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# Time optimal boundary controls for the heat equation

Sorin Micu  
Universitatea din Craiova  
Craiova, 200585, Romania  
sd\_micu@yahoo.com

Ionel Roventa  
Universitatea din Craiova  
Craiova, 200585, Romania  
roventaionel@yahoo.com

Marius Tucsnak  
Institut Elie Cartan  
Université Nancy I/CNRS/INRIA  
BP 70239, 54506 Vandoeuvre-lès-Nancy, France  
Marius.Tucsnak@iecn.u-nancy.fr

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## Abstract

The fact that the time optimal controls for parabolic equations have the bang-bang property has been recently proved for controls distributed inside the considered domain. The aim of this article consists in showing that the boundary controls for the heat equation have the same property, at least in rectangular domains. The main result is proved by combining results and methods from traditionally distinct fields: the Lebeau-Robbiano strategy for null controllability and estimates of the controllability cost in small time for parabolic systems, on one side, and a Remez-type inequality for Müntz spaces and a generalization of Turán's inequality, on the other side.

## 1 Introduction and main result

Let  $m \in \mathbb{N}$ , let  $\Omega \subset \mathbb{R}^m$  be an open and bounded set and let  $\Gamma$  be a non-empty open subset of  $\partial\Omega$ . We consider the heat equation

$$\frac{\partial z}{\partial t}(x, t) = \Delta z(x, t) \text{ for } (x, t) \in \Omega \times (0, \infty), \quad (1.1)$$

with the initial and boundary conditions

$$z(x, t) = u(x, t) \quad \text{on } \Gamma \times (0, \infty), \quad (1.2)$$

$$z(x, t) = 0 \quad \text{on } (\partial\Omega \setminus \Gamma) \times (0, \infty), \quad (1.3)$$

$$z(x, 0) = z_0(x) \text{ for } x \in \Omega. \quad (1.4)$$

It is known (see, for instance, Tucsnak and Weiss [17, Section 10.7]) that if  $\partial\Omega$  is of class  $C^2$  or  $\Omega$  is a rectangular domain then, for every  $u \in L^2([0, \infty), L^2(\Gamma))$  and  $z_0 \in H^{-1}(\Omega)$ , there exists a unique solution  $z \in C([0, \infty), H^{-1}(\Omega))$  of (1.1)-(1.4). It is also known that the system defined by (1.1)-(1.4) is null controllable in any time  $\tau > 0$ , in the sense that for

every  $z_0 \in H^{-1}(\Omega)$  there exists an input  $u \in L^2([0, \tau], L^2(\Gamma))$  such that the corresponding solution of (1.1)-(1.4) verifies

$$z(\cdot, \tau) = 0. \quad (1.5)$$

Our aim consists in studying the associated time optimal control problems. Since these problems are generally considered for inputs lying in  $L^\infty$  type spaces, it is natural to restrict the inputs to sets which are bounded in  $L^\infty$ . More precisely, given  $M > 0$ , we set

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Gamma \times [0, \infty)) \mid |u(x, t)| \leq M \text{ a. e. in } \Gamma \times [0, \infty)\}. \quad (1.6)$$

Given  $z_0 \in H^{-1}(\Omega)$ , we define the set of reachable states from  $z_0$  as

$$\mathcal{R}(z_0, \mathcal{U}_{ad}) = \{z(\tau) \mid \tau > 0 \text{ and } z \text{ is the solution of (1.1)-(1.4) with } u \in \mathcal{U}_{ad}\}.$$

For  $z_0 \in H^{-1}(\Omega)$  and  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ , the time optimal control problem for system (1.1)-(1.4) consists in determining an input  $u^* \in \mathcal{U}_{ad}$  such that the corresponding solution  $z^*$  of (1.1)-(1.4)

$$z^*(\tau^*(z_0, z_1)) = z_1, \quad (1.7)$$

where  $\tau^*(z_0, z_1)$  is the minimal time needed to steer the initial data  $z_0$  towards target  $z_1$  with controls in  $\mathcal{U}_{ad}$ ,

$$\tau^*(z_0, z_1) = \min_{u \in \mathcal{U}_{ad}} \{\tau \mid z(\cdot, \tau) = z_1\}. \quad (1.8)$$

General conditions ensuring the existence of at least one solution for the time optimal control problem will be recalled in Section 2. The main result in this work asserts that, if  $\Omega$  is a rectangular domain, then this solution is bang-bang and it is unique. More precisely, we have:

**Theorem 1.1.** *Suppose that  $\Omega$  is a rectangular domain in  $\mathbb{R}^m$  and that  $\Gamma$  is a nonempty open set of  $\partial\Omega$ . Then, for every  $z_0 \in H^{-1}(\Omega)$  and  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ , there exists a unique solution  $u^*$  of the time optimal control problem (1.8). This solution  $u^*$  has the bang-bang property:*

$$|u^*(x, t)| = M \quad \text{a.e. in } \Gamma \times [0, \tau^*(z_0, z_1)]. \quad (1.9)$$

Time optimal control problems for linear parabolic partial differential equations and the bang-bang property of the corresponding controls have been intensively studied during the last decades, beginning with Fattorini's paper [4]. The progress made in this field has been successively reported in the books of Lions [11] and of Fattorini [5]. The bang-bang property of time optimal controls has been quite rapidly established for invertible input operators (which means, roughly speaking, that the control is active in the entire spatial domain where the parabolic equation is considered).

Several important extensions of the classical results of Fattorini have been obtained during the last decades. We first recall those corresponding to the case of an input operator which is active only in a proper subset of the domain where the heat equation holds. Firstly, in Wang [19], the set of admissible inputs is defined (unlike in (1.6)) by bounding the  $L^\infty([0, \tau]; L^2(\Omega))$  norm of  $u$ . A strategy which has been introduced by Lebeau and Robbiano in [9] is adapted to establish a bang-bang property of the time optimal controls. This property is different from the one in Theorem 1.1, in the sense that, instead of (1.9), it is shown that  $\|u^*(\cdot, t)\|_{L^2(\Omega)} = 1$  for almost every  $t \in [0, \tau^*]$ . The strategy in [19] does

not seem directly applicable to the boundary control case. The results in [19] have been recently extended by Phung and Wang [14] to a system governed by a perturbed heat equation. In the case in which the target is an open ball in the state space instead of a point, the corresponding time optimal control problem, with control distributed inside the domain and pointwise control constraints, has been studied in Kunisch and Wang [8]. The main tools of their approach are Pontryagin's maximum principle and a special kind of property concerning the measure of the set where a nontrivial solution of the linear heat equation vanishes.

In the case of boundary control, with the control constraint  $|u(x, t)| \leq M$ , the first result establishing the bang-bang property has been obtained by Schmidt [15], under a *slackness* condition on the target state. More precisely, the assumption in [15] is that there exists  $M' < M$  such that the target is actually reachable (in some time) subject to  $|u| \leq M'$ . In the case of the heat equation in one space dimension, this condition has been removed by Mizel and Seidman in [12], by using in an essential manner previous results of Borwein and Erdelyi [2].

The main novelty of Theorem 1.1 consists in showing that, in the case of rectangular domains in several space dimensions, the bang-bang property holds for the time optimal boundary control for the heat equation. Like in [12], the only requirement for the targets is to be reachable in some time.

The outline of the paper is as follows. In Section 2 we present some background on null controllability and time optimal controls for infinite dimensional systems. Most of the included material is well-known, although not necessarily in the  $L^\infty$ -setting presented here. Proposition 2.7 gives a general sufficient condition for the existence, uniqueness and bang-bang property of time optimal controls. In Section 3 the Lebeau-Robbiano strategy (see, for instance, [9, 10]) to study the null controllability of the heat equation is adapted to prove the  $L^\infty$  null controllability over a positive measure set property. The main novelty we bring in into this section is that we replace the Lebeau-Robbiano assumption on the observability of finite combinations of eigenvectors with an assumption of observability of the truncation of the dynamical system to a finite number of modes (see inequality (3.8) in Theorem 3.2). The latter property involves the time variable and it is, in general, weaker than the observability of finite combinations of eigenvectors. The last two sections of the paper are devoted to prove that, in our case, this observability property holds. In Section 4, following an idea from Nazarov [13], an estimate for finite combination of eigenfunctions of the Laplace operator in a measurable  $m$ -dimensional set is deduced. Finally, in Section 5, by combining a result from [2, 3] for real exponential functions defined on a measurable set with the one obtained in the previous section, we conclude the proof of Theorem 1.1.

## 2 Some background on null controllability and time optimal controls for infinite dimensional systems

We first introduce some notation. If  $P \in \mathcal{L}(X; Y)$  then the *null-space* and the *range* of  $P$  are the subspaces of  $X$  and  $Y$  respectively defined by

$$\text{Ker } P = \{x \in X : Px = 0\}, \quad \text{Ran } P = \{Px : x \in X\}.$$

Throughout this section,  $X$  and  $U$  are complex Hilbert spaces, identified with their duals. The inner product and the norm in  $X$  are denoted by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ , respectively.

We denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  a strongly continuous semigroup on  $X$  generated by an operator  $A : \mathcal{D}(A) \rightarrow X$  with resolvent set  $\varrho(A)$ . The notation  $X_1$  stands for  $\mathcal{D}(A)$  equipped with the norm  $\|z\|_1 := \|(\beta I - A)z\|$ , where  $\beta \in \varrho(A)$  is fixed, while  $X_{-1}$  is the completion of  $X$  with respect to the norm  $\|z\|_{-1} := \|(\beta I - A)^{-1}z\|$ . We use the notation  $A$  and  $\mathbb{T}$  also for the extensions of the original generator to  $X$  and of the original semigroup to  $X_{-1}$ . Recall that  $X_{-1}$  is the dual of  $\mathcal{D}(A^*)$  with respect to the pivot space  $X$ .

Now, let  $B \in \mathcal{L}(U; X_{-1})$ ,  $z_0 \in X$  and  $u \in L^2([0, \infty), U)$ . We consider the infinite dimensional system described by the equation

$$\dot{z}(t) = Az(t) + Bu(t) \quad (t \geq 0), \quad z(0) = z_0. \quad (2.10)$$

With the above notation, the solution  $z$  of (2.10) is defined by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \quad (t \geq 0). \quad (2.11)$$

where  $\Phi_t \in \mathcal{L}(L^2([0, t], U); X_{-1})$  is given by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma. \quad (2.12)$$

Recall the following classical definition (see, for instance, [17, Section 4.2]):

**Definition 2.1.** With the above notation, the operator  $B \in \mathcal{L}(U; X_{-1})$  is called an *admissible control operator* for  $\mathbb{T}$  if the operator  $\Phi_\tau$  defined by (2.12) satisfies  $\text{Ran } \Phi_\tau \subset X$  for some  $\tau > 0$ .

