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Abstract

In this paper, we introduce a multivariate extension of the classical univariate *Value-at-Risk* (VaR). This extension may be useful to understand how solvency capital requirement computed for a given financial institution may be affected by the presence of additional risks. We also generalize the bivariate *Conditional-Tail–Expectation* (CTE), previously introduced by Di Bernardino *et al.* (2011), in a multivariate setting and we study its behavior. Several properties have been derived. In particular, we show that these two risk measures both satisfy the positive homogeneity and the translation invariance property. Comparison between univariate risk measures and components of multivariate VaR and CTE are provided. We also analyze how they are impacted by a change in marginal distributions, by a change in dependence structure and by a change in risk level. Interestingly, these results turn to be consistent with existing properties on univariate risk measures. Illustrations are given in the class of Archimedean copulas.

Introduction

During the last decades, researchers joined efforts to properly compare, quantify and manage risk. Regulators edict rules for bankers and insurers to improve their risk management and to avoid crises, not always successfully as illustrated by recent events.

Traditionally, risk measures are thought of as mappings from a set of real-valued random variables to the real numbers. However, it is often insufficient to consider a

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single real measure to quantify risks created by business activities, especially if the latter can be affected by other external risk factors. Let us consider for instance the problem of solvency capital allocation for financial institutions with multi-branch businesses confronted to risks with specific characteristics. Under Basel II and Solvability II, a bottom-up approach is used to estimate a "top-level" solvency capital. This is done by using risk aggregation techniques who may capture risk mitigation or risk diversification effects. Then this global capital amount is re-allocated to each subsidiaries or activities for internal risk management purpose ("top-down approach"). Note that the solvability of each individual branch may strongly be affected by the degree of dependence amongst all branches. As a result, the capital allocated to each branch has to be computed in a multivariate setting where both marginal effects and dependence between risks should be captured. In this respect, the "Euler approach" (e.g., see Tasche, 2008) involving vector-valued risk measures has already been tested by risk management teams of some financial institutions.
Whereas the previous risk allocation problem only involves internal risks associated with businesses in different subsidiaries, the solvability of financial institutions could also be affected by external risks whose sources cannot be controlled. These risks may also be strongly heterogeneous in nature and difficult to diversify away. One can think for instance of systemic risk or contagion effects in a strongly interconnected system of financial companies. In this regard, micro-prudential regulation has been criticized because of its failure to limit the systemic risk within the system. This question has been dealt with recently by among others, Gauthier *et al.* (2010) and Zhou (2010) who highlights the benefit of a "macro-prudential" approach as an alternative solution to the existing "micro-prudential" one (Basel II) which does not take into account interactions between financial institutions.

In this paper, we introduce a multivariate extension of the classical univariate *Value-at-Risk* (VaR). This extension may be useful to understand how solvency capital requirement (SCR) computed for a given financial institution may be affected by the presence of additional risks. We also study the multivariate extension of the *Conditional-Tail–Expectation* (CTE), previously introduced by Di Bernardino *et al.* (2011) in a bivariate setting. Several properties have been derived. In particular, we show that these two risk measures both satisfy the positive homogeneity and the translation invariance property. Comparison results between univariate risk measures and components of multivariate risk measures are provided. We also analyze how they are impacted by a change in marginal distributions, by a change in dependence structure and by a change in risk level. Interestingly, these results turn to be consistent with existing properties on univariate risk measures. In particular, we prove that, for different Archimedean families of copulas, an increase in dependence amongst risks tends to lower the amount of solvency capital required for these risks. At the extreme case where risks are perfectly dependent (i.e., comonotonic), each component of the multivariate VaR (resp. CTE) are equal to the corresponding univariate VaR (resp. CTE). If we think of univariate risk measures as solvency capital

computed in a micro-prudential regulated system, this feature is in line with the observation made by Zhou (2010): *"When regulating a system consisting of similar institutions, or in other words, the system is highly interconnected, considering a micro-prudential regulation can be sufficient for reducing the overall systemic risk."* (Zhou, 2010).
Furthermore, our multivariate risk measures can be considered as a "fair" allocation of solvency capital with respect to individual risk-taking behavior in an economy with d interconnected financial institutions. We prove that capital required for an institution is affected both by its own marginal risk and by the degree to which its business is connected to the activity of the other institutions. However, the solvency capital required for one particular institution does not depend on the marginal risks bearing by the others. This can be considered as an invariance property with respect to a change in external risk's marginal distributions, as far as the dependence structure is being fixed.

From the years 2000 onward, much research has been devoted to risk measures and many extension to multidimensional settings have been suggested (see, e.g., Jouini *et al.*, 2004 and Bentahar, 2006). Unsurprisingly, the main difficulty regarding multivariate generalizations of quantile-based risk measures (as the VaR and the CTE) is the fact that vector preorders are, in general, partial preorders. Then, what can be considered in a context of multidimensional portfolios as the analogous of a "worst case" scenario and a related "tail distribution"? This is the first question we shall address by suggesting a suitable definition of quantiles for multi-risk portfolios. In the last decade, several attempts have been made to provide a multidimensional generalization of the univariate quantile function. For example, Massé and Theodorescu (1994) defined multivariate quantiles as half-planes and Koltchinskii (1997) provided a general treatment of multivariate quantiles as inversions of mappings. Another approach is to use geometric quantiles (see for example Chaouch *et al.*, 2009). Along with the geometric quantile, the notion of depth function has been developed in recent years to characterize the quantile of multidimensional distribution functions (for further details see, for instance, Chauvigny *et al.*, 2011). We refer to Serfling (2002) for a large review on multivariate quantiles. When it turns to generalize the *Value-at-Risk* measure, Embrechts and Puccetti (2006), Nappo and Spizzichino (2009) use the notion of quantile curve (formally introduced in Section 2). Contrarily to the latter approach, the multivariate *Value-at-Risk* proposed in this paper quantifies multivariate risks in a more parsimonious and synthetic way. This feature can be relevant from an operational point of view.

In the literature, several generalizations of the classical univariate CTE have been proposed, mainly using as conditioning events the total risk or some univariate extreme risk in the portfolio. This kind of measures are suitable to model problems of capital allocation in a portfolio of dependent risks. More precisely, let $\mathbf{X} = (X_1, \ldots, X_d)$ be a risk vector, where, for any $i = 1, \ldots, d$, the component X_i denotes risk (usually claim or loss) associated with subportfolio i. Then, $S = X_1 + \cdots + X_d$ corresponds to the total risk of this portfolio, $X_{(1)} = \min\{X_1, ..., X_d\}$

and $X_{(d)} = \max\{X_1, ..., X_d\}$ are the extreme risks. In capital allocation problems, we are not only interested in the "stand-alone" risk measures $\mathrm{CTE}_\alpha(X_i) = \mathbb{E}[\, X_i \,|\, X_i > Q_{X_i}(\alpha)\,]$, where $Q_{X_i}(\alpha)$ is the univariate quantile function of X_i at risk level α, but also in

$$\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i) = \mathbb{E}[\, X_i \,|\, S > Q_S(\alpha)\,], \tag{1}$$

$$\mathrm{CTE}_\alpha^{\mathrm{min}}(X_i) = \mathbb{E}[\, X_i \,|\, X_{(1)} > Q_{X_{(1)}}(\alpha)\,], \tag{2}$$

$$\mathrm{CTE}_\alpha^{\mathrm{max}}(X_i) = \mathbb{E}[\, X_i \,|\, X_{(n)} > Q_{X_{(d)}}(\alpha)\,], \tag{3}$$

for $i = 1, \ldots, d$. The interested reader is referred to Cai and Li (2005) for further details. For explicit formulas of $\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i)$ in the case of Fairlie-Gumbel-Morgenstern family of copulas, see Bargès *et al.* (2009). Landsman and Valdez obtain an explicit formula for $\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i)$ in the case of elliptic distribution functions (see Landsman and Valdez, 2003); Cai and Li in the case of *phase-type* distributions (see Cai and Li, 2005). Furthermore, we recall that $\mathrm{CTE}_\alpha^{\mathrm{sum}}(X_i)$ is a key tool to calculate the amount of solvency capital in the "Euler approach" (see, e.g., Tasche, 2008).

The multivariate version of *Conditional-Tail-Expectation*, studied in this paper, is essentially based on a "*distributional approach*". It is constructed as the conditional expectation of a multivariate random vector given that the latter is located in a particular set corresponding to the α-upper level set of the associated multivariate distribution function (in a bivariate setting see Di Bernardino *et al.*, 2011). More precisely we consider a non-negative multivariate random vector $\mathbf{X} = (X_1, \ldots, X_d)$ and the associated α-upper level set, i.e., $L(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d \,:\, F_{\mathbf{X}}(\mathbf{x}) \geq \alpha\}$, for $\alpha \in (0, 1)$, where $F_{\mathbf{X}}$ is the multivariate distribution function of $\mathbf{X}$. As the total information of the vector $\mathbf{X}$ is completely described by its multivariate distribution function, the set $L(\alpha)$ captures the information coming both from the marginal distributions and from the multivariate dependence structure. Then contrarily to existing generalizations of the univariate CTE presented above, our $\mathrm{CTE}_\alpha(\mathbf{X})$ does not use an arbitrary real-valued aggregate transformation (sum, min, max,...). Indeed, using an aggregate procedure between the risks can be inappropriate to measure risks with heterogeneous characteristics especially in an external risks problem. Moreover, as opposed to our proposal of multivariate risk measures, the multivariate risk measures defined in (1)-(3) do not satisfy the invariance property with respect to a change in external risk's marginal distributions. This means that for a given risk component i in the portfolio, changes in marginal distributions of the other risks lead to changes in risk measures (1)-(3) for name i, even if the dependence structure does not change.

The paper is organized as follows. In Section 1, we introduce some notations, tools and technical assumptions. In Section 2, we propose a new multivariate generalization of *Value-at-Risk*. In Section 3, we generalize in a multivariate setting the *Conditional-Tail-Expectation*, previously introduced by Di Bernardino *et al.* (2011) in dimension two. We study the properties of our multivariate VaR and CTE in

terms of Artzner *et al.* (1999)'s invariance properties of risk measures (see Sections 2.1 and 3.1). We also compare the components of these multivariate risk measure with the associated univariate risk measures (see Sections 2.2 and 3.2). The behavior of $\text{VaR}_\alpha(\mathbf{X})$ (resp. $\text{CTE}_\alpha(\mathbf{X})$) with respect to a change in marginal distributions, a change in dependence structure and a change in risk level α is discussed in Sections 2.3-2.5 (resp. Sections 3.3-3.4). Further illustrations in some Archimedean copula cases, are presented both for the multivariate *Value-at-Risk* and for the *Conditional-Tail-Expectation*. In the conclusion, we discuss possible directions for future work.

1 Basic notions and preliminaries

In this section, we first introduce some notation and tools which will be used later on.

Stochastic orders

From now on, let $Q_X(\alpha)$ be the univariate quantile function of a risk X at level $\alpha \in (0,1)$. More precisely, given an univariate continuous and strictly monotonic loss distribution function F_X, $Q_X(\alpha) = F_X^{-1}(\alpha)$, $\forall\, \alpha \in (0,1)$. We recall here the definition and some properties of useful univariate and multivariate stochastic orders.

Definition 1 (Stochastic dominance order)**.** *Let X and Y be two random variables. Then X is said to be smaller than Y in stochastic dominance, denoted as $X \preceq_{st} Y$, if the inequality $Q_X(\alpha) \leq Q_Y(\alpha)$ is satisfied for all $\alpha \in (0,1)$.*

Definition 2 (Stop-loss order)**.** *Let X and Y be two random variables. Then X is said to be smaller than Y in the stop-loss order, denoted as $X \preceq_{sl} Y$, if for all $t \in \mathbb{R}$, $\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+]$, with $x_+ := \max\{x, 0\}$.*

Definition 3 (Increasing convex order)**.** *Let X and Y be two random variables. Then X is said to be smaller than Y in the increasing convex order, denoted as $X \preceq_{icx} Y$, if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$, for all non-decreasing convex function f such that the expectations exist.*

The stop-loss order and the increasing convex order are equivalent (see Theorem 1.5.7 in Müller and Stoyan, 2001). Note that stochastic dominance order implies stop-loss order. For more details about stop-loss order we refer the interested reader to Müller (1997). Moreover, a sufficient condition for the stop-loss order is the *dangerousness order relation* as stated in the following lemma.

