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Min Shen, Gabriel Turinici

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Min SHEN
CEREMADE, Université Paris Dauphine
and
Gabriel TURINICI*
CEREMADE, Université Paris Dauphine

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Abstract

We study in this work the liquidity, defined as the size of the trading volume, in a situation when an infinite number of agents with heterogeneous beliefs reach a trade-off between cost of a precise estimation (variable depending on the agent) and expected profit from trading at the resulting estimate price. The "true" asset price is not known and the market price is set at a level that clears the market. We show that under some technical assumptions the model has natural properties such as monotony of offer and demand functions with respect to the price, existence of an overall equilibrium and monotony with respect to cost of information. We also situate our approach within the Mean Field Games (MFG) framework of Lions and Lasry which allows to obtain an interpretation as a limit of Nash equilibrium for an infinite number of players.

1 Introduction

Liquidity risk is a concept that has been well illustrated by the worldwide financial crisis that started in 2007 (initially centered around "subprime" credits but then extended to the financial sphere since). The models used to price financial products did not take this risk into account and many well known institutions faced substantial looses (some leading to default).

More specifically, when one wants to measure the asset liquidity several notions have been taken into account:

- the bid-ask spread (which takes into account the difference between the price at which a security can be bought and sold based on real quotes available on the market. This notion is useful for operational purposes but is sometimes too short-sighted;

- market depth: Hachmeister [9] defines the market depth as the amount of a security that can be bought and sold at various bid-ask spreads.

- immediacy: it indicates the time needed to successfully trade a certain amount of an asset at a prescribed cost.

*corresponding author Gabriel.Turinici@dauphine.Fr

- resilience: Hachmeister refers to this as the speed with which prices return to former levels after a shock (e.g. a large transaction, etc.) ; this measure requires a time window.

Several modeling approaches have been proposed, such as a limit order book modeling and optimal order submission [3] where the authors study the optimal submission strategies of bid and ask orders in a limit order book. They consider an agent optimizing his utility with a finite and infinite horizon and obtain results such as optimal bid/ask spread etc.

Other authors considered not one but several (types of) agents that hold non-identical estimations (also called heterogeneous beliefs) on the future price of the asset: Gallmeyer and Hollifield ([5]) study the effects of a market-wide short-sale constraint in a dynamic economy with heterogeneous beliefs and analyze the impact on the stock price as generated by the optimistic investors' intertemporal elasticity of substitution. Another paper by Emilio Osambela ([19]) presents a dynamic general-equilibrium economy in which one population of optimistic investors is subject to endogenous liquidity constraints. On the other hand the importance of heterogeneous beliefs on asset pricing has been recognized widely in works by e.g., Jouini et al. [11, 10].

In all these situations the typologies of agents are intrinsically finite as the authors are not interested in what happens when an infinite number of different agent are present. We will suppose here that an infinite number of agents are acting in the market, each having his own methodology to arrive at an estimation of the "true" price of some security. We take the paradigm of heterogeneous beliefs i.e. we suppose that all agents receive the same (costly, see latter) information but they differ in the way to interpret it, more precisely in the way to obtain an estimation out of it. The estimation is obtained in the form of a random variable with a known mean and variance; the agent cannot change the result obtained by his methodology; the particularity of our approach is that he can diminish its variance by paying a price. Each agent optimizes an utility functional. Also, contrary to some previous works we are not interested in the dynamics of the price itself (that we will suppose constant to simplify); instead our focus is on the trading volume (the number of assets traded at the market price) that we will consider a proxy for liquidity. Such a substitute for liquidity is adapted to our setting which is a one period game with no dynamics.

Considering an infinite number of optimizing agents is not technically trivial and we resort to the "Mean Field Games" approach pioneered by Lasry and Lions [16, 14, 15, 17] where an Nash equilibrium with an infinite number of agents is analyzed. Mathematical properties for special cases of functionals (quadratic etc.) and examples of applications and numerical approaches are to be found in several works: in [2] the authors present a finite difference discretization in a finite an infinite time horizon and prove approximation properties, existence and uniqueness, bounds on the solutions; they also introduce a Newton method for the coupled direct-adjoint critical point equations for the finite horizon problem in a convex setting. In [7] the author studies a prototypical case and its stability properties. In [12] the authors present a numerical method and apply to an example of technology change; another modeling example is given in [13]. In [6] MFG are stated in a finite state space. Finally, the so-called "planning problem" where the final density of agents is prescribed is treated in [1].

Our analysis here needs to take into account a dimension which is particular to this setting: the "mean field" that couples the actions of all agents appears

as an equilibrium constraint. Although we only treat a single situation in this paper we expect that the MFG approach can be coupled with constraints on the density of agents and refer to future work for technical details.

