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The Journal of Symbolic Logic
Volume 00, Number 0, XXX 0000

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Abstract. We consider ω^n-automatic structures which are relational structures whose domain and relations are accepted by automata reading ordinal words of length ω^n for some integer $n \geq 1$. We show that all these structures are ω-tree-automatic structures presentable by Muller or Rabin tree automata. We prove that the isomorphism relation for ω^2-automatic (resp. ω^n-automatic for $n > 2$) boolean algebras (respectively, partial orders, rings, commutative rings, non commutative rings, non commutative groups) is not determined by the axiomatic system ZFC. We infer from the proof of the above result that the isomorphism problem for ω^n-automatic boolean algebras, $n \geq 2$, (respectively, rings, commutative rings, non commutative rings, non commutative groups) is neither a Σ^1_2-set nor a Π^1_2-set. We obtain that there exist infinitely many ω^n-automatic, hence also ω-tree-automatic, atomless boolean algebras $\mathcal{B}_n$, $n \geq 1$, which are pairwise isomorphic under the continuum hypothesis CH and pairwise non isomorphic under an alternate axiom AT, strengthening a result of [14].

§1. Introduction. An automatic structure is a relational structure whose domain and relations are recognizable by finite automata reading finite words. Automatic structures have very nice decidability and definability properties and have been much studied in the last few years, see [7, 8, 26, 33, 38, 39]. They form a subclass of the class of (countable) recursive structures where "recursive" is replaced by "recognizable by finite automata". Blumensath considered in [6] more powerful kinds of automata. If we replace automata by tree automata (respectively, Büchi automata reading infinite words, Muller or Rabin tree automata reading infinite labelled trees) then we get the notion of tree-automatic (respectively, ω-automatic, ω-tree-automatic) structures. Notice that an ω-automatic or ω-tree-automatic structure may have uncountable cardinality. All these kinds of automatic structures have the two following fundamental properties. (1) The class of automatic (respectively, tree-automatic, ω-automatic, ω-tree-automatic) structures is closed under first-order interpretations. (2) The first-order theory of an automatic (respectively, tree-automatic, ω-automatic, ω-tree-automatic) structure is decidable.

On the other hand, automata reading words of ordinal length had been firstly considered by Büchi in his investigation of the decidability of the monadic second

Key words and phrases. Automata reading ordinal words; ω^n-automatic structures; ω-tree-automatic structures; boolean algebras; partial orders; rings; groups; isomorphism relation; models of set theory; independence results.

0022-4812/00/0000-0000/$00.00

order theory of a countable ordinal, see [1, 16] and also [45, 46, 2, 4, 3] for further references on the subject. We investigate in this paper ω^n-automatic structures which are relational structures whose domain and relations are accepted by automata reading ordinal words of length ω^n for some integer $n \geq 1$. All these structures are ω-tree-automatic structures presentable by Muller or Rabin tree automata.

A fundamental question about classes of automatic structures is the following: "what is the complexity of the isomorphism problem for some class of automatic structures?" The isomorphism problem for the class of automatic structures, or even for the class of automatic graphs, is Σ_1^1-complete, [26]. On the other hand, the isomorphism problem is decidable for automatic ordinals or for automatic boolean algebras, see [26, 39]. Some more results about other classes of automatic structures may be found in [30]: in particular, the isomorphism problem for automatic linear orders is not arithmetical. Hjorth, Khoussainov, Montalbán, and Nies proved that the isomorphism problem for ω-automatic structures is not a Σ_2^1-set, [17]. More Recently, Kuske, Liu, and Lohrey proved in [29] that the isomorphism problem for ω-automatic structures (respectively, partial orders, trees of finite height) is not even analytical, i.e. is not in any class Σ_n^1 where $n \geq 1$ is an integer. In [14] we recently proved that the isomorphism relation for ω-tree-automatic structures (respectively, ω-tree-automatic boolean algebras, partial orders, rings, commutative rings, non commutative rings, non commutative groups) is not determined by the axiomatic system ZFC. This showed the importance of different axiomatic systems of Set Theory in the area of ω-tree-automatic structures.

We prove here that the isomorphism relation for ω^2-automatic (resp. ω^n-automatic for $n > 2$) boolean algebras (respectively, partial orders, rings, commutative rings, non commutative rings, non commutative groups) is not determined by the axiomatic system ZFC. We infer from the proof of the above result that the isomorphism problem for ω^n-automatic boolean algebras, $n \geq 2$, (respectively, rings, commutative rings, non commutative rings, non commutative groups) is neither a Σ_2^1-set nor a Π_2^1-set. We obtain that there exist infinitely many ω^n-automatic, hence also ω-tree-automatic, atomless boolean algebras $\mathcal{B}_n$, $n \geq 1$, which are pairwise isomorphic under the continuum hypothesis CH and pairwise non isomorphic under an alternate axiom AT (for "almost trivial"). This way we improve our result of [14], where we used the open coloring axiom OCA instead, in two ways:

(1) by constructing infinitely many structures with independent isomorphism problem, instead of only two such structures, and
(2) by finding such structures which are much simpler than the ω-tree-automatic ones, because they are even ω^n-automatic.

The paper is organized as follows. In Section 2 we recall definitions and first properties of automata reading ordinal words and of tree automata. In Section 3 we define ω^n-automatic structures and ω-tree-automatic structures and we prove simple properties of ω^n-automatic structures. We introduce in Section 4 some particular ω^n-automatic boolean algebras $\mathcal{B}_n$. We recall in Section 5 some results of Set Theory and recall in particular the Axiom AT ('almost trivial') and some related notions. We investigate in Section 6 the isomorphism relation for ω^n-automatic structures and for ω-tree-automatic structures. Some concluding remarks are given in Section 7.

§2. Automata.

2.1. ω^n-Automata. When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_0.a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 0. The *length* of x is $k+1$. The *empty word* has no letter and is denoted by ε; its length is 0. For $x = a_0.a_1 \ldots a_k$, we write $x(i) = a_i$. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.

We assume the reader to be familiar with the elementary theory of countable ordinals. Let Σ be a finite alphabet, and α be an ordinal; a word of length α (or α-word) over the alphabet Σ is an α-sequence $(x(\beta))_{\beta<\alpha}$ (or sequence of length α) of letters in Σ. The set of α-words over the alphabet Σ is denoted by Σ^α. The concatenation of an α-word $x = (x(\beta))_{\beta<\alpha}$ and of a γ-word $y = (y(\beta))_{\beta<\gamma}$ is the $(\alpha+\gamma)$-word $z = (z(\beta))_{\beta<\alpha+\gamma}$ such that $z(\beta) = x(\beta)$ for $\beta < \alpha$ and $z(\beta) = y(\beta')$ for $\alpha \leq \beta = \alpha + \beta' < \alpha + \gamma$; it is denoted $z = x \cdot y$ or simply $z = xy$.

We assume that the reader is familiar with the notion of Büchi automaton reading infinite words over a finite alphabet which can be found for instance in [42, 40]. Informally speaking an ω-word x over Σ is accepted by a Büchi automaton $\mathcal{A}$ iff there is an infinite run of $\mathcal{A}$ on x enterring infinitely often in some final state of $\mathcal{A}$. The ω-language $L(\mathcal{A}) \subseteq \Sigma^\omega$ accepted by the Büchi automaton $\mathcal{A}$ is the set of ω-words x accepted by $\mathcal{A}$. A Muller automaton is a finite automaton equipped with a set $\mathcal{F}$ of accepting sets of states. An ω-word x over Σ is accepted by a Muller automaton $\mathcal{A}$ iff there is an infinite run of $\mathcal{A}$ on x such that the set of states appearing infinitely often during this run is an accepting set of states, i.e. belongs to $\mathcal{F}$. It is well known that an ω-language is accepted by a Büchi automaton iff it is accepted by a Muller automaton.

