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Complex order van der Pol oscillator

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Abstract

In this paper it is considered a complex order van der Pol oscillator. The complex derivative $D^{\alpha \pm j\beta}$, with $\alpha, \beta \in \mathbf{R}^+$ is a generalization of the concept of integer derivative, where $\alpha = 1$, $\beta = 0$. By applying the concept of complex derivative, we obtain a high-dimensional parameter space. Amplitude and period values of the periodic solutions of the two versions of the complex order van der Pol oscillator, are studied for variation of these parameters. Fourier transforms of the periodic solutions of the two oscillators are also analyzed.

Keywords — *van der Pol oscillator, complex order derivative, dynamical behavior*

1 Introduction

The van der Pol (VDP) oscillator is an ordinary differential equation that has arisen as a model of electrical circuits containing vacuum tubes [45] (re-edited [8]). It produces self-sustaining oscillations in which energy is fed into small oscillations while is removed from large

oscillations. This is the first relaxation oscillator appearing in the literature [44, 46]. It is given by the following second order differential equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \quad (1)$$

Parameter μ controls the way the voltage flows through the system. For $\mu = 0$ this is just a simple linear oscillator. For large values of μ , that is for $\mu \gg 1$, the system exhibits a relaxation oscillation. This means that the oscillator has two distinct phases: a slow recovery phase and a fast release phase (vacuum tubes quickly release or relax their voltage after slowly building up tension).

This equation has been used in the design of various systems, from biology, with the modeling of the heartbeat [22, 30, 17], the generation of action potentials [21, 20], up to acoustic systems [2] and electrical circuits [3, 12]. The VDP oscillator has also been used in the context of chaos theory [9, 14, 15, 16, 12].

Fractional calculus (FC) has been an important research issue in the last few decades. FC is a generalization of the ordinary integer differentiation and integration to an arbitrary, real or complex, order [34, 43, 28]. Application of FC have been emerging in different and important areas of physics and engineering [35, 27, 26, 31, 39, 41, 10, 5, 42, 32]. Fractional order behavior has been found in areas such as fluid mechanics [29], mechanical systems [13], electrochemistry [33], and biology [11, 1], namely in the modeling of the central pattern generators for animal locomotor rhythms [36, 37].

There are several definitions of fractional derivatives of order $\alpha \in \mathbf{R}$ being three of the most important the Riemann - Liouville, the Grünwald - Letnikov, and the Caputo given by:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau, \quad n - 1 < \alpha < n \quad (2)$$

$$D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(t - kh) \quad (3)$$

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau, \quad n - 1 < \alpha < n \quad (4)$$

where $\Gamma()$ is the Euler's gamma function, $\lfloor \cdot \rfloor$ means the integer part of x , and h represents the step time increment. It is also possible to generalize results based on transforms, yielding:

$$L\{D_t^\alpha f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k D_t^{\alpha - k - 1} f(0^+) \quad (5)$$

where s and L represent the Laplace variable and operator, respectively.

The definitions demonstrate that fractional derivatives capture the history of the variable, or, by other words, that have memory, contrary to integer derivatives, that are local operators. The Grünwald - Letnikov formulation inspires the numerical calculation of the fractional derivative based on the approximation of the time increment h through the sampling period T and the series truncation at the r^{th} term. This method is often denoted as Power Series Expansion (PSE) yielding the equation in the z - domain:

$$Z\{D^\alpha x(t)\} \approx \left[\frac{1}{T^\alpha} \sum_{k=0}^r \frac{(-1)^k \Gamma(\alpha + 1)}{k! \Gamma(\alpha - k + 1)} z^{-k} \right] X(z) \quad (6)$$

