# Well-posedness results for the 3D Zakharov-Kuznetsov equation

Francis Ribaud, Stéphane Vento

# Well-posedness results for the 3D Zakharov-Kuznetsov equation

Francis Ribaud* Stéphane Vento†

**Abstract.** We prove the local well-posedness of the three-dimensional Zakharov-Kuznetsov equation $\partial_t u + \Delta \partial_x u + u \partial_x u = 0$ in the Sobolev spaces $H^s(\mathbb{R}^3)$, $s > 1$, as well as in the Besov space $B_2^{1,1}(\mathbb{R}^3)$. The proof is based on a sharp maximal function estimate in time-weighted spaces.

**Keywords:** KdV-like equations, Cauchy problem
**AMS Classification:** 35Q53, 35B65, 35Q60

## 1 Introduction and main results

In this paper we consider the local Cauchy problem for the three-dimensional Zakharov-Kuznetsov $(ZK)$ equation

$$\begin{cases} u_t + \Delta u_x + u u_x = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u = u(t, x, y, z)$, $u_0 = u_0(x, y, z)$, $t \in \mathbb{R}$ and $(x, y, z) \in \mathbb{R}^3$.

This equation was introduced by Zakharov and Kuznetsov in [11] to describe the propagation of ionic-acoustic waves in magnetized plasma. The formal derivation of $(ZK)$ from the Euler-Poisson system with magnetic field can be found in [9].

Clearly, the Zakharov-Kuznetsov equation can be considered as a multi-dimensional generalization of the well-known one-dimensional Korteweg-de Vries equation

$$\begin{cases} u_t + u_{xxx} + u u_x = 0, \\ u(0) = u_0. \end{cases} \tag{1.2}$$

*Laboratoire d'Analyse et de Mathématiques Appliquées - Université Paris-Est - 5 Bd. Descartes - Champs-Sur-Marne, 77454 Marne-La-Vallée Cedex 2 (FRANCE)

†Laboratoire Analyse, Géométrie et Applications - Université Paris 13 - Institut Galilée, 99 avenue J.B. Clément,93430 Villetaneuse (FRANCE)

We stress out the attention of the reader that contrary to some other generalizations of the Korteweg-de Vries equation (like the Kadomtsev-Petviasvili equations) the $(ZK)$ equation is not completely integrable and possesses only two invariant quantities by the flow. These two invariant quantities are the $L^2(\mathbb{R}^3)$ norm

$$N(t) = \int_{\mathbb{R}^3} u^2(t,x,y,z)\, dxdydz \ ,$$

and the Hamiltonian

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( (\nabla u(t,x,y,z))^2 - \frac{u(t,x,y,z)^2}{3} \right) dxdydz \, .$$

Hence, it is a natural issue to study the Cauchy problem for the $(ZK)$ equation in the Sobolev space $H^1(\mathbb{R}^3)$ since any local well-posedness result in this space would provide global well-posedness [1].

Another difference with usual generalization of the KdV equation is that the resonant function associated to the $(ZK)$ equation seems too complex to develop a Bourgain approach, see [7]. Indeed, this function is defined in the hyperplane $\bar{\xi_1} + \bar{\xi_2} + \bar{\xi_3} = 0$ by

$$h(\bar{\xi_1}, \bar{\xi_2}, \bar{\xi_3}) = \xi_1 |\bar{\xi_1}|^2 + \xi_2 |\bar{\xi_2}|^2 + \xi_3 |\bar{\xi_3}|^2, \quad \bar{\xi_j} = (\xi_j, \eta_j, \mu_j) \in \mathbb{R}^3,$$

and its zero set is not so easy to understand. This make again a sharp difference with the Kadomtsev-Petviasvili equations where the use of the resonant function allows to derive local well-posedness (for the KP-II equation, see [1]) or local ill-posedness (for the KP-I equation, see[10]). This explains why we follow the approach of Kenig, Ponce and Vega introduced in [6] for the study of the generalized KdV equation rather than a Bourgain approach.

In the the two-dimensional case, Faminskii [2] proved the local and global well-posedness of $(ZK)$ for initial data in $H^1(\mathbb{R}^2)$. This result was recently improved by Linares and Pastor who obtained in [7] the local well-posedness in $H^s(\mathbb{R}^2)$, $s > 3/4$. The main tool to derive those results is the following $L^2_x L^\infty_{yT}$ linear estimate [2],

$$\forall s > 3/4, \qquad \|U(t)\varphi\|_{L^2_x L^\infty_{yT}} \leq C \|\varphi\|_{H^s(\mathbb{R}^2)} \ ,$$

where $U(t)\varphi$ denotes the free operator associated to the linear part of the $(ZK)$ equation. Note also that in [8], Linares, Pastor and Saut recently

[1] Note that global existence occurs provided the local existence time only depends on the norm of the initial data in a suitable way. This is usually the case when solving the equation by a standard fixed point procedure.

obtained local well-posedness results in some spaces which contains the one-dimensional solitary-waves of $(ZK)$ as well as perturbations.

In the three-dimensional case, as far as we know, the only available result concerning the local well-posedness of $(ZK)$ in the usual Sobolev spaces goes back to Linares and Saut who proved in [9] the local well posedness in $H^s(\mathbb{R}^3)$, $s > 9/8$. In this paper we prove the following result,

**Theorem 1.1.** *For any $s > 1$ and $u_0 \in H^s(\mathbb{R}^3)$, there exist $T > 0$ and a unique solution $u$ of (1.1) in*

$$X_T^s \cap C_b([0,T], H^s(\mathbb{R}^3)).$$

*Moreover, the flow-map $u_0 \mapsto u$ is Lipschitz on every bounded set of $H^s(\mathbb{R}^3)$.*

To derive our result, the main issue is to prove the local in time linear estimate

$$\forall T < 1, \quad \forall s > 1, \quad \|U(t)\varphi\|_{L_x^2 L_{yzT}^\infty} \leq C\|\varphi\|_{H^s(\mathbb{R}^3)}, \tag{1.3}$$

where $U(t)\varphi$ denotes the free linear operator associated to the $(ZK)$ equation. Moreover, having a short look on the proof of Theorem 1.1, we note that any improvement of the linear estimate (1.3) will immediately give the corresponding improvement for Theorem 1.1. Unfortunately we prove in Section 3 that the $L_x^2 L_{yzT}^\infty$ linear estimate (1.3) fails when $s < 1$. Hence, this seems to indicate that the case $s = 1$ could be critical for the well-posedness of the $(ZK)$ equation.

Concerning the "critical case" $s = 1$, we have unfortunately not been able to prove the local well-posedness in the natural energy space $H^1(\mathbb{R}^3)$. Nonetheless, working with initial data in the Besov space $B_2^{1,1}(\mathbb{R}^3)$, we have the following

**Theorem 1.2.** *For any $u_0 \in B_2^{1,1}(\mathbb{R}^3)$, there exist $T > 0$ and a unique solution $u$ of (1.1) in*

$$X_T \cap C_b([0,T], B_2^{1,1}(\mathbb{R}^3)).$$

*Moreover, the flow-map $u_0 \mapsto u$ is Lipschitz on every bounded set of $B_2^{1,1}(\mathbb{R}^3)$.*

This result clearly improves Theorem 1.1 in view of the well-known embeddings

$$\forall s > 1, \quad H^s(\mathbb{R}^3) \hookrightarrow B_2^{1,1}(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3).$$

To get Theorem 1.2, our main ingredient is to prove the unusual weighted-in-time linear estimate for phase localized functions

$$\forall \alpha \geq 3/8, \qquad \|t^\alpha \Delta_k U(t)\varphi\|_{L^2_x L^\infty_{yzT}} \lesssim 2^k \|\Delta_k \varphi\|_{L^2}.$$

This estimate, combined with the standard Kato smoothing estimate (3.2), allows us to perform a fixed point argument on the Duhamel formulation of $(ZK)$.