We shall define ω^n-automatic structures as relational structures presentable by automata reading words of length ω^n, for some integer $n \geq 1$. In order to read some words of transfinite length greater than ω, an automaton must have a transition relation for successor steps defined as usual but also a transition relation for limit steps. After the reading of a word whose length is a limit ordinal, the state of the automaton will depend on the set of states which cofinally appeared during the run of the automaton. These automata have been firstly considered by Büchi, see [1, 16]. We recall now their definition and behaviour.

Definition 2.1 ([45, 46, 2]). *An ordinal Büchi automaton is a sextuple $(\Sigma, Q, q_0, \Delta, \gamma, F)$, where Σ is a finite alphabet, Q is a finite set of states, $q_0 \in Q$ is the initial state, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation for successor steps, and $\gamma \subseteq P(Q) \times Q$ is the transition relation for limit steps.*

A run of the ordinal Büchi automaton $\mathcal{A} = (\Sigma, Q, q_0, \Delta, \gamma, F)$ reading a word σ of length α, is an $(\alpha+1)$-sequence of states x defined by: $x(0) = q_0$ and, for $i < \alpha$, $(x(i), \sigma(i), x(i+1)) \in \Delta$ and, for i a limit ordinal, $(Inf(x,i), x(i)) \in \gamma$, where $Inf(x,i)$ is the set of states which cofinally appear during the reading of *the i first letters of σ, i.e.*

$$Inf(x,i) = \{q \in Q \mid \forall \mu < i, \exists \nu < i \text{ such that } \mu < \nu \text{ and } x(\nu) = q\}$$

A run x of the automaton $\mathcal{A}$ over the word σ of length α is called successful if $x(\alpha) \in F$. A word σ of length α is accepted by $\mathcal{A}$ if there exists a successful run of $\mathcal{A}$ over σ. We denote $L_\alpha(\mathcal{A})$ the set of words of length α which are accepted by $\mathcal{A}$. An α-language L is a regular α-language if there exists an ordinal Büchi automaton $\mathcal{A}$ such that $L = L_\alpha(\mathcal{A})$.

An ordinal Büchi automaton $(\Sigma, Q, q_0, \Delta, \gamma, F)$ is said to be deterministic iff $\Delta \subseteq Q \times \Sigma \times Q$ is in fact the graph of a function from $Q \times \Sigma$ into Q and $\gamma \subseteq P(Q) \times Q$ is the graph of a function from $P(Q)$ into Q. In that case there is at most one run of the automaton over a given word σ.

REMARK 2.2. *When we consider only finite words, the language accepted by an ordinal Büchi automaton is a rational language. If we consider only ω-words, the ω-languages acceped by ordinal Büchi automata are the ω-languages accepted by Muller automata and then also by Büchi automata.*

DEFINITION 2.3. *An ω^n-automaton is an ordinal Büchi automaton reading only words of length ω^n for some integer $n \geq 1$.*

We can obtain regular ω^n-languages from regular ω-languages and regular ω^{n-1}-languages by the use of the notion of substitution. The following result appeared in [16] and has been also proved in [13].

PROPOSITION 2.4. *Let $n \geq 2$ be an integer. An ω^n-language $L \subseteq \Sigma^{\omega^n}$ is regular iff it is obtained from a regular ω-language $R \subseteq \Gamma^\omega$ by substituting in every ω-word $\sigma \in R$ a regular ω^{n-1}-language $L_a \subseteq \Sigma^\omega$ to each letter $a \in \Gamma$.*

We now recall some fundamental properties of regular ω^n-languages.

THEOREM 2.5 (Büchi-Siefkes, see [1, 16, 2]). *Let $n \geq 1$ be an integer. One can effectively decide whether the ω^n-language $L(\mathcal{A})$ accepted by a given ω^n-automaton $\mathcal{A}$ is empty or not.*

THEOREM 2.6. *[see [2]] Let $n \geq 1$ be an integer. The class of regular ω^n-languages is effectively closed under finite union, finite intersection, and complementation, i.e. we can effectively construct, from two ω^n-automata $\mathcal{A}$ and $\mathcal{B}$, some ω^n-automata $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, such that $L(\mathcal{C}_1) = L(\mathcal{A}) \cup L(\mathcal{B})$, $L(\mathcal{C}_2) = L(\mathcal{A}) \cap L(\mathcal{B})$, and $L(\mathcal{C}_3)$ is the complement of $L(\mathcal{A})$.*

We assume the reader to be familiar with basic notions of topology that may be found in [32, 24, 34]. The usual Cantor topology on Σ^ω is the product topology obtained from the discrete topology on the finite set Σ, for which *open subsets* of Σ^ω are in the form $W \cdot \Sigma^\omega$, where $W \subseteq \Sigma^\star$.

Let $n \geq 1$ be an integer. Let $B : \omega \to \omega^n$ be a recursive bijection. Then we have a bijection ϕ from Σ^{ω^n} onto Σ^ω defined by $\phi(x)(n) = x(B(n))$ for each integer $n \geq 0$. Then for each ω^n-language $L \subseteq \Sigma^{\omega^n}$ we have the associated ω-language $\phi(L) = \{\phi(x) \mid x \in L\}$. Consider now a *regular* ω^n-language $L \subseteq \Sigma^{\omega^n}$. It is stated in [9] that $\phi(L)$ is Borel (in the class Σ^0_{2n+1}).

2.2. Tree automata. We introduce now languages of infinite binary trees whose nodes are labelled in a finite alphabet Σ.

A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where r means "right" and l means "left". Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T^ω_Σ. A tree language is a subset of T^ω_Σ, for some alphabet Σ. (Notice that we shall only consider in the sequel *infinite* trees so we shall often use the term tree instead of *infinite* tree).

Let t be a tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the prefix relation $\sqsubseteq$ and which is closed under this prefix relation, i.e. if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$ then $x \in B$. A branch B of a tree is said to be maximal iff there is not any other branch of t which strictly contains B.

Let t be an infinite binary tree in T^ω_Σ. If B is a maximal branch of t, then this branch is infinite. Let $(u_i)_{i\geq 0}$ be the enumeration of the nodes in B which is strictly increasing for the prefix order. The infinite sequence of labels of the nodes of such a maximal branch B, i.e. $t(u_0)t(u_1)\ldots t(u_n)\ldots$ is called a path. It is an ω-word over the alphabet Σ.

For a tree $t \in T^\omega_\Sigma$ and $u \in \{l, r\}^\star$, we shall denote $t_u : \{l, r\}^\star \to \Sigma$ the subtree defined by $t_u(v) = t(uv)$ for all $v \in \{l, r\}^\star$. It is in fact the subtree of t which is rooted in u.

We are now going to define tree automata and regular languages of infinite trees.

DEFINITION 2.7. *A (nondeterministic) tree automaton is a quadruple $\mathcal{A} = (Q, \Sigma, \Delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state and $\Delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation. A run of the tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T^\omega_\Sigma$ is an infinite binary tree $\rho \in T^\omega_Q$ such that:*
(a) $\rho(\varepsilon) = q_0$ and (b) for each $u \in \{l, r\}^\star$, $(\rho(u), t(u), \rho(ul), \rho(ur)) \in \Delta$.

A Muller (nondeterministic) tree automaton is a 5-tuple $\mathcal{A} = (Q, \Sigma, \Delta, q_0, \mathcal{F})$, where (Q, Σ, Δ, q_0) is a tree automaton and $\mathcal{F} \subseteq 2^Q$ is the collection of designated state sets. A run ρ of the Muller tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T^\omega_\Sigma$ is said to be accepting if for each path p of ρ, the set of states appearing infinitely often on this path is in $\mathcal{F}$. The tree language $L(\mathcal{A})$ accepted by the Muller tree automaton $\mathcal{A}$ is the set of infinite binary trees $t \in T^\omega_\Sigma$ such that there is (at least) one accepting run of $\mathcal{A}$ on t. A tree language $L \subseteq T^\omega_\Sigma$ is regular iff there exists a Muller automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$.