where $X(z) = Z\{x(t)\}$ and z and Z represent the z -transform variable and operator, respectively. In fact, expression (3) represents the Euler (or first backward difference) approximation in the $s \rightarrow z$ discretization scheme, being the Tustin approximation another possibility. The most often adopted generalization of the generalized derivative operator consists in $\alpha \in \mathbf{R}$. The case of having fractional derivative of complex-order $\alpha \pm j\beta \in \mathbf{C}$ leads to complex output valued results and imposes some restrictions before a practical application. To overcome this problem, it was proposed recently [18, 19, 6] the association of two complex-order derivatives. In fact, there are several arrangements that produce real valued results. For example, with the real part of two complex conjugate derivatives $D^{\alpha \pm j\beta}$ we get:

$$\begin{aligned} Z \left\{ \frac{1}{2} \left[D^{\alpha - j\beta} x(t) + D^{\alpha + j\beta} x(t) \right] \right\} \approx \frac{1}{T^\alpha} \left\{ \sin \left[\beta \ln \left(\frac{1}{T} \right) \right] \left[\beta z^{-1} + \frac{1}{2} \beta (1 - 2\alpha) z^{-2} + \dots \right] + \right. \\ \left. + \cos \left[\beta \ln \left(\frac{1}{T} \right) \right] \left[-1 + \alpha z^{-1} - \frac{1}{2} \beta (\alpha^2 - \alpha - \beta^2 + \dots) \right] \right\} X(z) \end{aligned} \quad (7)$$

Other combinations and the adoption of a Padé fraction, instead of the series for the approximation, are also possible. Nevertheless, in the sequel it is explored the case of expression (7).

We must remark that it is adopted a scheme of discretized fractional derivative commonly accepted in engineering, but not yet fully investigated from a mathematical view point. Let us recall the basic definitions of Riemann-Liouville (R-L) and Caputo (C) derivatives, in equations (2) and (4), based on integral expressions, and of Grünwald-Letnikov (G-L), in equation (3), based on a series expression suitable for discretization. The C derivative is a regularization of the R-L derivative and their G-L derivative is the series representation of the R-L derivative. In equation (5) the Laplace transform of the R-L deriva-

tives differs from that of the C derivative, as shown by Gorenflo and Mainardi [23, 24], subsequently pointed out in the well known treatises on Fractional Calculus by Podlubny [38] and Kilbas-Srivastava and Trujillo [25]. So, in practice it is generalized to complex orders the classical R-L derivative of real order. In case of adopting the Caputo derivative (in view of standard initial conditions), it should be adopted a different Laplace transform (see Gorenflo-Mainardi [23, 24]), and also a modified G-L series representation, because the C derivative is a regularization of the R-L derivative in the time origin. For real order less than one the G-L representation of the C derivative has been adopted in the paper by Gorenflo et al. [24]. As a consequence, the Z representation in equation (6) in the complex domain should be modified accordingly for the Caputo derivative. While tackling these matters is not straightforward, we must note that the simulations carried in the sequel correspond to steady state responses and that the initial conditions have a minor impact in the resulting charts.

Having these ideas in mind, this paper is organized as follows. In Section 2, we introduce the two approximations of the complex order van der Pol oscillator (CVDP) and we present results from numerical simulations. In Section 3 we outline the main conclusions of this study.

2 Complex order van der Pol system

Fractional VDP systems have been studied by many authors [9, 7, 40, 14, 4, 15, 16]. Their work differ in the approaches considered to express the fractional derivative. Chen *et al* [9] considered a forced van der Pol equation with fractional damping of the form:

$$\ddot{x} + \mu(x^2 - 1)D^\alpha x(t) + x(t) = a \sin(\omega t) \quad (8)$$

where μ is an endogenous damping parameter, a denotes the amplitude of a periodic forcing, and ω is the forcing frequency.

Barbosa *et al* [7] considered the following modified version of the van der Pol equation:

$$x^{(1+\lambda)} + \alpha(x^2 - 1)x^{(\lambda)} + x = 0 \quad (9)$$

with $0 < \lambda < 1$.

Tavazoei *et al* [40] determined the parametric range for which the fractional VDP system studied by Barbosa *et al* [7] can perform as an undamped oscillator. They also showed that, contrary to the integer order VDP, trajectories in a fractional VDP oscillator do not converge to a unique cycle.