The summary of the paper is the following: in Section 2 we explain the basic properties of the model and especially the specific investigation of this work which is the tradeoff between estimation cost and the trading volume. In Section 3 we compare our approach and situate it within the MFG model. Finally in Section 4 we prove the main properties of the model (offer/ demand monotonicity with respect to price, existence of an equilibrium, anti-monotony with respect to precision cost, etc.) and give some illustrative examples.

2 The liquidity model

Let us consider a traded security of "true" value V. The true value is unknown to market participants and will never be revealed. Instead, each agent x constructs his own estimation for V in the form of $V\tilde{A^x}$ where $\tilde{A^x}$ is a random variable; we will consider, for simplicity that $V\tilde{A^x}$ is normal, that $\tilde{A^x}$ and $\tilde{A^y}$ are independent as soon as $x \neq y$ and that the mean of $V\tilde{A^x}$ is VA^x and variance of $V\tilde{A^x}$ is $V^2(\sigma^x)^2$ and are known to agent x. It turns out that for technical reasons it is better to work with "precision" instead of the variance i.e. we introduce $B^x = 1/(\sigma^x)^2$.

We do not try in this model to explain how the agents construct their estimation $V\tilde{A^x}$ but will suppose that each agent has his own (deterministic) methodology that is specific to himself and fixed in advance; the agent cannot influence in any way the mean A^x during the process (but the mean can depend on time); in particular two different agents may (and will in practice) have different estimations (and average estimations A^x). This is not a collateral property of the model but the mere reason for which the agents trade: they trade because they have heterogeneous expectations about the final value of the security.

The only thing that the agent can do is to try to extract as much precision as possible from his methodology i.e. he can change B^x. However improving the precision comes at a cost i.e. the agent has to pay $f(b)$ to arrive at precision b. The precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined on positive numbers (we can take by convention $f(b) = \infty$ for any $b < 0$).

There are many reasons why such a modelisation is realistic, the cost can e.g. come from the cost of information sources (news broadcasting fees etc.), the pay of the research personnel, the need for more precise (but costly) computations, etc.

Based on his estimations the agent will decide to trade θ^x units i.e. the size of the position of the agent on the market is $V \cdot \theta^x$; note that when θ^x is positive this means that the agent is long (buys) and when it is negative it means that the agent is short (sells) the security.

Thus each agent is characterized by three quantities: his mean estimate VA^x, the precision B^x of the estimate and the quantity of units traded θ^x; denote $X = (A, \theta, B)^T$ (here ^{T} denotes the vectorial transposition); we set the investment horizon of all the agents to be the final time $T = 1$.

Remark 1 *The "time" here can be physical "wall-clock" time or "eductive*

time" (cf. [8]) i.e. a the mental time required by the agents to reach a decision.

We denote by $m(t, X)$ the distribution of the agents (a probability measure) at time t with $m(0, X) = m_0(X)$. We will also denote $\mathbb{E}^t$ the mean with respect to the measure $m(t, X)$.

Let us denote by $\rho(t, A)$ the marginal of $m(t, X)$ with respect to the variables θ and B at time t and $\rho_0(A) = \rho(0, A)$. Note that θ, B can (and will) depend on time. However the evolution of A^x is autonomous i.e. not related to B and θ but imposed by the estimation model chosen by the agent once for all at the beginning. It is not subject to decision or to control between the initial and final time. Thus, even when the mean estimation for each agent may depend on time, it is natural to consider an "ergodic" setting where the distribution $\rho(t, A)$, depending only on the autonomous evolution of A^x for each x, is stationary i.e. for all $t \in [0, T]$: $\rho(t, A) = \rho_0(A)$. In particular this is true when A^x is constant. We can then introduce the expectation value with respect to ρ_0 which will be denoted $\mathbb{E}^A$.

From a theoretical point of view it is interesting to consider the situation when the mean $\mathbb{E}^A(A) = 1$ which means that the average estimate is V i.e. the agents are neither overpricing nor underpricing the security with respect to its (unknown) true value. We will see however that this is not necessarily indicating that the market price will be V.

In order to describe the model for the market price, we will introduce the basic notions of total offer (and demand) for a price $p \geq 0$. Namely the total demand at final the time (which sole is of interest to us) will be denoted $D(p)$ and total offer $O(p)$ and are respectively defined as:

$$D(p) = \mathbb{E}^T(\theta_+), \quad O(p) = \mathbb{E}^T(\theta_-). \tag{1}$$

A price p^* such that $O(p^*) = D(p^*)$ will be said to clear the market. Indeed, from definitions of $D(\cdot)$ and $O(\cdot)$ this is equivalent to that $\mathbb{E}^T(\theta) = 0$ i.e. at the price p^* the overall (signed) demand is null. Note that such a price may not exist or may not be unique, cf. Remarks 3, 4 and Figures 1,2 below.

