The Aw-Rascle traffic model with locally constrained flow

Mauro Garavello, Paola Goatin

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Abstract

We consider solutions of the Aw-Rascle model for traffic flow fulfilling a constraint on the flux at $x = 0$. Two different kinds of solutions are proposed: at $x = 0$ the first one conserves both the number of vehicles and the generalized momentum, while the second one conserves only the number of cars. We study the invariant domains for these solutions and we compare the two Riemann solvers in terms of total variation of relevant quantities. Finally we construct *ad hoc* finite volume numerical schemes to compute these solutions.

Key Words: Aw-Rascle model, traffic models, unilateral constraint, Riemann problem, finite volume numerical scheme.

AMS Subject Classifications: 90B20, 35L65.

1 Introduction

The paper deals with solutions to the Aw-Rascle vehicular traffic model [2]

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t y + \partial_x (yv) = 0, \end{cases} \tag{1.1}$$

satisfying a constraint on the first component of the flux at $x = 0$:

$$\rho(t, 0)v(t, 0) \leq q, \tag{1.2}$$

where $q > 0$ is a given constant. Here ρ, v and y denote respectively the density, the average speed and a generalized momentum of cars in a road.

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Moreover $y = \rho\,(v + p(\rho))$, where $p \in C^2([0,+\infty[;[0,+\infty[)$ is a pressure function satisfying

$$\begin{cases} p(0) = 0, \\ p'(\rho) > 0 \text{ for every } \rho > 0, \\ \rho \mapsto \rho p(\rho) \text{ is strictly convex.} \end{cases} \tag{1.3}$$

Problem (1.1), (1.2) models the presence of a constraint on traffic flow at the point $x = 0$, such as a toll gate, a traffic light, a construction site, etc. All these situations limit the flow at a specific location along the road. Conservation laws with unilateral constraints as (1.2) were first introduced in [7], see also [1, 8, 9] for further analytical results and applications. In these papers, the scalar Lighthill-Whitham [17] and Richards [21] traffic model is coupled with a (possibly time-dependent) constraint on the flow, as in (1.2).

The model presented here constitutes the first example of a system of two equations with constrained flux. The Aw-Rascle model (1.1) belongs to the so-called "second order" traffic models, i.e. models consisting in two equations (see [6, 20, 23] for other examples). System (1.1) can also be written

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0. \end{cases} \tag{1.4}$$

The first equation in (1.4) states the conservation of the number of vehicles, moving with flow rate ρv. The second equation is derived from the former one and from the evolution equation of the quantity $w = v + p(\rho)$ (often referred to as "Lagrangian marker"), which moves with velocity v:

$$\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0.$$

The system in conservative form (1.4) belongs to the Temple class [22], i.e. systems for which shock and rarefaction curves in the unknowns' space coincide. In particular, for such systems the interaction of two waves of the same family can only give rise to a wave of the same family.

The Aw-Rascle model (1.4) has been widely studied in the mathematical literature. Concerning the model itself, various extensions have been proposed, see [3, 4, 12, 13, 19]. The model can also be used to describe traffic flow on a road network, as explained in [11, 14, 15].

In this paper we restrict the analysis to the Riemann problem for (1.1), (1.2), i.e. to the Cauchy problem with piecewise constant initial data of the form

$$(\rho, y)(0, x) = \begin{cases} (\rho^l, y^l), & \text{if } x < 0, \\ (\rho^r, y^r), & \text{if } x > 0. \end{cases}$$

We propose two Riemann solvers, described in Sections 2.1 and 2.2: the first one conserves at $x = 0$ both the number of cars and the generalized momentum, while the second one does not conserve the generalized momentum. In particular, the first Riemann solver produces a non-entropic shock wave at $x = 0$, which travels with zero velocity. In Section 3 we describe the invariant domains corresponding to the two Riemann solvers, and in Section 4 we compare the total variation of relevant quantities. Section 5 is devoted to the construction of *ad hoc* numerical schemes designed to capture the proposed solutions.

2 The Riemann problem

In this section we deal with the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \\ (\rho, v)(0, x) = \begin{cases} (\rho^l, v^l), & \text{if } x < 0, \\ (\rho^r, v^r), & \text{if } x > 0, \end{cases} \end{cases} \tag{2.1}$$

in the domain $\mathcal{D} = \mathbb{R}^+ \times \mathbb{R}^+$, and with the constraint (1.2).