**Remark 2.2.** System (1.1)-(1.4) may be written in the form (2.10). Indeed, let  $X = H^{-1}(\Omega)$ ,  $X_{-1} = (H^2(\Omega) \cap H_0^1(\Omega))'$  (the dual of  $H^2(\Omega) \cap H_0^1(\Omega)$  with respect to the pivot space  $L^2(\Omega)$ ),  $U = L^2(\Gamma)$ ,  $D(A) = H_0^1(\Omega)$  and  $A = \Delta$ . Also, let the control operator  $B \in \mathcal{L}(L^2(\Gamma), X_{-1})$  be defined by  $B = AD$ . In the formula of  $B$ ,  $D : L^2(\partial\Omega) \rightarrow L^2(\Omega)$  is the “Dirichlet map” defined by  $Dv = z$ , where  $z \in L^2(\Omega)$  is the unique solution of the nonhomogeneous elliptic equation

$$\begin{cases} \Delta z = 0 & \text{in } \Omega \\ z = v & \text{on } \partial\Omega, \end{cases} \quad (2.13)$$

and  $A$  still denotes the “extrapolation” of  $A$  to an operator from  $X$  to  $X_{-1}$ . With the above notation, (1.1)-(1.4) is equivalent to (2.10) and  $B$  is an admissible control operator for the semigroup  $\mathbb{T}$  generated by  $A$ .

Moreover, the operator  $B^* \in \mathcal{L}(X_1, L^2(\partial\Omega))$  is given by

$$B^* \varphi = - \frac{\partial(-A)^{-1} \varphi}{\partial \nu} \Big|_{\Gamma} \quad (\varphi \in L^2(\Omega)). \quad (2.14)$$

We refer the interested reader to [17, Sections 10.6-10.7] for a detailed derivation of the above setting.

The null controllability of the pair  $(A, B)$  in some time  $\tau > 0$  is usually defined by the property  $\text{Ran } \Phi_\tau \supset \text{Ran } \mathbb{T}_\tau$ . In this work we will mainly use a different concept of null controllability, which makes sense in the case  $U = L^2(\Gamma)$ , where  $\Gamma$  is a measurable set endowed with a measure  $\mu$ .

**Definition 2.3.** Given  $\tau > 0$ ,  $e \subset \Gamma \times [0, \tau]$  a set of positive measure and  $B \in \mathcal{L}(U; X_{-1})$ , an admissible control operator for  $\mathbb{T}$ , consider the operator  $\Phi_{\tau,e} \in \mathcal{L}(L^\infty(\Gamma \times [0, \tau]); X_{-1})$  defined by

$$\Phi_{\tau,e}u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B \chi_e(\sigma) u(\sigma) d\sigma, \quad (2.15)$$

where  $\chi_e$  is the characteristic function of  $e$ . The pair  $(A, B)$  is said  *$L^\infty$  null controllable in time  $\tau$  over  $e$*  if  $\text{Ran } \Phi_{\tau,e} \supset \text{Ran } \mathbb{T}_\tau$ . The above property for  $e = \Gamma \times [0, \tau]$  is simply called  *$L^\infty$  null controllability in time  $\tau$* . Given  $z_0 \in X_{-1}$ , a function  $u \in L^\infty(\Gamma \times [0, \tau])$  such that  $\Phi_{\tau,e}u = \mathbb{T}_\tau z_0$  is called  *$L^\infty$  null control for  $z_0$* .

If the pair  $(A, B)$  is  *$L^\infty$  null controllable in time  $\tau$  over  $e$*  then, for every  $z_0 \in X$ , the set

$$\mathcal{C}_{\tau,e,z_0} := \{u \in L^\infty(\Gamma \times [0, \tau]) \mid \Phi_{\tau,e}u + \mathbb{T}_\tau z_0 = 0\}$$

is non empty. The quantity

$$C_{\tau,e} := \sup_{\|z_0\|=1} \inf_{u \in \mathcal{C}_{\tau,e,z_0}} \|u\|_{L^\infty(e)} \quad (2.16)$$

is then called the *control cost in time  $\tau$  over  $e$* . If  $e = \Gamma \times [0, \tau]$  the control cost will be simply denoted by  $C_\tau$ .

**Remark 2.4.** Let  $\tau > 0$  and let  $e \subset \Gamma \times [0, \tau]$  of positive measure. Then, for every  $\tilde{e} \subset e$  with  $\tilde{e}$  of positive measure such that  $(A, B)$  is  *$L^\infty$  null controllable in time  $\tau$  over  $\tilde{e}$* , we have  $C_{\tau,\tilde{e}} \geq C_{\tau,e}$ . Indeed this obviously follows from the fact that every  *$L^\infty$  null control with support in  $\tilde{e}$*  can be extended by zero in  $e \setminus \tilde{e}$  to give a  *$L^\infty$  null control supported in  $e$* .

Let  $C \in \mathcal{L}(X_1, U)$  be an admissible observation operator for  $\mathbb{T}$ . The admissibility assumption means that for some  $\tau > 0$ , the operator  $\Psi_\tau$  defined by

$$(\Psi_\tau z_0)(t) = C \mathbb{T}_t z_0 \quad (z_0 \in X_1),$$

has an extension to an operator  $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau], U))$ . Equivalently, there is a positive number  $k$  such that  $\int_0^\tau \|C \mathbb{T}_t z_0\|^2 dt \leq k^2 \|z_0\|^2$  for all  $z_0 \in \mathcal{D}(A)$ . We refer to [17, 20, 21] for more material on this concept. Here we only mention that it follows from the admissibility assumption that  $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; U))$  holds for all  $\tau \geq 0$ . The operators  $\Psi_\tau$  are called output maps corresponding to the pair  $(A, C)$ . Denote by  $\Psi_\tau^d$  the output maps corresponding to the pair  $(A^*, B^*)$ . If  $e \subset \Gamma \times [0, \tau]$  is a set of positive measure, we consider the map

$$\Psi_{\tau,e}^d \in \mathcal{L}(X, L^1(\Gamma \times [0, \tau])), \quad \Psi_{\tau,e}^d = \chi_e \Psi_\tau^d.$$

We have the following duality result

**Proposition 2.5.** Suppose that  $B \in \mathcal{L}(U, X_{-1})$ . Then  $B$  is an admissible control operator for  $\mathbb{T}$  if and only if  $B^*$  is an admissible observation operator for the adjoint semigroup  $\mathbb{T}^*$ . If  $B$  is admissible, then

$$\Phi_{\tau,e} = (\Psi_{\tau,e'}^d)^* \mathfrak{H}_\tau, \quad (2.17)$$

where  $e' = \{(x, \tau - t) \mid (x, t) \in e\}$ ,  $(\Psi_{\tau,e'}^d)^* \in \mathcal{L}(L^\infty([0, \tau]; U), X)$  is the dual operator of  $\Psi_{\tau,e'}^d$  and  $\mathfrak{H}_\tau$  is the reflection operator on  $L^2([0, \tau]; U)$ , defined by  $\mathfrak{H}_\tau u(t) = u(\tau - t)$  (Notice that  $\mathfrak{H}_\tau$  is self-adjoint and also unitary.).

*Proof.* The first assertion in the statement of the proposition is well-known (see, for instance, [17, Section 4.4]). To check the second one we first note that for every  $v \in L^\infty(\Gamma \times [0, \tau])$  and  $\varphi \in X_1$  we have

$$\int_0^\tau \int_\Gamma v \overline{\Psi_{\tau, e'}^d \varphi} dx dt = \int_0^\tau \langle \chi_{e'}(\cdot, t) v(\cdot, t), B^* \mathbb{T}_t^* \varphi \rangle_U dt = \int_0^\tau \langle \mathbb{T}_t B \chi_{e'}(\cdot, t) v(\cdot, t), \varphi \rangle_{X_{-1}, X_1} dt.$$

By making the change of variable  $t = \tau - \sigma$  in the above integral and using the fact that  $B$  is admissible for  $\mathbb{T}$ , we obtain

$$\int_0^\tau \int_\Gamma v \overline{\Psi_{\tau, e'}^d \varphi} dx dt = \langle \Phi_{\tau, e} \mathbf{R}_\tau v, \varphi \rangle_{X_{-1}, X_1} = \langle \Phi_{\tau, e} \mathbf{R}_\tau v, \varphi \rangle \quad (v \in L^\infty(\Gamma \times [0, \tau]), \varphi \in X_1).$$

The above formula implies the conclusion by simply using the density of  $X_1$  in  $X$ .  $\square$

The following result shows the equivalence between the concepts of controllability and observability. Although this is a rather known property, since our framework escapes from the usual hilbertian setting, we have chosen to include it here.

**Proposition 2.6.** *Let  $e \subset \Gamma \times [0, \tau]$  a set of positive measure and  $K_{\tau, e} > 0$ . The following two properties are equivalent*

1. *The inequality*

$$K_{\tau, e} \left\| \Psi_{\tau, e'}^d \varphi \right\|_{L^1(\Gamma \times [0, \tau])} \geq \| \mathbb{T}_\tau^* \varphi \| \quad (2.18)$$

*holds for any  $\varphi \in X$ , where  $e' = \{(x, \tau - t) \mid (x, t) \in e\}$ .*

2. *The pair  $(A, B)$  is  $L^\infty$  null controllable in time  $\tau$  over  $e$  at cost less than  $K_{\tau, e}$ .*

*Proof.* “1.  $\Rightarrow$  2.” Consider the subspace  $\mathcal{X}$  of  $L^1(\Gamma \times [0, \tau])$  defined by

$$\mathcal{X} = \{ \Psi_{\tau, e'}^d \varphi \mid \varphi \in X \}.$$

Given  $z_0 \in X$ , consider the linear functional  $\mathcal{F}$  on  $\mathcal{X}$  defined by

$$\mathcal{F}(\Psi_{\tau, e'}^d \varphi) = -\langle z_0, \mathbb{T}_\tau^* \varphi \rangle \quad (\varphi \in X).$$

The fact that this functional is well defined follows from (2.18). Moreover, using again (2.18), it follows that

$$|\mathcal{F}v| \leq K_{\tau, e} \|z_0\| \|v\|_{L^1(\Gamma \times [0, \tau])} \quad (v \in \mathcal{X}).$$

By the Hahn-Banach Theorem  $\mathcal{F}$  can be extended to a bounded linear functional  $\tilde{\mathcal{F}}$  on  $L^1(\Gamma \times [0, \tau])$  such that

$$|\tilde{\mathcal{F}}v| \leq K_{\tau, e} \|z_0\| \|v\|_{L^1(\Gamma \times [0, \tau])} \quad (v \in L^1(\Gamma \times [0, \tau])).$$

By the Riesz representation theorem it follows that there exists  $w \in L^\infty(\Gamma \times [0, \tau])$  such that  $\|w\|_{L^\infty(\Gamma \times [0, \tau])} \leq K_{\tau, e} \|z_0\|$  and

$$\int_0^\tau \int_\Gamma w(\tau - \sigma, x) \overline{\Psi_{\tau, e'}^d \varphi} + \langle z_0, \mathbb{T}_\tau^* \varphi \rangle = 0 \quad (\varphi \in X).$$

By using (2.17) in the above formula, it follows that

$$\langle \Phi_{\tau,e} w, \varphi \rangle + \langle \mathbb{T}_\tau z_0, \varphi \rangle = 0 \quad (\varphi \in X).$$

which is equivalent to

$$\Phi_{\tau,e} u + \mathbb{T}_\tau z_0 = 0.$$

Since the above construction holds for every  $z_0 \in X$ , we get the desired result.