The transaction volume at some price p will be the number of units that can be exchanged at that price i.e is defined as

$$TV(p) = \min\{O(p), D(p)\}. \tag{2}$$

A price p^* where $TV(\cdot)$ attains its maximum is of interest because if will maximize the total number of shares exchanged. Note that such a price may not exists, cf. Remark 4 and Fig. 2 below. It also can be non-unique.

An elementary but important result gives information on the market price and its properties:

Theorem 2 *If $O(p), D(p)$ are continuous, $O(p)$ strictly increasing, $O(0) = 0$, $\lim_{p\to\infty} O(p) > 0$, $D(p)$ strictly decreasing , $D(0) > 0$, $\lim_{p\to\infty} D(p) = 0$, then*
1/ a unique p_1^* exists such that $O(p_1^*) = D(p_1^*)$;
2/ a unique p_2^* exists such that $TV(p_2^*) \geq TV(p)$ for all $p \geq 0$;
3/ $p_1^* = p_2^*$.

Proof. For 1/ let us note that the continuous, strictly monotone function $D - O$ is such that in zero its value is strictly positive and at infinity is strictly negative.

Thus there exists a unique p_1^* where the function vanishes which is the required result. We note that $(D-O)(p)$ is strictly positive for $p < p_1^*$ and strictly negative for $p > p_1^*$.

For 2/ note that

$$TV(p) = \begin{cases} O(p) \text{ for } p < p_1^* \\ D(p_1^*) = O(p_1^*) \text{ for } p = p_1^* \\ D(p) \text{ for } p > p_1^* \end{cases} \tag{3}$$

Then $TV(p_1^*) - TV(p) = O(p_1^*) - O(p)$ for $p \leq p_1^*$ and $D(p_1^*) - D(p)$ for $p \geq p_1^*$. In all situations $TV(p_1^*) - TV(p)$ is positive hence 2/ and 3/. ■

Remark 3 *Conditions on $D(0) = 0$ and similar are technical. But monotonicity is important for the equivalence between the two interpretations: as the maximum of the transaction volume and as matching offer and demand. Indeed it is enough to take as functions $p + \sin(\pi p)$ and $1/p$ (cf. Fig. 1) to understand that there can be several points that clear the market (there are three of them) and none maximize the trading volume. Such a situation is very ambiguous for a market and we want to avoid it.*

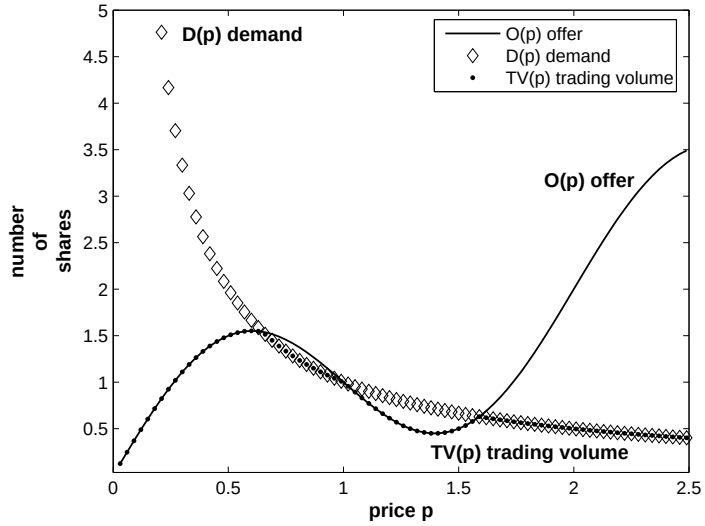

Figure 1: An illustration of the Remark 3. Here $O(p) = p + \sin(\pi p)$ and $D(p) = 1/p$ and $O(p)$ is not monotonic. Several prices exist that clear the market. The first price, situated at about $p = 0.684$, maximizes the trading volume among the points that clear the market (with value around 1.541) but it does not maximize the trading volume $TV(p)$ whose maximum value is around 1.551.