This paper is organized as follows. In Section 2 we introduce our notations and define the resolution spaces. Section 3 is devoted to estimates related to the linear part of the equation. Finally we prove the key bilinear estimates in Section 4.

## 2 Notation

For $A, B > 0$, $A \lesssim B$ means that there exists $c > 0$ such that $A \leq cB$. When $c$ is a small constant we use $A \ll B$. We write $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. For $u = u(t,x,y,z) \in \mathcal{S}'(\mathbb{R}^4)$, we denote by $\widehat{u}$ (or $\mathcal{F}u$) its Fourier transform in space. The Fourier variables corresponding to a vector $\bar{x} = (x,y,z)$ will be denoted by $\bar{\xi} = (\xi, \eta, \mu)$. We consider the usual Lebesgue spaces $L^p$, $1 \leq p \leq \infty$ and given a Banach space $X$ and a measurable function $u : \mathbb{R} \to X$, we define $\|u\|_{L^p X} = \|\|u(t)\|_X\|_{L^p}$. For $T > 0$, we also set $L^p_T = L^p([0,T])$. Let us define the Japanese bracket $\langle \bar{x} \rangle = (1 + |\bar{x}|^2)^{1/2}$ so that the standard non-homogeneous Sobolev spaces are endowed with the norm $\|f\|_{H^s} = \|\langle \nabla \rangle^s f\|_{L^2}$.

We use a Littlewood-Paley analysis. Let $p \in C^\infty_0(\mathbb{R}^d)$ be such that $p \geq 0$, $\operatorname{supp} p \subset B(0,2)$, $p \equiv 1$ on $B(0,1)$. We define next $p_k(\bar{\xi}) = p(\bar{\xi}/2^k)$ for $k \geq 0$. We set $\delta(\bar{\xi}) = p(\bar{\xi}/2) - p(\bar{\xi})$ and $\delta_k(\bar{\xi}) = \delta(\bar{\xi}/2^k)$ for any $k \in \mathbb{Z}$, and define the operators $P_k$ $(k \geq 0)$ and $\Delta_k$ $(k \in \mathbb{Z})$ by $\mathcal{F}(P_k u) = p_k \widehat{u}$ and $\mathcal{F}(\Delta_k) = \delta_k \widehat{u}$. When $d = 3$, we introduce the operators $P^x_k$, $P^y_k$, $P^z_k$, and $\Delta^x_k$, $\Delta^y_k$, $\Delta^z_k$ defined by

$$\begin{cases} P^x_k u(x,y,z) = \mathcal{F}^{-1}(p_k(\xi)\widehat{u}(\xi,\eta,\mu)), \\ P^y_k u(x,y,z) = \mathcal{F}^{-1}(p_k(\eta)\widehat{u}(\xi,\eta,\mu)), \\ P^z_k u(x,y,z) = \mathcal{F}^{-1}(p_k(\mu)\widehat{u}(\xi,\eta,\mu)) \end{cases}$$

and

$$\begin{cases} \Delta^x_k u(x,y,z) = \mathcal{F}^{-1}(\delta_k(\xi)\widehat{u}(\xi,\eta,\mu)), \\ \Delta^y_k u(x,y,z) = \mathcal{F}^{-1}(\delta_k(\eta)\widehat{u}(\xi,\eta,\mu)), \\ \Delta^z_k u(x,y,z) = \mathcal{F}^{-1}(\delta_k(\mu)\widehat{u}(\xi,\eta,\mu)) \end{cases}$$

Furthermore we define more general projections $P_{\lesssim k} = \sum_{j:2^j \lesssim 2^k} P_j$, $\Delta^x_{\gg j} = \sum_{j:2^j \gg 2k} \Delta^x_j$ etc. For future considerations, note that for $u \in \mathcal{S}'(\mathbb{R}^3)$ and $p \in [1, \infty]$ we have

$$\|\Delta_k u\|_{L^p} \lesssim \|\Delta^x_k P^y_{\lesssim k} P^z_{\lesssim k} u\|_{L^p} + \|P^x_{\lesssim k} \Delta^y_k P^z_{\lesssim k} u\|_{L^p} + \|P^x_{\lesssim k} P^y_{\lesssim k} \Delta^z_k u\|_{L^p}. \quad (2.1)$$

With these notations, it is well known that an equivalent norm on $H^s(\mathbb{R}^d)$ is given by

$$\|u\|_{H^s} \sim \|P_0 u\|_{L^2} + \left( \sum_{k \geq 0} 2^{2sk} \|\Delta_k u\|^2_{L^2} \right)^{1/2}.$$

For $s \in \mathbb{R}$, the Besov space $B^{s,1}_2(\mathbb{R}^d)$ denotes the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{B^{s,1}_2} = \|P_0 u\|_{L^2} + \sum_{k \geq 0} 2^{sk} \|\Delta_k u\|_{L^2}.$$

# 3 Linear estimates

Consider the linear ZK equation

$$u_t + \Delta u_x = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3. \quad (3.1)$$

Let $\omega(\bar{\xi}) = \xi(\xi^2 + \eta^2 + \mu^2)$ and $U(t) = \mathcal{F}^{-1} e^{it\omega(\bar{\xi})} \mathcal{F}$, be the associated linear operator.

First we prove a standard "Kato smoothing" estimate for the free evolution of (3.1).

**Proposition 3.1.** *For any $\varphi \in L^2(\mathbb{R}^3)$, it holds that*

$$\|\nabla U(t)\varphi\|_{L^\infty_x L^2_{yzt}} \lesssim \|\varphi\|_{L^2}. \quad (3.2)$$

*Proof.* The proof is modeled on the corresponding result for the KdV equation [4] (see also [5] and [2] for the two-dimensional case). We perform the change of variables $\theta = h(\xi) = \omega(\bar{\xi})$ and obtain

$$\begin{aligned} U(t)\varphi(\bar{x}) &= \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi} + t\omega(\bar{\xi})} \widehat{\varphi}(\bar{\xi}) d\bar{\xi} \\ &= \mathcal{F}^{-1}_{\theta\eta\mu} \left( e^{ixh(\theta)} (h^{-1})'(\theta) \widehat{\varphi}(h(\theta), \eta, \mu) \right)(t, y, z). \end{aligned}$$

Therefore, applying Plancherel theorem and returning to the $\xi$-variable yield

$$\begin{aligned}\|U(t)\varphi(x)\|_{L^2_{yzt}} &= \|(h^{-1})'(\theta)\widehat{\varphi}(h(\theta),\eta,\mu)\|_{L^2_{\theta\eta\mu}}\\ &= \||h'(\xi)|^{-1/2}\widehat{\varphi}(\xi,\eta,\mu)\|_{L^2_{\xi\eta\mu}}\\ &\sim \|\nabla^{-1}\varphi\|_{L^2}.\end{aligned}$$

□

We will use in a crucial way the following maximal estimate for the free evolution when acting on phase localized functions.