“2.  $\Leftarrow$  1.” Let  $\varphi \in X$  and  $z_0 = \mathbb{T}_\tau^* \varphi \in X$ . We have that there exists  $u \in L^\infty(\Gamma \times [0, \tau])$  such that  $\|u\|_{L^\infty(\Gamma \times [0, \tau])} \leq K_{\tau,e} \|z_0\|$  and  $\Phi_{\tau,e} u + \mathbb{T}_\tau z_0 = 0$ . It follows that

$$\begin{aligned} \|\mathbb{T}_\tau^* \varphi\|^2 &= -\langle \Phi_{\tau,e} u, \varphi \rangle = -\int_0^\tau \int_\Gamma \mathbf{R}_\tau u \overline{\Psi_{\tau,e'}^d \varphi} \leq \\ &\leq \|u\|_{L^\infty(\Gamma \times [0, \tau])} \|\Psi_{\tau,e'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} \leq K_{\tau,e} \|\mathbb{T}_\tau^* \varphi\| \|\Psi_{\tau,e'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} \end{aligned}$$

and the proof ends.  $\square$

Now we are ready to introduce the time optimal problem. Define the set of admissible controls

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Gamma \times [0, \infty)) \mid |u(x, t)| \leq M \text{ a. e. in } \Gamma \times [0, \infty)\}.$$

Given  $z_0 \in X$ , we define the set of targets which are reachable from  $z_0$

$$\mathcal{R}(z_0, \mathcal{U}_{ad}) = \cup_{t>0} \{z_1 = \mathbb{T}_t z_0 + \Phi_t u \in X \mid u \in \mathcal{U}_{ad}\}.$$

We consider the time optimal control problem which consists in determining, for every  $z_0 \in X$  and  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ , a control  $u^* \in \mathcal{U}_{ad}$  such that

$$\mathbb{T}_{\tau^*(z_0, z_1)} z_0 + \Phi_{\tau^*(z_0, z_1)} u^* = z_1 \quad (2.19)$$

where  $\tau^*(z_0, z_1)$  is the minimal time needed to steer the initial data  $z_0$  towards target  $z_1$  with controls in  $\mathcal{U}_{ad}$

$$\tau^*(z_0, z_1) = \min_{u \in \mathcal{U}_{ad}} \{t > 0 \mid \mathbb{T}_t z_0 + \Phi_t u = z_1\}.$$

As shown in [5, 11], the above problem admits at least one solution for every  $z_0 \in X$  and  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$  (see also the proof of Proposition 2.7 below).

The function  $(z_0, z_1) \mapsto \tau^*(z_0, z_1)$  is called the *minimal time function*. A natural question consists in investigating if the time optimal control  $u^*$  is bang-bang, in the sense that  $|u^*(x, t)| = M$  almost everywhere. A sufficient condition for this property is given in the following known result whose proof we present for the reader's convenience.

**Proposition 2.7.** *With the notation in Proposition 2.6, assume that the pair  $(A, B)$  is  $L^\infty$  null controllable in time  $\tau$  over  $e$  for every  $\tau > 0$  and for every set of positive measure  $e \subset \Gamma \times [0, \tau]$ . Then, for every  $z_0 \in X$  and  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ , the time optimal problem (2.19) has a unique solution  $u^*$  which is bang-bang.*



*Proof.* Let us first prove the existence of a solution  $u^*$  of the time optimal problem (2.19). Since  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ , there exists a minimizing sequence  $(\tau_n, u_n)_{n \geq 1}$  such that  $\lim_{n \rightarrow \infty} \tau_n = \tau^*(z_0, z_1)$  and  $(u_n)_{n \geq 1} \subset \mathcal{U}_{ad}$  has the property that  $\mathbb{T}_{\tau_n} z_0 + \Phi_{\tau_n} u_n = z_1$  for each  $n \geq 1$ .

Since  $(u_n)_{n \geq 1} \subset \mathcal{U}_{ad}$ , it follows that  $(u_n)_{n \geq 1}$  tends weakly- $*$  to some  $u^* \in \mathcal{U}_{ad}$  in  $L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])$ . We define  $\tilde{z}_1 = \mathbb{T}_{\tau^*(z_0, z_1)} z_0 + \Phi_{\tau^*(z_0, z_1)} u^*$  and we note that  $u^*$  is a time optimal control if  $\tilde{z}_1 = z_1$ . The latter equality follows if we prove that  $\langle \tilde{z}_1, \varphi \rangle = \langle z_1, \varphi \rangle$  for each  $\varphi \in X$  which, in view of the facts that  $\lim_{n \rightarrow \infty} \mathbb{T}_{\tau_n} z_0 = \mathbb{T}_{\tau^*(z_0, z_1)} z_0$ ,  $\lim_{n \rightarrow \infty} \|\Phi_{\tau_n} u_n - \Phi_{\tau^*(z_0, z_1)} u_n\| = 0$  and  $\mathbb{T}_{\tau_n} z_0 + \Phi_{\tau_n} u_n = z_1$  for each  $n \geq 1$ , is reduced to prove that

$$\langle \Phi_{\tau^*(z_0, z_1)} u, \varphi \rangle = \lim_{n \rightarrow \infty} \langle \Phi_{\tau^*(z_0, z_1)} u_n, \varphi \rangle \quad (\varphi \in X). \quad (2.20)$$

By noting that

$$\langle \Phi_{\tau^*(z_0, z_1)} u_n, \varphi \rangle = \int_0^{\tau^*(z_0, z_1)} \langle \mathbf{A}_{\tau^*(z_0, z_1)} u_n(t), \Psi_{\tau^*(z_0, z_1)}^d \varphi \rangle_U dt$$

and taking into account that  $\Psi_{\tau^*(z_0, z_1)}^d \varphi \in L^1([0, \tau^*(z_0, z_1)]; U)$ , we deduce from the weak- $*$  convergence of the sequence  $(u_n)_{n \geq 1}$  that (2.20) holds and the existence of a solution  $u^*$  for the time optimal problem is proved.

Now, let us show that  $u^* \in L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])$  is bang-bang. We denote by  $z^*$  the corresponding state trajectory. Assume that there exist  $\varepsilon > 0$  and a set of positive measure  $E \subset \Gamma \times [0, \tau^*(z_0, z_1)]$  such that

$$|u^*(x, t)| < M - \varepsilon \quad ((x, t) \in E). \quad (2.21)$$

Let  $\delta_0 > 0$  be small enough such that

$$\begin{cases} \tau_0 = \tau^*(z_0, z_1) - \delta_0 > 0 \\ \text{The set } e_0 = \{(x, t) \in \Gamma \times [\delta_0, \tau_0] \mid (x, t) \in E\} \text{ has positive measure.} \end{cases} \quad (2.22)$$

Since  $\lim_{t \rightarrow 0} z^*(t) = z_0$ , there exists  $\delta \in (0, \delta_0)$  such that

$$\|z_0 - z^*(\delta)\| \leq \frac{\varepsilon}{2C_{\tau_0, e_0}}. \quad (2.23)$$

Moreover, the  $L^\infty$  null controllability of  $(A, B)$  in time  $\tau_0$  over  $e_0$  implies that there exists  $v \in L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])$  with

$$\begin{cases} \text{supp } v \subset e_0 \\ \mathbb{T}_{\tau_0}(\mathbb{T}_{\delta_0 - \delta}(z_0 - z^*(\delta))) + \int_{\delta_0}^{\tau^*(z_0, z_1)} \mathbb{T}_{\tau^*(z_0, z_1) - s} Bv(s) ds = \\ = \mathbb{T}_{\tau^*(z_0, z_1) - \delta}(z_0 - z^*(\delta)) + \int_{\delta}^{\tau^*(z_0, z_1)} \mathbb{T}_{\tau^*(z_0, z_1) - s} Bv(s) ds = 0, \\ \|v\|_{L^\infty(e_0)} \leq 2C_{\tau_0, e_0} \|z_0 - z^*(\delta)\|, \end{cases} \quad (2.24)$$

where  $C_{\tau_0, e_0}$  is the cost constant defined in (2.16). From (2.23), together with the first and the third conditions in (2.24), it follows that

$$\|v\|_{L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])} \leq \varepsilon. \quad (2.25)$$

Now, let  $\tilde{u} \in L^\infty(\Gamma \times [0, \tau_0])$  be defined by

$$\tilde{u}(t) = u^*(t + \delta) + v(t + \delta) \quad (t \in [0, \tau_0]).$$

By combining (2.21), (2.25) and the fact that  $\text{supp } v \subset e_0$ , it follows that

$$\|\tilde{u}\|_{L^\infty(\Gamma \times [0, \tau_0])} \leq M. \quad (2.26)$$

Finally, the semi-group property, the above definition of  $\tilde{u}$  and (2.24), imply that

$$\begin{aligned} & \mathbb{T}_{\tau^*(z_0, z_1) - \delta} z_0 + \Phi_{\tau^*(z_0, z_1) - \delta} \tilde{u} = \mathbb{T}_{\tau^*(z_0, z_1) - \delta} (z_0 - z^*(\delta)) + \mathbb{T}_{\tau^*(z_0, z_1) - \delta} z^*(\delta) + \\ & \quad + \Phi_{\tau^*(z_0, z_1) - \delta} v(\cdot + \delta) + \Phi_{\tau^*(z_0, z_1) - \delta} u^*(\cdot + \delta) = \\ & = \mathbb{T}_{\tau^*(z_0, z_1) - \delta} (z_0 - z^*(\delta)) + \mathbb{T}_{\tau^*(z_0, z_1)} z_0 + \mathbb{T}_{\tau^*(z_0, z_1) - \delta} \int_0^\delta \mathbb{T}_{\delta - s} B u(s) ds + \\ & \quad + \int_\delta^{\tau^*(z_0, z_1)} \mathbb{T}_{\tau^*(z_0, z_1) - s} B v(s) ds + \int_\delta^{\tau^*(z_0, z_1)} \mathbb{T}_{\tau^*(z_0, z_1) - s} B u(s) ds = \\ & = \mathbb{T}_{\tau^*(z_0, z_1) - \delta} (z_0 - z^*(\delta)) + \int_\delta^{\tau_0} \mathbb{T}_{\tau^*(z_0, z_1) - s} B v(s) ds + \mathbb{T}_{\tau^*(z_0, z_1)} z_0 + \Phi_{\tau^*(z_0, z_1)} u^* = z_1. \end{aligned}$$

Hence,  $\tilde{u} \in \mathcal{U}_{ad}$  is a control which drives  $z_0$  to  $z_1$  in time  $\tau^*(z_0, z_1) - \delta$ . This contradicts the definition of  $\tau^*(z_0, z_1)$  and the bang-bang property is proved.