Lemma 1 (Ohlin, 1969)**.** *Let X and Y be random variables with finite means such that $\mathbb{E}[X] \leq \mathbb{E}[Y]$, and there exists some real number c such that $F_X(x) \leq F_Y(x)$, for all $x < c$ and $F_X(x) \geq F_Y(x)$, for all $x \geq c$. Then X precedes Y in dangerousness order, written $X \preceq_D Y$, and this implies the stop-loss order $X \preceq_{sl} Y$.*

According to Bühlmann's terminology, when $X \preceq_D Y$ the distribution function F_Y is said to be more dangerous than F_X. This terminology is essentially related to the variability of the random variables X and Y (see Section 3.4.2.2 in Denuit et al., 2005). For further details, the reader is referred to Bühlmann *et al.* (1977).
Finally, we introduce the definition of supermodular function and supermodular order for multivariate random vectors.

Definition 4 (Supermodular function)**.** *A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ it satisfies*

$$f(\boldsymbol{x}) + f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) + f(\boldsymbol{x} \vee \boldsymbol{y}),$$

where the operators $\wedge$ and $\vee$ denote coordinatewise minimum and maximum respectively.

Definition 5 (Supermodular order)**.** *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two $d-$dimensional random vectors such that $\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})]$, for all supermodular functions $f : \mathbb{R}^d \to \mathbb{R}$, provided the expectation exist. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ with respect to the supermodular order (denoted by $\boldsymbol{X} \preceq_{sm} \boldsymbol{Y}$).*

This will be a key tool to analyze the impact of dependence on our multivariate risk measures.

Kendall distribution function

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a $d-$dimensional random vector, $d \geq 2$. As we will see later on, our study of multivariate risk measures strongly relies on the key concept of Kendall distribution function (or multivariate probability integral transformation), that is, the distribution function of the random variable $F(\mathbf{X})$, where F is the multivariate distribution of random vector $\mathbf{X}$. From now on, the Kendall distribution will be denoted by K, so that $K(\alpha) = \mathbb{P}[F(\mathbf{X}) \leq \alpha]$, for $\alpha \in [0, 1]$. We also denote by $\overline{K}(\alpha)$ the survival distribution function of $F(\mathbf{X})$, i.e., $\overline{K}(\alpha) = \mathbb{P}[F(\mathbf{X}) > \alpha]$. For more details on the multivariate probability integral transformation, the interested reader is referred to Capéraà *et al.*, (1997), Genest and Rivest (2001), Nelsen *et al.* (2003), Genest and Boies (2003), Genest *et al.* (2006) and Belzunce *et al.* (2007).

In contrast to the univariate case, it is not generally true that the distribution function K of $F(\mathbf{X})$ is uniform on $[0, 1]$, even when F is continuous. Note also that it is not possible to characterize the joint distribution F or reconstruct it from the knowledge of K alone, since the latter does not contain any information about the marginal distributions $F_{X_1}, \ldots, F_{X_d}$ (see Genest and Rivest, 2001). Indeed, as a consequence of Sklar's Theorem, the Kendall distribution only depends on the dependence structure or the copula function C associated with $\mathbf{X}$ (see Sklar, 1959). Thus, we also have $K(\alpha) = \mathbb{P}[C(\mathbf{U}) \leq \alpha]$ where $\mathbf{U} = (U_1, \ldots, U_d)$ and $U_1 = F_{X_1}(X_1), \ldots, U_d = F_{X_d}(X_d)$.
Furthermore:

- For a $d-$dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with copula C, the Kendall distribution function $K(\alpha)$ is linked to the Kendall's tau correlation coefficient via: $\tau_C = \frac{2^d\, \mathbb{E}[C(\mathbf{U})]-1}{2^{d-1}-1}$, for $d \geq 2$ (see Section 5 in Genest and Rivest, 2001).
- The Kendall distribution can be obtain explicitly in the case of multivariate Archimedean copulas with generator ϕ. Table 1 provides the expression of Kendall distributions associated with Archimedean, independent, comonotonic and counter-monotonic $d-$dimensional random vectors (see Barbe *et al.*, 1996). Note that the Kendall distribution is uniform for comonotonic random vectors.

Copula	Kendall distribution $K(\alpha)$
Archimedean case	$\alpha + \sum_{i=1}^{d-1}(-1)^i \frac{\phi^i(\alpha)}{i!} f_{i-1}(\alpha)$
Counter-monotonic case	1
Independent case	$\alpha + \alpha \sum_{i=1}^{d-1} \left(\frac{\ln(1/\alpha)^i}{i!}\right)$
Comonotonic case	α

Table 1: Kendall distribution in some classical $d-$dimensional dependence structure.

In Table 1, $f_i(\alpha)$ stands for $\frac{\mathrm{d}^{i-1}\phi^{-1}(t)}{\mathrm{d}t^{i+1}}$, evaluated at $t = \phi(\alpha)$. For further details the interested reader is referred to Section 2 in Barbe *et al.* (1996) and Section 5 in Genest and Rivest (2001). For instance, in the bivariate case, the Kendall distribution function is equal to $\alpha - \frac{\phi(\alpha)}{\phi'(\alpha)}$, $\alpha \in (0,1)$, for Archimedean copulas with differentiable generator ϕ and $\alpha\,(1-\ln(\alpha))$, $\alpha \in (0,1)$, for the independence copula.

- It holds that $\alpha \leq K(\alpha) \leq 1$, for all $\alpha \in (0,1)$, i.e., the graph of the Kendall distribution function is above the first diagonal (see Section 5 in Genest and Rivest, 2001). This is equivalent to state that, for any random vector $\mathbf{U}$ with copula function C and uniform marginals, $C^{\mathrm{cc}}(\mathbf{U}^{\mathrm{cc}}) \preceq_{st} C(\mathbf{U}) \preceq_{st} C^{\mathrm{c}}(\mathbf{U}^{\mathrm{c}})$ where $\mathbf{U}^{\mathrm{cc}} = (U_1^{\mathrm{cc}}, \ldots, U_d^{\mathrm{cc}})$ (resp. $\mathbf{U}^{\mathrm{c}} = (U_1^{\mathrm{c}}, \ldots, U_d^{\mathrm{c}})$) is a counter-monotonic (resp. comonotonic) random vector with copula function C^{cc} (resp. C^{c}) and uniform marginals.

This last property suggests that when the level of dependence between $X_1, \ldots, X_d$ increases, the Kendall distribution also increases in some sense. The following result, using definitions of stochastic orders described above, investigates rigorously this intuition.

Proposition 2. *Let $\boldsymbol{U} = (U_1, \ldots, U_d)$ (resp. $\boldsymbol{U}^* = (U_1^*, \ldots, U_d^*)$) be a random vector with copula C (resp. C^*) and uniform marginals.*

$$\textit{If } \boldsymbol{U} \preceq_{sm} \boldsymbol{U}^*, \textit{ then } C(\boldsymbol{U}) \preceq_{sl} C^*(\boldsymbol{U}^*).$$

Proof: Trivially, $\mathbf{U} \preceq_{sm} \mathbf{U}^* \Rightarrow C(\mathbf{u}) \leq C^*(\mathbf{u})$, for all $\mathbf{u} \in [0,1]^d$ (see Section 6.3.3 in Denuit *et al.*, 2005). Let $f : [0,1] \to \mathbb{R}$ be a non-decreasing and convex function. It holds that $f(C(\mathbf{u})) \leq f(C^*(\mathbf{u}))$, for all $\mathbf{u} \in [0,1]^d$, and $\mathbb{E}[f(C(\mathbf{U}))] \leq \mathbb{E}[f(C^*(\mathbf{U}))]$. Remark that since C^* is non-decreasing and supermodular and f is non-decreasing and convex then $f \circ C^*$ is a non-decreasing and supermodular function (see Theorem 3.9.3 in Müller and Stoyan, 2001). Then, by assumptions, $\mathbb{E}[f(C(\mathbf{U}))] \leq \mathbb{E}[f(C^*(\mathbf{U}))] \leq \mathbb{E}[f(C^*(\mathbf{U}^*))]$. This implies $C(\mathbf{U}) \preceq_{sl} C^*(\mathbf{U}^*)$. Hence the result. □

From Proposition 2, we remark that $\mathbf{U} \preceq_{sm} \mathbf{U}^*$ implies an ordering relation between corresponding Kendall's tau : $\tau_C \leq \tau_{C^*}$. Note that the supermodular order between $\mathbf{U}$ and $\mathbf{U}^*$ does not necessarily yield the stochastic dominance order between $C(\mathbf{U})$ and $C^*(\mathbf{U}^*)$ (i.e., $C(\mathbf{U}) \preceq_{st} C^*(\mathbf{U}^*)$ does not hold in general). For a bivariate counter-example, the interested reader is referred to, for instance, Capéraà *et al.* (1997) or Example 3.1 in Nelsen *et al.* (2003).

Let us now focus on some classical families of bivariate Archimedean copulas. In Table 2, we obtain analytical expressions of the Kendall distribution function for Gumbel, Frank, Clayton and Ali-Mikhail-Haq families.

Copula	$\theta \in$	Kendall distribution $K(\alpha, \theta)$
Gumbel	$[1, \infty)$	$\alpha \left(1 - \frac{1}{\theta} \ln \alpha\right)$
Frank	$(-\infty, \infty) \setminus \{0\}$	$\alpha + \frac{1}{\theta} \left(1 - \mathrm{e}^{\theta\alpha}\right) \ln \left(\frac{1-\mathrm{e}^{-\theta\,\alpha}}{1-\mathrm{e}^{-\theta}}\right)$
Clayton	$[-1, \infty) \setminus \{0\}$	$\alpha \left(1 + \frac{1}{\theta}\left(1 - \alpha^{\theta}\right)\right)$
Ali-Mikhail-Haq	$[-1, 1)$	$\frac{\alpha\,-1+\theta+(1-\theta+\theta\alpha)(\ln(1-\theta+\theta\,\alpha)+\ln\alpha)}{\theta-1}$

Table 2: Kendall distribution in some bivariate Archimedean cases.

Remark 1. Bivariate Clayton copulas can be extended to d-dimensional copulas with $d > 2$ as far as the underlying dependence parameter is such that $\theta > -\frac{1}{d-1}$ (see Example 4.27 in Nelsen, 1999). Frank copulas can be extended to d-dimensional copulas ($d > 2$) for $\theta > 0$ (see Example 4.24 in Nelsen, 1999).

Note that parameter θ governs the level of dependence amongst components of the underlying random vector. Indeed, it can be shown that, for all Archimedean copulas in Table 2, an increase of θ yields an increase of dependence in the sense of the supermodular order, i.e., $\theta \leq \theta^* \Rightarrow \mathbf{U} \preceq_{sm} \mathbf{U}^*$ (see further examples in Joe, 1997 and Wei and Hu, 2002). Then, as a consequence of Proposition 2, the following comparison result holds

$$\theta \leq \theta^* \Rightarrow C(\mathbf{U}) \preceq_{sl} C^*(\mathbf{U}^*). \tag{4}$$

In fact, a stronger comparison result can be derived for Archimedean copulas of Table 2, as shown in the following remark.

Remark 2. For copulas in Table 2, one can check that $\frac{\partial K(\alpha,\theta)}{\partial \theta} \leq 0$, for all $\alpha \in (0,1)$. This means that, for these classical examples, the associated Kendall distributions actually increase with respect to the stochastic dominance order when the dependence parameter θ increases, i.e.,

$$\theta \leq \theta^* \Rightarrow C(\mathbf{U}) \preceq_{st} C^*(\mathbf{U}^*). \tag{5}$$

In order to illustrate this property we plot in Figure 1 the Kendall distribution function $K(\cdot,\theta)$ for different choices of parameter θ in the bivariate Clayton copula case and in the bivariate Gumbel copula case.