We now recall some fundamental closure properties of regular tree languages.

THEOREM 2.8 (Rabin, see [36, 42, 15, 34]). *The class of regular tree languages is effectively closed under finite union, finite intersection, and complementation, i.e. we can effectively construct, from two Muller tree automata $\mathcal{A}$ and*

$\mathcal{B}$, some Muller tree automata $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, such that $L(\mathcal{C}_1) = L(\mathcal{A}) \cup L(\mathcal{B})$, $L(\mathcal{C}_2) = L(\mathcal{A}) \cap L(\mathcal{B})$, and $L(\mathcal{C}_3)$ is the complement of $L(\mathcal{A})$.

§3. Automatic structures. Notice that one can consider a relation $R \subseteq \Sigma_1^{\omega^n} \times \Sigma_2^{\omega^n} \times \ldots \times \Sigma_k^{\omega^n}$, where $\Sigma_1, \Sigma_2, \ldots \Sigma_k$, are finite alphabets, as an ω^n-language over the product alphabet $\Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_k$. In a similar way, we can consider a relation $R \subseteq T^{\omega}_{\Sigma_1} \times T^{\omega}_{\Sigma_2} \times \ldots \times T^{\omega}_{\Sigma_k}$, as a tree language over the product alphabet $\Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_k$.

Let now $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq k})$ be a relational structure, where M is the domain, and for each $i \in [1, k]$ R_i^M is a relation of finite arity n_i on the domain M. The structure is said to be ω^n-automatic (respectively, ω-tree-automatic) if there is a presentation of the structure where the domain and the relations on the domain are accepted by ω^n-automata (respectively, by Muller tree automata), in the following sense.

DEFINITION 3.1 (see [6]). *Let $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq k})$ be a relational structure, where $k \geq 1$ is an integer, and each relation R_i is of finite arity n_i.*
An ω-tree-automatic presentation of the structure $\mathcal{M}$ is formed by a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq k})$, and a mapping h from $L(\mathcal{A})$ onto M, such that:

1. *The automaton $\mathcal{A}_=$ accepts an equivalence relation $E_\equiv$ on $L(\mathcal{A})$, and*
2. *For each $i \in [1, k]$, the automaton $\mathcal{A}_i$ accepts an n_i-ary relation R'_i on $L(\mathcal{A})$ such that $E_\equiv$ is compatible with R'_i, and*
3. *The mapping h is an isomorphism from the quotient structure $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq k})/E_\equiv$ onto $\mathcal{M}$.*

The ω-tree-automatic presentation is said to be injective if the equivalence relation $E_\equiv$ is just the equality relation on $L(\mathcal{A})$. In this case $\mathcal{A}_=$ and $E_\equiv$ can be omitted and h is simply an isomorphism from $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq k})$ onto $\mathcal{M}$. A relational structure is said to be (injectively) ω-tree-automatic if it has an (injective) ω-tree-automatic presentation.

Notice that sometimes an ω-tree-automatic presentation is only given by a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq k})$, i.e. *without the mapping h*. In that case we still get the ω-tree-automatic structure $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq k})/E_\equiv$ which is in fact equal to $\mathcal{M}$ up to isomorphism.

We get the definition of ω^n-automatic (injective) presentation of a structure and of ω^n-automatic structure by simply replacing Muller tree automata by ω^n-automata in the above definition.

Notice that, due to the good decidability properties of Muller tree automata and of ω^n-automata, we can decide whether a given automaton $\mathcal{A}_=$ accepts an equivalence relation $E_\equiv$ on $L(\mathcal{A})$ and whether, for each $i \in [1, k]$, the automaton $\mathcal{A}_i$ accepts an n_i-ary relation R'_i on $L(\mathcal{A})$ such that $E_\equiv$ is compatible with R'_i.

We denote ω^n-AUT the class of ω^n-automatic structures and ω-tree-AUT the class of ω-tree-automatic structures.

We state now two important properties of automatic structures.

THEOREM 3.2 (see [6]). *The class of ω-tree-automatic (respectively, ω^n-automa -tic) structures is closed under first-order interpretations. In other words if $\mathcal{M}$ is an ω-tree-automatic (respectively, ω^n-automatic) structure and $\mathcal{M}'$ is a relational structure which is first-order interpretable in the structure $\mathcal{M}$, then the structure $\mathcal{M}'$ is also ω-tree-automatic (respectively, ω^n-automatic).*

THEOREM 3.3 (see [18, 6]). *The first-order theory of an ω-tree-automatic (respectively, ω^n-automatic) structure is decidable.*

Notice that ω-tree-automatic (respectively, ω^n-automatic) structures are always *relational* structures. However we can also consider structures equipped with functional operations like groups, by replacing as usually a p-ary function by its graph which is a $(p+1)$-ary relation. This will always be the case in the sequel where all structures are viewed as relational structures.

Some examples of ω-automatic structures can be found in [38, 33, 26, 27, 8, 31, 17, 29].

A first one is the boolean algebra $\mathcal{P}(\omega)$ of subsets of ω.

The additive group $(\mathbb{R}, +)$ is ω-automatic, as is the product $(\mathbb{R}, +) \times (\mathbb{R}, +)$.

Assume that a finite alphabet Σ is linearly ordered. Then the set $(\Sigma^{\omega^n}, \leq_{lex})$ of ω^n-words over the alphabet Σ, equipped with the lexicographic ordering, is ω^n-automatic.

Is is easy to see that every (injectively) ω-automatic structure is also (injectively) ω-tree-automatic. Indeed a Muller tree automaton can easily simulate a Büchi automaton on the leftmost branch of an infinite tree.

The inclusions ω^n-AUT $\subseteq \omega^{n+1}$-AUT, $n \geq 1$, are straightforward to prove.

PROPOSITION 3.4. *For each integer $n \geq 1$, ω^n-$AUT \subseteq \omega$-tree-AUT.*

PROOF. We are first going to associate a tree t^x to each ω^n-word x in such a way that if $L \subseteq \Sigma^{\omega^n}$ is a regular ω^n-language then the tree language $\{t^x \in T_\Sigma^\omega \mid x \in L\}$ will be also regular. We make this by induction on the integer n. Let then Σ be a finite alphabet and $a \in \Sigma$ be a distinguished letter in Σ. We begin with the case $n = 1$. If $x \in \Sigma^\omega$ is an ω-word over the alphabet Σ then t^x is the tree in T_Σ^ω such that $t^x(l^k) = x(k)$ for every integer $k \geq 0$ and $t^x(u) = a$ for every word $u \in \{l, r\}^\star$ such that $u \notin \{l^k \mid k \geq 0\}$. It is clear that if $L \subseteq \Sigma^\omega$ is a regular ω-language then the tree language $\{t^x \in T_\Sigma^\omega \mid x \in L\}$ is a regular set of trees. Assume now that we have associated, for a given integer $n \geq 1$, a tree t^x to each ω^n-word $x \in \Sigma^{\omega^n}$ in such a way that if $L \subseteq \Sigma^{\omega^n}$ is a regular ω^n-language then the tree language $\{t^x \in T_\Sigma^\omega \mid x \in L\}$ is also regular. Consider now an ω^{n+1}-word x over Σ. It can be divided into ω subwords x_j, $0 \leq j < \omega$, of length ω^n. By induction hypothesis to each ω^n-word x_j is associated a tree $t^{x_j} \in T_\Sigma^\omega$. Recall that we denote by t_u the subtree of t which is rooted in u. We can now associate to the ω^{n+1}-word x the tree t^x which is defined by: $t^x_{l^k \cdot r} = t^{x_k}$ for every integer $k \geq 0$, and $t^x(l^k) = a$ for every integer $k \geq 0$. Let then now $L \subseteq \Sigma^{\omega^{n+1}}$ be a regular ω^{n+1}-language. By Proposition 2.4 the language L is obtained from a regular ω-language $R \subseteq \Gamma^\omega$ by substituting in every ω-word $\sigma \in R$ a regular ω^n-language $L_b \subseteq \Sigma^\omega$ to each letter $b \in \Gamma$. By induction hypothesis for each letter $b \in \Gamma$ there is a tree automaton $\mathcal{A}_b$ such that $L(\mathcal{A}_b) = \{t^x \in T_\Sigma^\omega \mid x \in L_b\}$. This implies easily that one can construct, from a Büchi automaton accepting the regular ω-language R and from the tree automata $\mathcal{A}_b$, $b \in \Gamma$, another tree automaton $\mathcal{A}$ such that $L(\mathcal{A}) = \{t^x \in T_\Sigma^\omega \mid x \in L\}$.