To show the uniqueness, let  $u$  and  $v$  be two time optimal controls in  $\mathcal{U}_{ad}$ . Note that in this case  $w = \frac{1}{2}(u + v)$  is also a time optimal control. From the proof above it follows that  $|u(x, t)| = |v(x, t)| = |w(x, t)| = M$  a. e. in  $\Gamma \times [0, \tau^*(z_0, z_1)]$ . If  $u(x, t) \neq v(x, t)$  in a set of positive measure  $E \subset \Gamma \times [0, \tau^*(z_0, z_1)]$  then

$$0 = u(x, t) + v(x, t) = 2w(x, t) \quad (x, t) \in E$$

which contradicts the fact that  $|w(x, t)| = M$  a. e. in  $\Gamma \times [0, \tau^*(z_0, z_1)]$ .  $\square$

### 3 A modified Lebeau-Robbiano strategy

In this section we propose a version of a method introduced by Lebeau and Robbiano [9] to study the null controllability of the heat equation. Roughly speaking, the Lebeau-Robbiano strategy combines the observability of finite combinations of eigenvectors (which is a property not involving the time variable) with the exponential decay of the heat semigroup to obtain the null controllability. The fact that this method can be adapted to  $L^\infty$  null controllability over a positive measure set  $e$  has been remarked by Wang [19]. The main novelty we bring in into this section is that we replace the Lebeau-Robbiano assumption on the observability of finite combinations of eigenvectors with an assumption of controllability of the truncation of the dynamical system to a finite number of modes.

This latter property involves the time variable and it is, in general, weaker than the observability of finite combinations of eigenvectors.

We continue to use in this section the notation and assumptions in Section 2 on the spaces  $X$ ,  $U$  and on the operators  $A$  and  $B$ . Moreover, we add some new notation and assumptions.

The operator  $A : \mathcal{D}(A) \rightarrow X$  is supposed to be a self-adjoint (possibly unbounded) operator on  $X$  such that

$$\langle A\psi, \psi \rangle \leq 0 \quad (\psi \in \mathcal{D}(A)).$$

Such an operator will be briefly called a *negative operator*. We also assume that  $A$  is diagonalizable with an orthonormal basis of eigenvectors  $\{\varphi_k\}_{k \geq 1}$  and corresponding family of eigenvalues  $\{-\lambda_k\}_{k \geq 1}$ , where the sequence  $\{\lambda_k\}$  is positive, non decreasing and satisfies  $\lambda_k \rightarrow \infty$  as  $k$  tends to infinity. According to classical results, this holds, in particular, if  $A$  has compact resolvents. With the above assumptions on  $A$ , we have

$$A\psi = - \sum_{k \geq 1} \lambda_k \langle \psi, \varphi_k \rangle \varphi_k \quad (\psi \in \mathcal{D}(A)), \quad (3.1)$$

so that the semigroup  $\mathbb{T}$  generated by  $A$  is a contraction semigroup on  $X$  satisfying

$$\mathbb{T}_t z = \sum_{k \geq 1} e^{-\lambda_k t} \langle z, \varphi_k \rangle \varphi_k \quad (t \geq 0, z \in X). \quad (3.2)$$

Moreover, the sets

$$X_\beta := \left\{ z \in X : \sum_{k \geq 1} (1 + \lambda_k^2)^\beta |\langle z, \varphi_k \rangle|^2 < \infty \right\} \quad (\beta > 0) \quad (3.3)$$

endowed with the inner product

$$\langle y, z \rangle_\beta = \sum_{k \geq 1} (1 + \lambda_k^2)^\beta \langle y, \varphi_k \rangle \overline{\langle z, \varphi_k \rangle} \quad (z, y \in X_\beta) \quad (3.4)$$

are Hilbert spaces. The scale  $\{X_\beta\}_{\beta \geq 0}$  of Hilbert spaces can be extended to a scale  $\{X_\beta\}_{\beta \in \mathbb{R}}$  by defining, for every  $\beta < 0$ ,  $X_\beta$  as the completion of  $X$  with respect to the norm associated to the inner product (3.4). Alternatively,  $X_{-\beta}$  may be defined, for  $\beta > 0$ , as the dual of  $X_\beta$  with respect to the pivot space  $X$ . For every  $\beta > 0$ , formulas (3.1) and (3.2), with  $\langle \cdot, \cdot \rangle$  standing this time for the duality between  $X_{-\beta}$  and  $X_\beta$ , provide canonical extensions for the operator  $A$  and the semigroup  $\mathbb{T}$  to a negative operator and a contraction semigroup on  $X_{-\beta}$ , respectively. These extensions will be still denoted by  $A$  and  $\mathbb{T}$ . Note that, for every  $\beta \in \mathbb{R}$ , the family  $\{(1 + \lambda_k^2)^{\beta/2} \varphi_k\}_{k \geq 1}$  is an orthonormal basis in  $X_\beta$ . Finally, in the sequel we shall need the following lemma.

**Lemma 3.1.** *Assume that  $U$  is a Hilbert space and  $B \in \mathcal{L}(U, X_{-\beta})$  for some  $\beta \in [0, 1/2)$ . Then  $B$  is an admissible control operator for  $\mathbb{T}$  with admissibility constant  $\frac{1}{\sqrt{2}} \|B\|_{L(U, X_{-\beta})}$ .*

*Proof.* Let  $u \in L^2([0, \tau]; U)$  and let  $z \in C([0, \tau]; X_{-1})$  be the mild solution of the equation

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = 0. \quad (3.5)$$

If  $(F_n)_{n \geq 1} \subset W^{1,1}([0, \tau]; X_1)$  is a sequence convergent to  $Bu$  in  $L^2([0, \tau]; X_{-\beta})$ , then let  $z_n \in C([0, \tau]; X_1) \cap C^1([0, \tau]; X)$  be the solution of

$$\dot{z}_n(t) = Az_n(t) + F_n(t), \quad z_n(0) = 0. \quad (3.6)$$

It follows that  $z_n$  verifies

$$\frac{1}{2} \frac{d}{dt} \|z_n(t)\|^2 = -\|z_n(t)\|_{\frac{1}{2}}^2 + \langle F_n(t), z_n(t) \rangle = -\|z_n(t)\|_{\frac{1}{2}}^2 + \langle F_n(t), z_n(t) \rangle_{-\frac{1}{2}, \frac{1}{2}} \leq \frac{1}{4} \|F_n(t)\|_{-\frac{1}{2}}^2.$$

By integrating the last inequality from 0 to  $\tau$ , we obtain that

$$\limsup_{n \rightarrow \infty} \|z_n(\tau)\|^2 \leq \frac{1}{2} \int_0^\tau \|Bu(s)\|_{-\frac{1}{2}}^2 ds \leq \frac{1}{2} \|B\|_{\mathcal{L}(U, X_{-\beta})}^2 \|u\|_{L^2([0, \tau]; U)}^2.$$

We deduce that  $(z_n(\tau))_{n \geq 1}$  converges weakly to some  $\tilde{z}$  in  $X$ . Since  $(z_n(\tau))_{n \geq 1}$  converges to  $z(\tau)$  in  $X_{-1}$ , it follows that  $z(\tau) = \tilde{z} \in X$  and verifies

$$\|z(\tau)\| \leq \frac{1}{\sqrt{2}} \|B\|_{\mathcal{L}(U, X_{-\beta})} \|u\|_{L^2([0, \tau]; U)}. \quad (3.7)$$

The proof of the Lemma is complete.  $\square$

For  $\gamma, \varsigma > 0$  we denote by

$$V_{\varsigma, \gamma} = \text{span} \{\varphi_k \mid \lambda_k^\gamma \leq \varsigma\},$$

and we denote by  $P_{\varsigma, \gamma}$  the orthogonal projection from  $X$  onto  $V_{\varsigma, \gamma}$ .

In the sequel we shall denote by  $\mu_k$  the  $k$ -dimensional Lebesgue measure. We now state the main result in this section.

**Theorem 3.2.** *Let  $\tau > 0$  and let  $e \subset \Gamma \times [0, \tau]$  be a set of positive measure. Assume  $B \in \mathcal{L}(U, X_{-\beta})$ , with  $\beta \in [0, 1/2)$ . Moreover, assume that there exist positive constants  $\gamma \in (0, 1)$ ,  $a \in (0, \tau)$ ,  $d_0, d_1$  and  $\kappa$  such that for every  $\varsigma > 0$ ,  $s, t > 0$ , with  $a \leq s < t \leq \tau$  and  $\mathcal{E} = \{(x, \sigma) \in e \mid s \leq \sigma \leq t\}$  of positive measure, we have*

$$\|\mathbb{T}_\tau^* \varphi\| \leq d_0 e^{d_1 \left[1 + \ln\left(\frac{1}{\mu_m(\mathcal{E})}\right)\right] \varsigma + \frac{\kappa}{\mu_m(\mathcal{E})}} \|\Psi_{\tau, \mathcal{E}'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} \quad (\varphi \in V_{\varsigma, \gamma}) \quad (3.8)$$

where  $\mathcal{E}' = \{(x, \tau - \sigma) \mid (x, \sigma) \in \mathcal{E}\}$ . Then the pair  $(A, B)$  is  $L^\infty$  null controllable in time  $\tau$  over  $e$ .

In order to prove Theorem 3.2 we need the following measure theoretic result whose proof may be found, for instance, in Lions [11, p. 275].

**Lemma 3.3.** *Let  $F$  be a set of positive measure. Then there exist positive constants  $\rho$  and  $c$  such that for almost every  $t \in F$  there exists an increasing sequence  $(t_n)_{n \geq 0}$  such that  $\lim_{n \rightarrow \infty} t_n = t$  and*

$$m([t_n, t_{n+1}] \cap F) \geq \rho(t_{n+1} - t_n), \quad \frac{t_{n+1} - t_n}{t_{n+2} - t_{n+1}} \leq c \quad (n \geq 0).$$

The following lemma will help us to separate the time and space variables in our estimates.

