An asymptotic study to explain the role of active transport in models with countercurrent exchangers

Magali Tournus

An asymptotic study to explain the role of active transport in models with countercurrent exchangers

Magali Tournus *†‡

November 9, 2011

Abstract

We study a solute concentrating mechanism that can be represented by coupled transport equations with specific boundary conditions. Our motivation for considering this system is urine concentrating mechanism in nephrons. The model consists in 3 tubes arranged in a countercurrent manner. Our equations describe a countercurrent exchanger, with a parameter V which quantifies the active transport. In order to understand the role of active transport in the mechanism, we consider the limit $V \longrightarrow \infty$. We prove that when V goes to infinity, the system converges to a profile which stays uniformly bounded in V and which presents a boundary layer at the border of the domain. The effect is that the solute is concentrated at a specific point in the tubes. When considering urine concentration, this is physilogically optimal because the composition of final urine is determined at this point.

Mathematics Subject Classification : $34B18, 34E99, 92C30$
Key-words : Countercurrent, active transport, asymptotic analysis, boundary layer, urine concentration, kidney physiology.

1 Introduction

Many problems occurring in biology or physiology [14, 9] can be described by transport equations with different propagation velocities that combine together to produce specific effects. For instance, typical cases are propagation of particules waves in neurones [5], or diphasic propagation arising in chemical engineering, which can describe chromatography or distillation [7]. Usually, the stationary states are of interest and their study constitute a classical field of analysis (see for instance [13] and the references therein).

Here, we study a model of a countercurrent exchanger combined with an active transport pump [6]. Countercurrent exchanges accross parallel tubes can be used for building up concentration or heat gradient. Our equations come from the modelisation of kidney nephrons, in which a concentration gradient is amplified by an active transport pump [16], which plays a fundamental role for urine concentration [11]. In this particular study, we investigate the effects of active transport using a limiting case.

The model consists in a fluid circulating at a constant velocity in 3 tubes arranged in a countercurrent architecture. The 3 tubes are bathing in a common bath in which no solute can accumulate. Each tube can exchange solute with the bath and solute transport accross tubes wall is driven by diffusion in all tubes and by an active pump in tube 3. This active pump extracts solute from tube 3 and carries it into the bath and is assumed to follow Michaelis-Menten kinetics. We call V the maximum rate achieved by the pump at saturating concentrations. We call $C^i(x)$ the solute concentration in tube i at depth x. The nonlinearity $V\dfrac{C_V^3}{1+C_V^3}$ represents the effect of active pumps along tube 3. The fluid enters tube 1 with a concentration value C_0^1 and tube 2 with a concentration value C_0^2. The outlet of tubes 1 and 3 are open at $x = L$ and we have $C^2(L) = C^3(L)$. See Figure 1 for a drawing of the system. The stationary state is of particular interest in renal physiology, given that the kidney acts to preserve homeostasis.

*Email: magali.tournus@ann.jussieu.fr
†UPMC Univ Paris 06, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris
‡INSERM UMRS872, and CNRS ERL7226, Laboratoire de génomique, physiologie et physiopathologie rénales, Centre de Recherche des Cordeliers, 75006, Paris, France;

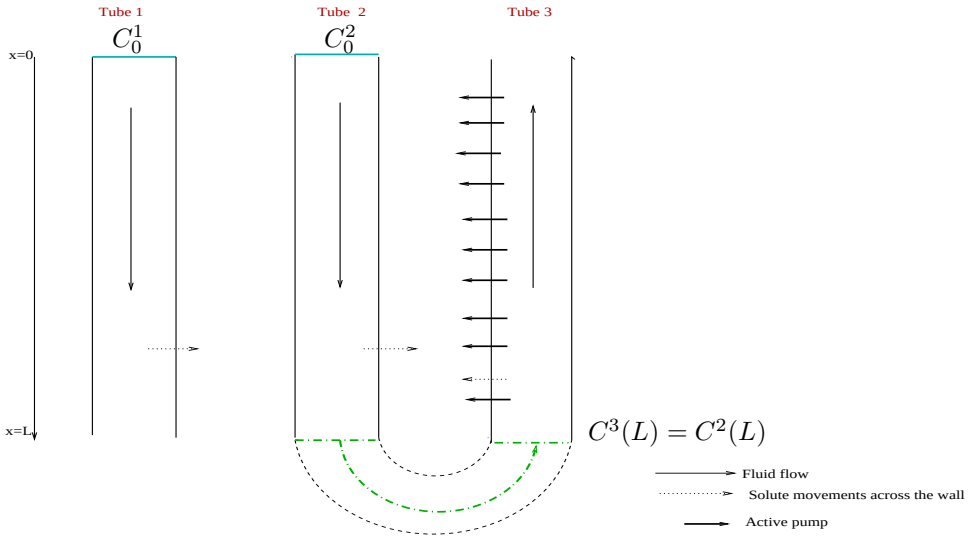

Figure 1: Representation of the 3-tube architecture in which the fluid circulates. Tubes are water-impermeable but can exchange solutes with the bath.

