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# ON THE WELL-POSEDNESS FOR KADOMTSEV-PETVIASHVILI-BURGERS I EQUATION.

DARWICH MOHAMAD

ABSTRACT. We prove local and global well-posedness in  $H^{s,0}(\mathbb{R}^2)$ ,  $s > -\frac{1}{2}$ , for the Cauchy problem associated with the Kadomtsev-Petviashvili-Burgers-I equation (KPBI) by working in Bourgain's type spaces. This result is almost sharp if one requires the flow-map to be smooth.

## 1. INTRODUCTION

We study the well-posedness of the initial value problem for the Kadomtsev-Petviashvili-Burgers (KPBI) equations in  $\mathbb{R}^2$  :

$$(1.1) \quad \begin{cases} (\partial_t u + u_{xxx} - u_{xx} + uu_x)_x - u_{yy} = 0, \\ u(0, x, y) = \varphi(x, y). \end{cases}$$

where  $u$  is a real-valued function of  $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$ . Note that if we replace  $-u_{yy}$  by  $+u_{yy}$ , (1.1) becomes the KPBI equation.

This equation, models in some regime the wave propagation in electromagnetic saturated zone( cf.[12]). More generally, be considered as a toys model for two-dimensional wave propagation taking into account the effect of viscosity. Note that since we are interested in an almost unidirectional propagation, the dissipative term acts only in the main direction of the propagation in KPBI. This equation is a dissipative version of the Kadomtsev-Petviashvili-I equation (KPI) :

$$(1.2) \quad (\partial_t u + u_{xxx} + uu_x)_x - u_{yy} = 0.$$

which is a "universal" model for nearly one directional weakly nonlinear dispersive waves, with weak transverse effects and strong surface tension effects. Bourgain had developed a new method, clarified by Ginibre in [5], for the study of Cauchy problem associated with non-linear dispersive equations. This method was successfully applied to the nonlinear Schrödinger, KdV as well as KPBI equations. It was shown by Molinet-Ribaud [14] that the Bourgain spaces can be used to study the Cauchy problems associated to semi-linear equations with a linear part containing both dispersive and dissipative terms (and consequently this applies to KPBI equations).

By introducing a Bourgain space associated to the usual KPI equation (related only to the dispersive part of the linear symbol in the KPBI equation), Molinet-Ribaud [14] had proved global existence for the Cauchy problem associated to the KPBI equation when the initial value in  $H^{s_1, s_2}(\mathbb{R}^2)$ ,  $s_1 > 0$  and  $s_2 \geq 0$ .

Kojok [9] had proved the local and global existence for (1.1) for small initial

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*Key words and phrases.* Dispersive PDEs, Bourgain spaces, Strichartz estimates.

data in  $L^2(\mathbb{R}^2)$ . In this paper, we improve the results obtained by Molinet-Ribaud, by proving the local existence for the KPBI equation, with initial value  $\varphi \in H^{s_1,0}$  when  $s_1 > -\frac{1}{2}$ .

The main new ingredient is a trilinear estimate for the KPI equation proved in [11]. Following [15], we introduce a Bourgain space associated to the KPBI equation. This space is in fact the intersection of the space introduced in [2] and of a Sobolev space linked to the dissipative effect. The advantage of this space is that it contains both the dissipative and dispersive parts of the linear symbol of (1.1).

This paper is organized as follows. In Section 2, we introduce our notations and we give an extension of the semi-group of the KPBI equation by a linear operator defined on the whole real axis. In Section 3 we derive linear estimates and some smoothing properties for the operator  $L$  defined by (2.15) in the Bourgain spaces. In Section 4 we state Strichartz type estimates for the KP equation which yield bilinear estimates. In Section 5, using bilinear estimates, a standard fixed point argument and some smoothing properties, we prove uniqueness and global existence of the solution of KPBI equation in anisotropic sobolev space  $H^{s,0}$  with  $s > -\frac{1}{2}$ . Finally, in section 6, we ensure that our local existence result is optimal if one requires the smoothness of the flow-map.

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## 2. NOTATIONS AND MAIN RESULTS

We will use  $C$  to denote various time independent constants, usually depending only upon  $s$ . In case a constant depends upon other quantities, we will try to make it explicit. We use  $A \lesssim B$  to denote an estimate of the form  $A \leq CB$ . Similarly, we will write  $A \sim B$  to mean  $A \lesssim B$  and  $B \lesssim A$ . We write  $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2} \sim 1 + |\cdot|$ . The notation  $a^+$  denotes  $a + \epsilon$  for an arbitrarily small  $\epsilon$ . Similarly  $a^-$  denotes  $a - \epsilon$ . For  $b \in \mathbb{R}$ , we denote respectively by  $H^b(\mathbb{R})$  and  $\dot{H}^b(\mathbb{R})$  the nonhomogeneous and homogeneous Sobolev spaces which are endowed with the following norms :

$$(2.1) \quad \|u\|_{H^b}^2 = \int_{\mathbb{R}} \langle \tau \rangle^{2b} |\hat{u}(\tau)|^2 d\tau, \quad \|u\|_{\dot{H}^b}^2 = \int_{\mathbb{R}} |\tau|^{2b} |\hat{u}(\tau)|^2 d\tau$$

where  $\hat{\cdot}$  denotes the Fourier transform from  $\mathcal{S}'(\mathbb{R}^2)$  to  $\mathcal{S}'(\mathbb{R}^2)$  which is defined by :

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{i\langle \lambda, \xi \rangle} f(\lambda) d\lambda, \quad \forall f \in \mathcal{S}'(\mathbb{R}^2).$$

Moreover, we introduce the corresponding space (resp space-time) Sobolev spaces  $H^{s_1, s_2}$  (resp  $H^{b, s_1, s_2}$ ) which are defined by :

$$(2.2) \quad H^{s_1, s_2}(\mathbb{R}^2) =: \{u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{H^{s_1, s_2}}(\mathbb{R}^2) < +\infty\},$$

$$(2.3) \quad H^{b, s_1, s_2}(\mathbb{R}^2) =: \{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{H^{b, s_1, s_2}}(\mathbb{R}^3) < +\infty\}$$

where,

$$(2.4) \quad \|u\|_{H^{s_1, s_2}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\nu)|^2 d\nu,$$

$$(2.5) \quad \|u\|_{H^{b,s_1,s_2}}^2 = \int_{\mathbb{R}^2} \langle \tau \rangle^b \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\tau, \nu)|^2 d\nu d\tau,$$

and  $\nu = (\xi, \eta)$ . Let  $U(\cdot)$  be the unitary group in  $H^{s_1,s_2}$ ,  $s_1, s_2 \in \mathbb{R}$ , defining the free evolution of the (KP-II) equation, which is given by

$$(2.6) \quad U(t) = \exp(itP(D_x, D_y)),$$

where  $P(D_x, D_y)$  is the Fourier multiplier with symbol  $P(\xi, \eta) = \xi^3 - \eta^2/\xi$ . By the Fourier transform, (2.6) can be written as :

$$(2.7) \quad \mathcal{F}_x(U(t)\phi) = \exp(itP(\xi, \eta))\hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \in \mathbb{R}.$$

Also, by the Fourier transform, the linear part of the equation (1.1) can be written as :

$$(2.8) \quad i(\tau - \xi^3 - \eta^2/\xi) + \xi^2 =: i(\tau - P(\eta, \xi)) + \xi^2.$$

We need to localize our solution, and the idea of Bourgain has been to consider this localisation, by defining the space  $X^{b,s}$  equipped by the

$$(2.9) \quad \|u\|_{X^{b,s_1,s_2}} = \|\langle i(\tau - P(\eta, \xi)) + \xi^2 \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \tilde{u}(\tau, \xi, \eta)\|_{L^2(\mathbb{R}^3)}.$$

We will need to define the decomposition of Littlewood-Paley. Let  $\eta \in C_0(\mathbb{R})$  be such that  $\eta \geq 0$ ,  $\text{supp } \eta \subset [-2, 2]$ ,  $\eta = 1$  on  $[-1, 1]$ . We define next  $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$ .