**Lemma 3.4.** *Let  $e \subset \Gamma \times [0, \tau]$  be a set of positive measure. For each  $t \in [0, \tau]$ , we define the  $t$ -section of  $e$  as  $e_t = \{x \in \Gamma \mid (x, t) \in e\}$ . If  $F = \left\{t \in [0, \tau] \mid \mu_{m-1}(e_t) > \frac{\mu_m(e)}{4\tau}\right\}$ , then*

$$\mu_1(F) > \frac{\mu_m(e)}{4\mu_{m-1}(\Gamma)}. \quad (3.9)$$

*Proof.* Let  $F = \left\{t \in [0, \tau] \mid \mu_{m-1}(e_t) > \frac{\mu_m(e)}{4\tau}\right\}$  and suppose that  $\mu_1(F) \leq \frac{\mu_m(e)}{4\mu_{m-1}(\Gamma)}$ . It follows that

$$\begin{aligned} \mu_m(e) &= \int_0^\tau \mu_{m-1}(e_t) dt = \int_F \mu_{m-1}(e_t) dt + \int_{[0, \tau] \setminus F} \mu_{m-1}(e_t) dt \leq \\ &\leq \frac{\mu_m(e)}{4\mu_{m-1}(\Gamma)} \mu_{m-1}(\Gamma) + \tau \frac{\mu_m(e)}{4\tau} = \frac{\mu_m(e)}{2} \end{aligned}$$

which is a contradiction.  $\square$

We are now in a position to prove Theorem 3.2.

*Proof of Theorem 3.2.* Since  $e \subset \Gamma \times [0, \tau]$  is a set of positive measure, we deduce from Lemma 3.4 that there exists  $F \subset [0, \tau]$  of measure greater than  $\frac{\mu_m(e)}{4\mu_{m-1}(\Gamma)}$  such that, for any  $t \in F$ , the sections  $e_t$  has measure greater than  $\frac{\mu_m(e)}{4\tau}$ . Let  $t' \in F$  be one of the points for which one can find a sequence  $(t_k)_{k \geq 0}$  as in Lemma 3.3. Without loss of generality we may suppose that  $t_0 := a > 0$ . Define the sequence  $(s_k)_{k \geq 0}$  by

$$s_k = \left[ \frac{1}{t_{2k+2} - t_{2k+1}} \right]^p \quad (k \geq 0) \quad (3.10)$$

where  $p = p(\gamma)$  will be conveniently chosen latter on.

For every  $k \geq 0$  we define the functions  $z_k \in C([t_{2k}, t_{2k+2}], X)$  and  $u_k \in L^\infty(\Gamma \times [t_{2k}, t_{2k+2}])$  recursively by the formulae

$$z_{-1}(t_0) = \mathbb{T}_{t_0} z_0. \quad (3.11)$$

$$u_k(x, t) = \begin{cases} \tilde{u}_k(x, t) & \text{for } (x, t) \in \Gamma \times [t_{2k}, t_{2k+1}], \\ 0 & \text{for } (x, t) \in \Gamma \times [t_{2k+1}, t_{2k+2}], \end{cases} \quad (k \geq 0), \quad (3.12)$$

$$z_k(\xi) = \mathbb{T}_{\xi - t_{2k}} z_{k-1}(t_{2k}) + \int_{t_{2k}}^\xi \mathbb{T}_{\xi - \sigma} B \chi_e u_k(\sigma) d\sigma \quad (k \geq 0, \xi \in [t_{2k}, t_{2k+2}]), \quad (3.13)$$

where  $\tilde{u}_k \in L^\infty(\Gamma \times [t_{2k}, t_{2k+1}])$  is chosen such that

$$P_{s_k, \gamma} z_k(t_{2k+1}) = 0, \quad \text{supp } \tilde{u}_k \subset e \cap \Gamma \times [t_{2k}, t_{2k+1}] \quad (k \geq 0), \quad (3.14)$$

$$\|\tilde{u}_k\|_{L^\infty(\Gamma \times [t_{2k}, t_{2k+1}])} \leq d_0 e^{d_1 \left[ 1 + \ln \left( \frac{1}{\rho(t_{2k+1} - t_{2k})} \right) \right] s_k + \frac{\kappa}{\rho(t_{2k+1} - t_{2k})}} \|z_{k-1}(t_{2k})\| \quad (k \geq 0). \quad (3.15)$$

The existence of  $\tilde{u}_k$  with the above properties follows from (3.8) with  $\mathcal{E} = e \cap \Gamma \times [t_{2k}, t_{2k+1}]$  by applying Proposition 2.6. Using (3.13) and (3.15) it follows that, for each  $k \geq 0$ ,

$$\|z_k(t_{2k+1})\| \leq \left( 1 + d_0 \|B\| \sqrt{t_{2k+1} - t_{2k}} e^{d_1 \left[ 1 + \ln \left( \frac{1}{\rho(t_{2k+1} - t_{2k})} \right) \right] \varsigma_k + \frac{\kappa}{\rho(t_{2k+1} - t_{2k})}} \right) \|z_{k-1}(t_{2k})\|. \quad (3.16)$$

In the above formula and in the remaining part of the proof,  $\|B\|$  denotes the norm in  $\mathcal{L}(U, X_{-\beta})$ . Note that formula (3.13) for  $z_k(t_{2k+2})$  and (3.14) imply that

$$\langle z_k(t_{2k+2}), \varphi_n \rangle = 0 \quad (k \geq 0, \lambda_n^\gamma \leq \varsigma_k) \quad (3.17)$$

and ensure the following estimate

$$\|z_k(t_{2k+2})\| \leq e^{-\varsigma_k^{1/\gamma}(t_{2k+2} - t_{2k+1})} \|z_k(t_{2k+1})\| \quad (k \geq 0). \quad (3.18)$$

From the above relation and (3.16) it follows that, for any  $k \geq 0$ ,

$$\|z_k(t_{2k+2})\| \leq e^{-\varsigma_k^{1/\gamma}(t_{2k+2} - t_{2k+1})} \left( 1 + d_0 \|B\| \sqrt{t_{2k+1} - t_{2k}} e^{d_1 \left[ 1 + \ln \left( \frac{1}{\rho(t_{2k+1} - t_{2k})} \right) \right] \varsigma_k + \frac{\kappa}{\rho(t_{2k+1} - t_{2k})}} \right) \|z_{k-1}(t_{2k})\|. \quad (3.19)$$

In order to show that the right-hand side of (3.15) forms a bounded sequence, we denote

$$a_k = d_0 e^{d_1 \left[ 1 + \ln \left( \frac{1}{\rho(t_{2k+1} - t_{2k})} \right) \right] \varsigma_k + \frac{\kappa}{\rho(t_{2k+1} - t_{2k})}} \|z_{k-1}(t_{2k})\| \quad (k \geq 0).$$

From (3.19) we deduce that

$$\begin{aligned} \frac{a_{k+1}}{a_k} &\leq e^{-\varsigma_k^{1/\gamma}(t_{2k+2} - t_{2k+1}) + d_1(\varsigma_{k+1} - \varsigma_k) + \frac{\kappa}{\rho} \left( \frac{1}{t_{2k+3} - t_{2k+2}} - \frac{1}{t_{2k+1} - t_{2k}} \right) + d_1 \varsigma_{k+1} \ln \left( \frac{1}{\rho(t_{2k+3} - t_{2k+2})} \right)} \\ &e^{-d_1 \varsigma_k \ln \left( \frac{1}{\rho(t_{2k+1} - t_{2k})} \right)} \left( 1 + d_0 \|B\| \sqrt{t_{2k+1} - t_{2k}} e^{d_1 \left[ 1 + \ln \left( \frac{1}{\rho(t_{2k+1} - t_{2k})} \right) \right] \varsigma_k + \frac{\kappa}{\rho(t_{2k+1} - t_{2k})}} \right) \quad (k \geq 0). \end{aligned}$$

The above estimate implies that there exists  $k_0 \in \mathbb{N}$  such that

$$\frac{a_{k+1}}{a_k} \leq d_0 \|B\| \sqrt{t_{2k+1} - t_{2k}} e^{-\varsigma_k^{1/\gamma}(t_{2k+2} - t_{2k+1}) + d_1 \varsigma_{k+1} \ln \left( \frac{1}{\rho(t_{2k+3} - t_{2k+2})} \right) + \frac{\kappa}{\rho(t_{2k+3} - t_{2k+2})}} \quad (k \geq k_0). \quad (3.20)$$

Using Lemma 3.3, formula (3.10) and the fact that  $\gamma \in (0, 1)$  it follows that there exists  $k_1 > 0$  and a constant  $C > 0$  such that

$$\begin{aligned} & -\varsigma_k^{1/\gamma}(t_{2k+2} - t_{2k+1}) + d_1 \varsigma_{k+1} \ln \left( \frac{1}{\rho(t_{2k+3} - t_{2k+2})} \right) + \frac{\kappa}{\rho(t_{2k+3} - t_{2k+2})} \\ & \leq - \left( \frac{1}{t_{2k+2} - t_{2k+1}} \right)^{\frac{p}{\gamma} - 1} + d_1 c^{2p} \left( \frac{1}{t_{2k+2} - t_{2k+1}} \right)^p \ln \left( \frac{c}{\rho(t_{2k+2} - t_{2k+1})} \right) + \frac{c\kappa}{\rho(t_{2k+2} - t_{2k+1})} \\ & \leq -C \left( \frac{1}{t_{2k+2} - t_{2k+1}} \right)^{\frac{p}{\gamma} - 1} \quad (k \geq k_1) \end{aligned}$$

for any  $p$  which verifies  $\frac{p}{\gamma} - 1 > \max\{1, p\}$ . Note that the last inequality is equivalent to  $p > \max\left\{2\gamma, \frac{\gamma}{1-\gamma}\right\}$  which has at least a solution for each  $\gamma \in (0, 1)$ .