Remark 4 *Continuity is also a crucial ingredient to this interpretation. Indeed, when the offer and demand functions are not continuous a price that maximizes trading volume may not exist, same for a price that clears the market. To illustrate this take for instance $O(p) = 2p^2$, $D(p) = \begin{cases} 4-p \\ 1/p \text{ for } p \geq 1 \end{cases}$ (cf. Fig. 2).*

In this situation the supremum of transaction volumes is 2 *but is not attained by any price; also, no price clears the market i.e. there does not exist p such that* $O(p) = D(p)$. *We enter in this situation the topic of market microstructure; a market maker is necessary on such a market that can smooth out offer and demand through a pricing rule or market making function, cf. [18] for details.*

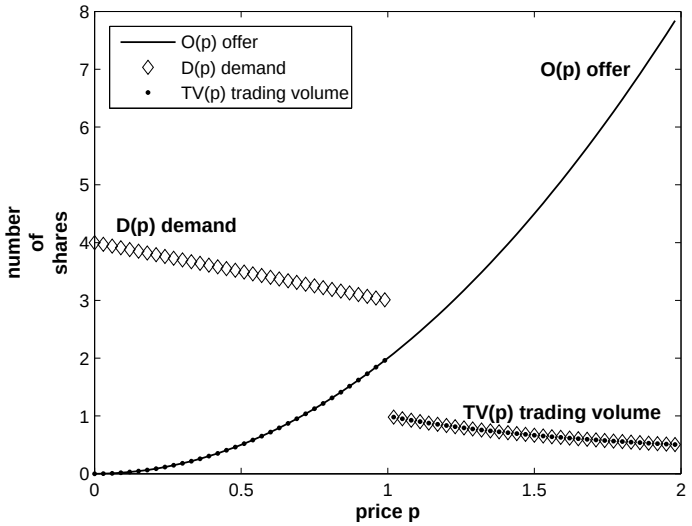

Figure 2: An illustration of the Remark 4: offer and demand functions are discontinuous: no price exists that clears the market; the maximum trading volume is not attained.

The market price at time T, denoted $V\overline{\mathcal{P}}$ equals total offer and demand i.e. the overall demand / offer balance is null; this will give an implicit equation for $\overline{\mathcal{P}}$:

$$\int \theta dm(T, A, \theta, B) = 0 \text{ or equivalently } \mathbb{E}^T(\theta) = 0. \tag{4}$$

In order the model the choices of the agents we will take the classical situation when the agent is maximizing a utility function. Since the uncertainty appears as a normal variable we have two alternatives which coincide: either consider that the utility is a function of the mean and variance of the profit or, equivalently, take an expected utility framework. To keep intuitive understanding we will keep the simplest situation of a utility function $U(u, v) = u - \frac{\lambda}{2}v$ where u is the expected profit and v its variance. Of course, this too simple utility function has several known drawbacks (not a coherent risk measure etc.) but this will not play an important role here and the simple choice above will considerably simplify the results.

Note that all agents have the same utility function.

Of course, the profit itself is a function of θ^x,B^x; the profit is computed under the assumption that the agent expects to buy/sell at market price and expects to sell at a price consistent with his estimation and the uncertainty

of this estimation. Thus the average profit for agent x, denoted u^x is $u^x = V\theta^x(A^x - \overline{\mathcal{P}}) - f(B^x)$; variance of the profit, denoted v^x is $v^x = \frac{(\theta^x)^2 V^2}{(B^x)_+}$ where $(B^x)_+$ is the positive part of B^x with convention that division by zero equals $+\infty$.

Thus agent x optimizes (at final time T):

$$J(X^x) = V\theta^x(A^x - \overline{\mathcal{P}}) - f(B^x) - \frac{\lambda}{2}\frac{(\theta^x)^2 V^2}{(B^x)_+}. \tag{5}$$

Let us say more on the precision cost function $f(b)$: it is the "research cost" to reach the precision b. Conditions for f that seem very natural are $f(0) = 0$, $f'(0) = 0$ (this is to fix the marginal cost at start; this is a non-trivial choice but its implications are not important for the technical results of the paper). We will also consider that f is increasing, strictly convex (this will be seen later to ensure well-posedness), C^2 and $\lim_{x\to\infty} f(x)/x = \infty$.

Now that the model has been set, several important questions are to be addressed to justify that the model corresponds to the intuitive picture one may have of it and also to justify the mathematical well-posedness of the overall problem:

- is the solution unique i.e. does there exist a unique $\overline{\mathcal{P}}$ that solves the equilibrium equation (4)

- are the total demand /offer $D(p)$ / $O(p)$ monotonic functions of p (in order to be within the framework of Thm. 2)?

Note that $\overline{\mathcal{P}}$ (given by Thm. 2) is not necessarily equal to 1 even if $\mathbb{E}^0(A) = 1$.