Any summations over capitalized variables such as  $N, L$  are presumed to be dyadic, i.e. these variables range over numbers of the form  $N = 2^j$ ,  $j \in \mathbb{Z}$  and  $L = 2^l$ ,  $l \in \mathbb{N}$ . We set  $\varphi_N(\xi) = \varphi(\frac{\xi}{N})$  and define the operator  $P_N$  by  $\mathcal{F}_x(P_N u) = \varphi_N \mathcal{F}_x(u)$ . We introduce  $\psi_L(\tau, \zeta) = \varphi_L(\tau - P(\zeta))$  and for any  $u \in \mathcal{S}(\mathbb{R}^2)$ ,

$$\mathcal{F}_x(P_N u(t))(\xi) = \varphi_N(\xi) \mathcal{F}_x(u)(t, \xi), \quad \mathcal{F}(Q_L u)(\tau, \xi, \eta) = \psi_L(\tau, \zeta) \mathcal{F}(u)(\tau, \xi); L > 1$$

and  $\mathcal{F}(Q_1 u)(\tau, \xi, \eta) = \eta(\tau - P(\zeta)) \mathcal{F}(u)(\tau, \xi)$ . Roughly speaking, the operator  $P_N$  localizes in the annulus  $\{|\xi| \sim N\}$  where as  $Q_L$  localizes in the region  $\{\langle \tau - P(\zeta) \rangle \sim L\}$ . We denote  $P_N u$  by  $u_N$ ,  $Q_L u$  by  $u_L$  and  $P_N(Q_L u)$  by  $u_{N,L}$ .

For  $T \geq 0$ , we consider the localized Bourgain spaces  $X_T^{b,s_1,s_2}$  endowed with the norm

$$\|u\|_{X_T^{b,s_1,s_2}} = \inf_{w \in X^{b,s_1,s_2}} \{ \|w\|_{X^{b,s_1,s_2}}, w(t) = u(t) \text{ on } [0, T] \}.$$

We also use the space-time Lebesgue space  $L_{t,x}^{p,q}$  endowed with the norm

$$\|u\|_{L_{t,x}^{q,r}} = \left\| \|u\|_{L_x^r} \right\|_{L_t^q},$$

and we will use the notation  $L_{t,x}^2$  for  $L_{t,x}^{2,2}$ .

We denote by  $W(\cdot)$  the semigroup associated with the free evolution of the KPB equations,

$$(2.10) \quad \mathcal{F}_x(W(t)\phi) = \exp(itP(\xi, \eta) - |\xi|^2 t) \hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \geq 0.$$

Also, we can extend  $W$  to a linear operator defined on the whole real axis by setting,

$$(2.11) \quad \mathcal{F}_x(W(t)\phi) = \exp(itP(\xi, \eta) - |\xi|^2 |t|) \hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \in \mathbb{R}.$$

By the Duhamel integral formulation, the equation (1.1) can be written as

$$(2.12) \quad u(t) = W(t)\phi - \frac{1}{2} \int_0^t W(t-t') \partial_x(u^2(t')) dt', \quad t \geq 0.$$

To prove the local existence result, we will apply a fixed point argument to the extension of (2.12), which is defined on whole the real axis by:

$$(2.13) \quad u(t) = \psi(t)[W(t)\phi - L(\partial_x(\psi_T^2 u^2))(x, t)],$$

where  $t \in \mathbb{R}$ ,  $\psi$  indicates a time cutoff function :

$$(2.14) \quad \psi \in C_0^\infty(\mathbb{R}), \quad \text{supp } \psi \subset [-2, 2], \quad \psi = 1 \text{ on } [-1, 1],$$

$\psi_T(\cdot) = \psi(\cdot/T)$ , and

$$(2.15) \quad L(f)(x, t) = W(t) \int e^{ix\xi} \frac{e^{it\tau} - e^{-|t|\xi^2}}{i\tau + \xi^2} \mathcal{F}(W(-t)f)(\xi, \tau) d\xi d\tau.$$

One easily sees that

$$(2.16) \quad \chi_{\mathbb{R}_+}(t)\psi(t)L(f)(x, t) = \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-\tau)f(\tau) d\tau.$$

Indeed, taking  $w = W(-\cdot)f$ , the right hand side of (2.16) can be rewritten as

$$W(t) \left( \chi_{\mathbb{R}_+}(t)\psi(t) \int e^{ix\xi} \frac{e^{it\tau} - e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \tau') d\xi d\tau' \right).$$

In [15], the authors performed the iteration process in the space  $X^{s,b}$  equipped with the norm:

$$\|u\|_{X^{b,s_1,s_2}} = \|\langle i(\tau - P(\nu)) + \xi^2 \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{u}(\tau, \nu)\|_{L^2(\mathbb{R}^3)},$$

which take advantage of the mixed dispersive-dissipative part of the equation. We will rather work in its Besov version  $X^{s,b,q}$  (with  $q = 1$ ) defined as the weak closure of the test functions that are uniformly bounded by the norm

$$\|u\|_{X^{b,s,0,q}} = \left( \sum_N \left[ \sum_L \langle L + N^2 \rangle^{bq} \langle N \rangle^{sq} \|P_N Q_L u\|_{L_{x,y,t}^2}^q \right]^{\frac{2}{q}} \right)^{\frac{1}{2}}.$$

**Remark 2.1.** *It is clear that if  $u$  solves (2.13) then  $u$  is a solution of (2.12) on  $[0, T]$ ,  $T < 1$ . Thus it is sufficient to solve (2.13) for a small time ( $T < 1$  is enough).*

**Definition 2.1.** *The Cauchy problem (1.1) is locally well-posed in the space  $X$  if for any  $\varphi \in X$  there exists  $T = T(\|\varphi\|_X) > 0$  and a map  $F$  from  $X$  to  $C([0, T]; X)$  such that  $u = F(\varphi)$  is the unique solution for the equation (1.1) in some space  $Y \hookrightarrow C([0, T]; X)$  and  $F$  is continuous in the sense that*

$$\|F(\varphi_1) - F(\varphi_2)\|_{L^\infty([0, T]; X)} \leq M(\|\varphi_1 - \varphi_2\|_X, R)$$

for some locally bounded function  $M$  from  $\mathbb{R}^+ \times \mathbb{R}^+$  to  $\mathbb{R}^+$  such that  $M(S, R) \rightarrow 0$  for fixed  $R$  when  $S \rightarrow 0$  and for  $\varphi_1, \varphi_2 \in X$  such that  $\|\varphi_1\|_X + \|\varphi_2\|_X \leq R$ .

**Remark 2.2.** *We obtain the global existence if we can extend the solutions to all  $t \in \mathbb{R}^+$ , by iterating the result of local existence, in this case we say that the Cauchy problem is globally well posed.*

*The global existence of the solution to our equation will be obtained thanks to the regularizing effect of the dissipative term and the fact that the  $L^2$  norm is not increasing.*

Let us now state our results:

**Theorem 2.2.** *Let  $s_1 > -1/2$ ,  $\beta \in ]-1/2, \min(0, s_1)]$  and  $\phi \in H^{s_1,0}$ . Then there exists a time  $T = T(\|\phi\|_{H^{\beta,0}}) > 0$  and a unique solution  $u$  of (1.1) in*

$$(2.17) \quad Y_T = X_T^{1/2, s_1, 0, 1}$$

*Moreover,  $u \in C(\mathbb{R}_+; H^{s_1,0})$  and the map  $\phi \mapsto u$  is  $C^\infty$  from  $H^{s_1,0}$  to  $Y_T$ .  $\square$*

**Theorem 2.3.** *Let  $s < -1/2$ . Then it does not exist a time  $T > 0$  such that the equation (1.1) admits a unique solution in  $C([0, T[, H^{s,0})$  for any initial data in some ball of  $H^{s,0}(\mathbb{R}^2)$  centered at the origin and such that the map*

$$(2.18) \quad \phi \mapsto u$$

*is  $C^2$ -differentiable at the origin from  $H^{s,0}$  to  $C([0, T], H^{s,0})$ .  $\square$*

**Remark 2.3.** *Note that these theorems holds also for all initial data belonging to  $H^{s_1, s_2}$  with  $s_2 \geq 0$ .*

The principle of the proof of local existence result holds in two steps:

**Step 1:** In order to apply a standard argument of fixed point, we want to estimate the two terms: free term and the forcing term of equation (2.13). A first step is to show using Fourier analysis, that the map  $\phi \mapsto \psi(t)W(t)\phi$  is bounded from  $H^{s,0}$  to  $X^{\frac{1}{2}, s, 0, 1}$  and the map  $L$  is also bounded from  $X^{-\frac{1}{2}, s, 0, 1}$  to  $X^{\frac{1}{2}, s, 0, 1}$ .