The above estimate, (3.20) and (3.15) show that the function

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t) \chi_{[t_{2k}, t_{2k+2}]} \quad ((x, t) \in \Gamma \times [0, \tau]) \quad (3.21)$$

belongs to  $L^\infty(\Gamma \times [0, \tau])$  and  $\text{supp } u \subset \mathcal{E}$ . Moreover, the function

$$z(t) = \sum_{k=0}^{\infty} z_k(t) \chi_{[t_{2k}, t_{2k+2}]} \quad (t \in [0, \tau]) \quad (3.22)$$

belongs to  $C([0, \tau]; X)$  and verifies

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0$$

and  $z(\tau) = \mathbb{T}_\tau z_0 + \Phi_{\tau, \mathcal{E}} u = 0$ . The proof of Theorem 3.2 is now complete.  $\square$

#### 4 A Turán type inequality

In this section we give an  $m$ -dimensional version of an inequality originally proved by Turán [18] for intervals and extended by Nazarov [13] for sets of positive measure in  $\mathbb{R}$ . Let  $m \geq 1$  be given and, for each  $1 \leq k \leq m$ , let  $I_k = \prod_{i=1}^k [0, l_i]$  be a  $k$ -dimensional rectangle of volume  $v_k = l_1 \dots l_k$ . For any  $\alpha \in (\mathbb{N}^*)^m$  we write  $\alpha = (\alpha', \alpha_m)$  with  $\alpha' = (\alpha_1, \dots, \alpha_{m-1}) \in (\mathbb{N}^*)^{m-1}$ ,  $\alpha_m \in \mathbb{N}$  and we denote

$$\Phi_\alpha(x) = \Phi_\alpha(x_1, \dots, x_m) = \sqrt{\frac{2^m}{v_m}} \prod_{j=1}^m \sin\left(\frac{\alpha_j x_j \pi}{l_j}\right), \quad (4.23)$$

$$\Phi_{\alpha'}(x') = \Phi_{\alpha'}(x_1, \dots, x_{m-1}) = \sqrt{\frac{2^{m-1}}{v_{m-1}}} \prod_{j=1}^{m-1} \sin\left(\frac{\alpha_j x_j \pi}{l_j}\right). \quad (4.24)$$

Now, let  $E \subset I_m$  be a set of positive measure. The aim of this section is to estimate from below the  $L^1(E)$ -norm of the finite linear combinations of functions  $\Phi_\alpha$ . We begin with the following simple lemma on sets of positive measure in product spaces.

**Lemma 4.1.** *Let  $E \subset I_m$  be a set of positive measure. For each  $k < m$  and  $(\xi_1, \xi_2, \dots, \xi_k) \in I_k$ , we define the  $(\xi_1, \xi_2, \dots, \xi_k)$ -section of  $E$  as the set*

$$E_{(\xi_1, \xi_2, \dots, \xi_k)} = \left\{ (\xi_{k+1}, \dots, \xi_m) \in \prod_{i=k+1}^m [0, l_i] \mid (\xi_1, \xi_2, \dots, \xi_k, \xi_{k+1}, \dots, \xi_m) \in E \right\}.$$

If, for each  $k < m$ ,  $F_k$  denotes the set

$$F_k = \left\{ (\xi_1, \xi_2, \dots, \xi_k) \in \prod_{i=1}^k [0, l_i] \mid \mu_{m-k}(E_{(\xi_1, \dots, \xi_k)}) > \frac{\mu_m(E)}{4l_1 \dots l_k} \right\},$$

then

$$\mu_k(F_k) > \frac{\mu_m(E)}{4l_{k+1} \dots l_m}. \quad (4.25)$$

*Proof.* Let us suppose that there exists  $k < m$  such that  $\mu_k(F_k) \leq \frac{\mu_m(E)}{4l_{k+1}\dots l_m}$ . It follows that

$$\begin{aligned} \mu_m(E) &= \int_0^{l_1} \int_0^{l_2} \dots \int_0^{l_k} \mu_{m-k}(E_{(\xi_1, \xi_2, \dots, \xi_k)}) d\xi_k \dots d\xi_1 = \\ &= \int_{F_k} \mu_{m-k}(E_{(\xi_1, \xi_2, \dots, \xi_k)}) d\xi_k \dots d\xi_1 + \int_{I_k \setminus F_k} \mu_{m-k}(E_{(\xi_1, \xi_2, \dots, \xi_k)}) d\xi_k \dots d\xi_1 \leq \\ &\leq \frac{\mu_m(E)}{4l_{k+1}\dots l_m} l_{k+1}\dots l_m + l_1 \dots l_k \frac{\mu_m(E)}{4l_1 \dots l_k} = \frac{\mu_m(E)}{2} \end{aligned}$$

which is a contradiction.  $\square$

The following theorem, proved in [13, Theorem I], will play an essential role in our study.

**Theorem 4.2.** *Let  $N \in \mathbb{N}$  be a nonnegative integer and  $p(x) = \sum_{|k| \leq N} a_k e^{i\nu_k x}$  ( $a_k \in \mathbb{C}$ ,  $\nu_k \in \mathbb{R}$ ) be an exponential polynomial. Let  $I \subset \mathbb{R}$  be an interval and  $E$  a measurable subset of  $I$  of positive measure. Then*

$$\sup_{x \in I} |p(x)| \leq \left( \frac{C\mu_1(I)}{\mu_1(E)} \right)^{2N} \sup_{x \in E} |p(x)| \quad (4.26)$$

where  $C > 0$  is an absolute constant.

The following result is a consequence of Theorem 4.2.

**Corollary 4.3.** *With the notations from Theorem 4.2 we have that the following inequality holds for any sequence  $(a_k)_{|k| \leq N} \subset \mathbb{C}$*

$$\left( \sum_{|k| \leq N} |a_k|^2 \right)^{\frac{1}{2}} \leq \frac{1}{C\mu_1(I)} \left( \frac{2C\mu_1(I)}{\mu_1(E)} \right)^{2N+1} \int_E \left| \sum_{|k| \leq N} a_k e^{i\nu_k x} \right| dx, \quad (4.27)$$

where  $C > 0$  is the constant from (4.26).

*Proof.* By denoting  $p(x) = \sum_{|k| \leq N} a_k e^{i\nu_k x}$  and by using the orthogonality of  $(e^{i\nu_k x})_{|k| \leq N}$  in  $L^2(I)$ , we have that

$$\mu_1(I) \sum_{|k| \leq N} |a_k|^2 = \int_I |p(x)|^2 dx \leq \mu_1(I) \sup_{x \in I} |p(x)|^2. \quad (4.28)$$

In order to bound  $\sup_{x \in I} |p(x)|$  with an integral over  $E$  we use (4.26) and an idea from [2, Theorem 5.6]. If we denote  $\mathcal{E} = \left\{ x \in E \mid |p(x)| \geq \frac{2}{\mu_1(E)} \|p\|_{L^1(E)} \right\}$ , we remark that  $\mu_1(\mathcal{E}) \leq \frac{\mu_1(E)}{2}$ . Thus  $\mu_1(E \setminus \mathcal{E}) \geq \frac{\mu_1(E)}{2}$  and, by applying (4.26) in  $E \setminus \mathcal{E}$ , we deduce that

$$\sup_{x \in I} |p(x)| \leq \left( \frac{C\mu_1(I)}{\mu_1(E \setminus \mathcal{E})} \right)^{2N} \sup_{x \in E \setminus \mathcal{E}} |p(x)| \leq \frac{2}{\mu_1(E)} \left( \frac{2C\mu_1(I)}{\mu_1(E)} \right)^{2N} \|p\|_{L^1(E)}$$

and the proof is complete.  $\square$



The following result generalizes Corollary 4.3 to several space dimensions.

**Corollary 4.4.** *Let  $m, N \geq 1$  and  $\mathcal{D} \subset [1, N]^m \cap \mathbb{N}^m$ . Let  $E \subset I_m = \prod_{k=1}^m [0, l_k]$  be a set of positive measure. If  $C > 0$  is the constant from (4.26), then the following inequality holds for any sequence  $(b_\alpha)_{\alpha \in \mathcal{D}} \subset \mathbb{C}$*

$$\left( \sum_{\alpha \in \mathcal{D}} |b_\alpha|^2 \right)^{\frac{1}{2}} \leq \left( \frac{N^{m-1}}{2^m C^{2m} v_m} \right)^{\frac{1}{2}} \left( \frac{2^{p_m} C v_m}{\mu_m(E)} \right)^{m(2N+1)} \int_E \left| \sum_{\alpha \in \mathcal{D}} b_\alpha \Phi_\alpha \right| dx \quad (4.29)$$

where the sequence  $(p_m)_{m \geq 1}$  is defined by  $p_1 = 1$  and  $p_m = \frac{2m+1+(m-1)p_{m-1}}{m}$  for  $m \geq 2$ .

*Proof.* We prove (4.29) by induction over  $m$ . For  $m = 1$ , (4.29) follows from Proposition 4.3. Indeed, it is sufficient to take in (4.27)  $a_0 = 0$ ,  $a_k = \frac{b_k}{2^i}$  and  $a_{-k} = -a_k$  for each  $k \geq 1$  to obtain (4.29) in the case  $m = 1$ .

Now, let us suppose that (4.29) holds in any dimension less or equal than  $m - 1$  and prove it for dimension  $m \geq 2$ . Let  $p(x) = \sum_{\alpha \in \mathcal{D}} b_\alpha \Phi_\alpha(x)$ , where  $\mathcal{D} \subset [0, N]^m \cap \mathbb{N}^m$ . With the notations from Lemma 4.1 note that

$$\int_E |p(x)| dx = \int_{I_{m-1}} \int_0^{l_m} \chi_E |p(x', x_m)| dx_m dx' \geq \int_{F_{m-1}} \int_{E_{x'}} |p(x', x_m)| dx_m dx'.$$

We have that  $p(x) = \sum_{k=1}^N \left( \sum_{\alpha' \in \mathcal{D}_k} a_{(\alpha', k)} \Phi_{\alpha'}(x') \right) \sqrt{\frac{2}{l_m}} \sin\left(\frac{k\pi x_m}{l_m}\right)$ , where  $\mathcal{D}_k = \{\alpha' \in [1, N]^{m-1} \mid (\alpha', k) \in \mathcal{D}\}$  for each  $1 \leq k \leq N$ . The induction's hypothesis implies that

$$\int_{E_{x'}} |p(x', x_m)| dx_m \geq \sqrt{2l_m} C \left( \frac{2Cl_m}{\mu_1(E_{x'})} \right)^{-(2N+1)} \left[ \sum_{k=1}^N \left| \sum_{\alpha' \in \mathcal{D}_k} a_{(\alpha', k)} \Phi_{\alpha'}(x') \right|^2 \right]^{\frac{1}{2}}$$

for each  $x' \in I_{m-1}$ . Integrating on  $F_m$  and taking into account that  $\mu_1(E_{x'}) \geq \frac{\mu_m(E)}{4v_{m-1}}$  for every  $x' \in F_m$ , we deduce that

$$\begin{aligned} \int_E |p(x)| dx &\geq \sqrt{2l_m} C \left( \frac{8C v_{m-1} l_m}{\mu_m(E)} \right)^{-(2N+1)} \int_{F_m} \left[ \sum_{k=1}^N \left| \sum_{\alpha' \in \mathcal{D}_k} a_{(\alpha', k)} \Phi_{\alpha'}(x') \right|^2 \right]^{\frac{1}{2}} dx' \geq \\ &\geq \sqrt{2l_m} C \left( \frac{8C v_m}{\mu_m(E)} \right)^{-(2N+1)} \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_{F_m} \left| \sum_{\alpha' \in \mathcal{D}_k} a_{(\alpha', k)} \Phi_{\alpha'}(x') \right| dx'. \end{aligned}$$