The differential system satisfied by C^1, C^2, C^3 is written as

$$\begin{cases} \dfrac{dC^1(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1+C^3(x)}\Big] - C^1(x), \\ \dfrac{dC^2(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1+C^3(x)}\Big] - C^2(x), \\ -\dfrac{dC^3(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1+C^3(x)}\Big] - C^3(x) - V\dfrac{C^3(x)}{1+C^3(x)}, \\ C^1(0) = C_0^1, \qquad C^2(0) = C_0^2, \qquad C^3(L) = C^2(L). \end{cases} \tag{1}$$

The specific boundary conditions relate the solutions at different points and make this system not a mere ordinary differential equation. We call $C_V = (C_V^1, C_V^2, C_V^3)$ the solution of (1). We wish to explain through the analysis of the system for large values of V, why this combination of active pump and boundary conditions (tube arrangement) is performing well the task of concentrating the solute at $x = L$, where the composition of final urine is determined. We already know [18] that each C_V^i is continuous and nonnegative on $[0, L]$. The question we want to answer is : How do the solutions of (1) behave when V tends to ∞?

Other asymptotic studies have been done for similar systems in the context of hyperbolic relaxation where a parameter is assumed to be small in comparison to the typical size of the problem [12, 8, 7]. This appraoch comes from the concept of mean free path in Boltzmann equation [3]. For example, in [5], the length of the domain is large, and they establish the asymtotic behavior of the solution in the limiting case of an infinite domain.

In our case, for answering our question, we prove that C_V converges toward a limit $C = (C^1, C^2, C^3)$ that we calculate. Our analysis uses only direct a priori estimates and weak limits obtained by compact injections which do not use the specific smooth form of the non-linearity and makes it very general. We identify completely the limit as $V \longrightarrow \infty$ including boundary layers. Compact injections give us the convergence of some particular subsequences, but as we point out that the limit only depends on the problem data, we are able to prove that the whole sequence converges. The boundary layers are coming from the particular boundary conditions in the model, which can be seen as reflection conditions and make the problem specific and interesting.

We state our main results in next section. Section 3 and 4 are devoted to the proofs of the asymptotic results. Numerical illustrations are given in section 5.

2 Main results

It is possible to identify completely the profiles of the limiting values for the solutions C^1, C^2, C^3 almost everywhere as $V \longrightarrow \infty$. This is stated in the

Theorem 1 (Asymptotics). *Solutions to* (1) *satisfy*

$$C_V^1 \underset{V\longrightarrow+\infty}{\longrightarrow} C^1, \qquad C_V^2 \underset{V\longrightarrow+\infty}{\longrightarrow} C^2, \qquad C_V^3 \underset{V\longrightarrow+\infty}{\longrightarrow} 0, \qquad L^p(1\le p<\infty),\ a.e., \tag{2}$$

with

$$C^1(x)=\frac{C_0^1+C_0^2}{2}+\frac{C_0^1-C_0^2}{2}e^{-x}, \qquad C^2(x)=\frac{C_0^1+C_0^2}{2}+\frac{C_0^2-C_0^1}{2}e^{-x}, \qquad C^3(x)=0 \qquad a.e. \tag{3}$$

This result is somewhat sharp since we will see that a boundary layer occurs and thus the convergence does not hold in L^∞. To state our next result, we need to define the quantity

$$M=\operatorname{ess\,inf}\{\frac{1}{C_V^3(x)}; x\in[0,L], V\in\mathbb{R}^+\}. \tag{4}$$

We prove in the next section that $M>0$. The second result is more accurate and states that C^3 decreases exponentially fast to zero. We describe also the boundary layer that appears at $x=L$.

Theorem 2 (The boundary layer). *The limits of the boundary values are*

$$\begin{aligned} C_V^1(L) &\underset{V\longrightarrow+\infty}{\longrightarrow} C_0^1+C_0^2 \\ C_V^2(L)=C_V^3(L) &\underset{V\longrightarrow+\infty}{\longrightarrow} C_0^1+C_0^2+(C_0^2-C_0^1)e^{-L} \end{aligned} \tag{5}$$

The behavior of C_V^3 for $x\simeq L$ is given by the inequalities

$$C_V^3(x)\le C_V^3(L)\exp\Big(-\frac{2}{3}VM(L-x)\Big)+\frac{K}{V}\Big[1-\exp\Big(-\frac{2}{3}VM(L-x)\Big)\Big] \tag{6}$$

$$C_V^3(x)\ge C_V^3(L)\exp\Big(-\frac{2}{3}V(L-x)\Big)+\frac{\overline{K}}{V}\Big[1-\exp\Big(-\frac{2}{3}V(L-x)\Big)\Big] \tag{7}$$

The next section is dedicated to the proof of these results.

3 Proof of the asymptotic results(Theorem 1)

First step: Uniform bounds on the solution.