**Step 2:** We treat the bilinear term, by proving that the map  $(u, v) \mapsto \partial_x(uv)$  is bounded from  $X^{\frac{1}{2}, s, 0, 1} \times X^{\frac{1}{2}, s, 0, 1}$  to  $X^{-\frac{1}{2}, s, 0, 1}$ .

### 3. LINEAR ESTIMATES

In this section, we mainly follow Molinet-Ribaud [15] ( see also [22] and [17] for the Besov version) to estimate the linear term in the space  $X^{\frac{1}{2}, s, 0, 1}$ . We start by the free term:

#### 3.1. Estimate for the free term.

**Proposition 3.1.** *Let  $s \in \mathbb{R}$ , then  $\forall \phi \in H^{s,0}$ , we have:*

$$\|\psi(t)W(t)\phi\|_{X^{\frac{1}{2}, s, 0, 1}} \lesssim \|\phi\|_{H^{s,0}}.$$

**Proof.** This is equivalent to prove that

$$(3.1) \quad \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|P_N Q_L(\psi(t)W(t)\phi)\|_{L^2_{x,y,t}} \lesssim \|P_N \phi\|_{L^2_{x,y}}$$

for each dyadic  $N$ . Using Plancherel, we obtain

$$\begin{aligned}
& \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|P_N Q_L(\psi(t)W(t)\phi)\|_{L^2_{x,y,t}} \\
& \lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\hat{\phi}(\xi)\mathcal{F}_t(\psi(t)e^{-|t|\xi^2}e^{itP(\nu)})(\tau)\|_{L^2_{\xi,\eta,\tau}} \\
(3.2) \quad & \lesssim \|P_N\phi\|_{L^2} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)P_L(\psi(t)e^{-|t|\xi^2})\|_{L^\infty_\xi L^2_\tau}.
\end{aligned}$$

Note that from Prop 4.1 in [17] we have:

$$(3.3) \quad \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)P_L(\psi(t)e^{-|t|\xi^2})\|_{L^\infty_\xi L^2_\tau} \lesssim 1.$$

Combining (3.3) and (3.2), we obtain the result.  $\square$

**3.2. Estimates for the forcing term.** Now we shall study in  $X^{\frac{1}{2},s,0,1}$  the linear operator  $L$  :

**Proposition 3.2.** *Let  $f \in \mathcal{S}(\mathbb{R}^2)$ , There exists  $C > 0$  such that:*

$$\|\psi(t)L(f)\|_{X^{\frac{1}{2},s,0,1}} \leq C\|f\|_{X^{-\frac{1}{2},s,0,1}}.$$

**Proof.** Let

$$w(\tau) = W(-\tau)f(\tau), \quad K(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau.$$

Therefore, by the definition, it suffices to prove that

$$(3.4) \quad \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\mathcal{F}_t(K)(\tau)\|_{L^2_{\xi,\eta,\tau}} \lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\hat{w}(\xi, \eta, \tau)\|_{L^2_{\xi,\eta,\tau}}.$$

We can break up  $K$  in  $K = K_{1,0} + K_{1,\infty} + K_{2,0} + K_{2,\infty}$ , where

$$K_{1,0} =: \psi(t) \int_{|\tau| \leq 1} \frac{e^{it\tau} - 1}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau, \quad K_{1,\infty} = \psi(t) \int_{|\tau| \geq 1} \frac{e^{it\tau}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau,$$

$$K_{2,0} = \psi(t) \int_{|\tau| \leq 1} \frac{1 - e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau, \quad K_{2,\infty} = \psi(t) \int_{|\tau| \geq 1} \frac{e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau.$$

Contribution of  $K_{2,\infty}$ .

Clearly we have

$$\begin{aligned}
\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)Q_L(K_{2,\infty})\|_{L^2_{\xi,\eta,\tau}} & \lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \sup_{\xi \in I_k} \|\varphi_N(\xi)Q_L(\psi(e^{-|t|\xi^2}))(t)\|_{L^2_{\xi,\tau}} \\
& \times \int \frac{\|\varphi_N(\xi)\hat{w}(\xi, \eta, \tau)\|_{L^2_{\xi,\eta}}}{\langle \tau \rangle} d\tau \\
& \lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\hat{w}(\xi, \eta, \tau)\|_{L^2_{\xi,\eta,\tau}},
\end{aligned}$$

where we use (3.3) in the last step.

Contribution of  $K_{2,0}$ .

We have for  $|\xi| \geq 1$

$$\begin{aligned} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{2,0})\|_{L_{\xi,\eta,\tau}^2} &\lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \sup_{\xi \in I_k} \|\varphi_N(\xi) P_L(\psi(1 - e^{-|t|\xi^2}))(t)\|_{L_t^2} \\ &\times \int \frac{\|\hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta}^2} d\tau}{\langle \tau \rangle} \\ &\lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta,\tau}^2}, \end{aligned}$$

where we used (3.3) in the last step.

For  $|\xi| \leq 1$ , using Taylor's expansion, we have

$$\begin{aligned} &\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{2,0})\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_n \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) \int_{|\tau| \leq 1} \frac{\hat{w}(\tau)}{i\tau + \xi^2} d\tau P_L(|t|^n \psi(t)) \frac{|\xi|^{2n}}{n!}\|_{L_{\xi,\eta,t}^2} \\ &\lesssim \sum_n \left\| \frac{t^n \psi(t)}{n!} \right\|_{B_{2,1}^{\frac{1}{2}}} \left\| \int_{|\tau| \leq 1} \frac{|\xi|^2 |\varphi_N(\xi) \hat{w}(\xi, \eta, \tau)|}{|i\tau + \xi^2|} d\tau \right\|_{L_{\xi,\eta}^2} \\ &\lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta,\tau}^2}, \end{aligned}$$

where in the last inequality we used the fact  $\| |t|^n \psi(t) \|_{B_{2,1}^{\frac{1}{2}}} \leq \| |t|^n \psi(t) \|_{H^1} \leq C 2^n$ .

Contribution of  $K_{1,\infty}$ .