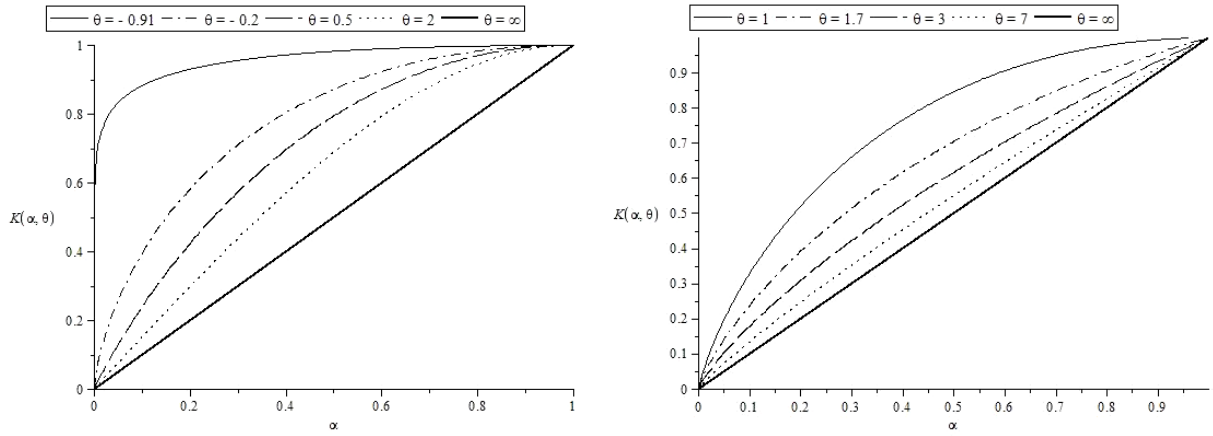

Figure 1: Kendall distribution $K(\cdot,\theta)$ for different values of θ in the bivariate Clayton copula case (left) and the bivariate Gumbel copula case (right). The dark full line represents the first diagonal and it corresponds to the comonotonic case.

2 Multivariate *Value-at-Risk*

From the usual definition in the univariate setting we know that the quantile function $Q_X(\alpha)$ provides a point which accumulates a probability α to the left tail and $1-\alpha$ to the right tail. The univariate quantile function Q_X is used in risk theory to define an univariate risk measure: the *Value-at-Risk*. This measure is defined as

$$\mathrm{VaR}_\alpha(\mathrm{X}) = \mathrm{Q}_\mathrm{X}(\alpha), \quad \forall\ \alpha \in (0,1).$$

In the recent literature, an intuitive and immediate generalization of the VaR measure in the case of a $d-$dimensional loss distribution function F is represented by its α-quantile curves. More precisely, let $F : \mathbb{R}_+^d \to [0,1]$ be a partially increasing multivariate distribution function[1] and for, any $\alpha \in (0,1)$, let $L(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) \geq \alpha\}$ be the *upper α-level set* of F. Tibiletti (1993), Embrechts and Puccetti (2006) and Nappo and Spizzichino (2009) propose to define the multivariate *Value-at-Risk* as $\partial L(\alpha)$, for $\alpha \in (0,1)$, where $\partial L(\alpha)$ denotes the boundary associated with the set $L(\alpha)$.

In the following, we will consider non-negative absolutely-continuous random vector[2] $\mathbf{X} = (X_1, \ldots, X_d)$ (with respect to Lebesgue measure λ on $\mathbb{R}^d$) with partially increasing multivariate distribution function F and such that $\mathbb{E}(X_i) < \infty$, for $i = 1, \ldots, d$. These conditions will be called *regularity conditions*.

However extensions of our results in the case of multivariate distribution function on the entire space $\mathbb{R}^d$ or in the presence of plateau in the graph of F are possible. Starting from these considerations, we introduce here a multivariate generalization of the VaR measure.

Definition 6. *Consider a random vector $\boldsymbol{X}$ satisfying the regularity conditions. For $\alpha \in (0,1)$, we define the multidimensional Value-at-Risk at probability level α by*

$$\mathrm{VaR}_\alpha(\boldsymbol{X}) = \mathbb{E}[\boldsymbol{X}|\ \boldsymbol{X} \in \partial L(\alpha)] = \begin{pmatrix} \mathbb{E}[\,X_1 \,|\, \boldsymbol{X} \in \partial L(\alpha)\,] \\ \vdots \\ \mathbb{E}[\,X_d \,|\, \boldsymbol{X} \in \partial L(\alpha)\,] \end{pmatrix}.$$

Analogously

$$\mathrm{VaR}_\alpha(\boldsymbol{X}) = \mathbb{E}[\boldsymbol{X}|\ F(\boldsymbol{X}) = \alpha] = \begin{pmatrix} \mathbb{E}[\,X_1 \,|\, F(\boldsymbol{X}) = \alpha\,] \\ \vdots \\ \mathbb{E}[\,X_d \,|\, F(\boldsymbol{X}) = \alpha\,] \end{pmatrix},$$

where $\partial L(\alpha)$ is the boundary of the α-level set of F.

[1] A function $F(x_1, \ldots, x_d)$ is partially increasing on $\mathbb{R}_+^d \setminus (0,0)$ if the functions of one variable $g(\cdot) = F(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d)$ are increasing. About properties of partially increasing multivariate distribution functions we refer the interested reader to Rossi (1973), Tibiletti (1991).

[2] We restrict ourselves to $\mathbb{R}_+^d$ because, in our applications, components of $d-$dimensional vectors correspond to random losses and are then valued in $\mathbb{R}_+$.

From now on, we denote by $\mathrm{VaR}_\alpha^1(\mathbf{X})$, ..., $\mathrm{VaR}_\alpha^d(\mathbf{X})$ the components of the vector $\mathrm{VaR}_\alpha(\mathbf{X})$. If $\mathbf{X}$ is an exchangeable vector, then $\mathrm{VaR}_\alpha^i(\mathbf{X}) = \mathrm{VaR}_\alpha^j(\mathbf{X})$, for $i, j = 1, \ldots, d$. Furthermore, given a random variable X, since $\mathbb{E}[X \,|X = \mathrm{VaR}_\alpha(\mathrm{X})] = \mathrm{Q_X}(\alpha)$, for all α in $(0, 1)$, Definition 6 can be viewed as a natural multivariate version of the univariate one. Moreover, the proposed generalization of *Value-at-Risk* for multivariate portfolio can be seen as a more parsimonious and synthetic measure compared to the Embrechts and Puccetti (2006)'s approach. Indeed, instead of considering the whole hyperspace $\partial L(\alpha)$ corresponding to the boundary of the α-level set of F, we only focus on the particular point in $\mathbb{R}_+^d$ that match the expectation of $\mathbf{X}$ given that $\mathbf{X}$ falls in $\partial L(\alpha)$. The latter feature could be relevant on practical grounds.

Note that, under the *regularity conditions*, $\partial L(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) = \alpha\}$ has Lebesgue-measure zero in $\mathbb{R}_+^d$ (e.g., see Property 3 in Tibiletti, 1990). Then we make sense of Definition 6 using the limit procedure in Feller (1966), Section 3.2:

$$\begin{aligned}\mathbb{E}[\, X_i \,|\, F(\mathbf{X}) = \alpha \,] &= \lim_{h \to 0} \mathbb{E}[\, X_i \,|\, \alpha < F(\mathbf{X}) \leq \alpha + h \,] \\ &= \lim_{h \to 0} \frac{\int_{Q_{X_i}(\alpha)}^{\infty} x \left(\int_\alpha^{\alpha+h} f_{(X_i, F(\mathbf{X}))}(x, y) \, \mathrm{d}y \right) \mathrm{d}x}{\int_\alpha^{\alpha+h} f_{F(\mathbf{X})}(y) \, \mathrm{d}y}, \quad (6)\end{aligned}$$

for $i = 1, \ldots, d$.
Dividing numerator and denominator in (6) by h, we obtain, as $h \to 0$

$$\mathbb{E}[X_i \,|F(\mathbf{X}) = \alpha] = \mathrm{VaR}_\alpha^i(\mathbf{X}) = \frac{\int_{Q_{X_i}(\alpha)}^{\infty} x \, f_{(X_i, F(\mathbf{X}))}(x, \alpha) \, \mathrm{d}x}{K'(\alpha)}, \quad (7)$$

for $i = 1, \ldots, d$, where $K'(\alpha) = \frac{\mathrm{d}K(\alpha)}{\mathrm{d}\alpha}$ is the Kendall distribution density function. This procedure gives a rigorous sense to our $\mathrm{VaR}_\alpha(\mathbf{X})$ in Definition 6. Remark that the existence of $f_{(X_i, F(\mathbf{X}))}$ and K' in (7) is guaranteed by the *regularity conditions* (for further details, see Proposition 1 in Imlahi *et al.*, 1999 or Proposition 4 in Chakak and Ezzerg, 2000).

Bivariate Archimedean copula case

In the bivariate case, the VaR introduced in Definition 6 can be computed analytically for random couple with an Archimedean copula dependence structure. Indeed, using a simple change of variable, formula (7) can be rewritten, for $i = 1$, as

$$\mathrm{VaR}_\alpha^1(X, Y) = \frac{\int_{Q_X(\alpha)}^{\infty} x \, f_{(U, C(U,V))}(F_X(x), \alpha) \, \mathrm{d}x}{K'(\alpha)},$$

where $f_{(U,C(U,V))}$ is the density associated with the random vector $(U, C(U, V))$. In the bivariate Archimedean case, the latter density can be obtained analytically by

using a change of variable transformation[3] from (U, V) to $(U, C(U, V))$:

$$F_{(U,C(U,V))}(s,t) = t - \frac{\phi(t)}{\phi'(t)} + \frac{\phi(s)}{\phi'(t)}, \tag{8}$$

for $0 < t < s < 1$. The previous change of variable involves a Jacobian matrix and it can be generalized to dimension $d \geq 2$ using a multivariate transformation. Therefore, in the class of Archimedean copulas, our multivariate extension of the VaR can be computed analytically, either through a closed-form formula (see, for instance, the bivariate Clayton example below) or at least by using numerical quadratures.

Let us now consider the Clayton family of bivariate copulas. This family is interesting since it contains the counter-monotonic, the independence and the comonotonic copulas as particular cases. Let (X, Y) be a random vector distributed as a Clayton copula with parameter $\theta \geq -1$. In particular, the marginal distribution of X and Y are uniform. Then, the distribution function C_θ of (X, Y) is such that

$$C_\theta(x,y) = (\max\{x^{-\theta} + y^{-\theta} - 1, 0\})^{-1/\theta}, \quad \text{for } \theta \geq -1, \quad (x,y) \in [0,1]^2. \tag{9}$$

Since X and Y are exchangeable, the two components of the multivariate VaR are identical. Table 3 gives analytical expressions for the first (equal to the second) component of the VaR, i.e., $\mathrm{VaR}^1_{\alpha,\theta}(X, Y)$. Note that the latter can be represented as a function of the risk level α and the dependence parameter θ. For $\theta = -1$ and $\theta = \infty$ we obtain the Fréchet-Hoeffding lower and upper bounds: $W(x, y) = \max\{x + y - 1, 0\}$ (counter-monotonic random variables) and $M(x, y) = \min\{x, y\}$ (comonotonic random variables) respectively. The settings $\theta = 0$ and $\theta = 1$ correspond to degenerate cases. For $\theta = 0$ we have the independence copula $\Pi(x, y) = x\,y$. For $\theta = 1$, we obtain the copula denoted by $\frac{\Pi}{\Sigma-\Pi}$ in Nelsen (1999) where $\frac{\Pi}{\Sigma-\Pi}(x, y) = \frac{x\,y}{x+y-x\,y}$.

Copula	θ	$\mathrm{VaR}^1_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1, \infty)$	$\frac{\theta}{\theta-1}\frac{\alpha^\theta-\alpha}{\alpha^\theta-1}$
Counter-monotonic W	-1	$\frac{1+\alpha}{2}$
Independent Π	0	$\frac{\alpha-1}{\ln\alpha}$
$\frac{\Pi}{\Sigma-\Pi}$	1	$\frac{\alpha\,\ln\alpha}{\alpha-1}$
Comonotonic M	∞	α

Table 3: $\mathrm{VaR}^1_{\alpha,\theta}(X, Y)$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \mathrm{VaR}^1_{\alpha,\theta}}{\partial \alpha} \geq 0$ and $\frac{\partial \mathrm{VaR}^1_{\alpha,\theta}}{\partial \theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0, 1)$. This proves that, for Clayton-distributed random couples, the

[3] In the book by Nelsen (1999) (Corollary 4.3.5), a geometrical argument is used instead to obtain the distribution function of $(U, C(U, V))$.

components of our multivariate VaR are increasing functions of the risk level α and decreasing functions of the dependence parameter θ. Note also that the multivariate VaR in the comonotonic case corresponds to the vector composed of the univariate VaR associated with each component. These properties are illustrated in Figure 2 where $\mathrm{VaR}^1_{\alpha,\theta}(X,Y)$ is plotted as a function of the risk level α for different values of the parameter θ. Observe that an increase of the dependence parameter θ tends to lower the VaR up to the perfect dependence case where $\mathrm{VaR}^1_{\alpha,\theta}(X,Y) = \mathrm{VaR}_\alpha(X) = \alpha$. These empirical behaviors will be formally confirmed in next sections.