Ge *et al* [14] studied the autonomous and non-autonomous fractional van der Pol oscillator. The non-autonomous system in state-space oscillator model is given by:

$$\begin{aligned} \frac{d^\alpha x_1}{dt^\alpha} &= x_2 \\ \frac{d^\beta x_2}{dt^\beta} &= -x_1 - \mu(1 - x_1^2)(c - ax_1^2)x_2 + b \sin t \end{aligned} \quad (10)$$

where α and β are fractional numbers.

To the best knowledge of the authors, little attention has been given to CVDP oscillators. In this paper, we consider the following two complex order state-space models of the VDP oscillator:

$$\begin{bmatrix} \frac{1}{2} (D^{\alpha+j\beta} + D^{\alpha-j\beta}) x_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu(x_1^2 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (11)$$

$$\begin{bmatrix} \dot{x}_1 \\ \frac{1}{2} (D^{\alpha+j\beta} + D^{\alpha-j\beta}) x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu(x_1^2 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (12)$$

where $D^{\alpha \pm j\beta}$, $\alpha, \beta \in \mathbf{R}^+$, is a generalization of the concept of the integer derivative, that corresponds to $\alpha = 1$ and $\beta = 0$.

We adopt the PSE method for the approximation of the complex-order derivative in the discrete time numerical integration. Several experiments demonstrated that it is required a slight adaption to the standard approach based on a simple truncation of the series. In fact, since our objective is to generate limit cycles, the truncation corresponds to a diminishing of the gain [42] and, consequently, leads to difficulties in the promotion of periodic orbits. Therefore, in order to overcome this limitation, we decided to include a gain adjustment factor corresponding to the sum of the missing truncated series coefficients.

The discretisation of the CVDP oscillators (11) and (12) leads to, respectively:

$$\begin{aligned} x_1(k+1) &= \frac{1}{\psi(\beta, \Delta t)} \left(H(x_1(k)) + (\Delta t)^\alpha x_2(k) \right) \\ x_2(k+1) &= x_2(k) + \Delta t \left(-x_1(k) - \mu(x_1^2(k) - 1)x_2(k) \right) \end{aligned} \quad (13)$$

$$\begin{aligned} x_1(k+1) &= x_1(k) + \Delta t x_2(k) \\ x_2(k+1) &= \frac{1}{\psi(\beta, \Delta t)} \left(H(x_2(k)) + (\Delta t)^\alpha \left(-x_1(k) - \mu(x_1^2(k) - 1)x_2(k) \right) \right) \end{aligned} \quad (14)$$

where $\Delta t = 0.0005$ is the time increment, $\psi(\beta, \Delta t) = \cos \left[b \log \left(\frac{1}{\Delta t} \right) \right]$ and function $H(x_i)$, $i = 1, 2$, results from the Taylor series expansion truncation.

In Table 1 we depict periodic solutions of systems (13)-(14) for $\alpha = 0.4, \beta = 0.8$ and $\mu = 0.5$. One can observe the appearance of the relaxation oscillation phenomena as α increases (first row of the table). Note that in the case of system (14) this phenomenon is already present for $\alpha = 0.4$ and is emphasized for $\alpha = 0.8$ (second row of the table).

In Figures 1-2, we show the phase portraits of solutions $(x_1(t), x_2(t))$ of systems (13)-(14) for $\beta = 0.8, \mu \in \{0.5, 2\}$ and different values of α . As expected, the larger the value of μ , the more nonlinear the oscillation becomes. We verify also that we can control the period of the oscillation by varying α .