We denote by $f(\rho, v)$ the flux for system (1.4), and with $f_1(\rho, v)$, $f_2(\rho, v)$ its components, i.e.

$$f(\rho, v) = \begin{pmatrix} f_1(\rho, v) \\ f_2(\rho, v) \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho v(v + p(\rho)) \end{pmatrix}. \tag{2.2}$$

For reader's comfort, we resume in the following tables the relevant quantities concerning systems (1.1), (1.4) respectively. In (ρ, y) plane they write:

$\lambda_1 = -p(\rho) + \frac{y}{\rho} - \rho p'(\rho)$	$\lambda_2 = -p(\rho) + \frac{y}{\rho}$
$r_1 = \begin{pmatrix} -1 \\ -\frac{y}{\rho} \end{pmatrix}$	$r_2 = \begin{pmatrix} 1 \\ \frac{y}{\rho} + \rho p'(\rho) \end{pmatrix}$
$\nabla\lambda_1 \cdot r_1 = 2p'(\rho) + \rho p''(\rho) > 0$	$\nabla\lambda_2 \cdot r_2 = 0$
$L_1(\rho; \rho_0, y_0) = \frac{y_0}{\rho_0}\rho$	$L_2(\rho; \rho_0, y_0) = \frac{y_0}{\rho_0}\rho + \rho\,(p(\rho) - p(\rho_0))$
$z = \frac{y}{\rho} - p(\rho)$	$w = \frac{y}{\rho}$

In (ρ, v) plane their expression is:

$\lambda_1 = v - \rho p'(\rho)$	$\lambda_2 = v$
$r_1 = \begin{pmatrix} -1 \\ p'(\rho) \end{pmatrix}$	$r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\nabla\lambda_1 \cdot r_1 = 2p'(\rho) + \rho p''(\rho) > 0$	$\nabla\lambda_2 \cdot r_2 = 0$
$L_1(\rho; \rho_0, v_0) = v_0 + p(\rho_0) - p(\rho)$	$L_2(\rho; \rho_0, v_0) = v_0$
$z = v$	$w = v + p(\rho)$

Above, λ_1 and λ_2 denote the eigenvalues of the Jacobian matrix Df, r_1 and r_2 the corresponding right eigenvectors, L_1 and L_2 the first and the second Lax curve, z and w the 1- and 2-Riemann invariant respectively.

We remark that the system is strictly hyperbolic away from $\rho = 0$ (i.e. $\lambda_1 < \lambda_2$). Moreover the first characteristic speed is genuinely nonlinear, with characteristic speed that can change sign, and the second one is linearly degenerate with strictly positive speed.

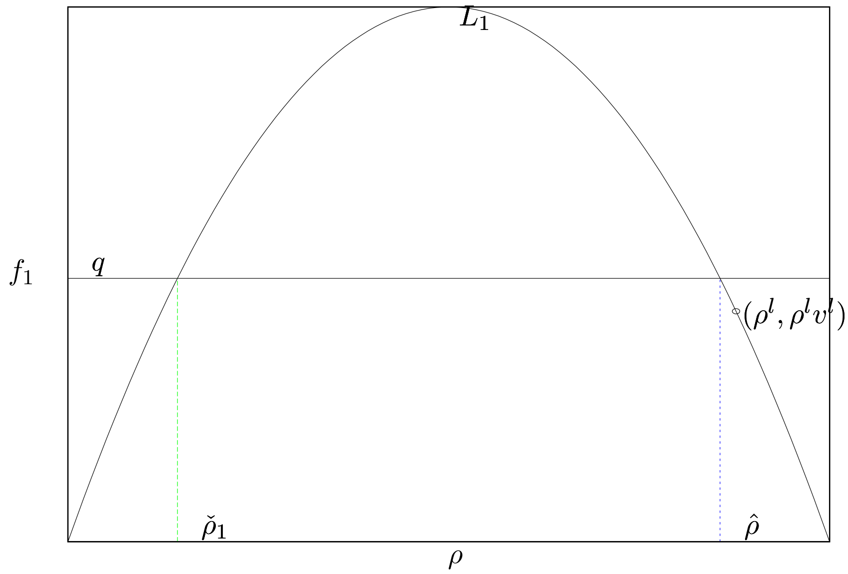

Figure 1: The set I_1 and the quantities of equation (2.4).