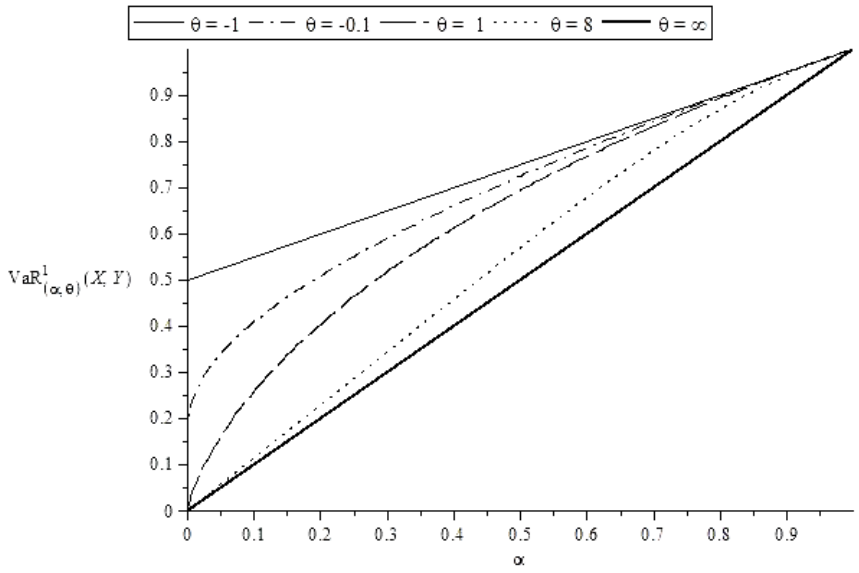

Figure 2: Behavior of $\mathrm{VaR}^1_{\alpha,\theta}(X,Y) = \mathrm{VaR}^2_{\alpha,\theta}(X,Y)$ with respect to risk level α for different values of dependence parameter θ. The random vector (X,Y) follows a Clayton copula distribution with parameter θ.

2.1 Invariance properties

Our aim in the present section is to analyze the multivariate *Value-at-Risk* introduced in Definition 6 in terms of classical invariance properties of risk measures (we refer the interested reader to Artzner *et al.*, 1999). The following proposition proves positive homogeneity and translation invariance for $\mathrm{VaR}_\alpha(\mathbf{X})$.

Proposition 3. *Consider a random vector* $\boldsymbol{X}$ *satisfying the regularity conditions. For* $\alpha \in (0,1)$, $\mathrm{VaR}_\alpha(\boldsymbol{X})$ *satisfies the following properties:*

Positive Homogeneity: $\forall\ \boldsymbol{c} \in \mathbb{R}^d_+$,

$$\mathrm{VaR}_\alpha(\boldsymbol{c}\,\boldsymbol{X}) = \boldsymbol{c}\,\mathrm{VaR}_\alpha(\boldsymbol{X}) = \begin{pmatrix} c_1\,\mathbb{E}[\,X_1 \,|\, F(\boldsymbol{X}) = \alpha\,] \\ \vdots \\ c_d\,\mathbb{E}[\,X_d \,|\, F(\boldsymbol{X}) = \alpha\,] \end{pmatrix}.$$

Translation Invariance: $\forall\ \boldsymbol{c} \in \mathbb{R}^d_+$,

$$\mathrm{VaR}_\alpha(\boldsymbol{c} + \boldsymbol{X}) = \boldsymbol{c} + \mathrm{VaR}_\alpha(\boldsymbol{X}) = \begin{pmatrix} c_1 + \mathbb{E}[\,X_1 \,|\, F(\boldsymbol{X}) = \alpha\,] \\ \vdots \\ c_d + \mathbb{E}[\,X_d \,|\, F(\boldsymbol{X}) = \alpha\,] \end{pmatrix}.$$

The proof is based on invariance properties of level sets. We refer the interested reader to Proposition 1 in Tibiletti (1993).

Proof: Let $h : \mathbb{R}^d_+ \to \mathbb{R}^d_+$, with $h(x_1, x_2, \ldots, x_d) = (h_1(x_1), h_2(x_2), \ldots, h_d(x_d))$ and $h_1, h_2, \ldots, h_d$ be increasing functions. Then we obtain
$\partial L_{h(\mathbf{X})}(\alpha) = \{\mathbf{x} \in \mathbb{R}^d_+ : F_{h(\mathbf{X})}(\mathbf{x}) = \alpha\} = \{\mathbf{x} \in \mathbb{R}^d_+ : F_{(h_1(X_1), h_2(X_2), \ldots, h_d(X_d))}(\mathbf{x}) = \alpha\}$
$= \{h(\mathbf{x}) \in \mathbb{R}^d_+ : F_{\mathbf{X}}(\mathbf{x}) = \alpha\} = \{(h_1(x_1), h_2(x_2), \ldots, h_d(x_d)) \in \mathbb{R}^d_+ : F_{\mathbf{X}}(\mathbf{x}) = \alpha\} = h(\partial L(\alpha))$.
In particular we take $h(\mathbf{x}) = \mathbf{c}+\mathbf{x}$, with $\mathbf{c} \in \mathbb{R}^d_+$. Then $\partial L_{(\mathbf{c}+\mathbf{X})}(\alpha) = \{(\mathbf{c}+\mathbf{x}) \in \mathbb{R}^d_+ : F_{\mathbf{X}}(\mathbf{x}) = \alpha\}$. Analogously with $h(\mathbf{x}) = \mathbf{c}\,\mathbf{x}$, with $\mathbf{c} \in \mathbb{R}^d_+$, $\partial L_{(\mathbf{c}\,\mathbf{X})}(\alpha) = \{(\mathbf{c}\,\mathbf{x}) \in \mathbb{R}^d_+ : F_{\mathbf{X}}(\mathbf{x}) = \alpha\}$. Then,

$$\begin{aligned} &\mathbb{E}[\,c_i + X_i \,|\, (\mathbf{c} + \mathbf{X}) \in \partial L_{(\mathbf{c}+\mathbf{X})}(\alpha)\,] = c_i + \mathbb{E}[\,X_i \,|\, h(\mathbf{X}) \in \partial L_{h(\mathbf{X})}(\alpha)\,] \\ &= c_i + \mathbb{E}[\,X_i \,|\, h(\mathbf{X}) \in h(\partial L_{\mathbf{X}})(\alpha)\,] = c_i + \mathbb{E}[\,X_i \,|\, h^{-1}(h(\mathbf{X})) \in \partial L_{\mathbf{X}}(\alpha)\,], \end{aligned}$$

for $i = 1, \ldots, d$. Hence the result. □

2.2 Comparison of univariate and multivariate VaR

Note that, using a change of variable, each component of the multivariate *Value-at-Risk* can be represented as an integral transformation of the associated univariate *Value-at-Risk*. Let us denote by F_{X_i} the marginal distribution functions of X_i for $i = 1, \ldots, d$ and by C the copula associated with $\mathbf{X}$. Thanks to Sklar's theorem we have $F(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$ (see Sklar, 1959). Then the random

variables U_i defined by $U_i = F_{X_i}(X_i)$, for $i = 1, \ldots, d$, are uniformly distributed and their joint distribution is equal to C. Using these notations, we get:

$$\mathrm{VaR}^i_\alpha(\mathbf{X}) = \frac{1}{K'(\alpha)} \int_\alpha^1 \mathrm{VaR}_\gamma(X_i) f_{(U_i, C(\mathbf{U}))}(\gamma, \alpha)\, d\gamma, \tag{10}$$

for $i = 1, \ldots, d$, where $f_{(U_i, C(\mathbf{U}))}$ is the density function associated with the multivariate vector $(U_i, C(\mathbf{U}))$. The following proposition allows us to compare univariate and multivariate *Value-at-Risk*.

Proposition 4. *Consider a random vector $\boldsymbol{X}$ satisfying the regularity conditions. Assume that its multivariate distribution function F is a quasi concave*[4]*. Then, for all $\alpha \in (0, 1)$, the following inequality holds*

$$\mathrm{VaR}^i_\alpha(\boldsymbol{X}) \geq \mathrm{VaR}_\alpha(X_i), \tag{11}$$

for $i = 1, \ldots, d$.

Proof: Let $\alpha \in (0, 1)$. From the definition of the accumulated probability, it is easy to show $\partial L(\alpha)$ is inferiorly bounded by the marginal univariate quantile functions. Moreover, recall that $L(\alpha)$ is a convex set in $\mathbb{R}^d_+$ from the quasi concavity of F (see Section 2 in Tibiletti, 1995). Then, for all $\mathbf{x} = (x_1, \ldots, x_d) \in \partial L(\alpha)$, $x_1 \geq \mathrm{VaR}_\alpha(X_1), \cdots, x_d \geq \mathrm{VaR}_\alpha(X_d)$ and trivially, $\mathrm{VaR}^i_\alpha(\mathbf{X})$ is greater than $\mathrm{VaR}_\alpha(X_i)$, for $i = 1, \ldots, d$. Hence the result. □

Proposition 4 states that the multivariate $\mathrm{VaR}_\alpha(\mathbf{X})$ is a more conservative measure than the vector composed with the univariate α-*Value-at-Risk* of marginals. Furthermore, we can prove that the previous lower bound in (11) is reached for comonotonic random vectors.

Proposition 5. *Consider a comonotonic non-negative random vector $\boldsymbol{X}$. Then, for all $\alpha \in (0, 1)$, it holds that*

$$\mathrm{VaR}^i_\alpha(\boldsymbol{X}) = \mathrm{VaR}_\alpha(X_i),$$

for $i = 1, \ldots, d$.

Proof: Let $\alpha \in (0, 1)$. If $\mathbf{X} = (X_1, \ldots, X_d)$ is a comonotonic non-negative random vector then there exist a random variable Z and d increasing functions $g_1, \ldots, g_d$ such that $\mathbf{X}$ is equal to $(g_1(Z), \ldots, g_d(Z))$ in distribution. So the set $\{(x_1, \ldots, x_d) : F(x_1, \ldots, x_d) = \alpha\}$ becomes $\{(x_1, \ldots, x_d) : \min\{g_1^{-1}(x_1), \ldots, g_d^{-1}(x_d)\} = Q_Z(\alpha)\}$, where Q_Z is the quantile function of Z. Then, trivially, $\mathrm{VaR}^i_\alpha(\mathbf{X}) = \mathbb{E}[\, X_i \,|\, F(\mathbf{X}) = \alpha\,] = Q_{X_i}(\alpha)$, for $i = 1, \ldots, d$. Hence the result. □

[4] A function F is quasi concave if the upper level sets of F are convex sets. Tibiletti (1995) points out families of distribution functions which satisfy the property of quasi concavity. For instance all the Archimedean copulas are quasi concave functions (see Nelsen, 1999).

Remark 3. Using (8) for bivariate independent random couple (X, Y), formula (10) becomes

$$\mathrm{VaR}^1_\alpha(X,Y) = \frac{1}{-\ln(\alpha)} \int_\alpha^1 \frac{\mathrm{VaR}_\gamma(X)}{\gamma}\, d\gamma,$$

then, obviously, in this case the X-related component only depends on the marginal behavior of X.

2.3 Behavior of the multivariate VaR with respect to marginal distributions

In this section we study the behavior of our risk measure with respect to a variation of marginals. Results presented below provide natural multivariate extensions of some classical results in the univariate setting (see, e.g., Denuit and Charpentier, 2004).

Proposition 6. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d–dimensional continuous random vectors satisfying the regularity conditions and with the same copula structure C. If $X_i \overset{d}{=} Y_i$ then it holds that*

$$\mathrm{VaR}^i_\alpha(\boldsymbol{X}) = \mathrm{VaR}^i_\alpha(\boldsymbol{Y}), \qquad \textit{for all } \alpha \in (0,1).$$

The proof of the previous Proposition directly comes down from formula (10). From Proposition 6, we remark that, for a fixed copula structure C, the i-th component $\mathrm{VaR}^i_\alpha(\mathbf{X})$ does not depend on marginal distributions of the other components j with $j \neq i$.

In order to derive the next result, we use the definitions of stochastic orders presented in Section 1.