The inclusion ω^n-AUT $\subseteq$ ω-tree-AUT holds because any element of the domain of an ω^n-automatic structure, represented by an ω^n-word x, can also be represented by a tree t^x. The relations of the structure are then also presentable by tree automata. $\dashv$

Notice that the strictness of the inclusion $\bigcup_{n\geq 1} \omega^n$-AUT $\subsetneq$ ω-tree-AUT follows easily from the existence of an ω-tree-automatic structrure without Borel presentation, proved in [17], and the fact that every ω^n-automatic structrure has a Borel presentation (see the end of Section 2.1).

On the other hand we can easily see that the inclusion ω-AUT $\subsetneq$ ω^2-AUT is strict by considering ordinals. Firstly, Kuske recently proved in [28] that the ω-automatic ordinals are the ordinals smaller than ω^ω. Secondly, it is easy to see that the ordinal ω^ω is ω^2-automatic. The ordinal ω^ω is the order-type of finite sequences of integers ordered by (1) increasing length of sequences and (2) lexicographical order for sequences of integers of the same length n. A finite sequence of integers $x = (n_1, n_2, \ldots, n_p)$ can be represented by the following ω^2-word α_x over the alphabet $\{a, b\}$:

$$\alpha_x = (a^{n_1+1} \cdot b^\omega) \cdot (a^{n_2+1} \cdot b^\omega) \cdots (a^{n_p+1} \cdot b^\omega) \cdot (b^\omega) \cdot (b^\omega) \cdots$$

it is then easy to see that there is an ω^2-automaton accepting exactly the ω^2-words of the form α_x for a finite sequence of integers x. Moreover there is an ω^2-automaton recognizing the pairs $(\alpha_x, \alpha_{x'})$ such that $x < x'$.

§4. Some ω^n-automatic boolean algebras. We have seen that the boolean algebra $\mathcal{P}(\omega)$ of subsets of ω is ω-automatic. Another known example of ω-automatic boolean algebra is the boolean algebra $\mathcal{P}(\omega)/\text{Fin}$ of subsets of ω modulo *finite sets*. The set Fin of finite subsets of ω is an ideal of $\mathcal{P}(\omega)$, i.e. a subset of the powerset of ω such that:

1. $\emptyset \in \text{Fin}$ and $\omega \notin \text{Fin}$.
2. For all $B, B' \in \text{Fin}$, it holds that $B \cup B' \in \text{Fin}$.
3. For all $B, B' \in \mathcal{P}(\omega)$, if $B \subseteq B'$ and $B' \in \text{Fin}$ then $B \in \text{Fin}$.

For any two subsets A and B of ω we denote $A\Delta B$ their symmetric difference. Then the relation $\approx$ defined by: "$A \approx B$ iff the symmetric difference $A\Delta B$ is finite" is an equivalence relation on $\mathcal{P}(\omega)$. The quotient $\mathcal{P}(\omega)/\approx$ denoted $\mathcal{P}(\omega)/\text{Fin}$ is a boolean algebra. It is easy to see that this boolean algebra is ω-automatic, see for example [31, 17, 14].

More generally we now consider the boolean algebras $\mathcal{P}(\omega^n)/I_{\omega^n}$ for integers $n \geq 1$. We first give the definition of the sets $I_{\omega^n} \subseteq \mathcal{P}(\omega^n)$. For $P \subseteq \omega^n$ we denote $o.t.(P)$ the order type of $(P, <)$ as a suborder of the order $(\omega^n, <)$. The set I_{ω^n} is defined by:

$$I_{\omega^n} = \{P \subseteq \omega^n \mid o.t.(P) < \omega^n\}.$$

For each integer $n \geq 1$ the set I_{ω^n} is an ideal of $\mathcal{P}(\omega^n)$.

For any two subsets A and B of ω^n we denote $A\Delta B$ their symmetric difference. Then the relation $\approx_n$ defined by: "$A \approx_n B$ iff the symmetric difference $A\Delta B$

is in I_{ω^n}" is an equivalence relation on $\mathcal{P}(\omega^n)$. The quotient $\mathcal{P}(\omega^n)/\approx_n$, also denoted $\mathcal{P}(\omega^n)/I_{\omega^n}$, is a boolean algebra.

We are going to show that this boolean algebra $\mathcal{P}(\omega^n)/I_{\omega^n}$ is ω^n-automatic.

We first notice that each set $P \subseteq \omega^n$ can be represented by an ω^n-word x_P over the alphabet $\{0,1\}$ by setting $x_P(\alpha) = 1$ if and only if $\alpha \in P$ for every ordinal $\alpha < \omega^n$. Let then

$$L_n = \{x_P \in \{0,1\}^{\omega^n} \mid P \in I_{\omega^n}\}.$$

THEOREM 4.1. *Let $n \geq 1$ be an integer. Then the set $L_n \subseteq \{0,1\}^{\omega^n}$ is a regular ω^n-language.*

PROOF. We reason by induction on the integer n. Firstly it is easy to see that L_1 is the set of ω-words over the alphabet $\{0,1\}$ having only finitely many letters 1. It is a well known example of a regular ω-language, see [34, 42]. Notice that its complement $L_1^- = \{0,1\}^\omega \setminus L_1$ is then also regular since the class of regular ω-languages is closed under complementation.

We now assume that we have proved that for each integer $k \leq n$ the set $L_k \subseteq \{0,1\}^{\omega^k}$ is a regular ω^k-language. In particular the language L_n is a regular ω^n-language. Moreover its complement $L_n^- = \{0,1\}^{\omega^n} \setminus L_n$ is then also regular since the class of regular ω^n-languages is closed under complementation.

Consider now a set $P \subseteq \omega^{n+1}$. It is easy to see that P belongs to $I_{\omega^{n+1}}$ if and only if there are only finitely many integers $k \geq 0$ such that $P \cap [\omega^n.k; \omega^n.(k+1)[$ has order type ω^n.

Thus the ω^{n+1}-language L_{n+1} is obtained from the regular ω-language $L_1 = \{0,1\}^\star \cdot 0^\omega$ by substituting the ω^n-language L_n^- to the letter 1 and the ω^n-language L_n to the letter 0. We can conclude, using Proposition 2.4, that the ω^{n+1}-language L_{n+1} is regular. $\dashv$

We can now state the following result.

THEOREM 4.2. *For every integer $n \geq 1$ the boolean algebra $\mathcal{P}(\omega^n)/I_{\omega^n}$ is ω^n-automatic.*

PROOF. Let $n \geq 1$ be an integer. We denote $[A]_{I_{\omega^n}}$, or simply $[A]$ when there is no confusion from the context, the equivalence class of a set $A \subseteq \omega^n$ for the equivalence relation $\approx_n$. Let $\Sigma = \{0,1\}$ and $L(\mathcal{A}) = \Sigma^{\omega^n}$ and for any $x \in \Sigma^{\omega^n}$, $h(x) = [\{\alpha < \omega^n \mid x(\alpha) = 1\}]$. Then it follows easily from the preceding Theorem 4.1 that $\{(u,v) \in (\Sigma^{\omega^n})^2 \mid h(u) = h(v)\}$ is accepted by an ω^n-automaton.