Lemma 1. *There is a constant K depending only on C_0^1, C_0^2 but not on V such that*

$$C_V^1(L)\le K, \qquad C_V^3(0)\le C_0^1+C_0^2, \tag{8}$$

$$\int_0^L C_V^i(x)dx\le K; \qquad V\int_0^L C_V^3(x)dx\le K; \qquad \int_0^L|\frac{dC_V^3}{dx}(x)|dx\le K; \qquad 0\le C_V^i\le K. \tag{9}$$

Proof. To prove (8), we sum the three lines of (1), and we obtain a quantity which does not depend on x,

$$C_V^1(x)+C_V^2(x)-C_V^3(x)=:K(V). \tag{10}$$

Using the boundary values, we find uniform bounds on $K(V)$

$$K(V)=C_0^1+C_0^2-C_V^3(0)\le C_0^1+C_0^2, \qquad K(V)=C_V^1(L)+C_V^2(L)-C_V^3(L)=C_V^1(L)\ge 0, \tag{11}$$

and thus

$$0\le K(V)\le C_0^1+C_0^2. \tag{12}$$

The combination of (11) and (12) proves (8).

Then, we prove the first two bounds in (9). The first equation can be written

$$\frac{dC_V^1(x)}{dx}+C_V^1(x)=Q_V(x)\ge 0, \tag{13}$$

with

$$Q_V(x) = \frac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\frac{C^3(x)}{1+C^3(x)}\Big].$$

Therefore we also have

$$\frac{d}{dx}\Big(C_V^1(x)e^x\Big) = Q_V(x)e^x.$$

By integration over $[0, L]$, we obtain

$$\int_0^L Q_V(x)dx \leq \int_0^L Q_V(x)e^x dx = C_V^1(L)e^L - C_0^1 \leq (C_0^1 + C_0^2)e^L. \tag{14}$$

We conclude that

$$\int_0^L V\frac{C_V^3(x)}{1+C_V^3(x)}dx,\ \int_0^L C_V^i(x)dx,\ i = 1, 2, 3, \text{ are uniformly bounded by } (C_0^1 + C_0^2)e^L. \tag{15}$$

Then, by injecting equation (14) in (13) and because the C_V^i are positive, we have

$$\int_0^L |\frac{dC_V^i}{dx}(s)|ds \leq (C_0^1 + C_0^2)e^L \qquad i = 1, 2, 3. \tag{16}$$

We finally prove that the functions $\Big(C_V^i(x)\Big)_V$ are uniformly bounded in V. We write

$$|C_V^i(x)| = |C_V^i(0) + \int_0^x \frac{dC_V^i}{dx}(s)ds| \leq |C_V^i(0)| + \int_0^L |\frac{dC_V^i}{dx}(s)|ds.$$

Thanks to (16) and (8), we conclude

$$\|C_V^i\|_\infty \leq (C_0^1 + C_0^2)(e^L + 1). \tag{17}$$

□

The upper bound (17) on C_V^3 gives us that $M > 0$.

Second step: The behaviour of C_V^i when $V \longrightarrow \infty$

Lemma 2. *After extraction of a subsequence,*

$$C_V^1 \underset{V\longrightarrow+\infty}{\longrightarrow} C^1, \qquad C_V^2 \underset{V\longrightarrow+\infty}{\longrightarrow} C^2, \qquad C_V^3 \underset{V\longrightarrow+\infty}{\longrightarrow} 0, \qquad L^p(1 \leq p < \infty),\ a.e., \tag{18}$$

and $C^1 + C^2 = K^0$ for some constant K^0.

Proof. We know from Lemma 1 that $\Big(C_V^i\Big)_V$ is bounded in BV, then, using the Rellich-Kondrachov compact injection [4]

$$C_V^i \underset{V\longrightarrow+\infty}{\longrightarrow} C^i, \qquad \text{in } L^p(1 \leq p < \infty) \text{ and } a.e. \tag{19}$$

On the other hand, we have thanks to (15),

$$\int_0^L \frac{C_V^3(x)}{1+C_V^3(x)}dx \underset{V\longrightarrow+\infty}{\longrightarrow} 0,$$

and thus

$$C^3 \equiv 0 \qquad a.e. \tag{20}$$

Combining (10) with (20), we have $C^1 + C^2 = K^0$ for some constant K^0. □

Third step : The behavior of $\frac{dC_V^3}{dx}$. We define $\mathcal{M}^1[0, L]$ the set of Radon measures on $[0, L]$, taken with the weak convergence of measures.

Lemma 3. *After extraction,*

$$C_V^3(L) \underset{V\longrightarrow+\infty}{\longrightarrow} B, \qquad \frac{dC_V^3}{dx} \underset{V\longrightarrow+\infty}{\longrightarrow} B\delta_{x=L} \qquad \text{in } \mathcal{M}^1[0,L],$$

Proof. The information (16) implies that $\Big(\frac{dC_V^3}{dx}\Big)_V$ is bounded in $L^1[0,L]$, then [2] there exists $\mu \in \mathcal{M}^1[0,L]$ a Radon measure so that, after extraction,

$$\frac{dC_V^3}{dx} \underset{V\longrightarrow+\infty}{\longrightarrow} \mu \qquad \text{in the sense of measures.} \tag{21}$$