Proposition 7. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d–dimensional continuous random vectors satisfying the regularity conditions and with the same copula structure C. If $X_i \preceq_{st} Y_i$ then it holds that*

$$\mathrm{VaR}^i_\alpha(\boldsymbol{X}) \leq \mathrm{VaR}^i_\alpha(\boldsymbol{Y}), \qquad \textit{for all } \alpha \in (0,1).$$

The proof comes down from formula (10) and Definition 1. Note that, the result in Proposition 7 is consistent with the one-dimensional setting (see Section 3.3.1 in Denuit *et al.*, 2005). Indeed, as in dimension one, a stochastic order on the coordinates implies an order relation on the coordinates of *Value-at-Risk*.

As a result, in an economy with several interconnected financial institutions, capital required for one particular institution is affected by its own marginal risk. But, for a fixed dependence structure, the solvency capital required for this specific institution does not depend on marginal risks bearing by the others. Then, our multivariate VaR implies a "fair" allocation of solvency capital with respect to individual risk-taking behavior. In other words, individual financial institutions may not have to pay more for risky business activities undertook by the others.

2.4 Behavior of multivariate VaR with respect to the dependence structure

In this section we study the behavior of our risk measure with respect to a variation of the dependence structure, with unchanged marginal distributions.

Proposition 8. *Let $\boldsymbol{X}$ and $\boldsymbol{X}^*$ be two d–dimensional continuous random vectors satisfying the regularity conditions and with the same margins F_{X_i} and $F_{X_i^*}$, for $i = 1, \ldots, d$, and let C (resp. C^*) denote the copula function associated with $\boldsymbol{X}$ (resp. $\boldsymbol{X}^*$). Let $U_i = F_{X_i}(X_i)$, $U_i^* = F_{X_i^*}(X_i^*)$, $\boldsymbol{U} = (U_1, \ldots, U_d)$ and $\boldsymbol{U}^* = (U_1^*, \ldots, U_d^*)$.*

$$\text{If } [U_i | C(\boldsymbol{U}) = \alpha] \preceq_{st} [U_i^* | C^*(\boldsymbol{U}^*) = \alpha] \text{ then } \mathrm{VaR}_\alpha^i(\boldsymbol{X}) \leq \mathrm{VaR}_\alpha^i(\boldsymbol{X}^*).$$

Proof: Let $U_1 \overset{d}{=} [U_i | C(\mathbf{U}) = \alpha]$ and $U_2 \overset{d}{=} [U_i^* | C^*(\mathbf{U}^*) = \alpha]$. We recall that $U_1 \preceq_{st} U_2$ if and only if $\mathbb{E}[f(U_1)] \leq \mathbb{E}[f(U_2)]$, for all non-decreasing function f, such that the expectations exist (see Denuit *et al.*, 2005; Proposition 3.3.14). We now choose $f(u) = Q_{X_i}(u)$, for $u \in (0, 1)$. Then we obtain

$$\mathbb{E}[\, Q_{X_i}(U_i) | C(\mathbf{U}) = \alpha \,] \leq \mathbb{E}[\, Q_{X_i}(U_i^*) | C^*(\mathbf{U}^*) = \alpha \,],$$

But the right-hand side of the previous inequality is equal to $\mathbb{E}[\, Q_{X_i^*}(U_i^*) | C^*(\mathbf{U}^*) = \alpha \,]$ since X_i and X_i^* have the same distribution. Finally, from formula (10) we obtain $\mathrm{VaR}_\alpha^i(\mathbf{X}) \leq \mathrm{VaR}_\alpha^i(\mathbf{X}^*)$. Hence the result. □

We now provide an illustration of Proposition 8 in the case of bivariate Archimedean copulas.

Example 1. We consider a bivariate Archimedean copula with differentiable generator ϕ. In this case, using formula (8) we obtain

$$\mathbb{P}[U > u \,|\, C(U, V) = \alpha] = \frac{\phi\,(u)}{\phi\,(\alpha)}, \quad \text{for } \ 0 < \alpha < u < 1. \tag{12}$$

Let C_θ and C_{θ^*} be two Archimedean copula with dependence parameter θ and θ^*. We consider that (X, Y) and (X^*, Y^*) are two bivariate continuous random vectors with the same marginals F_X and F_Y and with copula C_θ and C_{θ^*} respectively. Let $U = F_X(X)$, $V = F_Y(Y)$, $U^* = F_X(X^*)$ and $V^* = F_Y(Y^*)$. It can be shown that an increase of dependence ($\theta \leq \theta^*$) yields a decrease of the bivariate VaR ($\mathrm{VaR}_\alpha^1(X^*, Y^*) \leq \mathrm{VaR}_\alpha^1(X, Y)$). Using formula (12), the condition $[U^* | C_\theta(U^*, V^*) = \alpha] \preceq_{st} [U | C_\theta(U, V) = \alpha]$ is equivalent to

$$\frac{\phi_{\theta^*}\,(u)}{\phi_{\theta^*}\,(\alpha)} \leq \frac{\phi_\theta\,(u)}{\phi_\theta\,(\alpha)}, \quad \text{for } \ 0 < \alpha < u < 1. \tag{13}$$

Interestingly, the previous condition is satisfied for all Archimedean families for which the relation $\theta \leq \theta^* \Rightarrow C_\theta(U, V) \preceq_{st} C_{\theta^*}(U^*, V^*)$ holds (see, e.g., copulas in Table 2). Indeed, starting from the expression of Kendall distribution function for

Archimedean copulas (see Table 1), one can readily show that the previous relation yields (13). As a result, from Proposition 8, we have

$$\theta \leq \theta^* \Rightarrow \text{VaR}^1_\alpha(X^*, Y^*) \leq \text{VaR}^1_\alpha(X, Y). \tag{14}$$

Then, for copulas in Table 2, the multivariate VaR is non-increasing with respect to the dependence parameter θ (coordinate by coordinate). In particular, this means that limit behaviors of dependence parameters are associated with bounds for our multivariate risk measure in the case of Archimedean copula. For instance, if we denote by $\text{VaR}^1_{(\alpha,\theta)}(X, Y)$ the first component of the bivariate VaR for a vector (X, Y) with a Clayton dependence structure with parameter θ, we get from Table 3 the following comparison result:

$$\text{VaR}^1_{(\alpha,\infty)}(X, Y) \leq \text{VaR}^1_{(\alpha,\theta)}(X, Y) \leq \text{VaR}^1_{(\alpha,-1)}(X, Y), \tag{15}$$

for all $\alpha \in (0, 1)$ and all $\theta \in (-1, \infty)$. Note that the lower bound corresponds to comonotonic random variables, so that $\text{VaR}^1_{(\alpha,\infty)}(X, Y) = \text{VaR}_\alpha(X) = \alpha$ for random variables X, Y with uniform margins (see Table 3). The upper bound corresponds to counter-monotonic random variables, so that $\text{VaR}^1_{(\alpha,-1)}(X, Y) = \frac{1+\alpha}{2}$ for random variables X, Y with uniform margins, which turns to be also equal to $\text{CTE}_\alpha(X)$ in that case, where CTE stands for the univariate *Conditional-Tail-Expectation* defined in Section 3 (see (17)).

2.5 Behavior of multivariate VaR with respect to risk level

In order to study the behavior of the multivariate *Value-at-Risk* with respect to risk level α, we need to introduce the *positive regression dependence* concept. For a bivariate random vector (X, Y) we mean by positive dependence that X and Y are likely to be large or to be small together. An excellent presentation of positive dependence concepts can be found in Chapter 2 of the book by Joe (1997). A positive dependence concept that will be useful in the following was called by Lehmann (1966) *positive regression dependence* (PRD) but most of the authors use the term *stochastically increasing* (SI) (see Nelsen, 1999; Section 5.2.3).

Definition 7 (Positive regression dependence)**.** *A bivariate random vector* (X, Y) *is said to admit positive regression dependence with respect to* X, $\text{PRD}(Y|X)$, *if*

$$[Y|X = x_1] \preceq_{st} [Y|X = x_2], \quad \forall\, x_1 \leq x_2. \tag{16}$$

Clearly condition in (16) is a positive dependence notion (see Section 2.1.2 in Joe, 1997). In the following, a few examples are used to illustrate the dependence concept in Definition 7.

Let $f(x_1, x_2) = \frac{1}{(2\pi)(1-\rho^2)^{1/2}} \, \mathrm{e}^{\frac{-\frac{1}{2}(x_1^2+x_2^2-2\rho\, x_1\, x_2)}{(1-\rho^2)}}$, be the bivariate normal density with $\rho \in (-1, 1)$. Then if $\rho > 0$ we obtain PRD$(X_2|X_1)$ (see Example 2.1 in Joe, 1997).

We consider the bivariate copula (4.2.6) in Table 4.1 in Nelsen (1999): $C(u, v) = 1 - [(1-u)^\theta + (1-v)^\theta - (1-u)^\theta\,(1-v)^\theta]^{\frac{1}{\theta}}$, with $\theta \in [1, \infty)$. Then PRD$(V|U)$, for each $\theta \in [1, \infty)$ (see Example 2.3 in Joe, 1997).

From Definition 7, it is straightforward to derive the following result.

Proposition 9. *Consider a d–dimensional random vector $\boldsymbol{X}$, satisfying the regularity conditions, with marginal distributions F_{X_i}, for $i = 1, \ldots, d$, and copula C. Let $U_i = F_{X_i}(X_i)$ and $\boldsymbol{U} = (U_1, \ldots, U_d)$. Then it holds that :*

If $(U_i, C(\boldsymbol{U}))$ is PRD$(U_i|C(\boldsymbol{U}))$ then $\mathrm{VaR}^i_\alpha(\boldsymbol{X})$ is a non-decreasing function of α.

The proof of Proposition 9 essentially comes down from formula (10) and Definition 7.

Proof: We have $[U_i|C(\mathbf{U}) = \alpha_1] \preceq_{st} [U_i|C(\mathbf{U}) = \alpha_2]$, $\forall\, \alpha_1 \leq \alpha_2$. As in the proof of Proposition 8,

$$\mathbb{E}[\, Q_{X_i}(U_i)|C(\mathbf{U}) = \alpha_1\,] \leq \mathbb{E}[\, Q_{X_i}(U_i)|C(\mathbf{U}) = \alpha_2\,].$$

Then $\mathrm{VaR}^i_{\alpha_1}(\mathbf{X}) \leq \mathrm{VaR}^i_{\alpha_2}(\mathbf{X})$, for $\alpha_1 \leq \alpha_2$. Then $\mathrm{VaR}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α. □

Note that behavior of the multivariate VaR with respect to a change in the risk level does not depend on marginal distributions of $\mathbf{X}$.

Here is an illustration of Proposition 9 in the bivariate Archimedean case.

Remark 4. Note that each component of VaR is increasing with respect to α for all random couples with an Archimedean copula dependence structure. Indeed, from formula (12), it is straightforward to show that $\mathbb{P}[U > u|C(U, V) = \alpha]$ is a non-decreasing function of α, for all u, then PRD$(U|C(U, V))$ and PRD$(V|C(U, V))$ properties are satisfied in the bivariate Archimedean case.

3 Multivariate *Conditional-Tail-Expectation*

As well as in the univariate case, the multivariate VaR at a predetermined level α does not give any information about the thickness of the upper tail of the distribution function. This is a considerable shortcoming of the VaR measure because in practice we are not only concerned with the frequency of the default but also with the severity of loss in case of default. In order to overcome this problem, another risk measure has recently received growing attentions in the insurance and finance literature: the *Conditional-Tail-Expectation* (CTE). Following Artzner *et al.* (1999) and Dedu and

Ciumara (2010), for a continuous loss distribution function F_X the CTE at level α is defined by

$$\mathrm{CTE}_\alpha(X) = \mathbb{E}[\, X \,|\, X \geq \mathrm{VaR}_\alpha(X)\,], \tag{17}$$

where $\mathrm{VaR}_\alpha(X)$ is the univariate *Value-at-Risk* introduced above. For a comprehensive treatment and for references to the extensive literature on $\mathrm{VaR}_\alpha(X)$ and $\mathrm{CTE}_\alpha(X)$ one may refer to Denuit *et al.* (2005).

In the following, we propose a multivariate generalization of the bivariate *Conditional-Tail-Expectation*, previously introduced by Di Bernardino *et al.* (2011).