The operations $\cap, \cup, \neg$, of intersection, union, and complementation, on $\mathcal{P}(\omega^n)/I_{\omega^n}$ are defined by: $[B] \cap [B'] = [B \cap B']$, $[B] \cup [B'] = [B \cup B']$, and $\neg[B] = [\neg B]$, see [20].

Thus the operations of intersection, union, (respectively, complementation), considered as ternary relations (respectively, binary relation) are also given by regular ω^n-languages. On the other hand, $\mathbf{0} = [\emptyset]$ is the equivalence class of the empty set and $\mathbf{1} = [\omega^n]$ is the class of ω^n.

This proves that the structure $(\mathcal{P}(\omega^n)/I_{\omega^n}, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ is ω^n-automatic. $\dashv$

Notice that, as in the above proof, we can see that the relation $\{(u,v) \in (\Sigma^{\omega^n})^2 \mid h(u) \subseteq_n h(v)\}$ is a regular ω^n-language because the "almost inclusion" relation $\subseteq_n$ is defined by: $h(u) \subseteq_n h(v)$ iff $\{\alpha < \omega^n \mid u(\alpha) > v(\alpha)\} \in I_{\omega^n}$. Thus we can also state the following result.

THEOREM 4.3. *For each integer $n \geq 1$ the structure $(\mathcal{P}(\omega^n)/I_{\omega^n}, \subseteq_n)$ is ω^n-automatic.*

From now on we shall denote $\mathcal{B}_n = (\mathcal{P}(\omega^n)/I_{\omega^n}, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$. The boolean algebra $\mathcal{B}_n$ is ω^n-automatic hence also ω-tree-automatic.

Recall now the definition of an atomless boolean algebra.

DEFINITION 4.4. *Let $\mathcal{B} = (B, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ be a boolean algebra and $\subseteq$ be the inclusion relation on B defined by $x \subseteq y$ iff $x \cap y = x$ for all $x, y \in B$. Then the boolean algebra $\mathcal{B}$ is said to be an atomless boolean algebra iff for every $x \in B$ such that $x \neq \mathbf{0}$ there exists an element $z \in B$ such that $\mathbf{0} \subset z \subset x$.*

We can now recall the following known result.

PROPOSITION 4.5. *For each integer $n \geq 1$ the boolean algebra $\mathcal{B}_n$ is an atomless boolean algebra.*

PROOF. Let $n \geq 1$ be an integer. Consider the boolean algebra $\mathcal{B}_n = (\mathcal{P}(\omega^n)/I_{\omega^n}, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$. Let $A \subseteq \omega^n$ be such that the equivalence class $[A]$ is different from the element $\mathbf{0}$ in $\mathcal{B}_n$. Then the set A has order type ω^n and it can be splitted in two sets A_1 and A_2 such that $A = A_1 \cup A_2$ and both A_1 and A_2 have still order type ω^n. The element $[A_1]$ is different from the element $\mathbf{0}$ in $\mathcal{B}_n$ because A_1 has order type ω^n, and $[A_1] \subset [A]$ because $A - A_1 = A_2$ has order type ω^n. Thus the following strict inclusions hold in $\mathcal{B}_n$: $\mathbf{0} \subset [A_1] \subset [A]$. This proves that the boolean algebra $\mathcal{B}_n$ is atomless. ⊣

The following result is also well known but we give a proof for completeness.

PROPOSITION 4.6. *For each integer $n \geq 1$ the boolean algebra $\mathcal{B}_n$ has the cardinality $2^{\aleph_0}$ of the continuum.*

PROOF. By recursion on $n \geq 1$ we define a partition of ω^n into a sequence F^n_k $(k < \omega)$ of nonempty finite sets such that for every infinite $X \subseteq \omega$, the union $\Phi^n(X) = \bigcup_{k \in X} F^n_k$ does not belong to the ideal I_{ω^n}. For $n = 1$, let $F^1_k = \{k\}$. For $n > 1$, set $F^n_k = \bigcup_{\ell < k} F^{n-1}_k(\ell)$, where for each $\ell < \omega$ we have fixed a decomposition $F^{n-1}_k(\ell)$ $(k < \omega)$ of the interval $[\ell\omega^{n-1}, (\ell+1)\omega^{n-1})$ of ordinals into a sequence of finite nonempty pairwise disjoint sets with the property that $\bigcup_{k \in X} F^{n-1}_k(\ell)$ has order type ω^{n-1} for every infinite $X \subseteq \omega$.

Clearly $\Phi^n : \mathcal{P}(\omega) \to \mathcal{P}(\omega^n)$ is a complete Boolean algebra embedding which induces also an embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$ into $\mathcal{P}(\omega^n)/I_{\omega^n}$. Since $\mathcal{P}(\omega)/\mathrm{Fin}$ has cardinality continuum the conclusion follows. ⊣

In the sequel we shall often identify the powerset $\mathcal{P}(A)$ of a countable set A with the Cantor space $2^\omega = \{0,1\}^\omega$. Then $\mathcal{P}(A)$ can be equipped with the standard metric topology obained from this identification, and the topological notions like open, closed, Σ^0_2, Borel, analytic, can be applied to families of subsets of A.

REMARK 4.7. In fact a similar result holds for an arbitrary Σ^1_1-ideal of subsets of ω. More precisely, by a well-known result of Talagrand ([41] ; Théorème 21), for every proper Σ^1_1-ideal I of subsets of ω there is a partition of ω into a sequence F_n $(n < \omega)$ of nonempty finite sets such that for every infinite $X \subseteq \omega$, the union $\bigcup_{n \in X} F_n$ does not belong to the ideal I. Therefore, as above, there is an embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$ into $\mathcal{P}(\omega^n)/I$.

REMARK 4.8. Recall that two subsets X and Y are said to be *almost disjoint* if their intersection is finite. Recall that while a countable index set does not admit an uncountable family of pairwise disjoint subsets it does admit an uncountable family of subsets that are pairwise almost disjoint. So fix an uncountable family $\mathcal{F}$ of pairwise almost disjoint infinite subsets of ω. For $n \geq 1$ fix a sequence F^n_k $(k < \omega)$ of nonempty finite subsets of ω^n such that for every infinite $X \subset \omega$, the union $\Phi^n(X) = \bigcup_{k \in X} F^n_k$ does not belong to the ideal I_{ω^n} (see the proof of Proposition 4.6). For $X \in \mathcal{F}$ let $A_X = \bigcup_{k \in X} F^n_k$. Then A_X $(X \in \mathcal{F})$ is an uncountable family of infinite subsets of ω^n which is also almost disjoint (i.e., $A_X \cap A_Y \in \mathrm{Fin}$ for $X \neq Y$ in $\mathcal{F}$) but it has the additional property that $A_X \notin I_{\omega^n}$ for all $X \in \mathcal{F}$.

§5. Axioms of set theory. We now recall some basic notions of set theory which will be useful in the sequel, and which are exposed in any textbook on set theory, like [20].

The usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. A model $(\mathbf{V}, \in)$ of the axiomatic system ZFC is a collection $\mathbf{V}$ of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of ZFC. We shall often say " the model $\mathbf{V}$" instead of "the model $(\mathbf{V}, \in)$".

The axioms of ZFC express some natural facts that we consider to hold in the universe of sets.

The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$

We recall that Cantor's Continuum Hypothesis CH states that the cardinality of the continuum $2^{\aleph_0}$ is equal to the first uncountable cardinal $\aleph_1$. Gödel and Cohen have proved that the continuum hypothesis CH is independent from the axiomatic system ZFC. This means that, assuming ZFC is consistent, there are some models of ZFC + CH and also some models of ZFC + $\neg$ CH, where $\neg$ CH denotes the negation of the continuum hypothesis, [20].

If $\mathbf{V}$ is a model of ZF and $\mathbf{L}$ is the class of *constructible sets* of $\mathbf{V}$, then the class $\mathbf{L}$ forms a model of ZFC + CH.