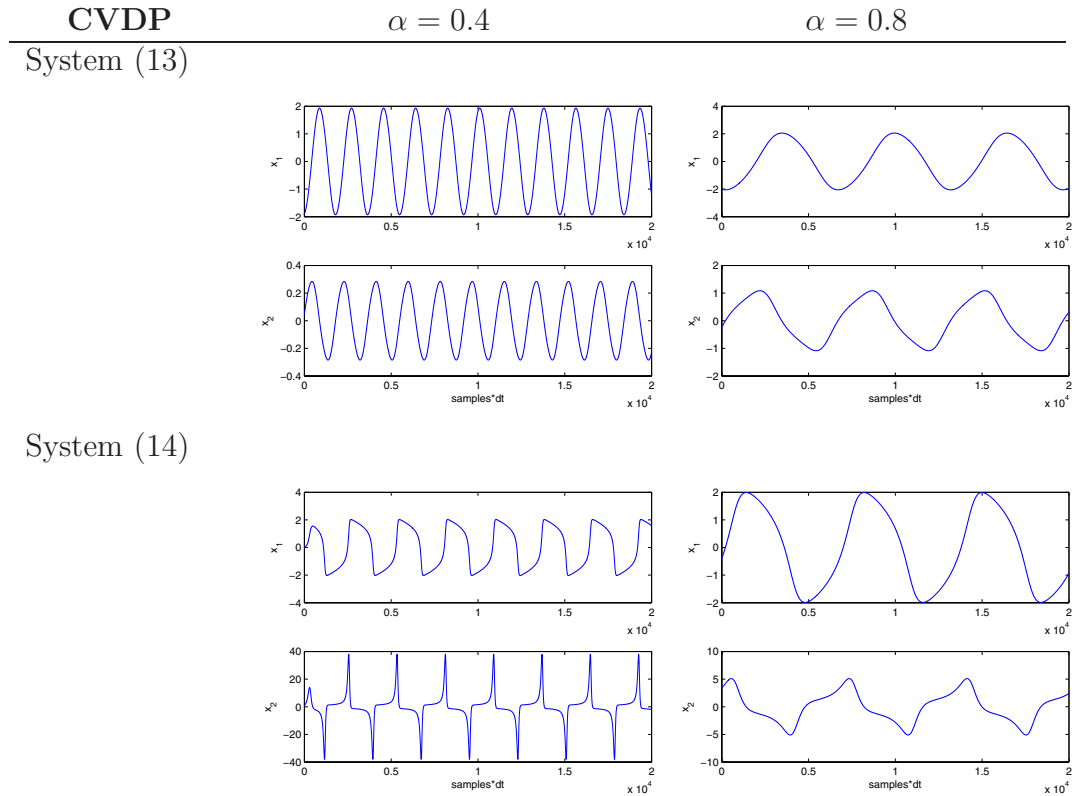


Table 1: Periodic solutions of the CVDP systems (13)-(14) for $\beta = 0.8, \mu = 0.5$ and $\alpha \in \{0.4, 0.8\}$.

We now simulate the ordinary differential systems given by expressions (13)-(14) for $\beta = 0.8, \alpha \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}, \mu = \{0.5, 1.0, 1.5, 2\}$, and we measure the amplitude and the period of the solutions. Values of $\alpha \in [0.0, 0.2[$ were also considered in the

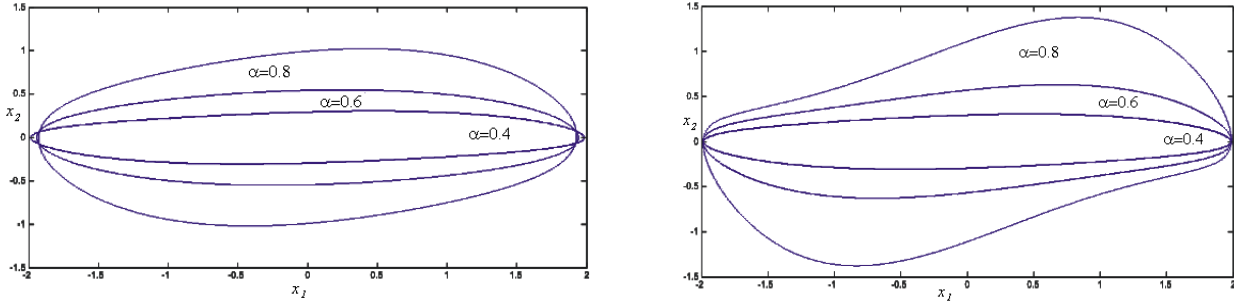


Figure 1: Phase-space solutions $(x_1(t), x_2(t))$ of the CVDP system (13) for $\alpha \in \{0.4, 0.6, 0.8\}$, $\beta = 0.8$ and $\mu = 0.5$ (left) and $\mu = 2.0$ (right).