For all functions $\phi \in C^1[0,L]$ such as $\phi(0)=\phi(L)=0$, we have using (18)

$$\int_0^L \phi(x)\frac{dC_V^3}{dx}(x)dx = \int_0^L C_V^3(x)\frac{d\phi}{dx}(x)dx \underset{V\longrightarrow+\infty}{\longrightarrow} 0, \tag{22}$$

which means,

$$\mu = 0 \qquad \text{on }]0,L[. \tag{23}$$

Therefore, we can write in the sense of measures

$$\frac{dC_V^3}{dx} \underset{V\longrightarrow+\infty}{\longrightarrow} \beta\delta_{x=L} + \alpha\delta_{x=0}. \tag{24}$$

It remains to compute α and β. To do so, we notice that $\Big(C_V^3(L)\Big)_V$ and $\Big(C_V^3(0)\Big)_V$ are both real value bounded sequences, so, there are two nonnegative real numbers A, B such that, after extraction,

$$\lim_{V\to\infty} C_V^3(L) = B \geq 0, \qquad \lim_{V\to\infty} C_V^3(0) = A \geq 0. \tag{25}$$

For $\phi \in C^1\Big([0,L]\Big)$, we compute

$$\int_0^L \phi(x)\frac{dC_V^3}{dx}(x)dx = C_V^3(L)\phi(L) - C_V^3(0)\phi(0) - \int_0^L C_V^3(x)\frac{d\phi}{dx}(x)dx \underset{V\longrightarrow+\infty}{\longrightarrow} B\phi(L) - A\phi(0),$$

which means

$$\frac{dC_V^3}{dx} \underset{V\longrightarrow+\infty}{\longrightarrow} B\delta_{x=L} - A\delta_{x=0} \qquad \text{in the sense of measures.} \tag{26}$$

We still have to prove $A=0$. To do so we use the system of equations (1) which gives us

$$\frac{dC_V^3}{dx} \geq -\frac{1}{3}(C_V^1 + C_V^2).$$

As we know from Lemma 1 that C_V^i is uniformly bounded from above by K, we also have,

$$\frac{dC_V^3}{dx} \geq -\frac{2}{3}K,$$

which implies $A=0$. □

Fourth step: The limiting equation.

Lemma 4. *In the limit $V \longrightarrow \infty$, we have*

$$C^1(x) + C^2(x) = C_0^1 + C_0^2, \qquad V\frac{C_V^3}{1+C_V^3} - \frac{C_V^1 + C_V^2}{2} \longrightarrow \frac{3}{2}B\delta_L \qquad \textit{in the sense of measures.}$$

Proof. We deduce from (19) and (26), by injecting in the third line of (1) that

$$V\frac{C_V^3}{1+C_V^3}-\frac{C_V^1+C_V^2}{2}-\frac{3}{2}B\delta_L\longrightarrow 0\qquad\text{in the sense of measures.}\tag{27}$$

Reinjecting in the first lines of (1), we find the limit equations on C^1 et C^2

$$\begin{cases}\dfrac{dC^1}{dx}=-\dfrac{1}{2}C^1+\dfrac{1}{2}C^2+\dfrac{1}{2}B\delta_L,\\ \dfrac{dC^2}{dx}=-\dfrac{1}{2}C^2+\dfrac{1}{2}C^1+\dfrac{1}{2}B\delta_L,\\ C^1(0)=C_0^1,\qquad C^2(0)=C_0^2.\end{cases}\tag{28}$$

Then, summing the two lines,

$$\begin{cases}\dfrac{d(C^2+C^1)}{dx}=B\delta_L,\\ (C^1+C^2)(0)=C_0^1+C_0^2.\end{cases}\tag{29}$$

By integrating this differential equation, we deduce [15] that

$$C^1(x)+C^2(x)=C_0^1+C_0^2\qquad\text{a.e.}$$

Indeed, the weak formulation of (29) is

$$\forall\phi\in\mathcal{C}^1[0,L],\qquad\int_0^L\frac{d\phi}{dx}(x)[C^1+C^2](x)dx+\phi(0)[C_0^1+C_0^2]=0.\tag{30}$$

By choosing ϕ such as $\phi(0)=0$, we obtain $C^1+C^2\equiv\alpha$ a.e., for some constant α. and then, by choosing any $\phi\in\mathcal{C}^1[0,L]$, we have that $\alpha=C_0^1+C_0^2$. □

The limit equation on C^i then becomes

$$\begin{cases}\dfrac{dC^i}{dx}(x)=-C^i(x)+\dfrac{C_0^1+C_0^2}{2}+\dfrac{1}{2}B\delta_L(x),\qquad i=1,2,\\ C^i(0)=C_0^i.\end{cases}$$

Fifth step: Explicit solution for the limit Using the variation of parameters, we compute easily C^1 and C^2. We find

$$C^1(x)=\frac{C_0^1+C_0^2}{2}+\frac{C_0^1-C_0^2}{2}e^{-x},\qquad C^2(x)=\frac{C_0^1+C_0^2}{2}+\frac{C_0^2-C_0^1}{2}e^{-x}.\tag{31}$$

In particular, (C^1,C^2,C^3) are $\mathcal{C}^\infty$ functions.