By using again the induction's hypothesis and the fact that  $\mu_{m-1}(F_m) > \frac{\mu_m(E)}{4l_m}$ , we deduce that

$$\begin{aligned} \int_E |p(x)| dx &\geq \sqrt{2l_m} C \left( \frac{8C v_m}{\mu_m(E)} \right)^{-(2N+1)} \frac{1}{\sqrt{N}} \times \\ &\sum_{k=1}^N \left( \frac{2^{m-1} C^{2m-2} v_{m-1}}{N^{m-2}} \right)^{\frac{1}{2}} \left( \frac{2^{p_{m-1}} C v_{m-1}}{\mu_{m-1}(F_m)} \right)^{-(m-1)(2N+1)} \left( \sum_{\alpha' \in \mathcal{D}_k} |a_{\alpha'}|^2 \right)^{\frac{1}{2}} \geq \end{aligned}$$

$$\begin{aligned}
&\geq \left( \frac{2^m C^{2m} v_m}{N^{\frac{m-1}{2}}} \right)^{\frac{1}{2}} \left( \frac{8 C v_m}{\mu_m(E)} \right)^{-(2N+1)} \left( \frac{2^{2+p_{m-1}} C v_m}{\mu_m(E)} \right)^{-(m-1)(2N+1)} \left( \sum_{\alpha \in \mathcal{D}} |a_\alpha|^2 \right)^{\frac{1}{2}} = \\
&= \left( \frac{2^m C^{2m} v_m}{N^{\frac{m-1}{2}}} \right)^{\frac{1}{2}} \left( \frac{2^{\frac{2m+1+(m-1)p_{m-1}}{m}} C v_m}{\mu_m(E)} \right)^{-m(2N+1)} \left( \sum_{\alpha \in \mathcal{D}} |a_\alpha|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

and the proof is complete.  $\square$

To end this section we give a simple consequence of the above estimates to an interior controllability problem for the heat equation. Let  $m \in \mathbb{N}$ , let  $\Omega \subset \mathbb{R}^m$  be an open and bounded set and let  $\Gamma$  be a non-empty open subset of  $\Omega$ . We consider the heat equation

$$\frac{\partial z}{\partial t}(x, t) = \Delta z(x, t) + u(x, t)\chi_\Gamma(x) \text{ for } (x, t) \in \Omega \times (0, \infty), \quad (4.30)$$

with the boundary and initial conditions

$$z(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (4.31)$$

$$z(x, 0) = z_0(x) \text{ for } x \in \Omega. \quad (4.32)$$

In (4.30)  $u$  is an interior control acting only on the subset  $\Gamma$  of  $\Omega$ . We have the following result for the interior controllability of the heat equation (4.30)-(4.32).

**Proposition 4.5.** *Suppose that  $\Omega$  is a rectangular domain in  $\mathbb{R}^m$  and that  $\Gamma$  is a nonempty open subset of  $\Omega$ . Then, for every  $z_0 \in L^2(\Omega)$  and  $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ , there exists a unique solution  $u^*$  of the time optimal control problem (1.8). This solution  $u^*$  has the bang-bang property:*

$$|u^*(x, t)| = M \quad \text{a.e. in } \Gamma \times [0, \tau^*(z_0, z_1)]. \quad (4.33)$$

*Proof.* System (4.30)-(4.32) may be written in the form (2.10). Indeed, let  $X = L^2(\Omega)$ ,  $U = L^2(\Gamma)$ ,  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$  and  $A = \Delta$ . Also, let the control operator  $B \in \mathcal{L}(L^2(\Gamma), X)$  be defined by

$$Bu = u\chi_\Gamma. \quad (4.34)$$

With the above notation, (4.30)-(4.32) is equivalent to (2.10) and  $B$  is an admissible control operator for the semigroup  $\mathbb{T}$  generated by  $A$ .

For any  $\gamma \in (0, 1)$  and  $\varsigma > 0$ , let  $V_{\varsigma, \gamma} = \{\varphi \in X \mid \varphi = \sum_{\lambda_\alpha^\gamma \leq \varsigma} a_\alpha \Phi_\alpha\}$ , where  $(\Phi_\alpha)_{\alpha \in (\mathbb{N}^*)^m}$  are the eigenfunctions of the operator  $-A$  defined in (4.23). The eigenvalue  $\lambda_\alpha$  corresponding to the eigenfunction  $\Phi_\alpha$  is given by  $\lambda_\alpha = \alpha_1^2 + \dots + \alpha_m^2$ .

For any  $0 < a \leq s < t \leq \tau$  let  $\mathcal{E} = \{(x, \sigma) \in e \mid s \leq \sigma \leq t\}$  be a set of positive measure and let  $\mathcal{E}' = \{(x, \tau - \sigma) \mid (x, \sigma) \in \mathcal{E}\}$ . By using Lemma 3.4 we have that there exists a set  $F \subset \{t \in [0, \tau] \mid (t, x) \in \mathcal{E}'\}$  such that  $\mu_1(F) > \frac{\mu_m(\mathcal{E})}{4\mu_{m-1}(\Gamma)}$  and the  $\sigma$ -section  $E_\sigma$  has the property that  $\mu_{m-1}(E_\sigma) > \frac{\mu_m(\mathcal{E})}{4\tau}$  for every  $\sigma \in F$ .

According to Proposition 2.7 and Theorem 3.2, the conclusion of the theorem follows if we show that the pair  $(A, B)$  verifies (3.8). In this particular case, (3.8) is a direct consequence of Corollary 4.4. Indeed, we have that

$$\|\Psi_{\tau, \mathcal{E}'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} = \int_0^\tau \int_\Gamma \chi_{\mathcal{E}'} |B^* \mathbb{T}_\sigma^* \varphi| dx d\sigma \geq$$

$$\geq \int_F \int_{E_\sigma} |B^* \mathbb{T}_\sigma^* \varphi| dx d\sigma = \int_F \int_{E_\sigma} \left| \sum_{\lambda_\alpha^\gamma \leq \zeta} a_\alpha \Phi_\alpha(x) e^{-\lambda_\alpha \sigma} \right| dx d\sigma.$$

By using Corollary 4.4 with  $N = \zeta^{\frac{1}{2\gamma}}$ , we deduce that there exists a constant  $C = C(\Gamma, m)$  such that

$$\begin{aligned} \|\Psi_{\tau, \mathcal{E}'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} &\geq \int_F \left\{ C \zeta^{-\frac{m-1}{4\gamma}} \left( \frac{C}{\mu_m(E_\sigma)} \right)^{-m \left( 2\zeta^{\frac{1}{2\gamma}} + 1 \right)} \left( \sum_{\lambda_\alpha^\gamma \leq \zeta} |a_\alpha|^2 e^{-2\lambda_\alpha \sigma} \right)^{\frac{1}{2}} \right\} d\sigma \geq \\ &\geq C \zeta^{-\frac{m-1}{4\gamma}} \left( \frac{4C}{\mu_m(\mathcal{E})} \right)^{-m \left( 2\zeta^{\frac{1}{2\gamma}} + 1 \right)} \mu_1(F) \left( \sum_{\lambda_\alpha^\gamma \leq \zeta} |a_\alpha|^2 e^{-2\lambda_\alpha \tau} \right)^{\frac{1}{2}}. \end{aligned}$$

Now, by taking into account that there exists  $\kappa > 0$  such that

$$\mu_1(F) \geq \frac{\mu_m(\mathcal{E})}{4\mu_{m-1}(\Gamma)} \geq e^{-\frac{\kappa}{\mu_m(\mathcal{E})}},$$

and by choosing  $\gamma = \frac{1}{2}$ , it follows that (3.8) holds, which ends the proof.  $\square$

**Remark 4.6.** *The generalization of the results in Theorem 1.1 and Proposition 4.5 to a domain  $\Omega$  of arbitrary shape is an interesting open question. A result which may allow to tackle this issue has been recently obtained in Apraiz and Escauriaza [1]. In this work, the authors prove the null controllability of the heat equation by means of controls supported in a subset of positive measure of the domain  $\Omega$  or of its boundary. The main new tool introduced in [1] is an inequality of the same nature as (4.29), in which the eigenfunctions  $\Phi_\alpha$  of the Dirichlet Laplace operator in  $\Omega$  are replaced by the solutions  $e^{\pm\sqrt{\lambda_\alpha}y}\Phi_\alpha(x)$  of the elliptic equation*

$$\Delta_x z(x, y) + \partial_y^2 z(x, y) = 0 \quad (x, y) \in \Omega \times \mathbb{R}.$$

## 5 Proof of the main result

The aim of this section is to prove our main result, Theorem 1.1. In order to achieve our objective we need to show that (3.8) holds. One of the key ingredients in the proof of (3.8) is the following Remez-type inequality, which has been proved in Theorem 2.1 from Borwein and Erdélyi [3] (see, also, [2]). Recall from the previous section that, for every  $k \geq 1$ ,  $\mu_k$  stands for the Lebesgue measure in  $\mathbb{R}^k$ .

**Theorem 5.1.** *Let  $\nu_k := k^\alpha$ ,  $k \in \{1, 2, \dots\}$ ,  $\alpha > 1$ . Let  $\rho \in (0, 1)$ ,  $\varepsilon \in (0, 1 - \rho)$  and  $\varepsilon \leq 1/2$ . Then there exists a constant  $c_\alpha > 0$ , depending only on  $\alpha$ , such that*

$$\|p\|_{[0, \rho]} \leq \exp\left(c_\alpha \varepsilon^{1/(1-\alpha)}\right) \|p\|_E$$

for every  $p \in \text{span}\{x^{\nu_1}, x^{\nu_2}, \dots\}$  and for every set  $E \subset [\rho, 1]$  of Lebesgue measure at least  $\varepsilon > 0$ .