Definition 8. *Consider a d–dimensional random vector $\boldsymbol{X}$ satisfying the regularity conditions. For $\alpha \in (0,1)$, we define the multivariate α-Conditional-Tail-Expectation by*

$$\mathrm{CTE}_\alpha(\boldsymbol{X}) = \mathbb{E}[\boldsymbol{X}|\boldsymbol{X} \in L(\alpha)] = \begin{pmatrix} \mathbb{E}[\, X_i \,|\, \boldsymbol{X} \in L(\alpha)\,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \boldsymbol{X} \in L(\alpha)\,] \end{pmatrix}.$$

Analogously,

$$\mathrm{CTE}_\alpha(\boldsymbol{X}) = \mathbb{E}[\boldsymbol{X}|F(\boldsymbol{X}) \geq \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\boldsymbol{X}) \geq \alpha\,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\boldsymbol{X}) \geq \alpha\,] \end{pmatrix}.$$

From now on, we denote by $\mathrm{CTE}^1_\alpha(\mathbf{X})$, ..., $\mathrm{CTE}^d_\alpha(\mathbf{X})$ the components of the vector $\mathrm{CTE}_\alpha(\mathbf{X})$. Note that this multivariate *Conditional-Tail-Expectation* is a natural extension of the univariate one. Moreover, if $\mathbf{X}$ is an exchangeable vector, $\mathrm{CTE}^i_\alpha(\mathbf{X}) = \mathrm{CTE}^j_\alpha(\mathbf{X})$, for $i, j = 1, \ldots, d$.

From Definition 8 and formula (7) we write, for $\alpha \in (0,1)$

$$\begin{aligned} \mathrm{CTE}^i_\alpha(\mathbf{X}) &= \int_{\mathbf{x}\in L(\alpha)} x_i \, \frac{f_{\mathbf{X}}(x_1,\ldots,x_d)}{K(\alpha)} \, \mathrm{d}x_1 \cdots \mathrm{d}x_d \\ &= \frac{\int_\alpha^1 \left(\int_{Q_{X_i}(\gamma)}^\infty x f_{(X_i, F(\mathbf{X}))}(x,\gamma) \mathrm{d}x \right) \mathrm{d}\gamma}{\int_\alpha^1 K'(\gamma) \mathrm{d}\gamma}. \end{aligned} \tag{18}$$

Formula (18) will be useful in Proposition 12 and Corollary 16 below.

Bivariate Archimedean copula case

We consider here a random couple (X, Y) which follows a Clayton copula distribution with parameter $\theta \geq -1$ as in (9). In that case, X and Y are uniformly distributed. We obtain in Table 4 a closed-form expression for the multivariate CTE in that case.

Copula	θ	$\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1,\infty)$	$\frac{1}{2}\,\frac{\theta}{\theta-1}\,\frac{\theta-1-\alpha^2(1+\theta)+2\,\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic W	-1	$\frac{1}{4}\,\frac{1-\alpha^2+2\,\ln\alpha}{1-\alpha+\ln\alpha}$
Independent Π	0	$\frac{1}{2}\,\frac{(1-\alpha)^2}{1-\alpha+\alpha\ln\alpha}$
$\frac{\Pi}{\Sigma-\Pi}$	1	$\frac{1}{2}\,\frac{1+\alpha^2(2\,\ln\alpha-1)}{(1-\alpha)^2}$
Comonotonic M	∞	$\frac{1+\alpha}{2}$

Table 4: $\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\alpha} \geq 0$ and $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0,1)$. This proves that, for Clayton-distributed random vectors, the components of our CTE are increasing functions of the risk level α and decreasing functions of the dependence parameter θ. Note also that, in the comonotonic case, our CTE corresponds to the vector composed of the univariate CTE associated with each component. These properties are illustrated in Figure 3 where $\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$ is plotted as a function of the risk level α for different values of θ. Note that we obtained exactly the same feature for the multivariate VaR (see Figure 2). The previous empirical behaviors will be formally investigated in next sections.

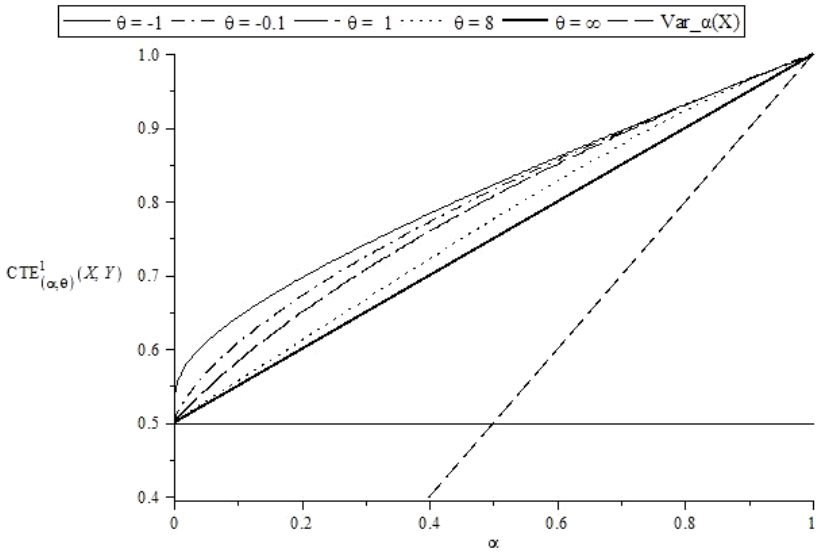

Figure 3: Behavior of $\mathrm{CTE}^1_{\alpha,\theta}(X,Y) = \mathrm{CTE}^2_{\alpha,\theta}(X,Y)$ with respect to risk level α for different values of dependence parameter θ. The random vector (X,Y) follows a Clayton copula distribution with parameter θ. The horizontal line corresponds to $\mathbb{E}[\,X\,] = \frac{1}{2}$.

3.1 Invariance properties

As in Section 2.1, we analyze here the multivariate *Conditional-Tail-Expectation* in terms of classical invariance properties of risk measures (we refer the interested reader to Artzner *et al.*, 1999). The following proposition proves positive homogeneity and translation invariance for $\mathrm{CTE}_\alpha(\mathbf{X})$.

Proposition 10. *Consider a random vector $\boldsymbol{X}$ satisfying the regularity conditions. For $\alpha \in (0,1)$, $\mathrm{CTE}_\alpha(\boldsymbol{X})$ satisfies the following properties:*

Positive Homogeneity: $\quad \forall\ \boldsymbol{c} \in \mathbb{R}^d_+$,

$$\mathrm{CTE}_\alpha(\boldsymbol{c}\,\boldsymbol{X}) = \boldsymbol{c}\,\mathrm{CTE}_\alpha(\boldsymbol{X}) = \begin{pmatrix} c_1\,\mathbb{E}[\,X_1 \,|\, F(\boldsymbol{X}) \geq \alpha\,] \\ \vdots \\ c_d\,\mathbb{E}[\,X_d \,|\, F(\boldsymbol{X}) \geq \alpha\,] \end{pmatrix}.$$

Translation Invariance: $\quad \forall\ \boldsymbol{c} \in \mathbb{R}^d_+$,

$$\mathrm{CTE}_\alpha(\boldsymbol{c} + \boldsymbol{X}) = \boldsymbol{c} + \mathrm{CTE}_\alpha(\boldsymbol{X}) = \begin{pmatrix} c_1 + \mathbb{E}[\,X_1 \,|\, F(\boldsymbol{X}) \geq \alpha\,] \\ \vdots \\ c_d + \mathbb{E}[\,X_d \,|\, F(\boldsymbol{X}) \geq \alpha\,] \end{pmatrix}.$$

Arguments completely analogous to those in the proof of Proposition 3 are used to prove Proposition 10.

Remark 5. For $\alpha = 0$, using both the definition of $\mathrm{CTE}_\alpha(\mathbf{X})$ and the definition of the α-upper level set $L(\alpha)$, we obtain

$$\mathrm{CTE}_0(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\, X_1\,] \\ \vdots \\ \mathbb{E}[\, X_d\,] \end{pmatrix} = \mathbb{E}[\mathbf{X}].$$

Then, as in the univariate case, the multivariate CTE is equal to the expected value of the underlying random vector for $\alpha = 0$.

3.2 Comparison with VaR and univariate CTE

In the following, we prove a comparison result between $\mathrm{CTE}_\alpha(\mathbf{X})$ and univariate *Value-at-Risk* (see Proposition 11).

Proposition 11. *Consider a d–dimensional random vector $\boldsymbol{X}$ satisfying the regularity conditions. Assume that its multivariate distribution function F is quasi concave. Then, for any $i = 1, \ldots, d$, the following inequality holds:*

$$\mathrm{CTE}^i_\alpha(\boldsymbol{X}) \geq \mathrm{VaR}_\alpha(X_i), \qquad \textit{for all } \alpha \in (0,1).$$

The proof is completely analogous to the proof of Proposition 4.

Let us now compare multivariate CTE (see Definition 8) with multivariate VaR (see Definition 6).

Proposition 12. *Consider a d–dimensional random vector $\boldsymbol{X}$ satisfying the regularity conditions. If $\mathrm{VaR}^i_\alpha(\boldsymbol{X})$ is a non-decreasing function of α, then it holds that*

$$\mathrm{CTE}^i_\alpha(\boldsymbol{X}) \geq \mathrm{VaR}^i_\alpha(\boldsymbol{X}), \qquad \textit{for all } \alpha \in (0,1).$$

Proof: Let us remark that, as in the univariate case, the multivariate *Conditional-Tail-Expectation* can be represented as an integral transformation of the multivariate *Value-at-Risk*:

$$\mathrm{CTE}^i_\alpha(\mathbf{X}) = \frac{1}{\overline{K}(\alpha)} \int_\alpha^1 K'(\gamma)\mathrm{VaR}^i_\gamma(\mathbf{X})d\gamma. \tag{19}$$

Since K' is non-negative and $\mathrm{VaR}^i_\gamma(\mathbf{X})$ is assumed to be non-decreasing with respect to γ, we obtain

$$\mathrm{CTE}^i_\alpha(\mathbf{X}) \geq \tfrac{1}{\overline{K}(\alpha)} \textstyle\int_\alpha^1 K'(\gamma)\mathrm{VaR}^i_\alpha(\mathbf{X})d\gamma = \tfrac{K(1)-K(\alpha)}{\overline{K}(\alpha)}\mathrm{VaR}^i_\alpha(\mathbf{X}) = \mathrm{VaR}^i_\alpha(\mathbf{X}),$$

for $\alpha \in (0,1)$. □

Analogously to Proposition 5, we can prove that, for comonotonic random vectors, components of the multivariate CTE are equal to univariate CTE of the corresponding marginals.

Proposition 13. *Consider a comonotonic non-negative d–dimensional random vector $\boldsymbol{X}$. Then, for all $\alpha \in (0,1)$, it holds that*

$$\mathrm{CTE}^i_\alpha(\boldsymbol{X}) = \mathrm{CTE}_\alpha(X_i),$$

for $i = 1, \ldots, d$.

Proof: Let $\alpha \in (0,1)$. If $\mathbf{X} = (X_1, \ldots, X_d)$ is a comonotonic non-negative random vector then there exist a random variable Z and d increasing functions $g_1, \ldots, g_d$ such that $\mathbf{X}$ is equal to $(g_1(Z), \ldots, g_d(Z))$ in distribution. So the set $\{(x_1, \ldots, x_d) : F(x_1, \ldots, x_d) \geq \alpha\}$ becomes $\{(x_1, \ldots, x_d) : \min\{g_1^{-1}(x_1), \ldots, g_d^{-1}(x_d)\} \geq Q_Z(\alpha)\}$. Then, trivially, $\mathrm{CTE}^i_\alpha(\mathbf{X}) = \mathbb{E}[\, X_i \,|\, X_i \geq Q_{X_i}(\alpha)\,]$, for $i = 1, \ldots, d$. Hence the result. □

Remark 6. Consider a random couple (X, Y) with standard uniform marginals and independent copula. For all $\alpha \in (0,1)$, it holds that

$$\mathrm{CTE}^1_\alpha(X, Y) \geq \mathrm{CTE}_\alpha(X), \tag{20}$$

$$\mathrm{CTE}^2_\alpha(X, Y) \geq \mathrm{CTE}_\alpha(Y). \tag{21}$$

Indeed $\mathrm{CTE}^1_\alpha(X, Y) - \mathrm{CTE}_\alpha(X) = -\frac{1}{2}\, \frac{\alpha\,(-2\,\alpha + \alpha\,\ln\alpha + 2 + \ln\alpha)}{-\alpha + \alpha\,\ln\alpha + 1} \geq 0$, for all $\alpha \in (0,1)$. Then in this particular case, coordinate by coordinate, $\mathrm{CTE}_\alpha(X, Y)$ is a more conservative measure than the usual univariate CTE of marginals.