4 Proof of theorem 2

The limiting profiles are $\mathcal{C}^\infty$ in $[0,L]$, nevertheless, the Dirac mass at $x=L$ indicates a boundary layer. The derivatives of the profiles for $V=\infty$ are given by

$$\begin{cases}\dfrac{dC^1}{dx}=\dfrac{1}{2}\Big[(C_0^2-C_0^1)e^{-x}+B\delta_L\Big],\\ \dfrac{dC^2}{dx}=\dfrac{1}{2}\Big[(C_0^1-C_0^2)e^{-x}+B\delta_L\Big],\qquad\text{in the sense of measures,}\\ \dfrac{dC^3}{dx}=B\delta_L.\end{cases}\tag{32}$$

First Step: The limiting values of C at $x=L$

Lemma 5.

$$\begin{aligned}C_V^1(L)&\underset{V\longrightarrow+\infty}{\longrightarrow}C_0^1+C_0^2,\\ C_V^2(L)=C_V^3(L)&\underset{V\longrightarrow+\infty}{\longrightarrow}C_0^1+C_0^2+(C_0^2-C_0^1)e^{-L}.\end{aligned}\tag{33}$$

Proof. We already have defined in Lemma 3

$$B = \lim_{V\to\infty} C_V^2(L) = \lim_{V\to\infty} C_V^3(L).$$

We know that the $C_V^i(L)$ are bounded real numbers, then we define

$$B' = \lim_{V\to\infty} C_V^1(L).$$

Our first task is to determine B. We compute for all $\phi \in \mathcal{C}^1[0,L]$

$$\int_0^L \phi(x)\frac{dC_V^2(x)}{dx}(x)dx = \phi(L)C_V^2(L) - \phi(0)C_V^2(0) - \int_0^L \frac{\phi}{dx}(x)C_V^2(x)dx$$

which converges when $V \longrightarrow +\infty$ toward

$$\begin{aligned}
&B\phi(L) - C_0^2\phi(0) - \int_0^L \frac{\phi}{dx}(x)[\frac{C_0^1+C_0^2}{2} + \frac{C_0^2-C_0^1}{2}e^{-x}]dx \\
&= B\phi(L) - C_0^2\phi(0) - \frac{C_0^1+C_0^2}{2}\phi(L) + \frac{C_0^1+C_0^2}{2}\phi(0) - \frac{C_0^2-C_0^1}{2}\Big[e^{-L}\phi(L) - \phi(0) + \int_0^L e^{-x}\phi(x)\Big] \\
&= B\phi(L) - \frac{C_0^1+C_0^2}{2}\phi(L) + \frac{C_0^2-C_0^1}{2}e^{-L}\phi(L) + \frac{C_0^2-C_0^1}{2}\int_0^L e^{-x}\phi(x)dx. \qquad (34)
\end{aligned}$$

On the other hand, thanks to (32)

$$\int_0^L \phi(x)\frac{dC_V^2(x)}{dx}(x)dx \underset{V\longrightarrow+\infty}{\longrightarrow} \frac{C_0^2-C_0^1}{2}\int_0^L e^{-x}\phi(x)dx + \frac{B}{2}\phi(L). \qquad (35)$$

By equalizing (34) and (35), we find

$$B = C_0^1 + C_0^2 + (C_0^2 - C_0^1)e^{-L}, \qquad (36)$$

which is the unique limit of $C_V^2(L)$ and $C_V^3(L)$. In particular, $B > 0$. Our second task is to obtain B'. We perform the same computation for C_V^1. On the one hand, for all $\phi \in \mathcal{C}^1[0,L]$,

$$\int_0^L \phi(x)\frac{dC_V^1(x)}{dx}(x)dx = \phi(L)C_V^1(L) - \phi(0)C_V^1(0) - \int_0^L \frac{\phi}{dx}(x)C_V^1(x)dx$$

which converges when $V \longrightarrow +\infty$ toward

$$= B'\phi(L) - \frac{C_0^1+C_0^2}{2}\phi(L) + \frac{C_0^1-C_0^2}{2}e^{-L}\phi(L) + \frac{C_0^1-C_0^2}{2}\int_0^L e^{-x}\phi(x)dx, \qquad (37)$$

and on the other hand,

$$\int_0^L \phi(x)\frac{dC_V^1(x)}{dx}(x)dx$$

converges toward

$$\frac{C_0^2-C_0^1}{2}\int_0^L e^{-x}\phi(x)dx + \frac{B}{2}\phi(L). \qquad (38)$$

This gives us

$$B' = C_0^1 + C_0^2, \qquad (39)$$

and ends the proof of Lemma 5. □

We proved in passing that the limits of the subsequences we deal with are only determined by the problem data and do not depend on the subsequence we choose. Thus, the whole sequences converge.

Second step: The boundary layer.