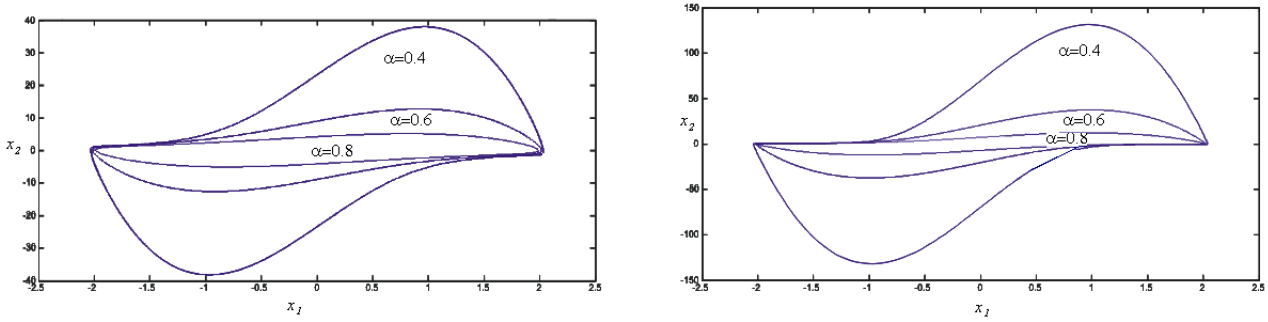


Figure 2: Phase-space solutions $(x_1(t), x_2(t))$ of the CVDP system (14) for $\alpha \in \{0.4, 0.6, 0.8\}$, $\beta = 0.8$ and $\mu = 0.5$ (left) and $\mu = 2.0$ (right).

simulations. For system (13) we found stable periodic solutions nevertheless for system (14) simulation results led to unstable solutions, so we have decided to omit these values.

Are adopted the initial conditions $x_1(1) = 0.0$, $x_1(2) = 0.005$, $x_1(3) = 0.010$, $x_1(4) = 0.015$, $x_1(5) = 0.02$, $x_2(1) = 1.0$, $x_2(2) = 1.005$, $x_2(3) = 1.010$, $x_2(4) = 1.015$, $x_2(5) = 1.02$.

Each simulation is executed until a stable periodic solution is found. The amplitude and the period of the solutions versus α are depicted in Figures 3-4. We find that the period increases as α goes from 0.2 to 1.0, in both systems (13)-(14). On the other hand, the amplitude is almost constant. To be precise, the amplitude shows a very tiny increase in system (13) and a very tiny decrease in system (14), as α increases to one.

We now compute the Fourier transforms $|\mathcal{F}\{x_1(t)\}|$ of the periodic solutions of systems (13) and (14), for $\beta = 0.8$ and $\alpha = \{0.3, 0.5, 0.7, 1.0\}$. Figures 5-6 depict the amplitude of the Fourier transforms vs the fre-