3.3 Behavior of multivariate CTE with respect to marginal distributions

In this section we study the behavior of the multivariate *Conditional-Tail-Expectation* in Definition 8 with respect to a variation of marginals. Results presented below provide a natural multivariate extension of results in the univariate setting (see, e.g., Denuit and Charpentier, 2004).

Analogously to Proposition 6, we can state the following result.

Proposition 14. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d–dimensional random vector satisfying the regularity conditions and with the same copula structure C. If $X_i \overset{d}{=} Y_i$, then it holds that*

$$\mathrm{CTE}^i_\alpha(\boldsymbol{X}) = \mathrm{CTE}^i_\alpha(\boldsymbol{Y}), \qquad \textit{for all } \alpha \in (0,1).$$

Proof: Let $F_1(\mathbf{x}) = F_1(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$. We recall that

$$\mathrm{CTE}^i_\alpha(\mathbf{X}) = \int_{\mathbf{x} \in L(\alpha)} x_i\, \frac{f_{\mathbf{X}}(x_1, \ldots, x_d)}{\overline{K}(\alpha)}\, \mathrm{d}x_1 \cdots \mathrm{d}x_d.$$

We remark that $\overline{K}(\alpha)$ depends solely on the copula structure (for further details see Theorem 9 in Belzunce *et al.*, 2007). Then, for a fixed copula C, $\overline{K}(\alpha)$ is invariant

to a change of marginals (see Section 1). Let $\mathbf{u} := (u_1, \ldots, u_d)$. Using a change of variables, we obtain:

$$\mathrm{CTE}^1_\alpha(\mathbf{X}) = \left(\int_{\mathbf{u}:\, C(\mathbf{u}) \geq \alpha} Q_{X_i}(u_i) \frac{\partial^d}{\partial_1 \cdots \partial_d} C(\mathbf{u})\, \mathrm{d}\mathbf{u} \right) \frac{1}{\overline{K}(\alpha)},$$

for $\alpha \in (0,1)$. Hence the result. □

In the following we analyze how the $\mathrm{CTE}_\alpha(\mathbf{X})$ behaves when the marginal random variables increase with respect to some particular stochastic orders (see, e.g., Section 3.3 in Denuit *et al.*, 2005). Using Definitions 1-2 and Lemma 1 in Section 1 we can state the following result.

Proposition 15. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d–dimensional random vector satisfying the regularity conditions and with the same copula structure C. If $X_i \preceq_D Y_i$ then it holds that*

$$\mathrm{CTE}^i_\alpha(\boldsymbol{X}) \leq \mathrm{CTE}^i_\alpha(\boldsymbol{Y}), \qquad \textit{for all } \alpha \in (0,1). \tag{22}$$

Proof: Let us consider the function Δ defined by

$$\begin{aligned} \alpha \mapsto \Delta(\alpha) &= \overline{K}(\alpha) \left(\mathrm{CTE}^i_\alpha(\mathbf{Y}) - \mathrm{CTE}^i_\alpha(\mathbf{X}) \right) \\ &= \left(\int_{\mathbf{u}:\, C(\mathbf{u}) \geq \alpha} (Q_{Y_i}(u_i) - Q_{X_i}(u_i)) \frac{\partial^d}{\partial_1 \ldots \partial_d} C(\mathbf{u})\, \mathrm{d}\mathbf{u} \right) \frac{1}{\overline{K}(\alpha)}, \end{aligned}$$

for $\alpha \in (0,1)$. Since $X_i \preceq_D Y_i$ yields $\mathbb{E}[X_i] \leq \mathbb{E}[Y_i]$ then $\Delta(0) = \mathbb{E}[Y_i] - \mathbb{E}[X_i] \geq 0$ and obviously $\lim_{t \to 1} \Delta(t) = 0$. Furthermore, from the definition of the dangerousness order relation, we observe that there exists some real number c such that $Q_{X_i}(t) \geq Q_{Y_i}(t)$, for all $t \in (0, F_{X_i}(c))$ and $Q_{X_i}(t) \leq Q_{Y_i}(t)$, for all $t \in (F_{X_i}(c), 1)$. So Δ first increases on $(0, F_{X_i}(c))$ and then decreases on $(F_{X_i}(c), 1)$. It thus remains non-negative and ensures that $\mathrm{CTE}^i_\alpha(\mathbf{X}) \leq \mathrm{CTE}^i_\alpha(\mathbf{Y})$, for all $\alpha \in (0,1)$. Hence the result. □

From Proposition 15, if we analyze the marginal behavior without varying the dependence structure we have that $\mathrm{CTE}_\alpha(\mathbf{X})$ associated with a more dangerous marginal distribution will be greater. This means that this measure induces a greater coverage in the case of more dangerous risk (in the sense of the dangerousness order). Note that if $X_i \preceq_D Y_i$ for all $i = 1, \ldots, d$, we obtain $\mathrm{CTE}_\alpha(\mathbf{X}) \leq \mathrm{CTE}_\alpha(\mathbf{Y})$, for all $\alpha \in (0,1)$.

In the following example we provide an illustration of Propositions 14 and 15.

Example 2. We consider a bivariate Clayton copula with parameter 1 and five different bivariate random vectors (X, Y_i), for $i = 1, \ldots, 5$. Let $X \sim \mathrm{Exp}(1)$, $Y_1 \sim \mathrm{Exp}(2)$, $Y_2 \sim \mathrm{Burr}(2,1)$, $Y_3 \sim \mathrm{Exp}(1)$, $Y_4 \sim$ Fréchet(4) and $Y_5 \sim \mathrm{Burr}(4,1)$. We assume the same copula structure for all vectors. We calculate $\mathrm{CTE}_\alpha(X, Y_i)$, for $i = 1, \ldots, 5$. The results are gathered in Table 5. As proved in Proposition 14

we obtain an invariant property on the first coordinate of all $\mathrm{CTE}_\alpha(X, Y_i)$, for $i = 1, \ldots, 5$.
Remark that $\mathrm{CTE}_\alpha(X, Y_3)$ is a case of an exchangeable continuous random vector then we have $\mathrm{CTE}^1_\alpha(X, Y_3) = \mathrm{CTE}^2_\alpha(X, Y_3)$. Furthermore, as $Q_X(\alpha) = 2\,Q_{Y_1}(\alpha)$, then $\mathrm{CTE}^1_\alpha(X, Y_1) = 2\,\mathrm{CTE}^2_\alpha(X, Y_1)$, for $\alpha \in (0, 1)$ (see Table 5).

Since $Y_1 \preceq_D Y_5 \preceq_D Y_4 \preceq_D Y_2$, then, for any level $\alpha \in (0, 1)$,

$$\mathrm{CTE}^2_\alpha(X, Y_1) \leq \mathrm{CTE}^2_\alpha(X, Y_5) \leq \mathrm{CTE}^2_\alpha(X, Y_4) \leq \mathrm{CTE}^2_\alpha(X, Y_2). \ .$$

Notice that the relation $\preceq_D$ is not transitive, since if F and G cross once, and G and H cross once, then F and H may cross twice. However, in the simple cases considered in Example 2, this situation does not happen and therefore the transitive relation is valid. Conversely Y_3, Y_4 and Y_3, Y_5 are not ordered in dangerousness sense, and also $(\mathrm{CTE}^2_\alpha(X, Y_3), \mathrm{CTE}^2_\alpha(X, Y_4))$ and $(\mathrm{CTE}^2_\alpha(X, Y_3), \mathrm{CTE}^2_\alpha(X, Y_4))$ are not ordered for any level $\alpha \in (0, 1)$. As before we can also verify that $\mathrm{CTE}_\alpha(X, Y_1) \leq \mathrm{CTE}_\alpha(X, Y_3) \leq \mathrm{CTE}_\alpha(X, Y_2)$, for any level $\alpha \in (0, 1)$.

α	$\mathrm{CTE}_\alpha(X, Y_1)$	$\mathrm{CTE}_\alpha(X, Y_2)$	$\mathrm{CTE}_\alpha(X, Y_3)$	$\mathrm{CTE}_\alpha(X, Y_4)$	$\mathrm{CTE}_\alpha(X, Y_5)$
0.10	(1.188, 0.594)	(1.188, 1.838)	(1.188, 1.188)	(1.188, 1.315)	(1.188, 1.229)
0.24	(1.449, 0.724)	(1.449, 2.218)	(1.449, 1.449)	(1.449, 1.431)	(1.449, 1.366)
0.38	(1.727, 0.864)	(1.727, 2.661)	(1.727, 1.727)	(1.727, 1.555)	(1.727, 1.506)
0.52	(2.049, 1.025)	(2.049, 3.235)	(2.049, 2.049)	(2.049, 1.704)	(2.049, 1.667)
0.66	(2.454, 1.227)	(2.454, 4.074)	(2.454, 2.454)	(2.454, 1.902)	(2.454, 1.876)
0.80	(3.039, 1.519)	(3.039, 5.591)	(3.039, 3.039)	(3.039, 2.219)	(3.039, 2.202)
0.90	(3.768, 1.884)	(3.768, 8.175)	(3.768, 3.768)	(3.768, 2.675)	(3.768, 2.665)
0.99	(6.102, 3.059)	(6.102, 26.59)	(6.102, 6.102)	(6.102, 4.813)	(6.102, 4.811)

Table 5: $\mathrm{CTE}_\alpha(X, Y_i)$, for $i = 1, \ldots, 5$, with the same copula Clayton copula with parameter 1, $X \sim \mathrm{Exp}(1)$ and $Y_1 \sim \mathrm{Exp}(2)$; $Y_2 \sim \mathrm{Burr}(2, 1)$; $Y_3 \sim \mathrm{Exp}(1)$; $Y_4 \sim$ Fréchet(4); $Y_5 \sim \mathrm{Burr}(4, 1)$.

3.4 Behavior of multivariate CTE with respect to risk level

As in Section 2.5 we study the behavior of the multivariate *Conditional-Tail-Expectation* with respect to risk level α. Also in this case the relationship between $\mathrm{CTE}_\alpha(\mathbf{X})$ and the level α will be connected to some positive dependence concepts.

Corollary 16. *Consider a d–dimensional random vector* $\boldsymbol{X}$ *satisfying assumptions of Proposition 12, then* $\mathrm{CTE}^i_\alpha(\boldsymbol{X})$ *is a non-decreasing function of risk level* α.

Proof: Let us consider the i-th coordinate $\mathrm{CTE}^i_\alpha(\mathbf{X})$. From (19) we have

$$\frac{d}{d\alpha}\mathrm{CTE}^i_\alpha(\mathbf{X}) = \frac{K'(\alpha)}{\overline{K}(\alpha)}\left[\mathrm{CTE}^i_\alpha(\mathbf{X}) - \mathrm{VaR}^i_\alpha(\mathbf{X})\right].$$

Using Proposition 12 the latter expression is non-negative for any level $\alpha \in (0,1)$. □

Corollary 16 is consistant with the univariate risk measure theory. Indeed in the continuous univariate framework, the *Conditional-Tail-Expectation* is non-decreasing with respect to α (see, e.g., Denuit and Charpentier, 2004).

Proofs of Proposition 12 and Corollary 16 use direct calculations. We get below an alternative approach based to the positive dependence concepts (see Joe, 1997).

Definition 9 (Right-tail increasing)**.** *A bivariate random vector* (X,Y) *is right-tail increasing with respect to* X, $\mathrm{RTI}(Y|X)$, *if*

$$[Y|X > x_1] \preceq_{st} [Y|X > x_2], \quad \forall\, x_1 \leq x_2. \tag{23}$$

Remark that in (23), Y is more likely to take large values if X increases.

Definition 10 (Right conditional increasing)**.** *A bivariate random vector* (X,Y) *is right conditional increasing with respect to* X, $\mathrm{RCI}(Y|X)$, *if*

$$[Y|X = x] \preceq_{st} [Y|X > x], \quad \forall\, x.$$

For more details on these positive dependence concepts, the reader is referred to Belzunce *et al.* (2007). In particular in the case of absolutely continuous random vector (X,Y) it holds that

$$\mathrm{PRD}(Y|X) \Rightarrow \mathrm{RTI}(Y|X) \Longleftrightarrow \mathrm{RCI}(Y|X), \tag{24}$$

(see Lemma 15 in Belzunce *et al.*, 2007). Then, using Definitions 9-10, we can alternatively prove Proposition 12 and Corollary 16.