Lemma 6. *For all $x \in [0, L]$ the inequalities hold*

$$C_V^3(x) \le C_V^3(L) exp\Big(-\frac{2}{3} VM(L-x)\Big) + \frac{K}{V}\Big[1 - exp\Big(-\frac{2}{3} VM(L-x)\Big)\Big], \tag{40}$$

$$C_V^3(x) \ge C_V^3(L) exp\Big(-\frac{2}{3} V(L-x)\Big) + \frac{\overline{K}}{V}\Big[1 - exp\Big(-\frac{2}{3} V(L-x)\Big)\Big], \tag{41}$$

which means that $C_V^3(x)$ relaxes exponentially fast with V to zero, away from the boundary layer at $X = L$.

Proof. We can write the third line of (1) as

$$-\frac{dC_V^3(x)}{dx} + \frac{2}{3} V \frac{C_V^3(x)}{1 + C_V^3(x)} = \frac{1}{3}\Big[C^1(x) + C^2(x) - 2C^3(x)\Big]. \tag{42}$$

We multiply this equation by the exponential factor

$$F(x) = \exp\Big(-\frac{2}{3} V \int_L^x \frac{1}{1 + C_V^3(s)} ds\Big), \tag{43}$$

$$-\frac{dC_V^3(x)}{dx} F(x) + \frac{2}{3} V \frac{C_V^3(x)}{1 + C_V^3(x)} F(x) = \frac{1}{3}\Big[C_V^1(x) + C_V^2(x) - 2C_V^3(x)\Big] F(x), \tag{44}$$

and we obtain

$$\frac{d}{dx}\Big[C_V^3(x) F(x)\Big] = -\frac{1}{3}\Big[C_V^1(x) + C_V^2(x) - 2C_V^3(x)\Big] F(x).$$

Integrating this equation between L and x, we find,

$$C_V^3(x)F(x) - C_V^3(L) = -\int_L^x \frac{1}{3}\Big[C_V^1(x) + C_V^2(x) - 2C_V^3(x)\Big] \exp\Big(-\frac{2}{3} V \int_L^u \frac{1}{1 + C_V^3(s)} ds\Big) du.$$

By Lemma 1, we have,

$$C_V^3(x) \exp\Big(-\frac{2}{3} V \int_L^x \frac{1}{1 + C_V^3(s)} ds\Big) \le C_V^3(L) + K' \int_x^L \exp\Big(-\frac{2}{3} V \int_L^u \frac{1}{1 + C_V^3(s)} ds\Big) du.$$

We can now complete our calculation. We estimate

$$\begin{aligned}
C_V^3(x) \le & C_V^3(L) \exp\Big(-\frac{2}{3} V \int_x^L \frac{1}{1 + C_V^3(s)} ds\Big) \\
& + K'\Big[\int_x^L \exp\Big(-\frac{2}{3} V \int_L^u \frac{1}{1 + C_V^3(s)} ds\Big) \exp\Big(-\frac{2}{3} V \int_x^L \frac{1}{1 + C_V^3(s)} ds\Big)\Big] du \\
= & C_V^3(L) \exp\Big(-\frac{2}{3} V \int_x^L \frac{1}{1 + C_V^3(s)} ds\Big) + K' \int_x^L \exp\Big(-\frac{2}{3} V \int_x^u \frac{1}{1 + C_V^3(s)} ds\Big) du \\
\le & C_V^3(L) \exp\Big(-\frac{2}{3} V \int_x^L \frac{1}{1 + C_V^3(s)} ds\Big) + K' \int_x^L \exp\Big(-\frac{2}{3} VM(u-x)\Big) du \qquad \Big(\text{using } 0 < M < \frac{1}{1 + C_V^3}\Big) \\
= & C_V^3(L) \exp\Big(-\frac{2}{3} V \int_x^L \frac{1}{1 + C_V^3(s)} ds\Big) + \frac{K}{V}\Big[1 - \exp\Big(-\frac{2}{3} VM(L-x)\Big)\Big].
\end{aligned}$$

and (40) is proved.

We can prove in the same way the second part of Lemma 6. □

5 Numerical results

The numerical algorithm. Numerical simulations illustrate the solute concentration mechanism at $x = L$ that is proved in the theoretical result. We obtain the numerical solution of (1) by using a dynamic algorithm, as presented in [17], which is based on a finite volume method [1, 10]. This algorithm solves efficiently the dynamic problem associated to (1) [18], and it has been proved [18] that this dynamic

problem relaxes exponentially fast toward the stationary state described by the equations of (1). The CFL condition is given by

$$\Delta t \leq \frac{3\Delta x}{3 + 2\Delta x(1+V)}, \tag{45}$$

which becomes a tough constraint on Δt when we choose V large. The constraint on Δx also depends on V because it is function of the size of the boundary layer. For each V, to be accurate enough around the boundary layer point L, we discretize the space in N_V cells and we validate a posteriori that this number of cells is high enough since we know from the analytical solution the behaviour of the solution for large values of V.

Concentration profiles for different V. We present in figure 2 concentration profiles for $V = 1, V = 10, V = 50$ and $V = 100$.