quency ω . The charts demonstrate that the main part of the

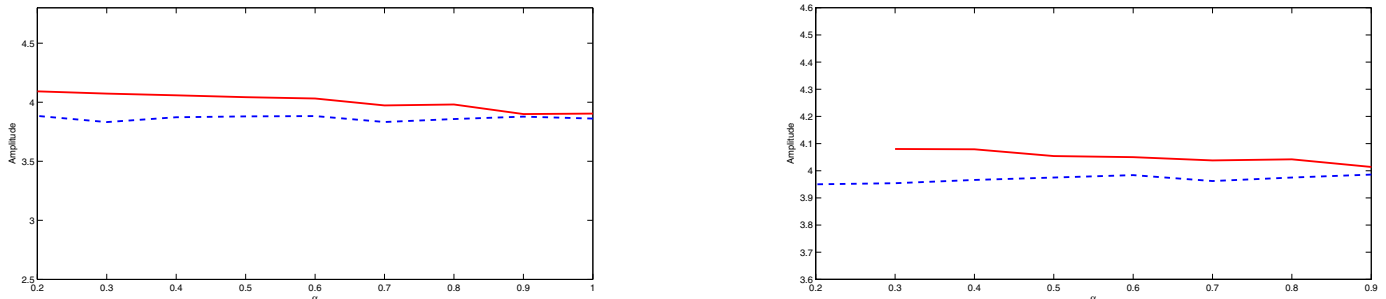


Figure 3: Amplitude of the periodic solutions $x_1(t)$ produced by the CVDP oscillators (13) (dashed) and (14) (line) for $\beta = 0.8$, $\alpha \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ and $\mu = 0.5$ (left) and $\mu = 2.0$ (right).

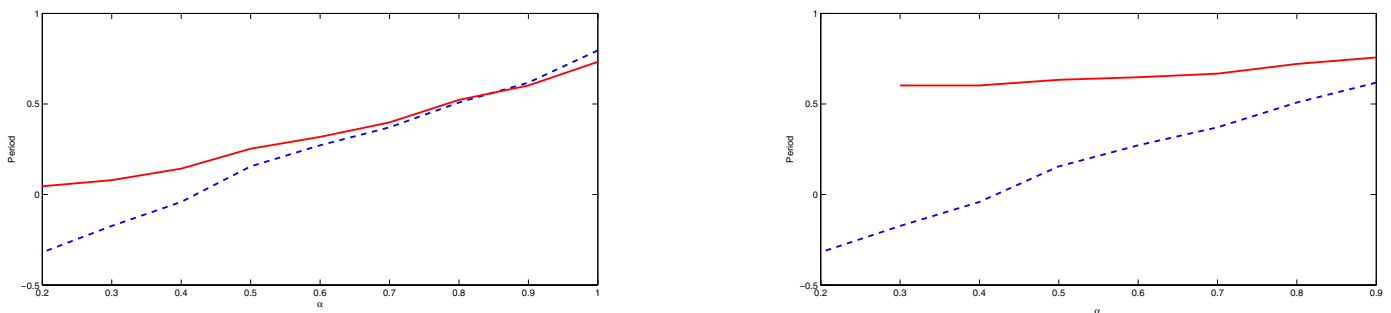


Figure 4: Period of the solutions $x_1(t)$ produced by the CVDP oscillators (13) (dashed) and (14) (line) for $\beta = 0.8$, $\alpha \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ and $\mu = 0.5$ (left) and $\mu = 2.0$ (right).

signal energy is concentrated in the fundamental frequency ω_0 . The remaining energy, located in the higher harmonics increases with μ . Furthermore, it is also observed that the fundamental frequency of the oscillations ω_0 varies with α and μ . For the range of tested values, the numerical fitting lead to exponential and rational fraction approximations, for systems (13) and (14), respectively, given by:

$$\omega_0 \approx 23.032 \exp(-3.0686\alpha) \quad (15)$$

and

$$\omega_0 \approx \frac{1}{0.0214 + 0.29\mu + (0.7437 - 0.1202\mu) \alpha^{2.3158+0.8522\mu}} \quad (16)$$