3 Comparison and interpretation as Mean Field Games (MFG)

The Mean Field Games framework (MFG) is a mathematical model for interaction among a large number of agent / players. An agent can control its situation, based on a set of preferences and by acting on some parameters. MFG can show the emergence of a collective behavior (fashion trends, financial crises, real estates valuation, etc.) out of individual optimization by each agent: while an agent by himself cannot influence the collective behavior (he only optimizes his own decisions given the environmental situation and his decisions have negligible impact on the collective parameters) the collective choices of all agents create an overall environment (the "mean field") that affects in return the individual decisions.

We refer to [16, 14, 15, 17] for further information. The MFG theory shows that a Nash equilibrium for a game of N players will tend, in some specified sense, when $N \to \infty$, to the so-called MFG equations.

Let X_t^x be the characteristics at time t of a agent/ player starting in x at time 0. It evolves with SDE:

$$dX_t^x = \alpha(t, X_t^x)dt + \sigma dW_t^x,\ X_0^x = x \tag{6}$$

where $\alpha(t, X_t^x)$ is the control and can be changed by the agent/ player.

Note that each agent has his own randomness modeled with independent Brownian. Denote by $m(t, x)$ the density of players at time t and position

$x \in E$; E is the state space. The optimization problem of the agent is: for a (fixed) finite horizon T optimize:

$$\inf_{\alpha} \mathbb{E}\left\{ \int_0^T L(X_t^x, \alpha(t, X_t^x)) + V(X_t^x; m(t,\cdot))dt + V_0(X_T^x; m(T,\cdot)) \right\} \tag{7}$$

Operator L encodes constraints or costs on the control while V and V_0 encode the goal. Define $H(x,\xi) = \sup_\alpha \langle \xi, \alpha \rangle - L(x,\alpha)$; $\nu = \sigma^2/2$.

For a finite number of agents (i.e; when $m(t,x)$ is a sum of N Dirac masses) critical point equations can be written that describe a Nash equilibrium; these equations converge (up to sub-sequences) to solutions of the following MFG system for $N \to \infty$:

$$\partial_t m + \operatorname{div}(\alpha m) - \nu \Delta m = 0, \tag{8}$$

$$m(0,x) = m_0(x), \ \int m = 1, \ m \geq 0 \tag{9}$$

$$\alpha = -\frac{\partial}{\partial p} H(x, \nabla u) \tag{10}$$

$$\partial_t u + \nu \Delta u - H(x, \nabla u) + V(x, m) = 0, \tag{11}$$

$$u(T,x) = V_0(x, m(T,\cdot)), \ \int u = 0. \tag{12}$$

To model the situation in Section 2 the evolution equations and the initial probability distribution will be:

$$dX_t^x = d\begin{pmatrix} A_t^x \\ \theta_t^x \\ B_t^x \end{pmatrix} = \begin{pmatrix} \alpha(t, A_t^x) \\ \alpha_\theta(t, X_t^x) \\ \alpha_B(t, X_t^x) \end{pmatrix} dt + \begin{pmatrix} \sigma_A(t, A_t^x) dW_t^A \\ \sigma_\theta(t, X_t^x) dW_t^\theta \\ \sigma_B(t, X_t^x) dW_t^B \end{pmatrix} \tag{13}$$

$$m(t,X)\Big|_{t=0} = m_0(X). \tag{14}$$

We will take operators L and V to be null. Recall that we supposed that autonomous evolution of A^x is defining a stationary distribution ρ_0. To simplify even more the setting we will take in fact A^x to be unchanged and θ^x and B^x to have a deterministic evolution.

$$d\begin{pmatrix} A_t^x \\ \theta_t^x \\ B_t^x \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_\theta(t, X_t^x) \\ \alpha_B(t, X_t^x) \end{pmatrix} dt \tag{15}$$

$$m(t,X)\Big|_{t=0} = m_0(X). \tag{16}$$

To this we add the equilibrium condition above (eqn.(4)). This framework allows to expect an interpretation of our setting: a Nash equilibrium for an infinite number of players. Note that we do not explicitly show the relationship between the Nash equilibrium of N agents and the results in the next section corresponding to an infinite number of agents.

4 Theoretical results

4.1 Existence of an equilibrium

Theorem 5 *Suppose that the precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that: $f(0) = 0$, $f'(0) = 0$; suppose also that f is increasing, strictly convex, of C^2 class and $\lim_{x\to\infty} f(x)/x = \infty$. Then:*

- *the optimal precision cost B^x and trading size θ^x are*

$$B^x = (f')^{-1}\left(\frac{(A^x - p)^2}{2\lambda}\right) \tag{17}$$

$$\theta^x = \frac{(A^x - p)B^x}{2\lambda V} = \frac{(A^x - p)}{2\lambda V}(f')^{-1}\left(\frac{(A^x - p)^2}{2\lambda}\right); \tag{18}$$

  In particular both are explicit functions of A^x.