**Proposition 3.2.** *Let $0 < T < 1$ and $\alpha \geq 3/8$.*

1. *For all $\varphi \in \mathcal{S}(\mathbb{R}^3)$ and $k \geq 0$, we have*

$$\|t^\alpha \Delta_k U(t)\varphi\|_{L^2_x L^\infty_{yzT}} \lesssim 2^k \|\Delta_k \varphi\|_{L^2}. \tag{3.3}$$

2. *For any $\varphi \in \mathcal{S}(\mathbb{R}^3)$, it holds that*

$$\|P_0 U(t)\varphi\|_{L^2_x L^\infty_{yzT}} \lesssim \|P_0\varphi\|_{L^2}. \tag{3.4}$$

Before proving Proposition 3.2, we first show some estimates related to the oscillatory integrals:

$$I_0(t,\bar{x}) = \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi}+t\omega(\bar{\xi}))} p_0(\bar{\xi})d\bar{\xi},$$

and for $k \geq 1$,

$$I_k(t,\bar{x}) = \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi}+t\omega(\bar{\xi}))} \psi_1(\xi)\psi_2(\eta)\psi_3(\mu)d\bar{\xi}$$

where $(\psi_1,\psi_2,\psi_3) = (\delta_k, p_k, p_k)$, $(p_k,\delta_k,p_k)$ or $(p_k,p_k,\delta_k)$.

**Lemma 3.1.**

$$\|I_0\|_{L^1_x L^\infty_{yzT}} \lesssim 1.$$

*Proof.* Since $|I_0| \lesssim 1$, it is clear that $\|I_0\|_{L^1_x L^\infty_{yzT}} \lesssim 1$ if we are in the region $|x| \lesssim 1$. Thus we may assume that $|x| \gg 1$. Define the phase function $\varphi_1$ by $\varphi_1(\xi) = t\omega(\bar{\xi}) + x\xi$ so that $\varphi_1'(\xi) = t(3\xi^2+\eta^2+\mu^2)+x$. On the support of $p_0$, we have $|\varphi_1'| \gtrsim |x|$. Thus, two integrations by parts yield the estimate

$$\left|\int_{\mathbb{R}} e^{\phi_1(\xi)} p_0(\bar{\xi})d\xi\right| \lesssim \int_{\mathbb{R}} \left|\frac{p_{0\xi\xi}}{\varphi_1''}\right| + \left|\frac{p_{0\xi}\varphi_1''^2}{\varphi_1'^3}\right| + \left|\frac{p_0\varphi_1'''}{\varphi_1'^3}\right| + \left|\frac{p_0\varphi_1''^2}{\varphi_1'^4}\right| \lesssim |x|^{-2}.$$

It follows that $|I_0| \lesssim |x|^{-2}$, which implies that $\|I_0\|_{L^1_x L^\infty_{yzT}} \lesssim 1$ as required. □

**Lemma 3.2.** *For any $\alpha \geq 3/8$ and $k \geq 0$, it holds that*

$$\|t^{2\alpha} I_k\|_{L^1_x L^\infty_{yzT}} \lesssim 2^{2k}.$$

*Proof.* We split $I_k$ into

$$\begin{aligned} I_k &= \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi}+t\omega(\bar{\xi}))}\psi_1(\xi)(1-p_0(\xi))\psi_2(\eta)\psi_3(\mu)d\bar{\xi} \\ &\quad + \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi}+t\omega(\bar{\xi}))}p_0(\xi)\psi_2(\eta)\psi_3(\mu)d\bar{\xi} \\ &:= I_k^1 + I_k^2. \end{aligned} \tag{3.5}$$

- Estimate for $I_k^1$.
  Since we have $|\xi| \gtrsim 1$, a rough estimate for $I_k^1$ yields $|I_k^1| \lesssim 2^{3k}$, which gives the desired bound in the region where $|x| \leq 2^{-k}$. Therefore we may assume $|x| \geq 2^{-k}$. If we have either $|x| \ll t2^{2k}$ or $|x| \gg t2^{2k}$, then using that $|\omega'| \sim 2^{2k}$ we infer $|\varphi_1'| \gtrsim \max(|x|, t2^{2k})$ where $\varphi_1$ is the phase function

  $$\varphi_1(\xi) = t\omega(\bar{\xi}) + x\xi. \tag{3.6}$$

  Integrating by parts twice with respect to $\xi$ we deduce

  $$\left| \int_{\mathbb{R}} e^{i\varphi_1}\psi_1(1-p_0) \right| \lesssim \max(|x|, t2^{2k})^{-2}.$$

  It follows that $|I_k^1| \lesssim 2^{2k}\max(|x|, t2^{2k})^{-2}$ and next

  $$t^{1/2}|I_k^1| \lesssim |x|^{-3/2}2^k,$$

  which is acceptable since we integrate in the region $|x| \geq 2^{-k}$. Now we consider the case $|x| \sim t2^{2k}$. Using that

  $$\int_{\mathbb{R}^2} e^{it\xi(\eta^2+\mu^2)+iy\eta+iz\mu}d\eta d\mu = \frac{\pi}{t|\xi|}e^{-i\frac{y^2+z^2}{4t\xi}}e^{i\frac{\pi}{2}\operatorname{sgn}(\xi)},$$

  we may rewrite $I_k^1$ as

  $$I_k^1 = \int_{\mathbb{R}^2} \check{\psi}_2(y-u)\check{\psi}_3(z-v)\left(\int_{\mathbb{R}} \frac{\pi i}{t\xi}e^{i\varphi_2(\xi)}\psi_1(\xi)(1-p_0(\xi))d\xi\right)dudv \tag{3.7}$$

  where we set $\varphi_2(\xi) = t\xi^3 + x\xi - \frac{u^2+v^2}{4t\xi}$. Since $|\varphi_2'''| \geq 6t$ on the support of $1-p_0$, Van der Corput's lemma implies that

  $$\left| \int_{\mathbb{R}} \frac{\pi i}{t\xi}e^{i\varphi_2(\xi)}\psi_1(\xi)(1-p_0(\xi))d\xi \right| \lesssim t^{-4/3}.$$

It follows that $|I_k^1| \lesssim t^{-4/3}$ and

$$t^{2\alpha}|I_k^1| \lesssim t^{2\alpha-4/3} \lesssim |x|^{2\alpha-4/3}2^{-2k(2\alpha-4/3)}.$$

Hence we obtain for $\alpha > 1/6$ and $|x| \geq 2^{-k}$ that

$$\|t^{2\alpha}I_k^1\|_{L_x^1 L_{yzT}^\infty} \lesssim 2^{2k(2\alpha-1/3)}2^{-2k(2\alpha-4/3)} \sim 2^{2k}.$$

