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# ERGODICITY OF CERTAIN COCYCLES OVER CERTAIN INTERVAL EXCHANGES

DAVID RALSTON AND SERGE TROUBETZKOY

ABSTRACT. We show that for odd-valued piecewise-constant skew products over a certain two parameter family of interval exchanges, the skew product is ergodic for a full-measure choice of parameters.

## 1. INTRODUCTION AND BACKGROUND

$\mathbb{Z}$ -valued (or more generally  $G$ -valued where  $G$  is a locally compact group) skew products are a natural construction of infinite-measure preserving transformations using ergodic sums over a finite-measure preserving transformation. For a thorough overview of constructing skew products over irrational rotations, see [3]. The natural generalization of an irrational rotation is an *interval exchange transformation*; recent work in studying generic skew products over generic interval exchanges may be found in [1]. We present here an alternate ‘hands-on’ approach to prove generic ergodicity for one specific construction.

Let  $X = \mathbb{S}^1 \times \{1, 2, \dots, k\}$ , endowed with Lebesgue measure  $\mu$  (scaled so  $\mu(X) = k$ ), and assume that  $k = 1 \pmod 2$ . Let  $T$  be a map on  $X$  defined by

$$T(s, \ell) = ((s + \alpha) \pmod 1, (\ell + I(s)) \pmod k),$$

where  $I(s)$  is the characteristic function of an interval of length  $\beta$ , and  $\alpha$  is irrational;  $\{X, T\}$  is a  $\mathbb{Z}/k\mathbb{Z}$ -valued skew product (in fact a cyclic extension) of the irrational rotation by  $\alpha$ . Let  $f$  be an odd-valued function on  $X$  (i.e. for all  $(s, \ell)$ ,  $f(s, \ell) = 1 \pmod 2$ ). The skew products we will consider are given by

$$T_f(s, \ell, m) = ((s + \alpha) \pmod 1, (\ell + I(s)) \pmod k, m + f(s, \ell)).$$

Denote by  $S_m(s, \ell)$  the  $\mathbb{Z}$ -coordinate of  $T_f^m(s, \ell, 0)$ :

$$S_m(s, \ell) = \sum_{i=0}^{m-1} f(T^i(s, \ell)).$$

Note that  $\{X \times \mathbb{Z}, T_f\}$  will *not* in general itself be a skew product over rotation by  $\alpha$ , as  $f(s, \ell)$  is not independent of  $\ell$ . We assume that  $f$  is of mean zero, and assume further that  $f$  is piecewise constant on finitely many intervals; let  $\text{Var}(f)$  be the sum over  $\ell$  of the (finite) variations of  $f$  restricted to each  $\mathbb{S}^1 \times \{\ell\}$ . Purely for convenience we furthermore assume that  $I$  and  $f$  are right-continuous; they are defined using intervals closed on the left and open on the right.

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An integer  $E$  is an *essential value* of our skew product if for every  $A \subset X$  of positive measure, there is some  $i$  such that

$$\mu(A \cap T^i A \cap \{(s, \ell) : S_n(s, \ell) = E\}) > 0.$$

If  $E$  is an essential value, the skew product is ergodic if and only if the skew product given by  $f$  into  $\mathbb{Z}/(E\mathbb{Z})$  is ergodic.

We will use *Koksma's inequality*: let  $P$  be a partition of  $\mathbb{S}^1$  into  $q$  intervals of equal length, let  $f$  be real-valued, of bounded variation on  $\mathbb{S}^1$ , and suppose that  $s_1$  through  $s_n$  are chosen such that each interval of  $P$  contains exactly one  $s_m$ . Then

$$\left| \sum_{m=1}^n f(s_m) \right| \leq \text{Var}(f).$$

Our interval exchanges are characterized by two choices:  $\alpha$  and  $\beta$ .

**Theorem 1.1.** *For  $f$  fixed, the set of  $\alpha, \beta$  for which the skew product is ergodic is of full measure.*

By [3], for any fixed  $\beta$  and any fixed  $E$ , the finite extension of  $X$  given by the  $\mathbb{Z}/E\mathbb{Z}$ -valued skew product (instead of  $\mathbb{Z}$ ) is ergodic for almost-all  $\alpha$ , so it will suffice to show that for generic  $\alpha$  and  $\beta$  there is some nonzero, finite essential value.