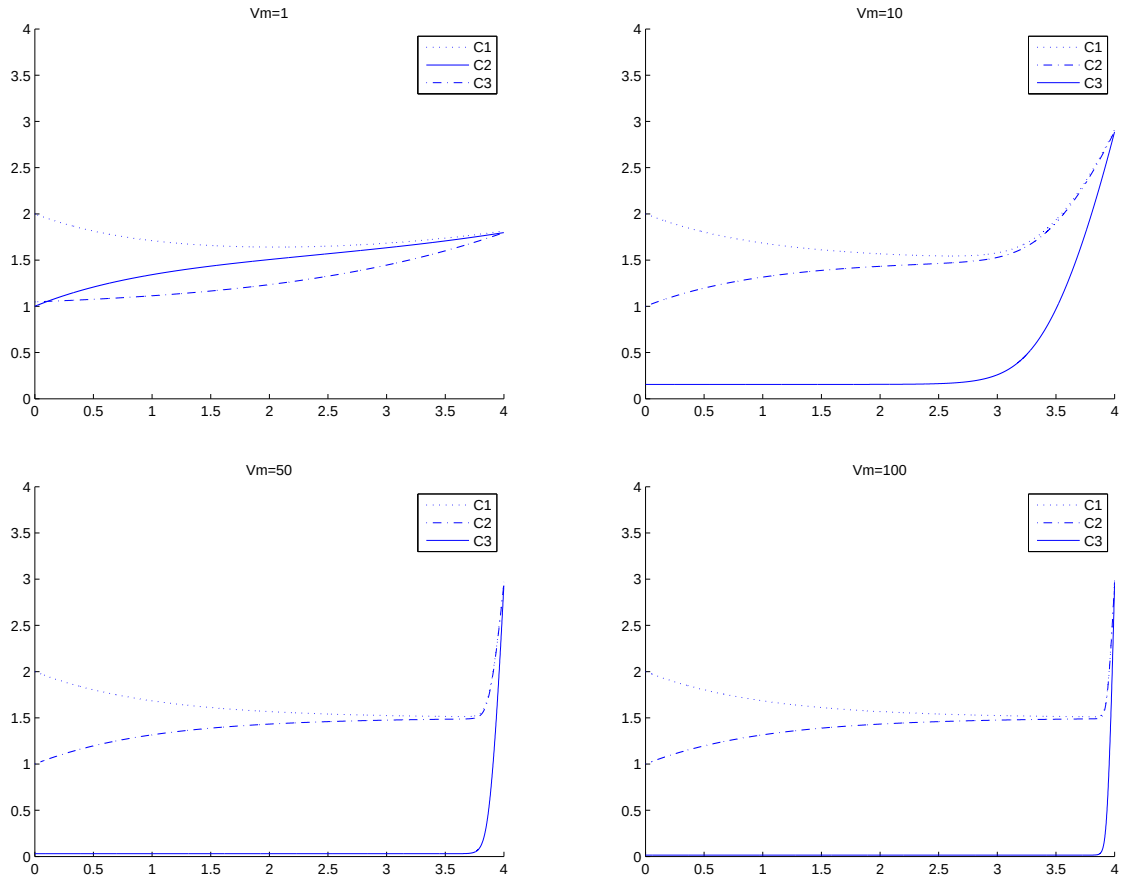

Figure 2: Concentration profiles for $V = 1, 10, 50, 100$, on a domain of length $L = 4$.

When the value of the rate V has the same order of magnitude as the parameters of (1), concentration is hardly increasing in tubes 2 and 3, but decreasing in tube 1. It comes from the fact that we chose $C_0^1 > C_0^2$, but it would have been the contrary in the opposite case. With low values of V, the difusive part of the system (1) is paramount and concentrations tend to homogenize along the tubes. If we increase the pump rate by a factor 10, the concentration tends to approach zero in tube 3 and is abruptly increasing from $L = 3$ and it achieves at $L = 4$ a value greater than $\max(C_0^1, C_0^2)$. We clearly observe the limit profiles and the boundary layer appear for $V \geq 50$.

Illustration of Theorem 2. We want to illustrate that the bounds from above and from below found in Theorem 2 give an accurate descrition of the qualitative behavior of C^3.

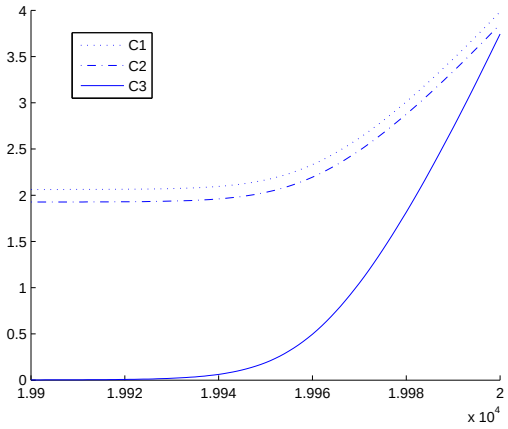

Figure 3: Zoom on the interval $[0.99L, L]$ of figure 2 for $V = 1000$.

