathcal{P}$  is a convex polyhedral cone in  $\mathbb{R}^n$ , with  $n$  facets.

If  $\mathbb{R}^n$  is endowed with the mixed-area form, and if we look only at polygons of area 1, we get a simplex of the  $n - 1$  dimensional sphere, denoted by  $\mathcal{P}(\varphi_1, \dots, \varphi_n)$ , or  $\mathcal{P}$ . Note that such a space can be constructed from any positive numbers  $\varphi_i$  with  $t$  satisfying (3). Note also that if  $s$  is a cyclic permutation, then  $\mathcal{P}(\varphi_{s(1)}, \dots, \varphi_{s(n)})$  is the same as  $\mathcal{P}(\varphi_1, \dots, \varphi_n)$ .

If  $n = 1$ ,  $\mathcal{P}$  is a point on a line, so until now we consider that  $n > 1$ . Let  $P \in \mathcal{P}$ . Because  $2A(\eta_i, P) = \ell_i(P)$ ,  $\eta_i$  (as the  $i$ th vector of the canonical basis of  $\mathbb{R}^n$ ) is an inward normal vector to the facet of  $\mathcal{P}$  defined by  $\ell_i = 0$ . When  $n = 2$ ,  $\mathcal{P}$  is an arc on the unit circle with length  $\theta$  satisfying

$$\cos \theta = \frac{\sinh \varphi_2}{\sinh(\varphi_1 + \varphi_2)}.$$

When  $n = 3$ ,  $\mathcal{P}$  is a spherical triangle with acute inner angles, whose cosines are given by:

$$(5) \quad \frac{A(\eta_k, \eta_{k+1})}{\sqrt{A(\eta_k, \eta_k)} \sqrt{A(\eta_{k+1}, \eta_{k+1})}} = \sqrt{\frac{\sinh \varphi_{k-1} \sinh \varphi_{k+1}}{\sinh(\varphi_{k-1} + \varphi_k) \sinh(\varphi_k + \varphi_{k+1})}}.$$

When  $n \geq 3$ , from (4) we see that each facet has an acute interior dihedral angle with exactly two other facets, and is orthogonal to the other facets. Such spherical simplices are called *spherical orthoschemes*, see [Debrunner, 1990, 5] for the history and main properties of these very particular simplices.

Configuration spaces of Euclidean polygons allow to construct the hyperbolic orthoschemes which are of Coxeter type (acute angles of the form  $\pi/k$ ,  $k \in \mathbb{N} \setminus \{0, 1, 2\}$ ) [Bavard and Ghys, 1992]. In our case the problem is empty as there is no spherical Coxeter orthoscheme (the Coxeter diagram of a spherical orthoscheme must be a cycle, and there is no cycle in the list of Coxeter diagrams of spherical Coxeter simplices. The list can be found for example in [Ratcliffe, 2006]).

## 5. SPHERICAL CONE-MANIFOLDS

Let  $n > 2$  and consider the orthoscheme  $\mathcal{P} = \mathcal{P}(\varphi_1, \dots, \varphi_n)$ . A facet of  $\mathcal{P}$  is isometric to the space of  $t$ -convex polygons with  $\eta_1, \dots, \hat{\eta}_i, \dots, \eta_n$  ( $\hat{\eta}_i$  means that  $\eta_i$  is deleted from the list) as normals to the fundamental edges. The angles between the normals are  $\varphi_1, \dots, \varphi_{i-2}, \varphi_{i-1} + \varphi_i, \varphi_{i+1}, \dots, \varphi_n$ . This orthoscheme is also isometric to a facet of the orthoscheme  $\mathcal{P}'$  obtained by permuting  $\varphi_{i-1}$  and  $\varphi_i$  in the list of angles. Hence we can glue  $\mathcal{P}$  and  $\mathcal{P}'$  isometrically along this common facet. We denote by  $C(\varphi_1, \dots, \varphi_n)$  the  $(n-1)$ -dimensional spherical cone-manifold obtained by gluing in this way all the  $(n-1)!$  orthoschemes obtained by permutations of the list  $\varphi_1, \dots, \varphi_n$ , up to cyclic permutations.

When  $n = 3$ ,  $C(\varphi_1, \varphi_2, \varphi_3)$  is isometric to a spherical cone-metric on the sphere with three conical singularities, with cone-angles  $< \pi$ , obtained by gluing two isometric spherical triangles along corresponding edges.

Let  $n \geq 4$ . Around the codimension 2 face of  $C$  isometric to

$$N := C(\varphi_1, \dots, \varphi_k + \varphi_{k+1}, \dots, \varphi_j + \varphi_{j+1}, \dots, \varphi_{n+3})$$

are glued four orthoschemes, corresponding to the four ways of ordering  $(\varphi_k, \varphi_{k+1})$  and  $(\varphi_j, \varphi_{j+1})$ . As the dihedral angle of each orthoscheme at such codimension 2 face is  $\pi/2$ , the total angle around  $N$  in  $C$  is  $2\pi$ . Hence metrically  $N$  is actually not a singular set. Around the codimension 2 face of  $C$  isometric to

$$S := C(\varphi_1, \dots, \varphi_k + \varphi_{k+1} + \varphi_{k+2}, \dots, \varphi_{n+3})$$

are glued six orthoschemes corresponding to the six ways of ordering  $(\varphi_k, \varphi_{k+1}, \varphi_{k+2})$ . Let  $\Theta$  be the cone-angle around  $S$ . It is the sum of the dihedral angles of the six orthoschemes glued around it. As formula (5) is symmetric for two variables,  $\Theta$  is two times the sum of three different dihedral angles. A direct computation gives ( $k = 1$  in the formula)

$$\cos(\Theta/2) = \frac{\sinh \varphi_1 \sinh \varphi_2 \sinh \varphi_3 - \sinh(\varphi_1 + \varphi_2 + \varphi_3)(\sinh \varphi_1 \sinh \varphi_2 + \sinh \varphi_2 \sinh \varphi_3 + \sinh \varphi_3 \sinh \varphi_1)}{\sinh(\varphi_1 + \varphi_2) \sinh(\varphi_2 + \varphi_3) \sinh(\varphi_3 + \varphi_1)}.$$

During the computation we used that

$$\sinh(a+b) \sinh(b+c) - \sinh a \sinh c = \sinh b \sinh(a+b+c)$$

which can be checked with  $\frac{1}{2}(\cosh(x+y) - \cosh(x-y)) = \sinh x \sinh y$ . The analog formula in the Euclidean convex polygons case was obtained in [Kojima et al., 1999].

For example when  $\varphi_i = \varphi \forall i$ , we have

$$\cos(\Theta/2) = -\frac{2 \cosh(\varphi)^2 + \sinh(\varphi)^2}{2 \cosh(\varphi)^3}.$$

The function on the right-hand side is a bijection from the positive numbers to  $] -1, 0[$ , hence all the  $\Theta \in ]2\pi, 3\pi[$  (the dihedral angle  $\theta \in ]\pi/3, \pi/2[$ ) are uniquely reached. In particular  $C$  is not an orbifold.

The cone-manifold  $C$  comes with an isometric involution which consists of reversing the order of the angles  $(\varphi_1, \dots, \varphi_n)$ .

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