- *offer $O(p)$ and demand $D(p)$ are strictly monotone with respect to p.*

- *an equilibrium price $\overline{\mathcal{P}}$ that clears the market (eqn.(4)) exists and is unique:*

$$\overline{\mathcal{P}} = \frac{\mathbb{E}^A(AB)}{\mathbb{E}^A(B)}. \tag{19}$$

Proof. An agent only sees the others through the market price $\overline{\mathcal{P}}$. If we consider now a possibly non-equilibrium price p as given then the agent optimizes the functional $J(X^x(T))$ which only depends on the final state $X^x(T)$ and not on the controls. Since the controls allow to obtain each possible configuration for $X^x(T)$ (compatible with the constraint that A^x is fixed) then the values $B^x(T)$ and $\theta^x(T)$ will be optimum of the function

$$\mathcal{J}(y, z) = Vy(A^x - p) - f(z) - \frac{\lambda}{2}\frac{y^2V^2}{z},\ for\ y > 0. \tag{20}$$

Let us denote y^*, z^* an optimum candidate. Asking that $\frac{\partial \mathcal{J}}{\partial y} = 0$ one obtains $y^* = \frac{(A^x-p)z^*}{\lambda V}$; then z^* optimizes the function $\frac{(A^x-p)^2}{2\lambda}z - f(z)$. If is straightforward to see that under hypothesis taken on f optimal points indeed exist and satisfy (17)-(18).

In order to prove the strict monotonicity of the offer and demand functions with respect to p it is enough to prove that e.g. $(\theta^x)_+$ is monotone with respect to p. This is a consequence of the fact that $(A^x - p)_+$ is (strictly) monotone with respect to $p < A^x$ and in the same domain $(f')^{-1}\left(\frac{(A^x-p)^2}{2\lambda}\right)$ is also monotone because of the assumptions on f, namely convexity, regularity and $f(0) = 0 = f'(0)$.

The monotonicity, by Thm 2, implies that a unique price that clears the market exists and this price also maximizes the trading volume.

Note that θ^x is a function of A^x, that can be written $\theta^x = \theta(A^x)$, same for $B^x = B(A^x)$. Thus equation (4) can be written $\mathbb{E}^T(\theta) = 0$ and also $\mathbb{E}^A\left(\frac{(A-p)B(A)}{2\lambda V}\right) = 0$ which gives the conclusion. ■

Remark 6 *Assumptions on f can be weakened (cf. [4]).*

In general the price $\overline{\mathcal{P}}$ depends on the cost function $f(\cdot)$. But for the particular case when a symmetry exists the following results proves independence:

Corollary 7 *Under assumptions in Thm. 5 on function f if ρ_0 is symmetric around p^1 then $\overline{\mathcal{P}} = p^1$ in particular is independent of $f(\cdot)$.*

Proof. The proof proceeds from the remark that $B(A)$ is a function symmetric around p thus $\theta(A)$ is antisymmetric. If the distribution ρ_0 is symmetric then for $p = p^1$ we have $\mathbb{E}^A(\theta(A)) = 0$ which means, by uniqueness, that $\overline{\mathcal{P}} = p^1$. ■

Remark 8 *Analog results holds for more general utility functions U (cf. [20]).*

Thus the relative market price $\overline{\mathcal{P}}$ is solution to the equation:

$$\mathbb{E}^A\left[(A-\overline{\mathcal{P}})(f')^{-1}\left(\frac{(A-\overline{\mathcal{P}})^2}{2\lambda}\right)\right] = 0 \tag{21}$$

We denote by TV_f the equilibrium trading volume for precision cost function f; it satisfies the relation:

$$TV_f = \frac{\overline{\mathcal{P}}}{2\lambda}\mathbb{E}^A\left[(A-\overline{\mathcal{P}})_+(f')^{-1}\left(\frac{(A-\overline{\mathcal{P}})^2}{2\lambda}\right)\right]. \tag{22}$$

4.2 Application for a power precision cost function

Let us take a particular case $f(b) = \mu\frac{b^\alpha}{\alpha}$ with $\alpha > 1$, $\mu > 0$. Then we have

$$B(A) = \left(\frac{(A-\overline{\mathcal{P}})^2}{2\lambda\mu}\right)^{\frac{1}{\alpha-1}} \tag{23}$$

and $\overline{\mathcal{P}}$ satisfies:

$$\frac{1}{2V\lambda(2\mu\lambda)^{\frac{1}{\alpha-1}}}\mathbb{E}^A(A-\overline{\mathcal{P}})|A-\overline{\mathcal{P}}|^{\frac{2}{\alpha-1}} = 0 \tag{24}$$

and

$$TV_f = \frac{1}{2\lambda(2\mu\lambda)^{\frac{1}{\alpha-1}}}\mathbb{E}^A(A-\overline{\mathcal{P}})_+|A-\overline{\mathcal{P}}|^{\frac{2}{\alpha-1}} \tag{25}$$

We note that the trading volume is inversely correlated with the risk aversion coefficient λ which means the more risk averse agents are, the less they trade. The same holds for the "cost of precision" μ: the more expensive the information is, the less transactions the market has, which is compatible with the situation of liquidity crisis where a sudden increase in the cost of precision can limit the market liquidity.