The above result has the following consequence:

**Corollary 5.2.** *For every  $\tau > 0$  and every  $F \subset [0, \tau]$  of positive Lebesgue measure there exist two constants  $C, \kappa > 0$  such that*

$$C e^{\kappa/\mu_1(F)} \int_F \left| \sum_{k \geq 1} a_k e^{-k^2 t} \right| dt \geq \left[ \sum_{k \geq 1} |a_k|^2 e^{-k^2 \tau} \right]^{\frac{1}{2}} \quad ((a_k) \in \ell^2(\mathbb{C})). \quad (5.35)$$

*Proof.* If  $(a_k)_{k \geq 1} \in \ell^2(\mathbb{C})$ , let us denote  $f(t) = \sum_{k \geq 1} a_k e^{-k^2 t}$ . Let  $\rho = e^{-\tau}$  and let

$$E = \{x = e^{-t} \mid t \in F\} \subset [\rho, 1].$$

Then

$$\mu_1(E) = \int_F e^{-t} dt \geq e^{-\tau} m(F).$$

We can thus apply Theorem 5.1 (with  $\alpha = 2$ ) to obtain that there exists an absolute constant  $c > 0$  such that

$$\|f\|_{[0, \tau]} \leq \exp(c e^\tau / \mu_1(F)) \|f\|_F. \quad (5.36)$$

Let  $\mathcal{S} = \left\{ t \in F \mid |f(t)| \geq \frac{2}{\mu_1(F)} \|f\|_{L^1(F)} \right\}$ . It is easily seen that  $\mu_1(\mathcal{S}) \leq \frac{\mu_1(F)}{2}$ . Thus  $\mu_1(F \setminus \mathcal{S}) \geq \frac{\mu_1(F)}{2}$  so that, by applying (5.36) in  $F \setminus \mathcal{S}$ , we deduce that

$$\|f\|_{[0, \tau]} \leq \exp(2c e^\tau / \mu_1(F)) \|f\|_{F \setminus \mathcal{S}} \leq \frac{2}{\mu_1(F)} \exp(2c e^\tau / \mu_1(F)) \|f\|_{L^1(F)}. \quad (5.37)$$

On the other hand, by using a classical result (see, for instance, [6, 7, 16, Corollary 3.6]), we deduce that there exists a constant  $C > 0$ , depending only of  $\tau$ , such that

$$\|f\|_{[0, \tau]}^2 \geq \frac{1}{\tau} \int_0^\tau \left| \sum_{k \geq 1} a_k e^{-k^2 t} \right|^2 dt \geq C \sum_{k \geq 1} |a_k|^2 e^{-\tau k^2}. \quad (5.38)$$

From (5.37) and (5.38) it follows that (5.35) holds with positive constants  $C$  and  $\kappa$  depending only of  $\tau$  and the proof ends.  $\square$

Now we have all the ingredients needed to prove our main result.

*Proof of Theorem 1.1.* Recall from Section 2 that (1.1)-(1.4) may be written in the form (2.10), with  $(A, B)$  defined in Remark 2.2 and with  $B$  admissible control operator for the semigroup  $\mathbb{T}$  generated by  $A$ . Therefore, according to Proposition 2.7, it suffices to show the  $L^\infty$  null controllability of the pair  $(A, B)$ . This will be done by using Theorem 3.2. We recall that  $B^* \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$  ( $\Gamma$  being endowed with the  $m - 1$  dimensional Hausdorff measure) is given by

$$B^* \varphi = - \frac{\partial(-A)^{-1} \varphi}{\partial \nu} \Big|_\Gamma \quad (\varphi \in L^2(\Omega)).$$

Note that  $(\Phi_\alpha)_{\alpha \in (\mathbb{N}^*)^m}$  defined by (4.23) is the complete family of orthonormal eigenfunctions of the operator  $-A$  and  $(\lambda_\alpha)_{\alpha \in (\mathbb{N}^*)^m}$  given by  $\lambda_\alpha = \alpha_1^2 + \dots + \alpha_m^2$ , are the corresponding eigenvalues. Now, for any  $\varsigma > 0$  and  $\gamma \in (0, 1)$ , let  $V_{\varsigma, \gamma} = \{\varphi = \sum_{\lambda_\alpha \leq \varsigma} a_\alpha \Phi_\alpha\}$ . For any  $0 < a \leq s < t \leq \tau$  let  $\mathcal{E} = \{(x, \sigma) \in e \mid s \leq \sigma \leq t\}$  be a set of positive measure and let  $\mathcal{E}' = \{(x, \tau - \sigma) \mid (x, \sigma) \in \mathcal{E}\}$ .

By using Lemma 3.4 we have that there exists a set  $F \subset \{t \in [0, \tau] \mid (t, x) \in \mathcal{E}'\}$  such that  $\mu_1(F) > \frac{\mu_m(\mathcal{E})}{4\mu_{m-1}(\Gamma)}$  and, if  $\sigma \in F$  then the  $\sigma$ -section  $E_\sigma$  has the property that  $\mu_{m-1}(E_\sigma) > \frac{\mu_m(\mathcal{E})}{4\tau}$ . It follows that

$$\begin{aligned} \|\Psi_{\tau, \mathcal{E}'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} &= \int_0^\tau \int_\Gamma \chi_{\mathcal{E}'} |B^* \mathbb{T}_\sigma^* \varphi| dx' d\sigma \geq \int_F \int_{E_\sigma} |B^* \mathbb{T}_\sigma^* \varphi| dx' d\sigma = \\ &= \int_F \int_{E_\sigma} \left| \sum_{\lambda_\alpha \leq \varsigma} a_\alpha \frac{(-1)^{\alpha_m} \alpha_m \pi}{l_m \lambda_\alpha} \Phi_{\alpha'}(x') e^{-\lambda_\alpha \sigma} \right| dx' d\sigma = \int_F \int_{E_\sigma} \left| \sum_{\alpha' \in \mathcal{D}_m} b_{\alpha'}(\sigma) \Phi_{\alpha'}(x') \right| dx' d\sigma, \end{aligned}$$

where  $\mathcal{D}_m = \{\alpha' \in (\mathbb{N}^*)^{m-1} \mid |\alpha'|^2 \leq \varsigma^{\frac{1}{\gamma}} - 1\}$  and

$$b_{\alpha'}(\sigma) = \sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}} - |\alpha'|^2}} a_\alpha \frac{(-1)^{\alpha_m} \alpha_m \pi}{l_m \lambda_\alpha} e^{-\lambda_\alpha \sigma} \quad (\alpha' \in \mathcal{D}_m). \quad (5.39)$$

From Corollary 4.4, by taking  $N = \varsigma^{\frac{1}{2\gamma}}$ , we deduce that there exists a constant  $C = C(\Gamma, m)$  such that

$$\begin{aligned} \int_0^\tau \int_\Gamma \chi_{\mathcal{E}'} |B^* \mathbb{T}_\sigma^* \varphi| dx' d\sigma &\geq \int_F \left\{ C \varsigma^{-\frac{m-2}{4\gamma}} \left( \frac{C}{\mu_{m-1}(E_\sigma)} \right)^{-(m-1)(2\varsigma^{\frac{1}{2\gamma}} + 1)} \left( \sum_{\alpha' \in \mathcal{D}_m} |b_{\alpha'}|^2 \right)^{\frac{1}{2}} \right\} d\sigma \geq \\ &\geq C \varsigma^{-\frac{m-2}{4\gamma}} \left( \frac{4C}{\mu_m(\mathcal{E})} \right)^{-(m-1)(2\varsigma^{\frac{1}{2\gamma}} + 1)} \int_F \left( \sum_{\alpha' \in \mathcal{D}_m} |b_{\alpha'}|^2 \right)^{\frac{1}{2}} d\sigma. \end{aligned}$$

We deduce that

$$\int_0^\tau \int_\Gamma \chi_{\mathcal{E}'} |B^* \mathbb{T}_\sigma^* \varphi| dx' d\sigma \geq C \varsigma^{-\frac{2m-3}{4\gamma}} \left( \frac{4C}{\mu_m(\mathcal{E})} \right)^{-(m-1)(2\varsigma^{\frac{1}{2\gamma}} + 1)} \sum_{\alpha' \in \mathcal{D}_m} \int_F |b_{\alpha'}| d\sigma. \quad (5.40)$$

In order to find a lower bound for the integral over the measurable set  $F \subset [0, \tau]$ , we use Corollary 5.2. We have that

$$\int_F |b_{\alpha'}| d\sigma \geq e^{-|\alpha'|^2 \tau} \int_F \left| \sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}} - |\alpha'|^2}} a_\alpha \frac{(-1)^{\alpha_m} \alpha_m \pi}{l_m \lambda_\alpha} e^{-\alpha_m^2 \sigma} \right| d\sigma \geq$$

$$\geq \frac{1}{C} \exp\left(-\frac{\kappa}{\mu_1(F)}\right) e^{-|\alpha'|^2\tau} \left[ \sum_{\alpha_m=1}^{\sqrt{\zeta^{\frac{1}{\gamma}}-|\alpha'|^2}} \frac{|a_\alpha|^2 \alpha_m^2 \pi^2}{l_m^2 \lambda_\alpha^2} e^{-\alpha_m^2 \tau} \right]^{\frac{1}{2}}.$$

By taking into account (3.9), we deduce that

$$\int_F |b_{\alpha'}| d\sigma \geq \frac{1}{C} \exp\left(-\frac{4\kappa\mu_{m-1}(\Gamma)}{\mu_m(\mathcal{E})}\right) \left[ \sum_{\alpha_m=1}^{\sqrt{\zeta^{\frac{1}{\gamma}}-|\alpha'|^2}} \left| \frac{a_\alpha}{\lambda_\alpha} \right|^2 e^{-2\lambda_\alpha \tau} \right]^{\frac{1}{2}}. \quad (5.41)$$

Finally, from (5.40) and (5.41) it follows that

$$\begin{aligned} \|\Psi_{\tau, \mathcal{E}'}^d \varphi\|_{L^1(\Gamma \times [0, \tau])} &\geq C \zeta^{-\frac{m-2}{4\gamma}} \left( \frac{4C}{\mu_m(\mathcal{E})} \right)^{-(m-1)} \left( 2\zeta^{\frac{1}{2\gamma}} + 1 \right) \times \\ &\exp\left(-\frac{4\kappa\mu_{m-1}(\Gamma)}{\mu_m(\mathcal{E})}\right) \zeta^{-\frac{1}{\gamma}} \sum_{\alpha' \in \mathcal{D}_m} \left[ \sum_{\alpha_m=1}^{\sqrt{\zeta^{\frac{1}{\gamma}}-|\alpha'|^2}} \left| \frac{a_\alpha}{\lambda_\alpha^{\frac{1}{2}}} \right|^2 e^{-2\lambda_\alpha \tau} \right]^{\frac{1}{2}} \\ &\geq \frac{1}{d_0} e^{-d_1 \left(1 + \ln\left(\frac{1}{\mu_m(\mathcal{E})}\right)\right)} \zeta^{\frac{1}{2\gamma} + \frac{\kappa}{\mu_m(\mathcal{E})}} \left[ \sum_{\lambda_\alpha^\gamma \leq \zeta} \left| \frac{a_\alpha}{\lambda_\alpha^{\frac{1}{2}}} \right|^2 e^{-2\lambda_\alpha \tau} \right]^{\frac{1}{2}}. \quad (5.42) \end{aligned}$$

Choosing  $\gamma = \frac{1}{2}$ , we conclude that  $(A, B)$  satisfies condition (3.8) in Theorem 3.2, which concludes the proof of Theorem 1.1.  $\square$

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