- Estimate for $I_k^2$.
We treat now the low frequencies term $I_k^2$. The case $|x| \lesssim 1$ is easily handled since we have the rough estimate $|I_k^2| \lesssim 2^{2k}$. Thus we only need to consider the region where $|x| \gg 1$. In the domain $|x| \ll t2^{2k}$ or $|x| \gg t2^{2k}$, we have $|\varphi_1'| \gtrsim \max(|x|, t2^{2k})$ where $\varphi_1$ is defined in (3.6), and thus

$$\left|\int_{\mathbb{R}} e^{i\varphi_1}p_0\right| \lesssim |x|^{-2}.$$

It follows that $|I_k^2| \lesssim 2^{2k}|x|^{-2}$ and

$$\|I_k^2\|_{L_x^1 L_{yzT}^\infty} \lesssim 2^{2k}.$$

Now we consider the most delicate case $|x| \sim t2^{2k}$ and rewrite $I_k^2$ as in (3.7) where $\psi_1(1-p_0)$ is replaced with $p_0$. Let us split $p_0$ as

$$p_0 = p_{-2k} + \sum_{j=-2k}^{0} \delta_j.$$

The part $p_{-2k}$ is straightforward since $|I_k^2| \lesssim 1$ and we get from $|x| \sim t2^{2k} \lesssim 2^{2k}$ that $\|I_k^2\|_{L_x^1 L_{yzT}^\infty} \lesssim 2^{2k}$. Thus we reduce to estimate

$$I_{k,j}^2 = \int_{\mathbb{R}^2} \check{\psi}_2(y-u)\check{\psi}_3(z-v)\left(\int_{\mathbb{R}} \frac{\pi i}{t\xi}e^{i\varphi_2(\xi)}\delta_j(\xi)d\xi\right)dudv$$

for $j = -2k, \ldots, 0$. First consider the case $|x| \ll t^{-1}2^{-2j}(u^2+v^2)$ or $|x| \gg t^{-1}2^{-2j}(u^2+v^2)$. Since $\varphi_2'(\xi) = 3t\xi^2 + x + \frac{u^2+v^2}{4t\xi^2}$, we have $|\varphi_2'| \gtrsim \max(|x|, t^{-1}2^{-2j}(u^2+v^2))$ and an application of the Van der Corput's lemma yields

$$\begin{aligned}|I_{k,j}^2| &\lesssim \int_{\mathbb{R}^2} |\check{\psi}_2(y-u)\check{\psi}_3(z-v)|t^{-1}|x|^{-3/4}(t^{-1}2^{-2j}(u^2+v^2))^{-1/4}2^{-j}dudv \\ &\lesssim |x|^{-3/4}t^{-3/4}2^{-j/2}\int_{\mathbb{R}} \frac{|\check{\psi}_2(y-u)|}{|u|^{1/2}}du.\end{aligned}$$

On the other hand, the change of variables $v = 2^k u$ leads to

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\check{\psi}_2(y-u)|}{|u|^{1/2}} du &= 2^k \int_{\mathbb{R}} \frac{|\check{\psi}(2^k y - 2^k u)|}{|u|^{1/2}} du \\ &= 2^{k/2} \int_{\mathbb{R}} \frac{|\check{\psi}(2^k y - v)|}{|v|^{1/2}} dv \\ &\lesssim 2^{k/2} \int_{|v|\leq 1} \frac{dv}{|v|^{1/2}} + 2^{k/2} \int_{|v|\geq 1} |\check{\psi}(2^k y - v)| dv \\ &\lesssim 2^{k/2}.\end{aligned}$$

Consequently, it is deduced that $t^{3/4}|I_{k,j}^2| \lesssim |x|^{-3/4} 2^{-j/2} 2^{k/2}$ and

$$\|t^{3/4} I_k^2\|_{L_x^1 L_{yzT}^\infty} \lesssim \sum_{j=-2k}^{0} 2^{-j/2} 2^{k/2} \int_{|x|\lesssim 2^{2k}} \frac{dx}{|x|^{3/4}} \lesssim 2^{2k}.$$

Finally assume that $|x| \sim t^{-1} 2^{-2j}(u^2+v^2)$ so that $|\varphi_2''| \gtrsim t 2^{2k-j}$. Then we get from Van der Corput's lemma that

$$|I_{k,j}^2| \lesssim (t 2^{2k-j})^{-1/2} t^{-1} 2^{-j} \sim t^{-3/2} 2^{-k} 2^{-j/2}.$$

Hence, we obtain

$$t^{2\alpha}|I_k^2| \lesssim t^{2\alpha-3/2} 2^{-k} \sum_{j=-2k}^{0} 2^{j/2} \lesssim |x|^{2\alpha-3/2} 2^{-2k(2\alpha-3/2)},$$

which is acceptable as soon as $\alpha > 1/4$. This concludes the proof of Lemma 3.2.

□

We are now in a position to prove Proposition 3.2.

*Proof of Proposition 3.2.* Let $k \geq 0$ and $U_k(t) = \Delta_k U(t)$. The proof will follow from a slight modification of the usual $AA^*$ argument. Let us define the operator $A_k : L_T^1 L_{\bar{x}}^2 \to L^2$ by

$$A_k g = \int_0^T t^\alpha U_k(-t) g(t) dt.$$

We easily check that $A_k^* h(t) = t^\alpha U_k(t) h$ for $h \in L^2(\mathbb{R}^3)$ and moreover,

$$A_k^* A_k g(t) = \int_0^T (tt')^\alpha U_k(t-t') g(t') dt'.$$

The previous integrand can be estimated by

$$|(tt')^\alpha U_k(t-t')g(t',\bar{x})| \lesssim \big||t-t'|^{2\alpha}U_k(t-t')g(t',\bar{x})\big| + \big||t+t'|^{2\alpha}U_k(t-t')g(t',\bar{x})\big| := I + II.$$

Using the Young inequality, the first term is bounded by

$$\begin{aligned} I &\lesssim \big|\big(|t-t'|^{2\alpha}I_k(t-t') *_{\bar{x}} g(t')\big)(\bar{x})\big| \\ &\lesssim \Big(\big(|t-t'|^{2\alpha}\|I_k(t-t')\|_{L^\infty_{yz}}\big) *_x \|g(t')\|_{L^1_{yz}}\Big)(x). \end{aligned}$$

Integrating this into $t' \in [0,T]$ and taking the $L^2_x L^\infty_{yzT}$ norm, this leads to

$$\begin{aligned} \Big\| \int_0^T \big||t-t'|^{2\alpha}U_k(t-t')g(t',\bar{x})\big|dt' \Big\|_{L^2_x L^\infty_{yzT}} &\lesssim \Big\| \|t^{2\alpha}I_k\|_{L^\infty_{yzT}} *_x \|g(t)\|_{L^1_{yzT}} \Big\|_{L^2_x} \\ &\lesssim \|t^{2\alpha}I_k\|_{L^1_x L^\infty_{yzT}} \|g\|_{L^2_x L^1_{yzT}}. \end{aligned} \tag{3.8}$$

To estimate $II$, we introduce $\check{g}(t,\bar{x}) = g(t,-\bar{x})$ and notice that

$$U_k(t-t')g(t',\bar{x}) = U_k(t+t')\check{g}(t',-\bar{x}).$$

We infer that

$$II \lesssim \Big(\big(|t+t'|^{2\alpha}\big)\|I_k(t+t')\|_{L^\infty_{yz}}\big) *_x \|\check{g}(t')\|_{L^1_{yz}}\Big)(x)$$

and we get

$$\begin{aligned} \Big\| \int_0^T \big||t+t'|^{2\alpha}U_k(t-t')g(t',\bar{x})dt' \Big\|_{L^2_x L^\infty_{yzT}} &\lesssim \Big\| \|t^{2\alpha}I_k\|_{L^\infty_{yzT}} *_x \|\check{g}\|_{L^1_{yzT}} \Big\|_{L^2_x} \\ &\lesssim \|t^{2\alpha}I_k\|_{L^1_x L^\infty_{yzT}} \|g\|_{L^2_x L^\infty_{yzT}}. \end{aligned} \tag{3.9}$$

Combining estimates (3.8)-(3.9) and Lemma 3.2 we deduce

$$\|A_k^* A_k g\|_{L^2_x L^\infty_{yzT}} \lesssim \|t^{2\alpha}I_k\|_{L^1_x L^\infty_{yzT}} \|g\|_{L^2_x L^{yzT}} \lesssim 2^{2k}\|g\|_{L^2_x L^\infty_{yzT}}.$$

The usual algebraic lemma (see Lemma 2.1 in [3]) applies and yields the first estimate in Proposition 3.2. The second one is obtained by following the same lines and using Lemma 3.1. □

In order to get the desired bounds for data in $H^s(\mathbb{R}^3)$, $s > 1$, we will use the following estimate.