By the identity  $\mathcal{F}(u \star v) = \hat{u} \hat{v}$  and the triangle inequality  $\langle i\tau + \xi^2 \rangle \leq \langle \tau_1 \rangle + |i(\tau - \tau_1) + \xi^2|$ , Let  $g(\xi, \eta, \tau) = \frac{|\hat{w}(\xi, \eta, \tau)|}{|i\tau + \xi^2|} \chi_{|\tau| \geq 1}$  we see that

$$\begin{aligned} &\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{1,\infty})\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\xi, \tau) \hat{\psi} \star_{\tau_1} g(\xi, \eta, \tau_1)\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_L \langle L \rangle^{\frac{1}{2}} \left\| \varphi_N(\xi) \varphi_L(\tau_1) |\hat{\psi}(\tau_1)| \star g(\xi, \eta, \tau_1) \right\|_{L_{\xi,\eta,\tau}^2} \\ &+ \sum_L \left\| \varphi_N(\xi) \varphi_L(\tau) \hat{\psi}(\tau_1) \star \left( \frac{\hat{w}(\xi, \eta, \tau_1)}{|i\tau + \xi^2|^{\frac{1}{2}}} \chi_{|\tau_1| \geq 1} \right) \right\|_{L_{\xi,\eta,\tau}^2}. \end{aligned}$$

Due to the convolution inequality  $\|u \star v\|_{L_\tau^2} \lesssim \|u\|_{L_\tau^1} \|v\|_{L_\tau^2}$ , we obtain

$$\begin{aligned} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{1,\infty})\|_{L_{\xi,\eta,\tau}^2} &\lesssim \sum_L L \|\hat{\psi}(t)\|_{L_\tau^1} \|\varphi_N(\xi) \varphi_L(\tau) \frac{|\hat{w}(\tau)|}{|i\tau + \xi^2|} \chi_{\{|\tau| \geq 1\}}\|_{L_{\xi,\eta,\tau}^2} \\ &+ \sum_L \|\psi(t)\|_{L_\tau^1} \|\varphi_N(\xi) \varphi_L(\tau) \frac{|\hat{w}(\tau)|}{|i\tau + \xi^2|^{1/2}} \chi_{\{|\tau| \geq 1\}}\|_{L_{\xi,\eta,\tau}^2} \\ &\leq C \sum_L \langle L + N^2 \rangle^{-1/2} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\tau)\|_{L_{\xi,\eta,\tau}^2}. \end{aligned}$$



Contribution of  $K_{1,0}$ .

Using Taylors expansion, we obtain that:

$$K_{1,0} = \psi(t) \int_{|\tau| \leq 1} \sum_{n \geq 1} \frac{(it\tau)^n}{n!(i\tau + \xi^2)} \hat{w}(\xi, \eta, \tau) d\tau.$$

Thus, we get

$$\begin{aligned} & \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{1,0})\|_{L_{\xi, \eta, \tau}^2} \\ & \lesssim \sum_{n \geq 1} \left\| \frac{t^n \psi(t)}{n!} \right\|_{B_{2,1}^{\frac{1}{2}}} \left\| \int_{|\tau| \leq 1} \frac{|\tau|}{|i\tau + |\xi|^2|} |\varphi_k(\xi) \hat{w}(\xi, \eta, \tau)| d\tau \right\|_{L_{\xi, \eta}^2} \\ & \lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi, \eta, \tau}^2}, \end{aligned}$$

where we used  $\| |t|^n \psi(t) \|_{B_{2,1}^{\frac{1}{2}}} \leq \| |t|^n \psi(t) \|_{H^1} \leq C2^n$  in the last step.

Therefore, we complete the proof of the proposition.

#### 4. STRICHARTZ AND BILINEAR ESTIMATES

The goal of this section is to establish the main bilinear estimate. This type of bilinear estimate is necessary to control the nonlinear term  $\partial_x(u^2)$  in  $X^{-\frac{1}{2}, s, 0, 1}$ .

First following [6] it is easy to check that for any  $u \in X^{\frac{1}{2}, 0, 0, 1}$  supported in  $[-T, T]$  and any  $\theta \in [0, \frac{1}{2}]$  it holds:

$$(4.1) \quad \|u\|_{X^{\theta, s, 0, 1}} \leq T^{\frac{1}{2} - \theta} \|u\|_{X^{1/2, s, 0, 1}}.$$

The following lemma is prepared by Molinet-Ribaud in [14].

**Lemma 4.1.** *Let  $2 \leq r$  and  $0 \leq \beta \leq 1/2$ . Then*

$$(4.2) \quad \left\| |D_x|^{-\frac{\beta \delta(r)}{2}} U(t) \varphi \right\|_{L_{t,x}^{q,r}} \leq C \|\varphi\|_{L^2}$$

where  $\delta(r) = 1 - \frac{2}{r}$ , and  $(q, r, \beta)$  fulfils the condition

$$(4.3) \quad 0 \leq \frac{2}{q} \leq \left(1 - \frac{\beta}{3}\right) \delta(r) < 1.$$

Now we will prove the following one:

**Lemma 4.2.** *Let  $v \in L^2(\mathbb{R}^3)$  with supp  $v \subset \{(t, x, y) : |t| \leq T\}$ ,  $\delta(r) = 1 - 2/r$  and  $\hat{v}_N = \varphi_N \hat{v}$  for some dyadic integer  $N$ . Then for all  $(r, \beta, \theta)$  with*

$$(4.4) \quad 2 \leq r < \infty, \quad 0 \leq \beta \leq 1/2, \quad 0 \leq \delta(r) \leq \frac{\theta}{1 - \beta/3},$$

$$(4.5) \quad \left\| \mathcal{F}_{t,x}^{-1} \left( |\xi|^{-\frac{\theta \beta \delta(r)}{2}} \langle \tau - P(\nu) \rangle^{-\frac{\theta}{2}} |\hat{v}_N(\tau, \nu)| \right) \right\|_{L_{t,x,y}^{q,r}} \leq C \|v_N\|_{L^2(\mathbb{R}^3)}$$

where  $q$  is defined by

$$(4.6) \quad 2/q = (1 - \beta/3)\delta(r) + (1 - \theta).$$

□

**Proof** Using Lemma 4.2 together with Lemma 3.3 of [5], we see that

$$(4.7) \quad \left\| |D_x|^{-\frac{\beta\delta(r)}{2}} u_N \right\|_{L_{t,x}^{q,r}} \leq C \|u_N\|_{X^{1/2,0,0,1}}.$$

By the definition of  $X^{b,s,0,1}$  we have

$$(4.8) \quad \|u_N\|_{L_{t,x}^2} = \|u_N\|_{X^{0,0,0,2}}.$$

Hence for  $0 \leq \theta \leq 1$ , by interpolation,

$$(4.9) \quad \left\| |D_x|^{-\frac{\theta\beta\delta(r)}{2}} u_N \right\|_{L_{t,x}^{q_1,r_1}} \leq C \|u_N\|_{X^{\frac{\theta}{2},0,0,1}}$$

where

$$\frac{1}{q_1} = \frac{\theta}{q} + \frac{1-\theta}{2}, \quad \frac{1}{r_1} = \frac{\theta}{r} + \frac{1-\theta}{2}.$$

Since  $\delta(r_1) = \theta\delta(r)$ , (4.4) follows from (4.3)

$$\frac{1}{q_1} = \left(1 - \frac{\beta}{3}\right) \delta(r_1) + (1-\theta),$$

which can be rewritten as

$$\left\| \mathcal{F}_{t,x}^{-1} \left( |\xi|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_N \right) \right\|_{L_{t,x}^{q_1,r_1}} \leq C \left\| \langle \tau - P(\nu) \rangle^{\frac{\theta}{2}} \hat{u}_N \right\|_{L^2}.$$

This clearly completes the proof.

Now, we will estimate the bilinear terms using the following Lemma (see [11]):

**Lemma 4.3.** *Let  $k_1, k_2, k_3 \in \mathbb{Z}$ ,  $j_1, j_2, j_3 \in \mathbb{Z}_+$ , and  $f_i : \mathbb{R}^3 \mapsto \mathbb{R}^+$  are  $L^2$  functions supported in  $D_{k_i, j_i}$ ,  $i = 1, 2, 3$ . Then*

$$(4.10) \quad \int (f_1 * f_2) f_3 \lesssim 2^{\frac{j_1+j_2+j_3}{2}} 2^{-\frac{(k_1+k_2+k_3)}{2}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}$$

Where  $D_{k,j} = \{(\xi, \mu, \tau) : |\xi| \in [2^{k-1}, 2^k], \mu \in \mathbb{R}, |\tau - P(\xi, \mu)| \leq 2^j\}$ .