Recall also that OCA denotes the *Open Coloring Axiom* (or *Todorcevic's axiom* as it is called in the more recent literature; see, for example, [10]), a natural alternative to CH that has been first considered by the second author in [43]. It is known that if the theory ZFC is consistent, then so are the theories (ZFC + CH) and (ZFC + OCA), see [20, pages 176 and 577]. In particular, if $\mathbf{V}$ is a model of (ZFC + OCA) and if $\mathbf{L}$ is the class of *constructible sets* of $\mathbf{V}$, then the class $\mathbf{L}$ forms a model of (ZFC + CH).

The axiom OCA was used in our previous paper [14] on tree-automatic structures but here we shall use another related axiom first considered by Just [21]. To introduce this axiom we need some definitions.

Let A and B be two infinite countable sets, a function $H : \mathcal{P}(A) \to \mathcal{P}(B)$ and an ideal I of $\mathcal{P}(B)$ containing all finite subsets of B but not the whole set B. Then the function H is said to *preserve intersections modulo* I whenever

(i) $H(X) \triangle H(Y) \in I$ for every $X, Y \subseteq A$ such that $X \triangle Y \in \mathrm{Fin}$, and
(ii) $H(X \cap Y) \triangle (H(X) \cap H(Y)) \in I$ for every $X, Y \subseteq A$

Recall that one can identify the powerset $\mathcal{P}(B)$, where B is a countable set, with the set 2^B equipped with the Cantor topology which is the product topology of the discrete topology on B. Thus one can also use notions like open, closed, Borel, analytic, for ideals of $\mathcal{P}(B)$, where $n \geq 1$ is an integer.

Then Just's axiom AT (where the shorthand stands for 'Almost Trivial') states that for every $\boldsymbol{\Sigma}^1_1$-ideal I of subsets of ω, for every $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ which preserves intersections modulo I, and for every uncountable family $\mathcal{A}$ of pairwise almost disjoint infinite subsets of ω there exist $A \in \mathcal{A}$ and a finite decomposition $A = \bigcup_{i<n} A_i$ such that for every $i < n$ there is a *continuous function*[1] $F_i : \mathcal{P}(A_i) \to \mathcal{P}(\omega)$ such that $F_i(X) \triangle H(X) \in I$ for every $X \subseteq A_i$.

The axiom AT implies also the following form which will be used in the sequel: For every $\boldsymbol{\Sigma}^1_1$-ideal I of subsets of ω^n, for every $H : \mathcal{P}(\omega^m) \to \mathcal{P}(\omega^n)$ which preserves intersections modulo I, and for every uncountable family $\mathcal{A}$ of pairwise almost disjoint infinite subsets of ω^m there exist $A \in \mathcal{A}$ and a finite decomposition $A = \bigcup_{i<k} A_i$ such that for every $i < k$ there is a *continuous function* $F_i : \mathcal{P}(A_i) \to \mathcal{P}(\omega^n)$ such that $F_i(X) \triangle H(X) \in I$ for every $X \subseteq A_i$.

In [21], Just showed that every model $\mathbf{V}$ of ZFC admits a forcing extension satisfying ZFC + AT. In another paper ([22]) he showed that OCA implies many instances of AT. In particular, it is shown in [22] that OCA implies AT restricted to the class of all $\boldsymbol{\Sigma}^0_2$-ideals of subsets of ω. This was later extended by Farah [12] to a larger class of $\boldsymbol{\Sigma}^1_1$-ideals of subsets of ω, however it is still not known if OCA implies the full AT. The motivation behind the axiom AT came from the theory of quotient Boolean algebras of the form $\mathcal{P}(\omega)/I$ where I is a proper (i.e., $\omega \notin I$) ideal on ω which we always assume to include the ideal Fin of all finite subsets of ω. Let $\pi_I : \mathcal{P}(\omega) \to \mathcal{P}(\omega)/I$ denotes the natural quotient map, i.e. $\pi_I(X) = \pi_I(Y)$ whenever $X \triangle Y \in I$. A homomorphism

$$\Phi : \mathcal{P}(\omega)/I \to \mathcal{P}(\omega)/J$$

between two such quotient Boolean algebras is usually given by its *lifting* $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ i.e., a map for which the following diagram

[1]Continuity here is interpreted when we make the standard identification of $\mathcal{P}(A_i)$ and $\mathcal{P}(\omega)$ with the Cantor cubes 2^{A_i} and 2^ω, respectively.

$$\begin{array}{ccc} \mathcal{P}(\omega) & \xrightarrow{H} & \mathcal{P}(\omega) \\ \downarrow \pi_I & & \downarrow \pi_J \\ \mathcal{P}(\omega)/\ I & \xrightarrow{\Phi} & \mathcal{P}(\omega)/J \end{array}$$

commutes. Note that any such lifting H preserves intersections modulo the range ideal J. Note also that in general $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ does not need to be a Boolean algebra homomorphism. It is therefore quite natural to ask for conditions on the given ideals I and J on ω and the homomorphism $\Phi : \mathcal{P}(\omega)/I \to \mathcal{P}(\omega)/J$ that would guarantee the existence of liftings $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ that preserve the Boolean algebra operations of the algebra $\mathcal{P}(\omega)$, even the infinitary ones. Such liftings $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ are called *completely additive liftings*. Note that such completely additive liftings are always given by maps $h : \omega \to \omega$ in such a way that

$$H(\{n\}) = h^{-1}(n) \text{ for all } n < \omega.$$

It follows that $H(X) = h^{-1}(X)$ for all $X \subseteq \omega$ and so from this we can conclude that every completely additive lifting $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is a *continuous map* when we make the natural identification of $\mathcal{P}(\omega)$ with the Cantor set 2^ω. Thus AT asserts the seemingly weak form of this, the *local continuity* of liftings between quotient algebras over Σ^1_1-ideals I of subsets of ω. While local continuity of liftings is a matter of additional axioms of set theory, the second author (see, for example, [44], Problem 1) has posed a problem about the natural mathematical counterpart of this asking under which conditions continuous liftings can be turned into completely additive ones. In subsequent work of Farah [11] and Kanovei-Reeken [23] this conjecture has been verified for a very wide class of ideals I of subsets of ω. We shall use the following particular result from this work, which is a reformulation of [23, Theorem 2], using the fact that a continuous homomorphism $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega^\xi)$ is actually completely additive. Notice that below the boolean algebras $\mathcal{P}(\omega^\xi)/I_{\omega^\xi}$ are a direct generalization of the boolean algebras $\mathcal{P}(\omega^n)/I_{\omega^n}$ and that we shall in fact only use in the sequel the case where the ordinal $\xi \geq 1$ is an integer.

THEOREM 5.1. *(see [23, Theorem 2]) For every countable ordinal $\xi \geq 1$, if a homomorphism*

$$\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega^\xi)/I_{\omega^\xi}$$

has a continuous lifting $H : \mathcal{P}(\omega) \to \mathcal{P}(\omega^\xi)$ then it also has a completely additive lifting, or in other words, there is a map $h : \omega^\xi \to \omega$ such that

$$\Phi(X) = [h^{-1}(X)]_{I_{\omega^\xi}} \text{ for all } X \subseteq \omega.$$

§6. The isomorphism relation. We had proved in [14] that there exist two ω-tree automatic boolean algebras $\mathcal{B}$ and $\mathcal{B}'$ such that: (1) (ZFC + CH) $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic. (2) (ZFC + OCA) $\mathcal{B}$ and $\mathcal{B}'$ are not isomorphic. We are going to prove a similar result for the class of ω^n-automatic structures, for any integer $n \geq 2$ using AT in place of OCA.

We first recall the following folklore result (see, for example, [11]).