**Proposition 3.3.** *For $0 < T < 1$, $s > 1$ and any $\varphi \in \mathcal{S}(\mathbb{R}^3)$, it holds that*

$$\|U(t)\varphi\|_{L^2_x L^\infty_{yzT}} \lesssim \|\varphi\|_{H^s}. \tag{3.10}$$

This proposition is a direct consequence of Lemma 3.1 together with the following result.

**Lemma 3.3.** *For any $\varepsilon > 0$ and $k \geq 0$, it holds that*

$$\|I_k\|_{L^1_x L^\infty_{yzT}} \lesssim 2^{(2+\varepsilon)k}.$$

*Proof.* Setting

$$I_{i,j,k}(t,\bar{x}) = \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi}+t\omega(\bar{\xi}))}\delta_i(\xi)\delta_j(\eta)\delta_k(\mu)d\bar{\xi},$$

we see that from the estimate

$$\|\Delta_n u\|_{L^1_x L^\infty_{yzT}} \lesssim \sum_{i,j,k:2^i,2^j,2^k\lesssim 2^n} \|\Delta^x_i \Delta^y_j \Delta^z_k u\|_{L^1_x L^\infty_{yzT}},$$

it suffices to show that

$$\|I_{i,j,k}\|_{L^1_x L^\infty_{yzT}} \lesssim (1+M)2^{2M} \tag{3.11}$$

for all $i,j,k \in \mathbb{Z}$ and with $M = \max(i,j,k) \geq 0$. From the straightforward bound $|I_{i,j,k}| \lesssim 2^{i+j+k}$, we see that we may assume $|x| \geq 2^{-m}$ where $m = \min(i,j,k)$. In the region $|x| \ll t2^{2M}$ or $|x| \gg t2^{2M}$, the phase function $\varphi_1$ defined in (3.6) satisfies $|\varphi_1'| \gtrsim \max(|x|, t2^{2M})$ and integrations by parts lead to

$$\left|\int_{\mathbb{R}} e^{i\varphi_1}\delta_i\right| \lesssim 2^{-i}|x|^{-2}.$$

Therefore we get $|I_{i,j,k}| \lesssim 2^{j+k-i}|x|^{-2}$ and then $\|I_{i,j,k}\|_{L^1_x L^\infty_{yzT}} \lesssim 2^{j+k-i+m} \lesssim 2^{2M}$. Finally consider the case $|x| \sim t2^{2M}$. Rewriting $I_{i,k,j}$ as

$$I_{i,j,k} = \int_{\mathbb{R}^2} \check{\delta}_j(y-u)\check{\delta}_k(z-v)\left(\int_{\mathbb{R}} \frac{\pi i}{t\xi} e^{i\varphi_2(\xi)}\delta_i(\xi)d\xi\right)dudv,$$

we immediately obtain $|I_{i,j,k}| \lesssim t^{-1} \lesssim 2^{2M}|x|^{-1}$ and thus $\|I_{i,j,k}\|_{L^1_x L^\infty_{yzT}} \lesssim M2^{2M}$, as desired.

□

Up to the end point $s = 1$, we show next that estimate (3.10) is sharp.

**Proposition 3.4.** *Suppose that for any $\varphi \in H^s(\mathbb{R}^3)$ we have*

$$\|t^\alpha U(t)\varphi\|_{L^2_x L^\infty_{yzT}} \lesssim \|\varphi\|_{H^s}, \tag{3.12}$$

*for some $\alpha \geq 0$. Then it must be the case that $s \geq 1$.*

*Proof.* Let us define the smooth functions $\varphi_k$ through their Fourier transforms by

$$\hat{\varphi}(\bar{\xi}) = \delta_{-2k}(\xi)\delta_k(\eta)\delta_k(\mu)$$

for $k \geq 0$. Then it is easy to see that

$$\|\varphi_k\|_{H^s} \sim 2^{ks}. \tag{3.13}$$

On the other hand, for $\varepsilon > 0$ small enough, we set $t = \varepsilon$ and $y = z = \varepsilon 2^{-k}$ so that $|y\eta + z\mu + t\omega(\bar{\xi})| \lesssim \varepsilon$. Choosing $|x| \ll 2^{2k}$, we obtain the lower bound

$$\begin{aligned}
|U(t)\varphi_k(\bar{x})| &= \left| \int_{\mathbb{R}^3} e^{i(\bar{x}\bar{\xi} + t\omega(\bar{\xi}))} \varphi_k(\bar{\xi}) d\bar{\xi} \right| \\
&= \Big| \int_{\mathbb{R}^3} [e^{i(y\eta + z\mu + t\omega(\bar{\xi}))} - 1] e^{ix\xi} \delta_{-2k}(\xi)\delta_k(\eta)\delta_k(\mu) d\bar{\xi} \\
&\quad + \int_{\mathbb{R}^3} e^{ix\xi} \delta_{-2k}(\xi)\delta_k(\eta)\delta_k(\mu) d\bar{\xi} \Big| \\
&\gtrsim 1.
\end{aligned}$$

It follows that $\|t^\alpha U(t)\varphi_k\|_{L^2_x L^\infty_{yzT}} \gtrsim 2^k$ where the implicit constant does not depend on $k$. Therefore, (3.12) and (3.13) imply

$$2^k \lesssim 2^{ks}.$$

From this, we get for large $k$ that $s \geq 1$. □

In the sequel of this section we prove retarded linear estimates which will be used later to perform the fixed point argument.

**Proposition 3.5.** *Let $f \in \mathcal{S}(\mathbb{R}^4)$. Then we have*

$$\left\| \nabla \int_0^t U(t - t')f(t')dt' \right\|_{L^\infty_T L^2_{\bar{x}}} \lesssim \|f\|_{L^1_x L^2_{yzT}}. \tag{3.14}$$

*Proof.* The dual estimate of (3.2) reads

$$\left\| \int_{-\infty}^{\infty} U(-t')\nabla f(t')dt' \right\|_{L^2} \lesssim \|f\|_{L^1_x L^2_{yzT}}. \tag{3.15}$$

Noticing that $U(t)$ is a unitary group on $L^2(\mathbb{R}^3)$ we obtain for any fixed $t$,

$$\left\| \int_{-\infty}^{\infty} U(t-t')\nabla f(t')dt' \right\|_{L^2_{\bar{x}}} \lesssim \|f\|_{L^1_x L^2_{yzT}}. \tag{3.16}$$

To conclude we substitute in (3.16) $f(t')$ by $\chi_{[0,t]}(t')f(t')$ and then take the supremum in time in the left-hand side of the resulting inequality. □

**Proposition 3.6.** *Let $f \in \mathcal{S}(\mathbb{R}^4)$. Then we have*

$$\left\| \nabla^2 \int_0^t U(t-t')f(t')dt' \right\|_{L^\infty_x L^2_{yzT}} \lesssim \|f\|_{L^1_x L^2_{yzT}}. \tag{3.17}$$