We are now in position to prove our main bilinear estimate:

**Proposition 4.4.** *For all  $u, v \in X^{1/2,s,0,1}(\mathbb{R}^3)$ ,  $s > -\frac{1}{2}$  with compact support in time included in the subset  $\{(t, x, y) : t \in [-T, T]\}$ , there exists  $\mu > 0$  such that the following bilinear estimate holds*

$$(4.11) \quad \|\partial_x(uv)\|_{X^{-1/2,s,0,1}} \leq CT^\mu \|u\|_{X^{1/2,s,0,1}} \|v\|_{X^{1/2,s,0,1}}.$$

□

**Remark 4.1.** *We will mainly use the following version of (4.11), which is a direct consequence of Proposition 4.4, together with the triangle inequality*

$$\forall \beta \in ]-\frac{1}{2}, 0], \forall s \geq \beta, \quad \langle \xi \rangle^s \leq \langle \xi \rangle^\beta \langle \xi_1 \rangle^{s-\beta} + \langle \xi \rangle^\beta \langle \xi - \xi_1 \rangle^{s-\beta},$$

$$(4.12) \quad \begin{aligned} \|\partial_x(uv)\|_{X^{-1/2,s,0,1}} &\leq CT^{\mu(\beta)} \left( \|u\|_{X^{1/2,\beta,0,1}} \|v\|_{X^{1/2,s,0,1}} \right. \\ &\quad \left. + \|u\|_{X^{1/2,s,0,1}} \|v\|_{X^{1/2,\beta,0,1}} \right). \end{aligned}$$

with  $\mu(\beta) > 0$ .

**Proof of Prop 4.4.** We proceed by duality. Let  $w \in X^{1/2, -s, 0, \infty}$ , we will estimate the following term

$$J = \sum_{N, N_1, N_2} \sum_{L, L_1, L_2} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \left| \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \right|$$

By symmetry we can assume that  $N_1 \leq N_2$ , note that  $|\xi| \leq |\xi_1| + |\xi_2|$  then  $N \lesssim N_2$ .

From Lemma 4.3, we have:

$$(4.13) \quad \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L^{\frac{1}{2}} N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N^{-\frac{1}{2}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}.$$

**Case 1.:**  $1 \leq N$ ,  $N_1 \geq 1$ , and  $N_2 \geq 1$ .

We have clearly:

$$(4.14) \quad \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim \|u_{N_1, L_1}\|_{L_{t, x, y}^4} \|v_{N_2, L_2}\|_{L_{t, x, y}^4} \|w_{N, L}\|_{L_{t, x, y}^2}$$

using Lemma 4.2 ( with  $\beta = \frac{1}{2}$ ,  $r = 4$ ) we obtain that there exists  $\alpha \in [\frac{6}{7}, \frac{12}{13}]$  such that:

$$(4.15) \quad \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim L_1^{\frac{\alpha}{2}} N_1^{\frac{\alpha}{8}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} L_2^{\frac{\alpha}{2}} N_2^{\frac{\alpha}{8}} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \tau}^2} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}.$$

By interpolating (4.13) with (4.15) we obtain that: there exist  $\beta = \frac{\theta\alpha}{2} + \frac{1-\theta}{2} \in [\frac{\alpha}{2}, \frac{1}{2}]$  and  $\theta = \frac{-8s+\alpha}{4+\alpha} \in ]0, 1[$  such that:

$$\begin{aligned} \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim N_1^s L_1^\beta \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} \\ &\times N_2^s L_2^\beta \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \\ &\times L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

Then

$$\begin{aligned} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim N_1^s L_1^\beta \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} \\ &\times N_2^s L_2^\beta \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \\ &\times L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

Now we have:

$$(4.16) \quad \begin{aligned} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim N_1^{(\frac{1}{2}-\beta)} N_1^s \langle L_1 + N_1^2 \rangle^{\frac{1}{2} - (\frac{1}{2}-\beta)} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} \\ &\times N_2^s \langle L_2 + N_2^2 \rangle^{\frac{1}{2}} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \\ &\times L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N N_2^{\beta-\frac{1}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

Note that:

$$\begin{aligned} \sum_{L < N^2} L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N N_2^{\beta - \frac{1}{2}} \|\hat{w}_{N,L}\|_{L_{\xi,\eta,\tau}^2} &\lesssim \sum_{L < N^2} \left(\frac{L}{N^2}\right)^{\frac{\theta}{2}} N^{\frac{\theta}{2} + s + \beta - \frac{1}{2}} \|\hat{w}_{N,L}\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_{L < N^2} \left(\frac{L}{N^2}\right)^{\frac{\theta}{2}} N^{\sigma} \|\hat{w}_{N,L}\|_{L_{\xi,\eta,\tau}^2} \end{aligned}$$

where  $\sigma = \frac{\alpha}{8} + \theta(\frac{3\alpha}{8} - \frac{1}{2}) < 0$ .

By summing in  $L_1, N_1, L_2, N_2$  and  $L < N^2$ , we get:

$$J \lesssim \|u\|_{X^{\frac{1}{2}-\mu, s, 0, 1}} \|v\|_{X^{\frac{1}{2}, s, 0, 1}} \|w\|_{L_{\xi,\eta,\tau}^2} \lesssim T^{\mu} \|u\|_{X^{\frac{1}{2}, s, 0, 1}} \|v\|_{X^{\frac{1}{2}, s, 0, 1}} \|w\|_{L_{\xi,\eta,\tau}^2},$$

where  $\mu = \frac{1}{2} - \beta > 0$ .

Now we have:

$$\begin{aligned} \sum_{L > N^2} L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N N_2^{\beta - \frac{1}{2}} \|\hat{w}_{N,L}\|_{L_{\xi,\eta,\tau}^2} &\lesssim \sum_{L > N^2} \left(\frac{L}{N^2}\right)^{\frac{\theta-1}{2}} N^{\frac{\theta}{2} + s + \beta - \frac{1}{2}} \|\hat{w}_{N,L}\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_{L > N^2} \left(\frac{N^2}{L}\right)^{\frac{1-\theta}{2}} N^{\sigma} \|\hat{w}_{N,L}\|_{L_{\xi,\eta,\tau}^2} \end{aligned}$$

where  $\sigma = \sigma(\alpha, \theta) < 0$ . Thus by summing (4.16) in  $L_1, N_1, L_2, N_2$  and  $L \geq N^2$ , we get the desired estimate.

**Case 2.:**  $N_1 \leq 1$  and  $N_2 \sim N \geq 1$ .

By Cauchy-Schwarz we obtain:

$$\begin{aligned} &\langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N,L} d\xi d\eta d\tau \\ &\leq \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \|u_{N_1, L_1}\|_{L_{t,x,y}^{4+, 4+}} \|v_{N_2, L_2}\|_{L_{t,x,y}^{4-, 4-}} \|w_{N,L}\|_{L_{t,x,y}^2}. \end{aligned}$$

But  $|\xi_1| \sim N_1 \leq 1$  thus

$$\|u_{N_1, L_1}\|_{L_{t,x,y}^{4+, 4+}} \lesssim N_1^{\frac{\theta\beta\delta(r)}{2}} \|\mathcal{F}_{t,x}^{-1}(|\xi_1|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_{N_1, L_1})\|_{L_{t,x,y}^{4+, 4+}}.$$

By applying Lemma 4.2 with  $r = 4^+$ ,  $\beta = \frac{1}{2}$  and  $\theta = 1$  we obtain that:

$$\begin{aligned} \|\mathcal{F}_{t,x}^{-1}(|\xi_1|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_{N_1, L_1})\|_{L_{t,x,y}^{4+, 4+}} &\lesssim N_1^{\epsilon} \|\mathcal{F}_{t,x}^{-1}(|\xi_1|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_{N_1, L_1})\|_{L_{t,x,y}^{q, 4+}} \\ &\lesssim N_1^{\epsilon} \|\langle \tau - P(\nu) + \xi^2 \rangle^{\frac{1}{2}} \hat{u}_{N_1, L_1}\|_{L_{t,x,y}^2} \\ &\lesssim N_1^{\epsilon} \langle N_1 \rangle^s \|\langle L_1 + N_1^2 \rangle^{\frac{1}{2}} \hat{u}_{N_1, L_1}\|_{L_{t,x,y}^2} \end{aligned}$$

where  $\epsilon = \frac{\theta\beta\delta(r)}{2}$ .