## 2. PROOF

Let  $\alpha$  be irrational with continued fraction expansion

$$\alpha = [a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

where each  $a_m$  is a positive integer; an excellent reference for the theory of continued fractions is [4]. Denote by  $p_n/q_n$  the convergents to  $\alpha$ , and by  $\|\cdot\|$  the distance to the nearest integer. Then it is well-known that

$$(1) \quad q_n \|q_n \alpha\| \leq \frac{1}{a_{n+1}}.$$

On  $X$  we also use  $\|\cdot\|$  for distance, with the convention that if  $\ell \neq \ell'$ ,  $\|(s, \ell) - (s', \ell')\| = 1$ . We denote by  $Q_n$  the periodic approximation to  $T$  given by

$$Q_n(s, \ell) = \left( s + \frac{p_n}{q_n} \bmod 1, \ell + I(s) \bmod k \right).$$

**Definition 2.1.** A point  $x = (s, \ell) \in X$  will be called *n-good for rational approximation* if for all  $i = 0, 1, \dots, kq_n - 1$  we have

$$f(T^i x) = f(Q_n^i(x)), \quad I(T^i x) = I(Q_n^i x).$$

That is, as far as the functions  $f$  and  $I$  are concerned, through time  $kq_n$  we may replace the orbit of  $x$  under  $T$  with the orbit of  $x$  under  $Q_n$ .

**Definition 2.2.** A point  $x \in X$  will be called *n-spread out* if the set  $\{T^i(x)\}$ ,  $i = 0, 1, \dots, kq_n - 1$ , has the property that

- there are exactly  $q_n$  points in each  $\mathbb{S}^1 \times \{\ell\}$ , and

- for each  $\ell$ , there is a partition of  $\mathbb{S}^1 \times \{\ell\}$  into disjoint intervals of length  $1/q_n$  such that there is exactly one of the  $T^i x$  in each partition element.

**Lemma 2.3.** *Suppose that  $x$  is  $n$ -spread out. Then*

$$\left| \sum_{i=0}^{kq_n-1} f(T^i x) \right| \leq \text{Var}(f).$$

*Proof.* The restriction of the orbit of  $x$  to each  $\mathbb{S}^1 \times \{\ell\}$  may be summed separately, and the  $n$ -spread out assumption allows us to use Koksma's inequality on each  $\mathbb{S}^1 \times \{\ell\}$ .  $\square$

Let  $D = \{d_1, \dots, d_N\}$  be the projection of all discontinuities of  $f$  onto  $\mathbb{S}^1$  together with the discontinuities of  $I(x)$ . For  $n = 0 \pmod 2$  define

$$A_n = \left( \mathbb{S}^1 \setminus \left( \bigcup_{i=0}^{kq_n-1} \bigcup_{j=1}^N [d_j - k\|q_n\alpha\| - i\alpha, d_j - i\alpha] \right) \right) \times \{1, 2, \dots, k\},$$

while for  $n = 1 \pmod 2$  we use the intervals

$$[d_j - i\alpha, d_j + k\|q_n\alpha\| - i\alpha].$$

**Lemma 2.4.** *Each  $x \in A_n$  is  $n$ -good for rational approximation, and*

$$\mu(A_n) \geq k \left( 1 - k^2 N q_n \|q_n\alpha\| \right) \geq k \left( 1 - \frac{k^2 N}{a_{n+1}} \right).$$

*Proof.* The first inequality is elementary (assume all removed intervals are disjoint), and the final inequality is simply due to (1); the only content to prove is that  $x \in A_n$  implies that  $x$  is  $n$ -good for rational approximation. Suppose that  $n = 0 \pmod 2$  so that  $p_n/q_n > \alpha$ . Let  $x = (s, \ell) \in A_n$ ; there is no  $i < kq_n$  such that

$$s + i\alpha \in [d_j - k\|q_n\alpha\|, d_j].$$

The distance between  $s + i\alpha$  and  $s + ip_n/q_n$  is no larger than  $k\|q_n\alpha\|$ , so we cannot have

$$s + i\alpha < d_j \leq s + i \frac{p_n}{q_n}$$

for any  $i, j$ . As  $p_n/q_n > \alpha$ , this completes the proof for  $n = 0 \pmod 2$ . For  $n = 1 \pmod 2$  the process is identical, but we remove intervals from the other side of the discontinuities  $d_j$ , and  $p_n/q_n < \alpha$ .  $\square$

**Definition 2.5.** The action of  $T^{kq_n}$  on  $A$  is *nearly-rigid* if  $\|x - T^{kq_n}(x)\| \leq k\|q_n\alpha\|$  for all  $x \in A$ .