*Proof.* We first observe that for any $g : \mathbb{R} \to \mathbb{R}$,

$$\int_0^t g(t')dt' = \frac{1}{2}\int_{\mathbb{R}} g(t')\operatorname{sgn}(t-t')dt' + \frac{1}{2}\int_{\mathbb{R}} g(t')\operatorname{sgn}(t')dt',$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function. Consequently we have

$$\begin{aligned}
\nabla^2 R(t,\bar{x}) &= \nabla^2 \int_0^t U(t-t')f(t')dt' \\
&= \frac{1}{2}\nabla^2 \int_{\mathbb{R}} U(t-t')f(t')\operatorname{sgn}(t-t')dt' + \frac{1}{2}\nabla^2 \int_{\mathbb{R}} U(t-t')f(t')\operatorname{sgn}(t')dt' \\
&= \frac{1}{2}\nabla^2 R_1(t,\bar{x}) + \frac{1}{2}\nabla^2 R_2(t,\bar{x}).
\end{aligned} \tag{3.18}$$

Taking the inverse space-time Fourier transform, it is clear that

$$R_1 = \mathcal{F}^{-1}_{\tau\bar{\xi}}\left(\widehat{\operatorname{sgn}}(\tau - \omega(\bar{\xi}))\hat{f}(\tau,\bar{\xi})\right).$$

Hence we get by Plancherel theorem

$$\begin{aligned}
\|\nabla^2 R_1\|_{L^2_{yzt}} &= \left\| \mathcal{F}^{-1}_{\xi}\left(|\bar{\xi}|^2\widehat{\operatorname{sgn}}(\tau - \omega(\bar{\xi}))\hat{f}(\tau,\bar{\xi})\right)\right\|_{L^2_{\eta\mu\tau}} \\
&= \|K(\tau,x,\eta,\mu) *_x \mathcal{F}_{yzt}(f(x))(\eta,\mu,\tau)\|_{L^2_{\eta\mu\tau}}
\end{aligned} \tag{3.19}$$

where $K$ is the inverse Fourier transform (in $\xi$) of the tempered distribution defined as the principal value of $|\bar{\xi}|^2/(\tau - \omega(\bar{\xi}))$. It follows that

$$\begin{aligned} K(\tau, x, \eta, \mu) &= \int_{\mathbb{R}} e^{ix\xi} \frac{|\bar{\xi}|^2}{\tau - \omega(\bar{\xi})} d\xi \\ &= \int_{\mathbb{R}} e^{ix(\eta^2+\mu^2)^{1/2}\xi} \frac{(\eta^2+\mu^2)^{3/2}(\xi^2+1)}{\tau - (\eta^2+\mu^2)^{3/2}\xi(\xi^2+1)} d\xi \\ &= \int_{\mathbb{R}} e^{iy\xi} \frac{\xi^2+1}{c - \xi(\xi^2+1)} d\xi \end{aligned}$$

with $y = (\eta^2+\mu^2)^{1/2}x$ and $c = \tau/(\eta^2+\mu^2)^{3/2}$. Next, a partial fraction expansion leads to

$$\frac{\xi^2+1}{c - \xi(\xi^2+1)} = -\frac{\alpha^2+1}{(3\alpha^2+1)(\xi-\alpha)} - \frac{\alpha}{3\alpha^2+1} \frac{2\alpha\xi + \alpha^2 - 1}{\xi^2 + \alpha\xi + \alpha^2 + 1}$$

where $\alpha$ is the unique real root of $c - X(X^2+1)$. Therefore, we get

$$\begin{aligned} |K(\tau, x, \eta, \mu)| &\lesssim \frac{\alpha^2+1}{3\alpha^2+1} \left| \int_{\mathbb{R}} \frac{e^{iy\xi}}{\xi - \alpha} d\xi \right| + \frac{2\alpha^2}{3\alpha^2+1} \left| \int_{\mathbb{R}} e^{iy\xi} \frac{\xi}{\xi^2 + \alpha\xi + \alpha^2 + 1} d\xi \right| \\ &\quad + \frac{|\alpha(\alpha^2-1)|}{3\alpha^2+1} \left| \int_{\mathbb{R}} \frac{e^{iy\xi}}{\xi^2 + \alpha\xi + \alpha^2 + 1} d\xi \right| \\ &= K_1 + K_2 + K_3. \end{aligned}$$

The first term $K_1$ is bounded by an integral witch is the Fourier transform of a function that behaves near the singular points like the kernel of the Hilbert transform $1/\xi$ (or its translates) whose Fourier transform is $\operatorname{sgn}(x)$. It follows that $K_1$ is bounded uniformly in $\alpha$ and $y$. In the same way, $K_2 \in L^\infty$ since

$$\begin{aligned} K_2 &\lesssim \left| \int_{\mathbb{R}} e^{iy\xi} \frac{\xi}{\left(\xi + \frac{\alpha}{2}\right)^2 + \frac{3\alpha^2}{4} + 1} d\xi \right| \lesssim \frac{1}{3\alpha^2+4} \left| \int_{\mathbb{R}} e^{iy\xi} \frac{\xi}{\left(\frac{\xi}{\sqrt{3\alpha^2+4}}\right)^2 + 1} d\xi \right| \\ &\lesssim \left| \int_{\mathbb{R}} e^{iz\xi} \frac{\xi}{\xi^2+1} d\xi \right| \end{aligned}$$

where $z = y\sqrt{3\alpha^2+4}$. Concerning $K_3$, it is easily estimated by

$$\begin{aligned} K_3 &\lesssim \frac{|\alpha(\alpha^2-1)|}{3\alpha^3+1} \int_{\mathbb{R}} \frac{d\xi}{\xi^2 + \frac{3\alpha^2}{4} + 1} \\ &\lesssim \frac{|\alpha(\alpha^2-1)|\sqrt{3\alpha^2+4}}{(3\alpha^2+1)(3\alpha^2+4)} \lesssim 1. \end{aligned}$$

We deduce that $K \in L^\infty(\mathbb{R}^4)$. Hence, (3.19) combining with the Young inequality yields

$$\|\nabla^2 R_1\|_{L^\infty_x L^2_{yzt}} \lesssim \|K\|_{L^\infty} \|\mathcal{F}_{yzt}(f(x))\|_{L^1_x L^2_{\eta\mu\tau}} \lesssim \|f\|_{L^1_x L^2_{yzt}}. \tag{3.20}$$

To treat the contribution of $R_2$, we use the smoothing bound (3.2) and its dual estimate (3.15) to get

$$\begin{aligned} \|\nabla^2 R_2\|_{L^\infty_x L^2_{yzt}} &\lesssim \left\| \nabla \int_{\mathbb{R}} U(-t') f(t') \operatorname{sgn}(t') dt' \right\|_{L^2} \\ &\lesssim \|f\|_{L^1_x L^2_{yzt}}. \end{aligned} \tag{3.21}$$

Estimates (3.20)-(3.21) together with (3.18) yield the desired bound. □

**Proposition 3.7.** *Let $T \leq 1$, $k \geq 0$ and $\alpha \geq 3/8$. Then for any $f \in \mathcal{S}(\mathbb{R}^4)$,*

$$\left\| \int_0^t U(t-t') P_0 f(t') dt' \right\|_{L^2_x L^\infty_{yzT}} \lesssim \|P_0 f\|_{L^1_x L^2_{yzT}}, \tag{3.22}$$

$$\left\| t^\alpha \int_0^t U(t-t') \Delta_k f(t') dt' \right\|_{L^2_x L^\infty_{yzT}} \lesssim \|\Delta_k f\|_{L^1_x L^2_{yzT}}. \tag{3.23}$$