It is also interesting to compute the (optimal) expected profit for an agent having average estimation A; this profit is:

$$\left(\frac{\alpha-1}{2\lambda\alpha(2\mu\lambda)^{\frac{1}{\alpha-1}}}\right)|A-\overline{\mathcal{P}}|^{\frac{2\alpha}{\alpha-1}} \tag{26}$$

Note that for $\alpha > 1$ the profit is strictly positive. For $\alpha = 1$ the formula is not valid and the profit is infinity.

If A tends to infinity then the expected profit also tends to infinity, which means that the larger A is the more the agent expects to win. But if the distribution of A decreases when A becomes large (this should necessary be the case in order for a first moment to exist) then the probability to be in this situation is small which means that large profits are only expected by a negligible amount of agents involved. Of course the real profit of each agent is zero because the price does not change in our model.

By using this strategy, total expected profit of the entire market is finite as soon as the distribution $\rho_0(A)$ has moments of order $\frac{2\alpha}{\alpha-1}$ i.e.

$$\mathbb{E}^A\left(\frac{\alpha-1}{2\lambda\alpha(2\mu\lambda)^{\frac{1}{\alpha-1}}}\right)|A-\overline{\mathcal{P}}|^{\frac{2\alpha}{\alpha-1}} < \infty \tag{27}$$

For the particular case $\alpha = 2$ we obtain (after simplifications) the equation for market price:

$$\mathbb{E}^A(A-\overline{\mathcal{P}})^3 = 0, \tag{28}$$

which tells us that if the third central moment of the distribution $\rho_0(A)$ is null then $\overline{\mathcal{P}} = 1$ and thus the price is exactly the true price V. The formula is interesting in itself and also because it shows that the mere condition $\mathbb{E}^A(A) = 1$ does not insure that the market will trade at the "true" price V.

Other information cost functions than a polynomial one can be proposed such as exponential function $f(b) = \mu\left(e^{\xi b} - 1 - b\xi\right)$, $\xi \in \mathbb{R}$.

4.3 Dependence of the trading volume on the precision cost function

A different set of questions refers to the precision cost function f. A result that investigated the properties of the trading volume in relation to f is the following:

Theorem 9 (anti-monotony of trading volume) *Let f, g be two precision cost functions fulfilling the hypothesis of Thm. 5. Let also f, g be such that $g'(b) \geq f'(b)$ for any $b \in \mathbb{R}_+$. Denote by TV_f and TV_g the equilibrium trading volumes for precision cost functions f and g respectively. Then $TV_f \geq TV_g$.*

Remark 10 *Each function will generate* **its own** *market price; the monotony is not necessarily true for other market price except the equilibrium ones (which may be different or not). We saw that if the distribution $\rho_0(A)$ is symmetric then the prices will be the same because they are independent of the cost functions.*

Proof. Let us note that if a function is monotone its inverse (when it exists) is also monotone and of the same type of monotonicity. Since f and g are convex it follows that f' and g' are monotone increasing.

Denote $F = (f')^{-1}$ and $G = (g')^{-1}$; from the hypothesis we obtain from $g' > f'$ that $F \geq G$ with both F and G being increasing functions. For any precision cost function h denoted by $D(h, p)$ the demand at price p given by the formula

$$D(h,p) = \frac{1}{2V\lambda}\mathbb{E}^A\left[(A-p)_+(h')^{-1}\left(\frac{(A-p)^2}{2\lambda}\right)\right] \tag{29}$$

and symmetrically the offer at price p

$$O(h,p) = \frac{1}{2V\lambda}\mathbb{E}^A\left[(A-p)_-(h')^{-1}\left(\frac{(A-p)^2}{2\lambda}\right)\right]. \tag{30}$$