THEOREM 6.1. (ZFC + CH) *The boolean algebras $\mathcal{B}_n$, $n \geq 1$, are pairwise isomorphic.*

Notice that this result is an immediate consequence of the simple fact that each of the Boolean algebras $\mathcal{B}_n$, $n \geq 1$, is $\aleph_1$-saturated. Therefore assuming CH, as the boolean algebras $\mathcal{B}_n$, $n \geq 1$, are all of cardinality $\aleph_1$, a well-known Cantor's back and forth argument will give us the isomorphisms. (The reader may find these notions in a textbook on Model Theory, like [35]).

Note that if $1 \leq m \leq n$ the equality $\omega^p \omega^m = \omega^n$ for $p = n - m$ transfers easily to the existence of a map $f : \omega^n \to \omega^m$ with the property that for every subset $X \subseteq \omega^m$, $X \in I_{\omega^m}$ if and only if $f^{-1}(X) \in I_{\omega^n}$. It follows that the corresponding map $X \mapsto f^{-1}(X)$ is a lifting of an isomorphic embedding $\Phi : \mathcal{P}(\omega^m)/I_{\omega^m} \to \mathcal{P}(\omega^n)/I_{\omega^n}$ and therefore we have the following fact.

PROPOSITION 6.2. (ZFC) *The algebra $\mathcal{B}_m$ is isomorphic to a subalgebra of $\mathcal{B}_n$ whenever m and n are positive integers such that $m \leq n$.*

We shall see that such an isomorphic embedding is not always possible if we have the inequality $m > n$. We shall use the following consequence of a well-known result of Rotman [37] which one can prove by an easy induction on $n \geq 1$.

LEMMA 6.3. *Suppose that n is an integer ≥ 1 and that β is some ordinal. Then for every mapping $f : \omega^n \to \beta$ there is $Y \subseteq \omega^n$ of order type ω^n such that the image $f(Y)$ is a subset of β of order type at most ω^n.*

PROOF. Consider firstly a mapping $f : \omega \to \beta$ where β is some ordinal. If the order-type of $f(\omega)$ is strictly greater than ω then there is a subset Z of $f(\omega)$ which has order-type ω. But then $Y = f^{-1}(Z)$ is a subset of ω which has also order-type ω and $f(Y) = Z$ has order-type ω. Assume now that the result is proved for every integer $1 \leq p < n$ and let $f : \omega^n \to \beta$ be a mapping where β is some ordinal. The ordinal ω^n can be decomposed into ω successive intervals $(I_k)_{k \geq 1}$ of length ω^{n-1}. We can now consider the restriction f_k of f to the interval I_k. By induction hypothesis, for each integer $k \geq 1$ there is a subset Y_k of I_k which has order-type ω^{n-1} and such that $f_k(Y_k)$ has order type at most ω^{n-1}. The set $Y = \bigcup_{1 \leq k} Y_k \subseteq \omega^n$ has order type ω^n and its image $f(Y)$ is a subset of β of order type at most ω^n.

$\dashv$

We are now ready to state the following result.

THEOREM 6.4. (ZFC + AT) *The ω^n-automatic boolean algebras $\mathcal{B}_n$, $n \geq 1$, are pairwise non isomorphic and in fact $\mathcal{B}_m$ is not isomorphic to a subalgebra of $\mathcal{B}_n$ whenever $m > n \geq 1$.*

PROOF. Suppose that for some $m > n \geq 1$ there is an isomorphic embedding

$$\Phi : \mathcal{P}(\omega^m)/I_{\omega^m} \to \mathcal{P}(\omega^n)/I_{\omega^n}.$$

Choose a lifting $H : \mathcal{P}(\omega^m) \to \mathcal{P}(\omega^n)$ for the isomorphic embedding, i.e., a map for which the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}(\omega^m) & \xrightarrow{H} & \mathcal{P}(\omega^n) \\ \downarrow \pi_{I_{\omega^m}} & & \downarrow \pi_{I_{\omega^n}} \\ \mathcal{P}(\omega^m)/\, I_{\omega^m} & \xrightarrow{\Phi} & \mathcal{P}(\omega^n)/I_{\omega^n} \end{array}$$

It follows that for every $X, Y \subseteq \omega^m$,

(1) $X \bigtriangleup Y \in I_{\omega^m}$ if and only if $H(X) \bigtriangleup H(Y) \in I_{\omega^n}$,
(2) $H(X) \in \Phi([X]_{I_{\omega^m}})$.

So, in particular, H preserves intersections modulo I_{ω^n}. Using the argument appearing in Remark 4.8 above, there is an uncountable family $\mathcal{A}$ of pairwise almost disjoint (i.e., $A \cap B \in \mathrm{Fin}$ for all $A \neq B$ from $\mathcal{A}$) infinite subsets of ω^m such that $A \notin I_{\omega^m}$ for all $A \in \mathcal{A}$. By AT there is $A \in \mathcal{A}$ and a finite decomposition $A = \bigcup_{i<k} A_i$ and for every $i < k$ a continuous function[2] $F_i : \mathcal{P}(A_i) \to \mathcal{P}(\omega^n)$ forming a lifting of Φ when restricted to $\mathcal{P}(A_i)/I_{\omega^m}$, or in other words a function for which the restricted diagram

$$\begin{array}{ccc} \mathcal{P}(A_i) & \xrightarrow{F_i} & \mathcal{P}(\omega^n) \\ \downarrow \pi_{I_{\omega^m}} & & \downarrow \pi_{I_{\omega^n}} \\ \mathcal{P}(A_i)/\, I_{\omega^m} & \xrightarrow{\Phi} & \mathcal{P}(\omega^n)/I_{\omega^n} \end{array}$$

commutes. In particular, for $X \subseteq A_i$, we have that $F_i(X) \in I_{\omega^n}$ if and only if $X \in I_{\omega^m}$. Since $A \notin I_{\omega^m}$ and the decomposition $A = \bigcup_{i<k} A_i$ is finite there is some $i < k$ such that $A_i \notin I_{\omega^m}$. Fix $i < k$ such that $A_i \notin I_{\omega^m}$. Let $B = H(A_i)$. By Theorem 5.1, there is a function $f_i : B \to A_i$ which induces the completely additive lifting $X \mapsto f_i^{-1}(X)$ of the restriction of Φ to $\mathcal{P}(A_i)/\, I_{\omega^m}$. Since F_i is also a lifting of this isomorphic embedding, we have that

$$F_i(X) \bigtriangleup f_i^{-1}(X) \in I_{\omega^n} \text{ for every } X \subseteq A_i.$$

It follows in particular that for $X \subseteq A_i$,

$$X \in I_{\omega^m} \text{ if and only if } f_i^{-1}(X) \in I_{\omega^n}.$$

By Lemma 6.3, we can find a set $Y \subseteq B$ of order type ω^n whose image $X = f_i(Y)$ is a subset of A_i of order type at most $\omega^n < \omega^m$. But then, we have a subset X of A_i of order type $< \omega^m$ whose preimage $f_i^{-1}(X)$ has order type ω^n as it contains the set Y, i.e. $X \in I_{\omega^m}$ and $f_i^{-1}(X) \notin I_{\omega^n}$, a contradiction. This completes the proof. $\dashv$

We can now state the following consequence of the above theorems.

Corollary 6.5. *The isomorphism relation for ω^2-automatic (respectively, ω^n-automatic for $n > 2$) boolean algebras (respectively, partial orders) is not determined by the axiomatic system* ZFC.

[2]Recall the identifications $\mathcal{P}(A_i) = 2^{A_i}$ and $\mathcal{P}(\omega^n) = 2^{\omega^n}$ which are giving us the topologies to which the continuity refers to.

Proof. The result for ω^n-automatic boolean algebras, $n \geq 2$, follows directly from Theorems 6.1 and 6.4 and the fact that the boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are ω^2-automatic, hence also ω^n-automatic for $n > 2$. For partial orders, we consider the ω^2-automatic structures $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq_1)$ and $(\mathcal{P}(\omega^2)/I_{\omega^2}, \subseteq_2)$. These two structures are isomorphic if and only if the two boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic, see [20, page 79]. Then the result for partial orders follows from the case of boolean algebras. ⊣

Reasoning as in [14] for ω-tree-automatic structures, we can now get similar results for other classes of ω^n-automatic structures.