*Proof.* Combining estimates (3.3) and (3.15) we obtain the non-retarded version of (3.23):

$$\left\| t^\alpha \int_0^T U(t-t') \Delta_k f(t') dt' \right\|_{L^2_x L^\infty_{yzT}} \lesssim \|\Delta_k f\|_{L^1_x L^2_{yzT}}. \tag{3.24}$$

We consider the function $H_k(t, \bar{x}) = \left| t^\alpha \int_0^t U(t-t') \Delta_k f(t') dt' \right|$ that we may always assume to be continuous on $[0,T] \times \mathbb{R}^3$. From (3.24) it follows that for all measurable function $t : \mathbb{R}^3 \to [0,T]$,

$$\left\| t(x)^\alpha \int_0^T U(t(x)-t') \Delta_k f(t', \bar{x}) dt' \right\|_{L^2_x L^\infty_{yz}} \lesssim \|\Delta_k f\|_{L^1_x L^2_{yzT}}.$$

Replacing now $f(t', \bar{x})$ by $f(t', \bar{x}) \chi_{[0, t(\bar{x})]}(t')$ we see that

$$\|H_k(t(\bar{x}), \bar{x})\|_{L^2_x L^\infty_{yz}} \lesssim \|\Delta_k f\|_{L^1_x L^2_{yzT}}. \tag{3.25}$$

Since the map $t \mapsto H_k(t, \bar{x})$ is continuous on the compact set $[0,T]$, there exists $\alpha \in [0,T]$ such that $H_k(\alpha, \bar{x}) = \sup_{t \in [0,T]} H_k(t, \bar{x})$, therefore the map

$$\bar{x} \mapsto t_0(\bar{x}) = \inf \left\{ \alpha \in [0,T] : H_k(\alpha, \bar{x}) = \sup_{t \in [0,T]} H_k(t, \bar{x}) \right\}$$

is well-defined and measurable on $\mathbb{R}^3$. Hence, choosing $t_0$ in (3.25) we infer that

$$\|H_k(t_0(\bar{x}),\bar{x})\|_{L^2_x L^\infty_{yz}} \lesssim \|\Delta_k f\|_{L^1_x L^2_{yzT}},$$

which yields

$$\left\| \sup_{t\in[0,T]} H_k(t,\bar{x}) \right\|_{L^2_x L^\infty_{yz}} \lesssim \|\Delta_k f\|_{L^1_x L^2_{yzT}}$$

and ends the proof of (3.23). The proof of (3.22) is similar and therefore will be omitted. □

# 4 Proofs of Theorems (1.2) and (1.1)

In this section we solve (1.1) in the spaces $B^{1,1}_2(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3)$, $s>1$. We consider the associated integral equation

$$u(t) = U(t)u_0 - \frac{1}{2}\int_0^t U(t-t')\partial_x(u^2)(t')dt'. \tag{4.1}$$

## 4.1 Well-posedness in $B^{1,1}_2(\mathbb{R}^3)$

For $u_0 \in B^{1,1}_2(\mathbb{R}^3)$, we look for a solution in the space

$$X_T = \{u \in C_b([0,T], B^{1,1}_2) : \|u\|_{X_T} < \infty\}$$

for some $T>0$ and where

$$\|u\|_{X_T} = N(u) + T(u) + M(u)$$

with

$$\begin{aligned}
N(u) &= \|P_0 u\|_{L^\infty_T L^2_{\bar{x}}} + \sum_{k\geq 0} 2^k \|\Delta_k u\|_{L^\infty_T L^2_{\bar{x}}},\\
T(u) &= \|P_0 u\|_{L^\infty_x L^2_{yzT}} + \sum_{k\geq 0} 2^{2k} \|\Delta_k u\|_{L^\infty_x L^2_{yzT}},\\
M(u) &= \|P_0 u\|_{L^2_x L^\infty_{yzT}} + \sum_{k\geq 0} \|t^\alpha \Delta_k u\|_{L^2_x L^\infty_{yzT}},
\end{aligned}$$

and $3/8 \leq \alpha < 1/2$. First from Propositions 3.1, 3.2 together with the obvious bound

$$\|U(t)u_0\|_{L^\infty_T L^2_{\bar{x}}} \lesssim \|u_0\|_{L^2}, \tag{4.2}$$

we get the following linear estimate:

$$\|U(t)u_0\|_{X_T} \lesssim \|u_0\|_{B_2^{1,1}}. \tag{4.3}$$

Now we need to estimate

$$\left\| \int_0^t U(t-t')\partial_x(u^2)(t')dt' \right\|_{X_T}$$

in terms of the $X_T$-norm of $u$. Using standard paraproduct rearrangements, we can rewrite $\Delta_k(u^2)$ as

$$\begin{aligned}
\Delta_k(u^2) &= \Delta_k \left[ \lim_{j\to\infty} P_j(u)^2 \right] \\
&= \Delta_k \left[ P_0(u)^2 + \sum_{j\geq 0} (P_{j+1}(u)^2 - P_j(u)^2) \right] \\
&= \Delta_k \left[ P_0(u)^2 + \sum_{j\gtrsim k} \Delta_{j+1}u(P_{j+1}u + P_j u) \right].
\end{aligned}$$

On the other hand, by similar considerations, we see that $P_0(u^2)$ can be rewritten as

$$P_0(u^2) = P_0[P_0(u)^2] + P_0 \left[ \sum_{j\geq 0} \Delta_{j+1}u(P_{j+1}u + P_j u) \right].$$

Hence, without loss of generality, we can restrict us to consider only terms of the form:

$$A = P_0[P_0(u)^2], \quad B = \Delta_k \left[ \sum_{j\gtrsim k} \Delta_j u P_j u \right], \quad C = P_0 \left[ \sum_{j\geq 0} \Delta_j u P_j u \right]$$

for $k \geq 0$, since the estimates for the other terms would be similar.

By virtue of (3.14)-(3.17)-(3.22) and (3.23), we infer that

$$\begin{aligned}
\left\| \int_0^t U(t-t')\partial_x(u^2)(t')dt' \right\|_{X_T} &\lesssim \|P_0(u)^2\|_{L_x^1 L_{yzT}^2} \\
&+ \sum_{k\geq 0} 2^k \left( \sum_{j\gtrsim k} \|\Delta_j u P_j u\|_{L_x^1 L_{yzT}^2} \right) + \sum_{j\geq 0} \|\Delta_j u P_j u\|_{L_x^1 L_{yzT}^2}.
\end{aligned} \tag{4.4}$$

The first term in the right hand side is bounded by

$$\begin{aligned}\|P_0(u)^2\|_{L^1_x L^2_{yzT}} &\lesssim \|P_0 u\|_{L^2}\|P_0 u\|_{L^2_x L^\infty_{yzT}}\\ &\lesssim T^{1/2}\|P_0 u\|_{L^\infty_T L^2_{\bar{x}}} M(u)\\ &\lesssim T^{1/2}\|u\|^2_{X_T}. \end{aligned} \tag{4.5}$$