**Lemma 2.6.** *The action of  $T^{kq_n}$  on  $A_n$  is nearly-rigid.*

*Proof.* Through time  $q_n$  the point  $x = (s, \ell)$  orbits into the interval defining  $I(s)$  some number of times. Under  $Q_n$ , however,  $x$  has returned exactly to the same  $\mathbb{S}^1$  coordinate. Over the next  $q_n$  times, the orbit of  $x$  will therefore intersect this interval *the same number of times* (recall that  $I(s, \ell)$  is independent of  $\ell$ ), and so on for each  $q_n$  steps in the orbit. Whatever this number of intersections is, once we have applied  $Q_n$  a total of  $kq_n$  times, the total number of points in these intervals must be zero modulo  $k$ :  $Q_n^{kq_n}(x) = x$ . As  $x \in A_n$ , we certainly have  $T^{kq_n}(x)$  belonging to the same copy of  $\mathbb{S}^1$  as  $x$ , then, and the distance in  $\mathbb{S}^1$  between  $x$  and  $T^{kq_n}(x)$  is equal to  $\|kq_n\alpha\|$ , which is no larger than  $k\|q_n\alpha\|$ .  $\square$

**Definition 2.7.** The set  $A$  is *nearly invariant* under  $T$  if

$$\mu(A \Delta T(A)) \leq 2k^2 N \|q_n \alpha\|.$$

**Lemma 2.8.** *The set  $A_n$  is nearly invariant under  $T$ .*

*Proof.* Recall that  $A_n$  is constructed by removing successive preimages of  $kN$  different intervals of length  $k\|q_n \alpha\|$  ( $N$  such intervals in each copy of  $\mathbb{S}^1$ ). Therefore  $A_n \Delta T(A_n)$  at most consists of the first image of these intervals and the next preimage.  $\square$

For  $x = (s, \ell)$  define

$$\sigma_n(x) = \sum_{i=0}^{q_n-1} I\left(s + \frac{i}{q_n} \bmod 1\right).$$

If  $x = (s, \ell) \in A_n$ , then

$$\sigma_n(x) = \sum_{i=0}^{q_n-1} I(T^i x).$$

**Lemma 2.9.** *If  $x \in A_n$ ,  $a_{n+1} \geq k$ , and  $\sigma_n(x)$  is relatively prime to  $k$ , then  $x$  is  $n$ -spread out.*

*Proof.* Note that  $\sigma_n(x)$  is exactly the number of times through time  $q_n$  that  $I(Q_n^i x) = 1$ . By the assumption that  $x \in A_n$ , this is also the number of times that  $T^i x$  will orbit into this interval, and furthermore this number will be repeated for each successive length- $q_n$  segment of the orbit we consider:

$$x \in A_n \implies \sigma_n(x) = \sigma_n(T^{q_n} x) = \dots = \sigma_n(T^{(k-1)q_n} x).$$

As  $\sigma_n(x)$  was assumed to be relatively prime to  $k$  (i.e.  $\sigma_n(x)$  generates  $\mathbb{Z}/k\mathbb{Z}$ ), it follows that for each  $i = 0, 1, \dots, q_n - 1$ , each of

$$\{T^{i+\ell q_n}(x)\} \quad (\ell = 0, 1, \dots, k-1)$$

belongs to a *different* copy of  $\mathbb{S}^1$ . Finally, the assumption that  $a_{n+1} \geq k$  implies (again via (1)) that

$$k\|q_n \alpha\| < \frac{1}{q_n},$$

so the intervals  $[s+i/q_n, s+(i+1)/q_n]$  in each circle (if  $n = 0 \bmod 2$ ; for  $n = 1 \bmod 2$  reverse which end is closed versus open) each contain one element of the orbit.  $\square$

**Lemma 2.10.** *For all  $x$ ,  $\sigma_n(x) \in \{M, M+1\}$ , where  $M = [q_n \beta]$ , the integer part of  $q_n \beta$ .*