Now taking  $r = 4^-$ ,  $\beta = \frac{1}{2}$ , and  $\theta = \frac{1}{2}$  and using again Lemma 4.2 we obtain that:

$$\begin{aligned} \|v_{N_2, L_2}\|_{L_{t,x,y}^{4-, 4-}} &\lesssim N_2^{\frac{\theta\beta\delta(r)}{2}} \|\mathcal{F}_{t,x}^{-1}(|\xi_2|^{-\frac{\theta\beta\delta(r)}{2}} \hat{v}_{N_2, L_2})\|_{L_{t,x,y}^{4-, 4-}} \\ &\lesssim N_2^{\frac{1}{16}+} \|\langle L_2 + N_2^2 \rangle^{\frac{1}{4}} \hat{v}_{N_2, L_2}\|_{L_{t,x,y}^2} \\ &\lesssim N^{-\gamma} \|\langle L_2 + N_2^2 \rangle^{\frac{1}{2}-\delta} \hat{v}_{N_2, L_2}\|_{L_{t,x,y}^2} \end{aligned}$$

where  $0 < \delta < \frac{1}{2}$ , and  $\gamma > 0$  small. Thus:

$$\begin{aligned} & \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \\ & \lesssim N_1^\epsilon (\langle N_1 \rangle^s \langle L_1 + N_1^2 \rangle^{\frac{1}{2}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2}) \\ & \times (\langle N_2 \rangle^s \langle L_2 + N_2^2 \rangle^{\frac{1}{2} - \delta} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2}) \\ & \times \langle L + N^2 \rangle^{-\frac{1}{2}} N N^{-\gamma} \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

But  $\langle L + N^2 \rangle^{-\frac{1}{2}} \leq L^{-\frac{\gamma}{4}} N^{-1 + \frac{\gamma}{2}}$ , then :

$$\sum_N \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} N N^{-\gamma} \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2} \leq \sum_N \sum_L N N^{-\gamma} L^{-\frac{\gamma}{4}} N^{-1 + \frac{\gamma}{2}} \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2} \lesssim \|w\|_{L_{\xi, \eta, \tau}^2}.$$

This yields:

$$J \lesssim \|u\|_{X^{1/2, s, 0, 1}} \|v\|_{X^{1/2 - \delta, s, 0, 1}} \|w\|_{L^2} \lesssim T^\delta \|u\|_{X^{1/2, s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \|w\|_{L^2}.$$

**Case 3.:**  $N_1, N_2$  and  $N \lesssim 1$ .

From (4.15) we have :

$$\int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim L_1^{\frac{\alpha}{2}} N_1^{\frac{\alpha}{8}} L_2^{\frac{\alpha}{2}} N_2^{\frac{\alpha}{8}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}.$$

Thus :

$$\begin{aligned} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau & \lesssim (\langle N_1 \rangle^s \langle L_1 + N_1^2 \rangle^{\frac{\alpha}{2}} N_1^{\frac{\alpha}{8}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2}) \\ & \times (\langle N_2 \rangle^s \langle L_2 + N_2^2 \rangle^{\frac{\alpha}{2}} N_2^{\frac{\alpha}{8}} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2}) \\ & \times \langle L + N^2 \rangle^{-\frac{1}{2}} N N^s \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

By summing we obtain that:

$$J \lesssim \|u\|_{X^{\frac{1}{2} - (\frac{1}{2} - \frac{\alpha}{2}), s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \|w\|_{L^2} \lesssim T^\mu \|u\|_{X^{\frac{1}{2}, s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \|w\|_{L^2},$$

where  $\mu = \frac{1}{2} - \frac{\alpha}{2} > 0$ . This completes the proof.  $\square$

## 5. PROOF OF THEOREM 2.2

**5.1. Existence result.** 5 Let  $\phi \in H^{s_1, 0}$  with  $s_1 > -1/2$ . For  $T \leq 1$ , if  $u$  is a solution of the integral equation (2.13), then  $u$  solve  $KPB - I$ - equation on  $[0, T/2]$ . We first prove the statement for  $T = T(\|\phi\|_{H^{s_1, 0}})$ .

Now we are going to solve (2.13) in a ball of the space  $X_T^{1/2, s_1, 0, 1}$ .

By Proposition 3.1 and Proposition 3.2, it results that,

$$(5.1) \quad \|L(u)\|_{X_T^{1/2, s_1, 0, 1}} \leq C \|\phi\|_{H^{s_1, 0}} + C \|\partial_x(u^2)\|_{X_T^{-1/2, s_1, 0, 1}}.$$

By the Proposition 4.4, we can deduce

$$(5.2) \quad \|L(u)\|_{X_T^{1/2, s_1, 0, 1}} \leq C \|\phi\|_{H^{s_1, 0}} + CT^\mu \|u\|_{X_T^{1/2, s_1, 0, 1}}^2.$$

Noticing that  $\partial_x(u^2) - \partial_x(v^2) = \partial_x[(u - v)(u + v)]$ , in the same way we get

$$(5.3) \quad \|L(u) - L(v)\|_{X_T^{1/2, s_1, 0, 1}} \leq CT^\mu \|u - v\|_{X_T^{1/2, s_1, 0, 1}} \|u + v\|_{X_T^{1/2, s_1, 0, 1}}.$$

Now take  $T = (4C^2\|\phi\|_{H^{s_1,0}})^{-1/\mu}$  we deduce from (5.2) and (5.3) that  $L$  is strictly contractive on the ball of radius  $2C(\|\phi\|_{H^{s_1,0}})$  in  $X_T^{\frac{1}{2},s_1,0,1}$ . This proves the existence of a unique solution  $u_1$  to (2.13) in  $X_T^{1/2,s_1,0,1}$  with  $T = T(\|\phi\|_{H^{s_1,0}})$ .

Note that our space  $X_T^{\frac{1}{2},s_1,0,1}$  is embedded in  $C([0, T], H^{s_1,0})$ , thus  $u$  belongs  $C([0, T_1], H^{s_1,0})$ .

**5.2. Uniqueness.** The above contraction argument gives the uniqueness of the solution to the truncated integral equation (2.13). We give here the argument of [15] to deduce easily the uniqueness of the solution to the integral equation (2.12).