To evaluate the contribution of $B$, note that

$$\begin{aligned}\|\Delta_j u P_j u\|_{L^1_x L^2_{yzT}} &= \|(t^{-\alpha}\Delta_j u)(t^\alpha P_j u)\|_{L^1_x L^2_{yzT}}\\ &\lesssim \|t^{-\alpha}\Delta_j u\|_{L^2}\|t^\alpha P_j u\|_{L^2_x L^\infty_{yzT}}\\ &\lesssim T^\mu \|\Delta_j u\|_{L^\infty_T L^2_{\bar{x}}}\|t^\alpha P_j u\|_{L^2_x L^\infty_{yzT}} \end{aligned} \tag{4.6}$$

where $\mu = 1/2 - \alpha > 0$. Therefore, since

$$\|t^\alpha P_j u\|_{L^2_x L^\infty_{yzT}} \lesssim \|P_0 u\|_{L^2_x L^\infty_{yzT}} + \sum_{k=0}^{j} \|t^\alpha \Delta_k u\|_{L^2_x L^\infty_{yzT}} \lesssim M(u), \tag{4.7}$$

we deduce from the discrete Young inequality that

$$\begin{aligned}\sum_{k\geq 0} 2^k \left(\sum_{j\gtrsim k} \|\Delta_j u P_j u\|_{L^1_x L^2_{yzT}}\right) &\lesssim T^\mu M(u) \sum_{k\geq 0}\left(\sum_{j\gtrsim k} 2^{k-j}(2^j \|\Delta_j u\|_{L^2_x L^\infty_{yzT}})\right)\\ &\lesssim T^\mu \|u\|_{X_T}\sum_{k\geq 0} 2^k \|\Delta_k u\|_{L^2_x L^\infty_{yzT}}\\ &\lesssim T^\mu \|u\|^2_{X_T}. \end{aligned} \tag{4.8}$$

Using again estimates (4.6)-(4.7), we easily get that the last term in the r.h.s. of (4.4) can be estimated by

$$\sum_{j\geq 0} \|\Delta_j u P_j u\|_{L^1_x L^2_{yzT}} \lesssim T^\mu M(u) \sum_{j\geq 0} \|\Delta_j u\|_{L^\infty_T L^2_{\bar{x}}} \lesssim T^\mu \|u\|^2_{X_T}. \tag{4.9}$$

Hence, gathering (4.3)-(4.4)-(4.5)-(4.8) and (4.9), we infer that

$$\|\mathcal{G}u\|_{X_T} \lesssim \|u_0\|_{B^{1,1}_2} + T^\mu \|u\|^2_{X_T},$$

and in the same way,

$$\|\mathcal{G}u - \mathcal{G}v\|_{X_T} \lesssim (\|u\|_{X_T} + \|v\|_{X_T})\|u - v\|_{X_T}.$$

Hence for $T > 0$ small enough, $\mathcal{G}$ is a strict contraction in some ball of $X_T$. Theorem 1.2 follows then from standard arguments.

### 4.2 Well-posedness in $H^s(\mathbb{R}^3)$, $s > 1$

Let $u_0 \in H^s(\mathbb{R}^3)$ with $s > 1$. For $T > 0$, we introduce the space

$$X_T^s = \{u \in C_b([0,T], H^s) : \|u\|_{X_T^s} < \infty\}$$

where

$$\|u\|_{X_T^s} = N(u) + T(u) + M(u)$$

with

$$N(u) = \|P_0 u\|_{L_T^\infty L_{\bar{x}}^2} + \left(\sum_{k\geq 0} 4^{sk} \|\Delta_k u\|^2_{L_T^\infty L_{\bar{x}}^2}\right)^{1/2},$$

$$T(u) = \|P_0 u\|_{L_x^\infty L_{yzT}^2} + \left(\sum_{k\geq 0} 4^{(s+1)k} \|\Delta_k u\|^2_{L_x^\infty L_{yzT}^2}\right)^{1/2},$$

$$M(u) = \|P_0 u\|_{L_x^2 L_{yzT}^\infty} + \left(\sum_{k\geq 0} 4^{(s-1-\varepsilon)k} \|\Delta_k u\|^2_{L_x^2 L_{yzT}^\infty}\right)^{1/2},$$

for some $\varepsilon > 0$ small enough. From Proposition 3.1 together with (3.10) and (4.2),

$$\|U(t)u_0\|_{X_T^s} \lesssim \|u_0\|_{H^s}.$$

Following the arguments given in Subsection 4.1, it is not too hard to see that

$$\left\|\int_0^t U(t-t')\partial_x(u^2)(t')dt'\right\|_{X_T^s} \lesssim \|P_0(u)^2\|_{L_x^1 L_{yzT}^2}$$
$$+ \left(\sum_{k\geq 0} 4^{sk}\left(\sum_{j\gtrsim k} \|\Delta_j u P_j u\|_{L_x^1 L_{yzT}^2}\right)^2\right)^{1/2} + \sum_{j\geq 0} \|\Delta_j u P_j u\|_{L_x^1 L_{yzT}^2}.$$

The estimates for these terms follow the same lines than the $B_2^{1,1}$ case, except that we use the Young inequality for $\ell^1 \star \ell^2$, as well as the bound

$$\begin{aligned}\|P_j u\|_{L_x^2 L_{yzT}^\infty} &\lesssim \|P_0 u\|_{L_x^2 L_{yzT}^\infty} + \sum_{k=0}^{j} \|\Delta_k u\|_{L_x^2 L_{yzT}^\infty} \\ &\lesssim M(u) + \left(\sum_{k\geq 0} 4^{(1-s+\varepsilon)k}\right)^{1/2} M(u) \\ &\lesssim M(u),\end{aligned}$$

as soon as $0 < \varepsilon < s - 1$. This leads to

$$\|\mathcal{G}u\|_{X_T^s} \lesssim \|u_0\|_{H^s} + T^\mu \|u\|_{X_T^s}^2,$$

and

$$\|\mathcal{G}u - \mathcal{G}v\|_{X_T^s} \lesssim (\|u\|_{X_T^s} + \|v\|_{X_T})\|u - v\|_{X_T^s}.$$

This proves the existence and uniqueness of a local solution $u$ in $X_T^s$ with $T = T(\|u_0\|_{H^s})$ small enough.

# References


[1] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation.* Geom. Funct. Anal. 3 (1993), no. 4, 315–341.

[2] A. V. Faminskii, *The Cauchy problem for the Zakharov-Kuznetsov equation.* Differential Equations 31 (1995), no. 6, 1002–1012

[3] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations.* Comm. Math. Phys. 144 (1992), no. 1, 163–188.

[4] C E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations.* Indiana Univ. Math. J. 40 (1991), no. 1, 33–69.

[5] C E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation.* J. Amer. Math. Soc. 4 (1991), no. 2, 323–347.

[6] C E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle.* Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.

[7] F. Linares and A. Pastor, *Well-posedness for the two-dimensional modified Zakharov-Kuznetsov equation.* SIAM J. Math. Anal. 41 (2009), no. 4, 1323–1339.

[8] F. Linares, A. Pastor, J.-C. Saut, *Well-posedness for the ZK equation in a cylinder and on the background of a KdV soliton.* Comm. Partial Differential Equations 35 (2010), no. 9, 1674–1689.

[9] F. Linares and J.-C. Saut, *The Cauchy problem for the 3D Zakharov-Kuznetsov equation.* Discrete Contin. Dyn. Syst. 24 (2009), no. 2, 547–565.

[10] L. Molinet, J.-C. Saut and N. Tzvetkov, *Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation.* Duke Math. J. 115 (2002), no. 2, 353–384.

[11] V.E. Zakharov and E.A. Kuznetsov, *On three dimensional solitons.* Sov. Phys. JETP., 39 (1974), 285–286.