Remark 7. We note that, if (X,Y) has one of the positive dependence properties mentioned above, then all random vectors with the same copula than (X,Y) have also the same property (see Nelsen, 1999; Corollary 5.2.11). For instance, if $\mathrm{PRD}(Y|X)$ then also $\mathrm{PRD}(a_2(Y)|a_1(X))$, for some increasing functions a_1 and a_2.

In the univariate setting the *Conditional-Tail-Expectation* contains a safety loading i.e., $\mathrm{CTE}_\alpha(\mathrm{X}) \geq \mathbb{E}[\,\mathrm{X}\,]$, $\forall\, \alpha \in (0,1)$ (see Section 2.4.3.3 in Denuit *et al.*, 2005). The safety loading should cover the fluctuations of loss experience. Corollary 17 below provides a similar property also for our multivariate CTE.

Corollary 17. *Under assumptions of Corollary 16, for all* $\alpha \in (0,1)$, *it holds that*

$$\mathrm{CTE}^i_\alpha(\boldsymbol{X}) \geq \mathbb{E}[X_i],$$

Remark 8. To summarize, under assumptions of Proposition 4 and Proposition 9 it holds that

$$\mathrm{CTE}^i_\alpha(\mathbf{X}) \geq \mathrm{VaR}^i_\alpha(\mathbf{X}) \geq \mathrm{VaR}_\alpha(X_i),$$

for $\alpha \in (0,1)$, with $\mathrm{CTE}^i_\alpha(X,Y)$ and $\mathrm{VaR}^i_\alpha(X,Y)$ non-decreasing functions of α. We remark that this non-decreasing property of our measures depends only on the dependence structure and not on the marginal distribution functions. Furthermore, from Remark 4 we know that, in the bivariate Archimedean class of copulas, these assumptions are automatically satisfied.

In the following we provide illustrations of Remark 8 in some particular bivariate Archimedean cases: Frank and Ali-Mikhail-Haq copulas (see Figures 4 and 5).

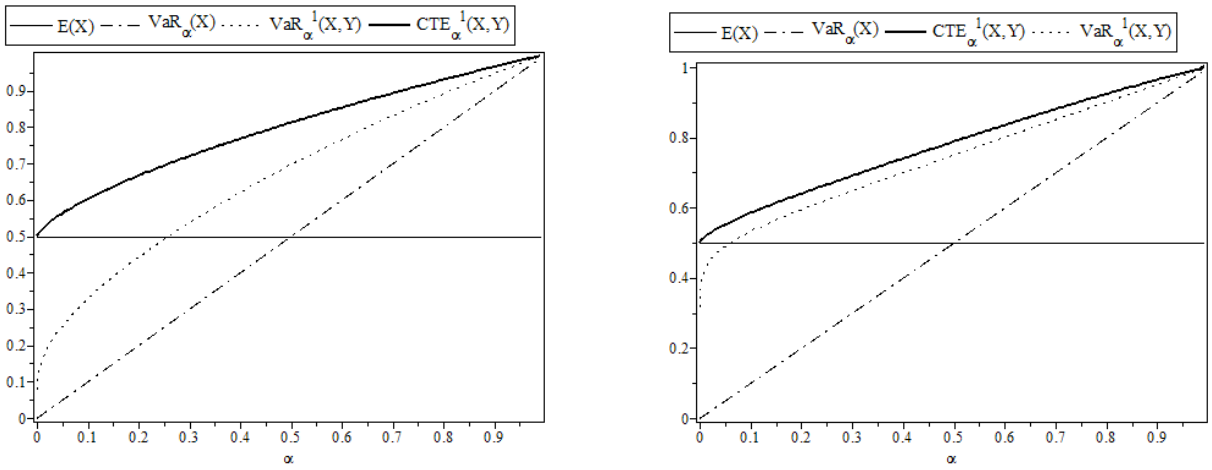

Figure 4: Frank copula with standard uniform marginals, parameter $\theta = 2$ (left), parameter $\theta = -10$ (right).

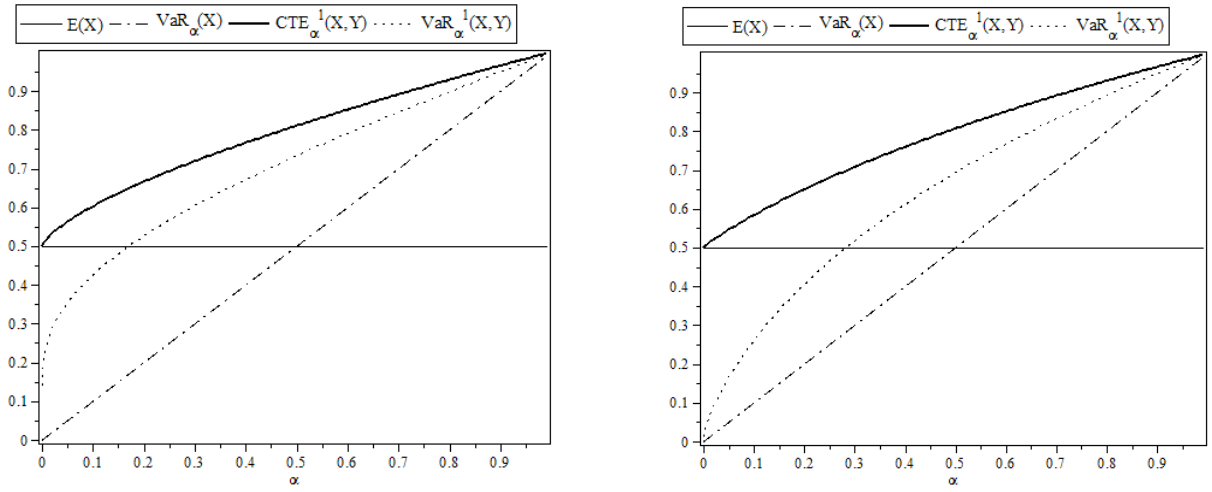

Figure 5: Ali-Mikhail-Haq copula with standard uniform marginals, parameter $\theta = -0.7$ (left), parameter $\theta = 0.99$ (right).

Conclusion and perspectives

In this paper, we proposed an extension of the classical Value-at-Risk and Conditional-Tail-Expectation risk measures for continuous random vectors. As in the Embrechts and Puccetti (2006)'s approach, the introduced risk measures are based on multivariate generalization of quantiles but they are able to quantify risks in a much more parsimonious and synthetic way: the risk of a d-dimensional portfolio is evaluated by a point in $\mathbb{R}_+^d$. Multivariate risk measures may be useful for some applications where risks are heterogeneous in nature or because they cannot be diversify away by an aggregation procedure.

We analyzed our multivariate risk measures in several directions. Interestingly, we shown that many properties satisfied by the univariate VaR and CTE can be translated to our proposed multivariate versions under some conditions. In particular, the proposed VaR and CTE both satisfy the positive homogeneity and the translation invariance property which are parts of the classical axiomatic properties of Artzner *et al.*, 1999. Thanks to the theory of stochastic orders, we also analyzed the effect of some risk perturbations on these measures. In the same vein as for the univariate VaR and CTE, we proved that an increase of marginal risks yield an increase of our multivariate VaR and CTE. We also gave the condition under which an increase of the risk level tends to increase components of the multivariate extensions and we show that these conditions are satisfied for bivariate families of Archimedean copulas. We also study the effect of dependence between risks on individual contribution to the multivariate risk measure and we prove that for different bivariate families of Archimedean copulas, an increase of the dependence parameter tends to lower the components of the VaR and the CTE. At the extreme case where risks

are perfectly dependent or comonotonic, our multivariate risk measures is equal to the vector composed of univariate risk measures associated with each component. This feature is in line with the observation made by Zhou (2010): *"When regulating a system consisting of similar institutions, or in other words, the system is highly interconnected, considering a micro-prudential regulation can be sufficient for reducing the overall systemic risk."* (Zhou, 2010).

In a future perspective, it should be interesting to discuss the possible extensions of our risk measures to the case of discrete distribution functions, using "discrete level sets" as multivariate definitions of quantiles. For further details the reader is referred, for instance, to Laurent, 2003. Another subject of future work should be to compare our multivariate *Conditional-Tail-Expectation* and *Value-at-Risk* with existing multivariate generalizations of these measures, both theoretically and experimentally.

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A Gumbel copula case

We consider Gumbel bivariate family of copulas:

$$C_\theta(x,y) = \mathrm{e}^{-\left((-\ln x)^\theta + (-\ln y)^\theta\right)^{\frac{1}{\theta}}},$$

for $\theta \in [1,\infty)$, $(x,y) \in [0,1]^2$ (e.g. see Section 3.3.1 in Nelsen, 1999) and X, Y standard uniform marginals. For $\theta = 1$ we have the independent copula $C_1(x,y) = \Pi(x,y) = x\,y$; for $\theta = \infty$ the Fréchet bound $M(x,y) = \min\{x,y\}$ (comonotonic random variables). In this case we get

$$\mathrm{CTE}_\alpha(X,Y) = \begin{pmatrix} t(\alpha,\theta) \\ t(\alpha,\theta) \end{pmatrix},$$

with

$$t(\alpha,\theta) = \frac{1}{2}\,\frac{\ln\alpha(\theta+\theta^2-\alpha^2\theta-\alpha^2\theta^2)+2\,\alpha^{\frac{3}{2}}(-\ln\alpha)^{-\frac{\theta}{2}+2}\mathrm{WM}\left(\frac{1}{2}\,\theta,\frac{1}{2}\,\theta+\frac{1}{2},-\ln\alpha\right)+2\,\alpha^2(\ln\alpha)^2+2\,\alpha^2(\ln\alpha)^2\theta}{(1+\theta)\ln\alpha(\theta-\alpha\,\theta+\alpha\,\ln\alpha)},$$

where WM is the Whittaker function $\mathrm{WM}(\mu,\nu,z) = \exp^{-\frac{1}{2}z}\,z^{\frac{1}{2}+\nu}\mathrm{Hpy}(\frac{1}{2}+\nu-\mu,1+2\nu;z)$ and Hpy is the Kummer's confluent hypergeometric function.

Remark that X and Y are exchangeable, then obviously $\mathrm{CTE}^1_{\alpha,\theta}(X,Y) = \mathrm{CTE}^2_{\alpha,\theta}(X,Y)$. Furthermore $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\alpha} \geq 0$ and $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\theta} \leq 0$, for $\theta \geq 1$ and $\alpha \in (0,1)$. In analogy with Figure 3, in Figure 6 we propose a graphical illustration of $\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$, in Gumbel copula case, for different values of the dependence parameter θ and with respect to risk level α.

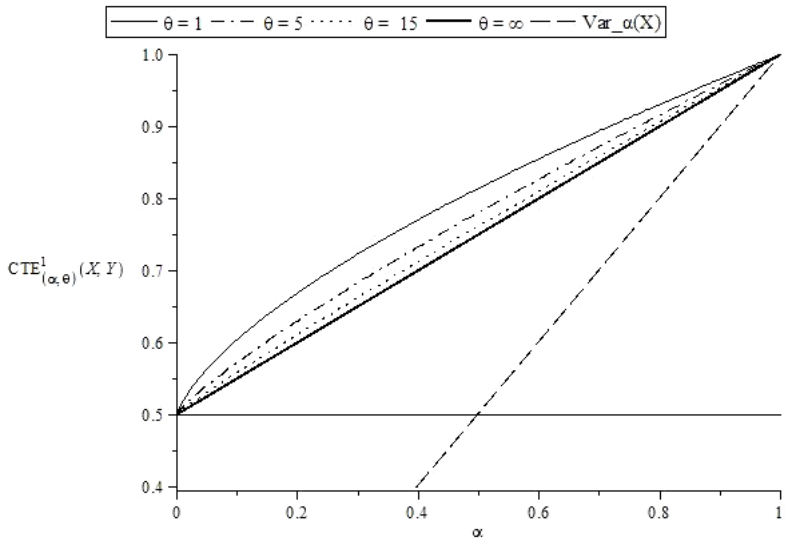

Figure 6: Behavior of $\mathrm{CTE}^1_{\alpha,\theta}(X,Y) = \mathrm{CTE}^2_{\alpha,\theta}(X,Y)$ with respect to risk level α for different values of dependence parameter θ. The random vector (X,Y) follows a Gumbel copula distribution with parameter θ. The horizontal line corresponds to $\mathbb{E}[\,X\,] = \frac{1}{2}$.