First a boolean algebra $(B, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ can be seen as a commutative ring with unit element $(B, \Delta, \cap, \mathbf{1})$, where Δ is the symmetric difference operation. The operations of union and complementation can be defined from the symmetric difference and intersection operations. Moreover two boolean algebras $(B, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ and $(B', \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ are isomorphic if and only if the rings $(B, \Delta, \cap, \mathbf{1})$ and $(B', \Delta, \cap, \mathbf{1})$ are isomorphic. For each integer $n \geq 1$, we denote $\mathcal{R}_n = (\mathcal{P}(\omega^n)/I_{\omega^n}, \Delta, \cap, \mathbf{1})$ the commutative ring associated with the boolean algebra $\mathcal{B}_n$.

Theorem 6.6.
1. (ZFC + CH) *The ω^n-automatic commutative rings $\mathcal{R}_n$, $n \geq 1$, are pairwise isomorphic.*
2. (ZFC + AT) *The ω^n-automatic commutative rings $\mathcal{R}_n$, $n \geq 1$, are pairwise non isomorphic.*

Recall that $M_k(R)$ is the set of square matrices with k columns and k rows and coefficients in a given ring R. If $k \geq 2$ then the set $M_k(R)$, equipped with addition and multiplication of matrices, is a non commutative ring. The ring $M_k(R)$ is first-order interpretable in the ring R; each matrix M being represented by a unique k^2-tuple of elements of R, the addition and multiplication of matrices are first order definable in R.

On the other hand, for each integer $n \geq 1$, the class of ω^n-automatic structures is closed under first order interpretations. Thus if R is an ω^n-automatic ring then the ring of matrices $M_k(R)$ is also ω^n-automatic. It is well known that two rings R and R' are isomorphic if and only if the rings $M_k(R)$ and $M_k(R')$ are isomorphic, (this is proved for instance in [14]). We now denote, for each integer $n \geq 1$, $\mathcal{M}_n = M_k(\mathcal{R}_n)$, where $k \geq 2$ is a fixed integer. So we can state the following result.

Theorem 6.7.
1. (ZFC + CH) *The ω^n-automatic non commutative rings $\mathcal{M}_n$, $n \geq 1$, are pairwise isomorphic.*
2. (ZFC + AT) *The ω^n-automatic non commutative rings $\mathcal{M}_n$, $n \geq 1$, are pairwise non isomorphic.*

Consider now the unitriangular group $UT_k(R)$ for some integer $k \geq 3$ and R a unitary ring. A matrix $M \in M_k(R)$ is in the group $UT_k(R)$ if and only if it is an upper triangular matrix which has only coefficients 1 on the diagonal, where 1 is the multiplicative unit of R. The group $UT_k(R)$ is also first order interpretable in the ring R. (It is actually a classical example of nilpotent group of class $k-1$, for more details, see [5, 14]).

We denote $\mathcal{U}_{n,k} = UT_k(\mathcal{R}_n)$ for each $n \geq 1$. The groups $\mathcal{U}_{n,k}$ are first order interpretable in the ring $\mathcal{R}_n$. Thus the groups $\mathcal{U}_{n,k}$ are ω^n-automatic. We can now state the following result.

Theorem 6.8. *Let $k \geq 3$ be an integer.*

1. (ZFC + CH) *The ω^n-automatic groups $\mathcal{U}_{n,k}$, $n \geq 1$, are pairwise isomorphic.*
2. (ZFC + AT) *The ω^n-automatic groups $\mathcal{U}_{n,k}$, $n \geq 1$, are pairwise non isomorphic.*

Proof. The result follows from Theorem 6.6, from the fact that if R and S are two isomorphic commutative rings then $UT_k(R)$ and $UT_k(S)$ are also isomorphic, and from a result of Belegradek who proved in [5] that if $UT_k(R)$ and $UT_k(S)$ are isomorphic, for some integer $k \geq 3$ and some commutative rings R and S, then the rings R and S are also isomorphic. $\dashv$

Then we can now state the following result.

Corollary 6.9. *The isomorphism relation for ω^2-automatic (respectively, ω^n-automatic for $n > 2$) commutative rings (respectively, non-commutative rings, groups, nilpotent groups of class $p \geq 2$) is not determined by the axiomatic system* ZFC.

An ω^n-automatic presentation of a structure is given by a tuple of ω^n-automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq k})$. The tuple of ω^n-automata can be coded by a finite sequence of symbols, hence by a unique integer N. If N is the code of the tuple of ω^n-automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq k})$ we shall denote $\mathcal{S}_N$ the ω^n-automatic structure $(L(\mathcal{A}), (R_i)_{1 \leq i \leq k}))/E_{\equiv}$.

The isomorphism problem for ω^n-automatic structures is:

$$\{(p, m) \in \omega^2 \mid \mathcal{S}_p \text{ is isomorphic to } \mathcal{S}_m\}.$$

We can now infer from above independence results the following one.

Theorem 6.10. *The isomorphism problem for ω^2-automatic (respectively, ω^n-automatic for $n > 2$) boolean algebras (respectively, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $p \geq 2$) is neither a Σ^1_2-set nor a Π^1_2-set.*

Proof. We prove first the result for ω^n-automatic boolean algebras. By Theorem 6.4 we know that if ZFC is consistent then there is a model $\mathbf{V}$ of (ZFC + AT) in which the two ω^2-automatic boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are not isomorphic. But the inner model $\mathbf{L}$ of constructible sets in $\mathbf{V}$ is a model of (ZFC + CH) so in this model the two boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic by Theorem 6.1.

On the other hand, Schoenfield's Absoluteness Theorem implies that every Σ_2^1-set (respectively, Π_2^1-set) is absolute for all inner models of ZFC, see [20, page 490].

In particular, if the isomorphism problem for ω^n-automatic boolean algebras, $n \geq 2$, was a Σ_2^1-set (respectively, a Π_2^1-set), then it could not be a different subset of ω^2 in the models $\mathbf{V}$ and $\mathbf{L}$ considered above. Thus the isomorphism problem for ω^2-automatic (respectively, ω^n-automatic for $n > 2$) boolean algebras is neither a Σ_2^1-set nor a Π_2^1-set.

The other cases of rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $p \geq 2$, follow in the same way from Theorems 6.6, 6.7, and 6.8. $\dashv$

REMARK 6.11. *We had proved in [14] that there exist two ω-tree-automatic atomless boolean algebras which are isomorphic under* OCA *but not under* CH. *But all the ω^n-automatic atomless boolean algebras $\mathcal{B}_n$, $n \geq 1$, we have considered in this paper are also ω-tree-automatic. Thus we have also in some sense improved our previous result by showing the following one.*

THEOREM 6.12. *There exist infinitely many ω-tree-automatic atomless boolean algebras $\mathcal{B}_n$, $n \geq 1$, which are pairwise isomorphic under* CH *and pairwise non isomorphic under* AT.

Notice that we have also a similar result for partial orders, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $p \geq 2$.

§7. Concluding remarks. Khoussainov, Nies, Rubin, and Stephan proved in [26] that the automatic infinite boolean algebras are the finite products $B_{fin-cof}^n$ of the boolean algebra $B_{fin-cof}$ of finite or cofinite subsets of the set of positive integers ω. An open problem is to characterize completely the ω^n-automatic (respectively, ω-tree-automatic) boolean algebras. A similar problem naturally arises for other classes of ω^n-automatic (respectively, ω-tree-automatic) structures, like groups, rings, linear orders, and so on.

Acknowledgements. We thank the anonymous referees for useful comments on a preliminary version of this paper.

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