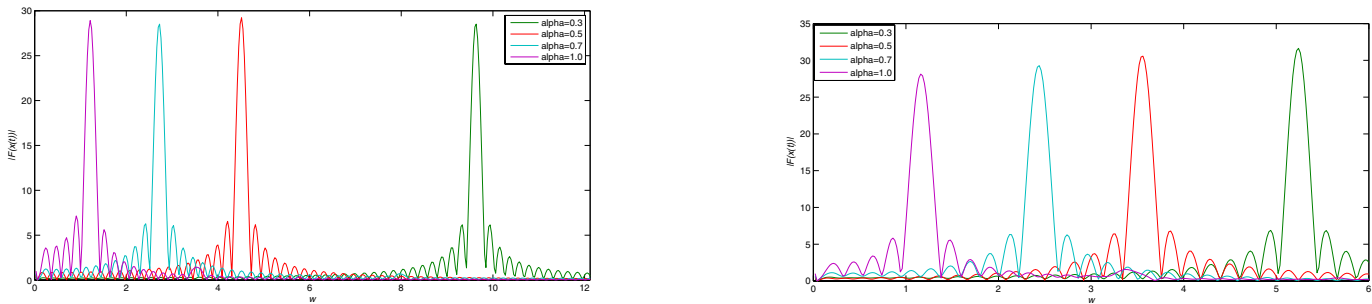


Figure 5: Amplitude of the Fourier transform $|\mathcal{F}\{x_1(t)\}|$ vs ω for the CVDP oscillators (13) (left) and (14) (right) with $\beta = 0.8$, $\alpha \in \{0.3, 0.5, 0.7, 1.0\}$ and $\mu = 0.5$.

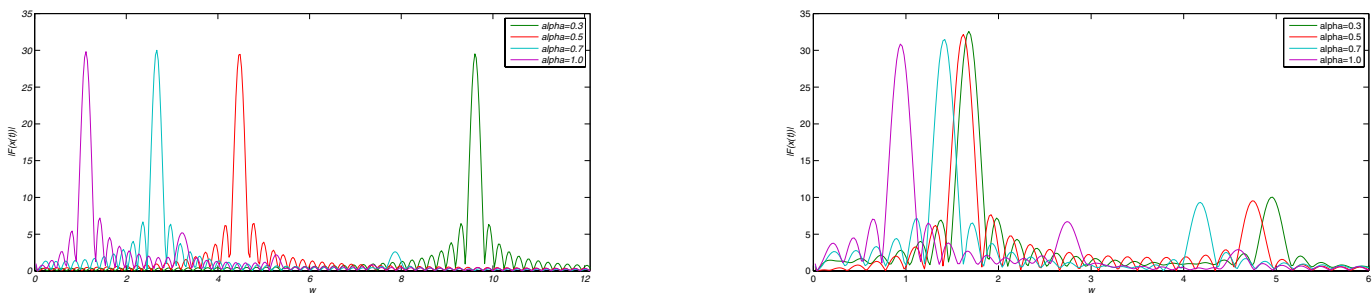


Figure 6: Amplitude of the Fourier transform $|\mathcal{F}\{x_1(t)\}|$ vs ω for the CVDP oscillators (13) (left) and (14) (right) with $\beta = 0.8$, $\alpha \in \{0.3, 0.5, 0.7, 1.0\}$ and $\mu = 2.0$.

We verify that in the first case the frequency of oscillation is independent of μ , while in the second case it is related both with α and μ .

3 Conclusions

In this paper two complex-order approximations to the well-known van der Pol oscillator were proposed. The amplitude and the period of solutions produced by these two approximations were then measured. The imaginary part was fixed while the real component was varied, for two distinct values of parameter μ . It was observed that the waveform period increases as α varies between 0.2 and one. On the other hand, the amplitude values are almost constant as α varies. Moreover, it seems there is a tiny increase in the amplitude of solutions

for system (13) and a tiny decrease for system (14) as α approaches one.

It was also calculated the Fourier transform $|\mathcal{F}\{x_1(t)\}|$ for systems (13) and (14). It was verified that the main part of the signal energy is concentrated in the fundamental frequency ω_0 . The remaining energy, located in the higher harmonics increases with μ . It is also observed that the fundamental frequency of the oscillations ω_0 varies with α and μ . For the range of tested values, the numerical fitting lead to exponential and rational fraction approximations for systems (13) and (14), respectively.

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