Note that $D(h,p)$ is a decreasing function of p and $O(h,p)$ is increasing. Recall that the equilibrium price $\overline{\mathcal{P}}_f$ is the one that equals offer and demand i.e. satisfies:

$$D(f,\overline{\mathcal{P}}_f) = O(f,\overline{\mathcal{P}}_f) \tag{31}$$

and the corresponding equation for g. Since $F \geq G$ one obtains that for any price p: $O(g,p) \leq O(f,p)$ and also $D(g,p) \leq D(f,p)$. In particular $O(g,\overline{\mathcal{P}}_g) = D(g,\overline{\mathcal{P}}_g) \leq D(f,\overline{\mathcal{P}}_g)$. Define P_1 as the solution of the equation : $O(g,P_1) = D(f,P_1)$ (such a solution exists because $O(g,\cdot) - D(f,\cdot)$ is continuous, the value in zero is strictly negative and value at infinity strictly positive). Then $P_1 \geq \overline{\mathcal{P}}_g$ because $O(g,p)$ is increasing and $D(f,p)$ is decreasing. In a symmetric way one can prove that $P_1 \geq \overline{\mathcal{P}}_f$.

Then $\frac{TV_g}{V} = O(g,\overline{\mathcal{P}}_g) \leq O(g,P_1) = D(f,P_1) \leq D(f,\overline{\mathcal{P}}_f) = \frac{TV_f}{V}$ hence the conclusion. ■

4.4 Results under weaker hypotheses on the precision cost function

The hypotheses accepted so far on the precision cost function $f(B)$ are rather strong: strictly convex of C^2 class. We relax in this section these assumptions but refer to [4] for optimal results.

Theorem 11 *Suppose that the precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is convex (thus continuous for $b > 0$), $f(0) = 0$ and f is continuous in 0. Also assume f to be coercive in the sense that $\liminf_{x\to\infty} f(x)/x = \infty$. Then for each given price p each agent x attains its optimum in at least a (possibly non-unique) configuration with precision B^x and order volume θ^x; moreover θ^x is monotone (decreasing) with respect to p. Finally, the overall demand and offer functions $D(p)$ and $O(p)$ are also monotone with respect to p.*

Proof.

As in the proof of Thm 5 we denote by $y^*(p), z^*(p)$ an optimum candidate where we explicitly mark the dependence on p. Since the functional $\mathcal{J}$ in equation (20) is differentiable with respect to z we obtain as before from $\frac{\partial \mathcal{J}}{\partial y} = 0$ that

$$y^*(p) = \frac{(A^x - p)z^*(p)}{\lambda V}; \tag{32}$$

then $z^*(p)$ optimizes the function $g_p(z) = \frac{(A^x-p)^2}{2\lambda}z - f(z)$. Since f is not necessarily differentiable, nor strictly convex the optimum exists (because $f(0) = 0$, f is continuous and coercive) but is not necessarily unique.

Take $A^x \leq p_1 \leq p_2$ and suppose that some choice of optimums $z^*(p_1)$ and $z^*(p_2)$ exists such that $z^*(p_1) > z^*(p_2)$. Using the optimality properties for $z^*(p_1)$ and $z^*(p_2)$ we obtain:

$$g_{p_2}(z^*(p_2)) - g_{p_1}(z^*(p_2)) \geq g_{p_2}(z^*(p_1)) - g_{p_1}(z^*(p_1))$$

thus

$$\frac{(p_2 - A^x)^2}{2\lambda} z^*(p_2) - \frac{(p_1 - A^x)^2}{2\lambda} z^*(p_2) \geq \frac{(p_2 - A^x)^2}{2\lambda} z^*(p_1) - \frac{(p_1 - A^x)^2}{2\lambda} z^*(p_1)$$

which implies

$$z^*(p_2) \geq z^*(p_1). \tag{33}$$

which contradicts our hypothesis.

Recall now that z^* stands for the optimal value of B^x thus we have monotonicity for B^x.

Recall also that the optimal value of θ^x is given by the formula (32); we obtain thus the monotonicity of θ^x for $p \geq A^x$. An analogous argument works on the branch $p \leq A^x$ and since the optimal θ^x for $p = A^x$ is zero we obtain the monotony of θ^x with respect to p. The monotony of overall offer and demand functions $D(p)$ and $O(p)$ follow. ■

Remark 12 *We do not claim that $O(p)$ and $D(p)$ are necessarily continuous functions nor that the monotonicity is strict. This precludes the use of Thm. 2. But it is obvious that convexity is better than just continuity, which means that additional properties of $D(p)$ and $O(p)$ can be proved. We refer to [4] for details.*

4.5 Further comments

As this model is concerned only with deriving a formula for the trading volume, the dynamics of the "true" price was not considered. Of course, it would be interesting to take this into account; also a further refinement concerns the estimation process $\tilde{A}$ and its costs that may possess stochastic dynamics; we refer to future work for such follow-ups [21].

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