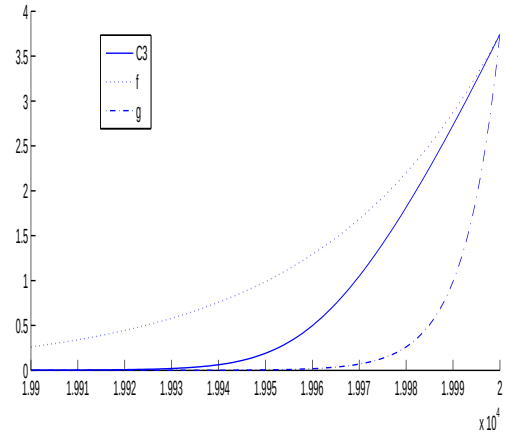

Figure 4: The curve in the middle represents C^3_{1000} on $[0.99L, L]$. The upper curve represents the upperbound f for C^3 found in Lemma 6 and the the lower curve represents g, the bound from below.

Figure 3 displays a zoom of the numerical aproximation of $C^3_{1000}(L)$ that we will denote $C^3_{num,1000}(L)$. This approximative value enables us to define $M_{num} = \dfrac{1}{C^3_{num,1000}(L)}$. To illustrate Lemma 6, we define

$$f(x) = C^3_{num,1000}(L) \exp\Big(- \frac{2 \times 1000}{3} M_{num}(L - x)\Big)$$

and

$$g(x) = C^3_{num,1000}(L) \exp\Big(- \frac{2}{3} \times 1000(L - x)\Big).$$

In Figure 4, we depict C^3_{1000} on the interval $[\dfrac{99L}{100}, L]$ and the profiles of the two functions f and g which control C^3_{1000}. We observe that $g \leq C^3_{num,1000} \leq f$, as expected in theorem 2, and then that the component C^3_V decreases exponentially to zero.

6 Conclusion

Motivated by renal flows, we have studied a concentration mechanism with an active pump characterized by a parameter V. As expected, for V large enough, a large axial solute concentration gradient appears in all tubes. The result of our analysis is that the concentrations are uniformly bounded in V for all $x \in [0, L]$, and so are their derivatives, except at $x = L$. In the limit $V = \infty$, the concentration gradient converges to a Dirac profile at $x = L$. We obtain a limit concentration profile in all tubes which presents a boundary layer at $x = L$. In the urine concentrating model, we are mostly interested on the behaviour

at $x = L$, because it is at this depth that the composition of final urine is determined. Therefore, our analysis explains why ctive transport plays a very specific role, which is to increase solute concentration at $x = L$ and only at $x = L$.

Acknowledgments. *I thank Benoît Perthame for numerous helpful discussions.*

Funding for this study was provided by the program EMERGENCE (EME 0918) of the Université Pierre et Marie Curie (Paris Univ. 6).

References

[1] F. Bouchut. *Non linear stability of finite volume methods for hyperbolic conservation laws and well balanced schemes for sources*. Birkhaüser-Verlag, 2004.

[2] H. Brezis. *Functional Analysis*. Collection of Applied Mathematics for the Master's Degree. Masson, Paris, 1983. Theory and applications.

[3] C. Cercignani. *The Boltzmann equation and its applications*, volume 67 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.

[4] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[5] A. Friedman and G. Craciun. A model of intracellular transport of particles in an axon. *J. Math. Biol.*, 51(2):217–246, 2005.

[6] B. Hargitay and W. Kuhn. The Multiplication Principle as the Basis for Concentrating Urine in the Kidney(with comments by Bart Hargitay and S. Randall Thomas). *J. Am. Soc. Nephrol.*, 12(7):1566–1586, 2001.

[7] François James. Convergence results for some conservation laws with a reflux boundary condition and a relaxation term arising in chemical engineering. *SIAM J. Math. Anal.*, 29(5):1200–1223 (electronic), 1998.

[8] M. A. Katsoulakis and A. E. Tzavaras. Contractive relaxation systems and the scalar multidimensional conservation law. *Comm. Partial Differential Equations*, 22(1-2):195–233, 1997.

[9] J. Keener and J. Sneyd. *Mathematical physiology. Vol. II: Systems physiology*, volume 8/ of *Interdisciplinary Applied Mathematics*. Springer, New York, second edition, 2009.

[10] R. J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge University Press, 2002.

[11] L. C. Moore and D. J. Marsh. How descending limb of henle's loop permeability affects hypertonic urine formation. *Am.J.Physiol*, (239):F57–F71, 1980.

[12] R. Natalini. Convergence to equilibrium for the relaxation approximations of conservation laws. *Comm. Pure Appl. Math.*, 49(8):795–823, 1996.

[13] K. Pakdaman, B. Perthame, and D. Salort. Dynamics of a structured neuron population. *Nonlinearity*, 23(1):55–75, 2010.

[14] B. Perthame. *Transport equations in biology*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.

[15] L. Schwartz. *Théorie des distributions*. Hermann, Paris, 1966.

[16] J. L. Stephenson. *Urinary concentration and dilution: Models*. Oxford University Press, New-York, 1992.

[17] M. Tournus, A. Edwards, and N. Seguin. A finite volume scheme for a kidney nephron model. *Preprint*, 2011.

[18] M. Tournus, A. Edwards, N. Seguin, and B. Perthame. Analysis of a simplified model of the urine concentration mechanism. *Preprint*, 2011.