Let  $u_1, u_2 \in X_T^{1/2,s_1,0,1}$  be two solution of the integral equation (2.13) on the time interval  $[0, T]$  and let  $\tilde{u}_1 - \tilde{u}_2$  be an extension of  $u_1 - u_2$  in  $X^{1/2,s_1,0,1}$  such that  $\tilde{u}_1 - \tilde{u}_2 = u_1 - u_2$  on  $[0, \gamma]$  and

$$\|\tilde{u}_1 - \tilde{u}_2\|_{X^{1/2,s_1,0,1}} \leq 2\|u_1 - u_2\|_{X_\gamma^{1/2,s_1,0,1}}$$

with  $0 < \gamma \leq T/2$ . It results by Proposition 3.1 and 3.2 that,

$$\begin{aligned} & \|u_1 - u_2\|_{X_\gamma^{1/2,s_1,0,1}} \\ & \leq \|\psi(t)L[\partial_x(\psi_\gamma^2(t')(\tilde{u}_1(t') - \tilde{u}_2(t'))(u_1(t') + u_2(t')))]\|_{X^{1/2,s_1,0,1}} \\ & \leq C\|\partial_x(\psi_\gamma^2(t)(\tilde{u}_1(t) - \tilde{u}_2(t))(u_1(t) + u_2(t)))\|_{X^{-1/2,s_1,0,1}} \\ & \leq C\gamma^{\mu/2}\|\tilde{u}_1 - \tilde{u}_2\|_{X^{1/2,s_1,0,1}}\|u_1 + u_2\|_{X_T^{1/2,s_1,0,1}} \end{aligned}$$

for some  $\mu > 0$ . Hence

$$\|u_1 - u_2\|_{X_\gamma^{1/2,s_1,0,1}} \leq 2C\gamma^{\mu/2} \left( \|u_1\|_{X_T^{1/2,s_1,0,1}} + \|u_2\|_{X_T^{1/2,s_1,0,1}} \right) \|u_1 - u_2\|_{X_\gamma^{1/2,s_1,0,1}}.$$

Taking  $\gamma \leq \left(4C(\|u_1(t)\|_{X_T^{1/2,s_1,0,1}} + \|u_2(t)\|_{X_T^{1/2,s_1,0,1}})\right)^{-\mu/2}$ , this forces  $u_1 \equiv u_2$  on  $[0, \gamma]$ . Iterating this argument, one extends the uniqueness result on the whole time interval  $[0, T]$ .  $\square$

Now proceeding exactly (with (4.12) in hand) in the same way as above but in the space

$$Z = \{u \in X_T^{1/2,s_1,0} / \|u\|_Z = \|u\|_{X_T^{1/2,\beta,0,1}} + \frac{\|\varphi\|_{H^{\beta,0}}}{\|\varphi\|_{H^{s_1,0}}} \|u\|_{X_T^{1/2,s_1,0,1}} < +\infty\},$$

where  $\beta$  is such that  $\beta \in ]-\frac{1}{2}, \min(0, s_1)[$ , we obtain that for  $T_1 = T_1(\|\varphi\|_{H^{\beta,0}})$ ,  $L$  is also strictly contractive on a ball of  $Z$ . It follows that there exists a unique solution  $\tilde{u}$  to KPBI in  $X_T^{1/2,s_1,0,1}$ . If we indicate by  $T_* = T_{max}$  the maximum time of the existence in  $X^{1/2,s_1,0,1}$  then by uniqueness, we have  $u = \tilde{u}$  on  $[0, \min(T_1, T_*)[$  and this gives that  $T_* \geq T(\|\phi\|_{H^{\beta,0}})$ .

The continuity of map  $\phi \mapsto u$  from  $H^{s_1,0}$  to  $X^{1/2,s_1,0,1}$  follows from classical argument, and in particular the map is continuous from  $H^{s_1,0}$  to  $C([0, T_1], H^{s_1,0})$ .

The analyticity of the flow-map is a direct consequence of the implicit function theorem.  $\square$

**5.3. Global existence .** Recalling that  $T = T(\|\phi\|_{H^{\delta,0}})$  with  $\delta \in ]-\frac{1}{2}, \min(0, s)]$ , and  $u \in X^{1/2,s,0,1} \subset L_t^2 H^{s+1,0}$ ,  $s+1 > 0$ , it follows that there exists  $t_0 \in ]0, T[$  such that  $u(t_0) \in L^2$ . Taking  $u(t_0) \in L^2$  as initial data, it is easy to show that  $\|u(t)\|_{L^2} \leq \|u(t_0)\|_{L^2}$ ,  $\forall t \geq t_0$ . Since the time of local existence  $T$  only depends on  $\|\phi\|_{H^{\delta,0}}$ , this clearly gives that the solution is global in time. By iteration, we obtain that  $u \in C(\mathbb{R}_+^*, H^{\infty,0})$ .  $\square$

## 6. PROOF OF THEOREM 2.3

Let  $u$  be a solution of (1.1), we have

$$(6.1) \quad u(\phi, t, x, y) = W(t)\phi(x, y) - \frac{1}{2} \int_0^t W(t-t') \partial_x(u^2(\phi, t', x, y)) dt'.$$

Suppose that the solution map is  $C^2$ . Since  $u(0, t, x, y) = 0$ , it is easy to check that

$$\begin{aligned} u_1(t, x, y) &:= \frac{\partial u}{\partial \phi}(0, t, x, y)[h] = W(t)h \\ u_2(t, x, y) &:= \frac{\partial^2 u}{\partial \phi^2}(0, t, x, y)[h, h] \\ &= - \int_0^t W(t-t') \partial_x(W(t')h)^2 dt'. \end{aligned}$$

The assumption of  $C^2$ -regularity of the solution map implies that

$$(6.2) \quad \|u_1(t, \cdot, \cdot)\|_{H^{s,0}} \lesssim \|h\|_{H^{s,0}}, \quad \|u_2(t, \cdot, \cdot)\|_{H^{s,0}} \lesssim \|h\|_{H^{s,0}}^2.$$

Now let  $P(\xi, \eta) = \xi^3 + \eta^2/\xi$ . A straightforward calculation reveals that

$$(6.3) \quad \begin{aligned} \mathcal{F}_{x \mapsto \xi, y \mapsto \eta}(u_2(t, \cdot, \cdot)) &= (i\xi) e^{itP(\xi, \eta)} \int_{\mathbb{R}^2} \hat{\phi}(\xi_1, \eta_1) \hat{\phi}(\xi - \xi_1, \eta - \eta_1) \\ &\quad \times \frac{e^{-t(\xi_1^2 + (\xi - \xi_1)^2)} e^{it\chi(\xi, \xi_1, \eta, \eta_1)} - e^{-\xi^2 t}}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 \end{aligned}$$

where  $\chi(\xi, \xi_1, \eta, \eta_1) = P(\xi_1, \eta_1) + P(\xi - \xi_1, \eta - \eta_1) - P(\xi, \eta)$ . Note that, from the definition of  $P(\xi, \eta)$ , we have that

$$\chi(\xi, \xi_1, \eta, \eta_1) = 3\xi\xi_1(\xi - \xi_1) - \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi\xi_1(\xi - \xi_1)}.$$

Let us first recall the counter-example constructed in [10]. We define the sequence of initial data  $(\phi_N)_N$ ,  $N > 0$  by

$$(6.4) \quad \hat{\phi}_N(\xi, \eta) = N^{-3/2-s} (\chi_{A_N}(|\xi|, \eta) + \chi_{B_N}(|\xi|, |\eta|))$$

where  $A_N, B_N$  are defined by

$$A_N = [N/2, N] \times [-6N^2, 6N^2], \quad B_N = [N, 2N] \times [\sqrt{3}N^2, (\sqrt{3} + 1)N^2].$$

It is simple to see that  $\|\phi_N\|_{H^{s,0}} \sim 1$ . We denote by  $u_{2,N}$  the sequence of the second iteration  $u_2$  associated with  $\phi_N$ . Hence it is readily seen that

$$\|u_{2,N}(t)\|_{H^{s,0}}^2 \geq CN^{-6-4s} \left[ \int_{\mathbb{R}^2} |\xi|^2 (1+|\xi|^2)^s \left| \int_{D(\xi, \eta)} K(t, \xi, \xi_1, \eta, \eta_1) d\xi_1 d\eta_1 \right|^2 d\xi d\eta \right]^{1/2}.$$

where

$$\begin{aligned}
D(\xi, \eta) &= \left\{ (\xi_1, \eta_1) : (\xi - \xi_1, \eta - \eta_1) \in A_N, (\xi_1, \eta_1) \in B_N \right\} \\
&\cup \left\{ (\xi_1, \eta_1) : (\xi_1, \eta_1) \in A_N, (\xi - \xi_1, \eta - \eta_1) \in B_N \right\} \\
(6.5) \quad &:= D^1(\xi, \eta) \cup D^2(\xi, \eta)
\end{aligned}$$

and

$$K(t, \xi, \xi_1, \eta, \eta_1) = \frac{e^{-t(\xi_1^2 + (\xi - \xi_1)^2)} e^{it\chi(\xi, \xi_1, \eta, \eta_1)} - e^{-\xi^2 t}}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)}.$$