*Proof.* The number  $M$  is the minimum number of abutting intervals of length  $1/q_n$  (closed on the left, open on the right, say) which will always be completely contained within an interval of length  $\beta$ :

$$\frac{M}{q_n} \leq \beta < \frac{M+1}{q_n}.$$

For any  $x = (s, \ell)$ , then, there are at least  $M$  successive  $I(s+i/q_n) = 1$ . On the other hand, as  $(M+1)/q_n > \beta$ , no  $x$  may have  $\sigma_n(x) \geq M+2$ .  $\square$

**Definition 2.11.** If  $T^{kq_n}$  is nearly rigid and there is some  $\epsilon > 0$  such that  $\mu(A_n) \geq \epsilon$  then  $T$  is called *quasi-rigid* and the  $A_n$  are called *quasi-rigidity sets*.

**Corollary 2.12.** *Suppose that for infinitely many  $n$  we have*

- $a_{n+1} > k^2N$ ,
- $q_n = 1 \pmod{2}$ ,
- $\sigma_n(x)$  is relatively prime to  $k$  for all  $x \in X$ .

*Then there is a finite nonzero essential value.*

*Proof.* The assumption that  $a_{n+1} > k^2N$  implies that the  $A_n$  are quasi-rigidity sets (via Lemmas 2.4 and 2.6). That  $\sigma_n(x)$  is relatively prime to  $k$  ensures that for each  $x \in A_n$ ,  $x$  is  $n$ -spread out, so by applying the Koksma inequality there is a uniform bound on the absolute value of the ergodic sums on  $A_n$ . We therefore apply [2, Corollary 2.6] (utilizing that the  $A_n$  are quasi-rigid and nearly invariant, which we have already established) to find an essential value (possibly zero) for the skew product; in short, as there is an upper bound on the sums from Koksma's inequality, we may pass to a sequence of subsets along which a single value is seen, and this value is therefore an essential value. As  $kq_n$  is odd and  $f$  takes only odd values, it follows that for all  $x \in A_n$  (and all  $m \in \mathbb{Z}$ ) we must have

$$\left| \sum_{i=0}^{kq_n-1} f(T_f^i(x, m)) \right| \geq 1,$$

so therefore the essential value we have found in this manner is not zero.  $\square$

It is therefore of interest to determine when  $\sigma_n(x)$  is relatively prime to  $k$ .

**Proposition 2.13.** *For almost every choice of  $\alpha, \beta$ , there are infinitely many  $n$  such that  $a_{n+1} > k^2N$ ,  $q_n = 1 \pmod{2}$ , and  $[q_n\beta] = 1 \pmod{k}$ .*

*Proof.* For generic  $\alpha$  there are infinitely many pairs  $a_{n+1}, a_{n+2}$  of arbitrarily large partial quotients, and no two consecutive  $q_n, q_{n+1}$  may be even, so the first two conditions are trivially satisfied. By passing to a subsequence, we may assume further that the  $\{q_{n_m}\}$  are *superlacunary*:

$$\lim_{i \rightarrow \infty} \frac{q_{n_{i+1}}}{q_{n_i}} = \infty.$$

For any superlacunary sequence  $\{t_m\}$ , for almost every  $\beta$  arbitrary residue classes of  $[t_m\beta]$  modulo any fixed  $k$  are achieved infinitely many times.  $\square$

This completes the proof of ergodicity: for generic choice of  $\alpha, \beta$  the skew product will have a nonzero essential value  $E$  by Corollary 2.12. As remarked at the end of the introduction, this information suffices for generic ergodicity.

## REFERENCES

- [1] J. Chaika and P. Hubert. under preparation. preprint.
- [2] J.-P. Conze and K. Fraczek. Cocycles over interval exchange transformations and multivalued Hamiltonian flows. *ArXiv e-prints*, March 2010.
- [3] Jean-Pierre Conze. Recurrence, ergodicity and invariant measures for cocycles over a rotation. *Contemporary Mathematics*, 485:45–70, 2009.
- [4] A. Ya. Khinchin. *Continued fractions*. Dover Publications Inc., Mineola, NY, Russian edition, 1997. With a preface by B. V. Gnedenko, Reprint of the 1964 translation.

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