Since we can write  $\xi = \xi_1 + (\xi - \xi_1)$ , it follows that

$$\begin{aligned}
\|u_2(t)\|_{H^{s,0}}^2 &\geq CN^{-4s-6} \int_{3N/2}^{3N} \int_{(\sqrt{3}-6)N^2}^{(\sqrt{3}+7)N^2} |\xi|^2 (1 + |\xi|^2)^s \\
(6.6) \quad &\times \left| \int_{D(\xi, \eta)} \frac{e^{-t(\xi_1^2 + (\xi - \xi_1)^2)} e^{it\chi(\xi, \xi_1, \eta, \eta_1)} - e^{-\xi^2 t}}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 \right|^2 d\xi d\eta.
\end{aligned}$$

We need to find a lower bound for the right-hand side of (6.6). We will prove the following lemma:

**Lemma 6.1.** *Let  $(\xi_1, \eta_1) \in D^1(\xi, \eta)$  or  $(\xi_1, \eta_1) \in D^2(\xi, \eta)$ . For  $N \gg 1$  we have*

$$|\chi(\xi, \xi_1, \eta, \eta_1)| \lesssim N^3.$$

□

**Proof of lemma 6.1.** Let  $\xi, \eta \in \mathbb{R}$  and  $(\xi_1, \eta_1) \in D^1(\xi, \eta)$  such that  $(\xi - \xi_1) \in [N/2, N]$ . Let

$$\Lambda(\xi, \xi_1, \eta_1) = \eta_1 + \frac{(\xi - \xi_1)(\eta_1 - \sqrt{3}\xi\xi_1)}{\xi_1}.$$

Thus

$$|\Lambda(\xi, \xi_1, \eta_1) - \eta_1| \leq \frac{|\xi - \xi_1|}{|\xi_1|} |\eta_1 - \sqrt{3}\xi\xi_1 - \sqrt{3}\xi_1(\xi - \xi_1)|.$$

We recall that  $\eta_1 \in [\sqrt{3}N^2, (\sqrt{3} + 1)N^2]$  and  $\xi_1 \in [N, 2N]$ . Therefore, it follows that

$$\sqrt{3}\xi_1^2 \in [\sqrt{3}N^2, 4\sqrt{3}N^2]$$

and we have

$$|\eta_1 - \sqrt{3}N^2| \leq N^2.$$

Since  $\xi_1 \in [N, 2N]$  and  $\xi - \xi_1 \in [\frac{N}{2}, N]$ , it results that

$$|\Lambda(\xi, \xi_1, \eta_1) - \eta_1| \leq 1/4 \left( 3\sqrt{3}N^2 + 2\sqrt{3}N^2 \right) \leq 6N^2.$$

Now by the mean value theorem we can write

$$\chi(\xi, \xi_1, \eta, \eta_1) = \chi(\xi, \xi_1, \Lambda(\xi, \xi_1, \eta_1), \eta_1) + (\eta - \Lambda(\xi, \xi_1, \eta_1)) \frac{\partial \chi}{\partial \eta}(\xi, \xi_1, \bar{\eta}, \eta_1)$$

where  $\bar{\eta} \in [\eta, \Lambda(\xi, \xi_1, \eta_1)]$ . Note that we choosed  $\Lambda$  such that  $\chi(\xi, \xi_1, \Lambda(\xi, \xi_1, \eta_1), \eta_1) = 0$ . Hence

$$|\chi(\xi, \xi_1, \eta, \eta_1)| = |\eta - \Lambda(\xi, \xi_1, \eta_1)| \left| \frac{2\xi_1(\bar{\eta}\xi_1 - \eta_1\xi)}{\xi\xi_1(\xi - \xi_1)} \right|.$$



Since  $|\eta - \Lambda(\xi, \xi_1, \eta_1)| \leq |\eta - \eta_1| + |\eta_1 - \Lambda(\xi, \xi_1, \eta_1)| \leq CN^2$ , it follows that

$$\begin{aligned} |\chi(\xi, \xi_1, \eta, \eta_1)| &\lesssim |\xi_1| |\eta - \Lambda(\xi, \xi_1, \eta_1)| \left| \frac{(\bar{\eta} - \eta_1)\xi_1 - \eta_1(\xi - \xi_1)}{\xi\xi_1(\xi - \xi_1)} \right| \\ &\lesssim N^3 \left( \frac{|(\bar{\eta} - \eta_1)\xi_1|}{|\xi\xi_1(\xi - \xi_1)|} + \frac{|\eta_1(\xi - \xi_1)|}{|\xi\xi_1(\xi - \xi_1)|} \right) \\ &\lesssim N^3 \left( \frac{(\sqrt{3} + 1)N^3}{N^3} + C \frac{N^3}{N^3} \right) \\ &\lesssim N^3. \end{aligned}$$

In the other case where  $(\xi_1, \eta_1) \in D^2(\xi, \eta)$  i.e.  $(\xi_1, \eta_1) \in A_N$  and  $(\xi - \xi_1, \eta - \eta_1) \in B_N$ , follows from first case since we can write  $(\xi_1, \eta_1) = (\xi - (\xi - \xi_1), \eta - (\eta - \eta_1)) \in A_N$  and that

$$\chi(\xi, \xi_1, \eta, \eta_1) = \chi(\xi, \xi - \xi_1, \eta, \eta - \eta_1).$$

This completes the proof of the Lemma.  $\square$

We return to the proof of the theorem, note that for any  $\xi \in [3N/2, 3N]$  and  $\eta \in [(\sqrt{3} - 6)N^2, (\sqrt{3} + 7)N^2]$ , we have  $\text{mes}(D(\xi, \eta)) \geq CN^3$ .

Now, for  $0 < \epsilon \ll 1$  fixed, we choose a sequence of times  $(t_N)_N$  defined by

$$t_N = N^{-3-\epsilon}.$$

For  $N \gg 1$  it can be easily seen that

$$(6.7) \quad e^{-\xi^2 t_N} \sim e^{-N^2 t_N} > C.$$

By Lemma 6.1 we have  $|-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)| \leq N^2 + N^3 \leq CN^3$ . Hence

$$(6.8) \quad \left| \frac{e^{(-2\xi_1(\xi - \xi_1)t + it\chi(\xi, \xi_1, \eta, \eta_1))} - 1}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} \right| = \frac{1}{N^{3+\epsilon}} + O\left(\frac{1}{N^{6+2\epsilon}}\right).$$

By combining the relations (6.7) and (6.8), we obtain

$$(6.9) \quad \left| \int_{D(\xi, \eta)} \frac{e^{-\xi^2 t} \left[ e^{(-2\xi_1(\xi - \xi_1)t + it\chi(\xi, \xi_1, \eta, \eta_1))} - 1 \right]}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 \right| \geq CN^{-\epsilon}.$$

It results that

$$\begin{aligned} 1 \sim \|\phi_N\|_{H^{s,0}}^2 &\geq \|u_{2,N}(t_N)\|_{H^{s,0}}^2 \gtrsim N^{-4s-6} \int_{3N/2}^{3N} \int_{(\sqrt{3}-6)N^2}^{(\sqrt{3}+7)N^2} |\xi|^2 (1 + |\xi|^2)^s d\xi d\eta \times N^{-2\epsilon} \\ &\gtrsim N^{-1-2\epsilon-2s}. \end{aligned}$$

This leads to a contradiction for  $N \gg 1$ , since we have  $-1 - 2\epsilon - 2s > 0$  for  $s \leq -1/2 - \epsilon$ . This completes the proof of Theorem 2.3.  $\square$

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