Definition 2.1 *A Riemann solver for system (2.1) is a function, which associates, for every initial condition $(\rho^l, v^l) \in \mathcal{D}$, $(\rho^r, v^r) \in \mathcal{D}$, a map belonging to $L^1(\mathbb{R})$ and representing a solution to (2.1) at time $t = 1$.*

By $\mathcal{RS}$ we denote the classical Riemann solver for (2.1), i.e. the Riemann solver without the constraint (1.2); see for example [2]. We introduce some more notation.

Given $(\rho^l, v^l) \in \mathcal{D}$ and $q > 0$, let us consider the set

$$\begin{aligned} I_1 &= \left\{\rho \in [0, +\infty[\ : \rho L_1(\rho; \rho^l, v^l) = q\right\} \\ &= \left\{\rho \in [0, +\infty[\ : \rho(v^l + p(\rho^l) - p(\rho)) = q\right\}. \end{aligned} \tag{2.3}$$

The set I_1 contains the densities of all the points $(\rho, v) \in \mathcal{D}$ belonging to the Lax curve of the first family passing through (ρ^l, v^l) and such that $f_1(\rho^l, v^l) = q$. If $I_1 \neq \emptyset$, then we denote by $\hat{\rho}$, $\hat{v}$, $\check{\rho}_1$, $\check{v}_1$ respectively

$$\hat{\rho} = \max I_1, \quad \hat{v} = \frac{q}{\hat{\rho}}, \quad \check{\rho}_1 = \min I_1, \quad \check{v}_1 = \frac{q}{\check{\rho}_1}; \tag{2.4}$$

see Figure 2.1. Given $(\rho^r, v^r) \in \mathcal{D}$ and $q > 0$, let $\check{\rho}_2$ and $\check{v}_2$ be defined by

$$\check{\rho}_2 L_2(\check{\rho}_2; \rho^r, v^r) = q, \quad \check{v}_2 = \frac{q}{\check{\rho}_2}; \tag{2.5}$$

i.e. $(\check{\rho}_2, \check{v}_2)$ belongs to the Lax curve of the second family passing through (ρ^r, v^r) and satisfies $f_1(\check{\rho}_2, \check{v}_2) = q$. In particular, note that $\check{v}_2 = v^r$ and

$\check{\rho}_2 = q/v^r$.
Given (ρ^l, v^l) and $(\rho^r, v^r) \in \mathcal{D}$, let us consider the set

$$\begin{aligned} I_2 &= \Big\{\rho \in [0, +\infty[\; : L_1(\rho; \rho^l, v^l) = L_2(\rho; \rho^r, v^r)\Big\} \\ &= \Big\{\rho \in [0, +\infty[\; : v^l + p(\rho^l) - p(\rho) = v^r\Big\} \end{aligned} \tag{2.6}$$

and define

$$\rho^m = \max I_2, \quad v^m = v^r, \tag{2.7}$$

which provide the intermediate state for the classical solution to (2.1).

Lemma 2.1 *Let $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$ and $q > 0$ be fixed. Assume (1.3) holds. If*

$$f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(0)) > q,$$

then the set I_1 is not empty and it consists in exactly two different points: $I_1 = \{\check{\rho}_1, \hat{\rho}\}$.

PROOF. Notice that the function $\rho \mapsto \rho L_1(\rho; \rho^l, v^l)$ is strictly concave by the hypotheses (1.3) on the pressure function $p(\rho)$ and so, by (2.3), the cardinality of I_1 is at most 2.
Denote with (ρ^M, v^M) the trace of $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))$ at $x = 0+$. Since the waves of the second family have strictly positive speed, then $v^M = L_1(\rho^M; \rho^l, v^l)$. Therefore, if $I_1 = \emptyset$ or it contains only one element, then $\rho^M v^M \leq q$ and therefore $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(0)) \leq q$, which is a contradiction. Thus the only possibility is that I_1 is composed exactly by two elements. $\square$

We propose two different ways of solving problem (2.1)-(1.2).

2.1 The Constrained Riemann Solver $\mathcal{RS}_1^q$

In this part, we introduce the Riemann solver $\mathcal{RS}_1^q$ for (2.1)-(1.2), which is characterized by the conservation of both the quantities ρ and $y = \rho(v + p(\rho))$ at $x = 0$.

Fix $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$. The Riemann solver $\mathcal{RS}_1^q$ is defined as follows.

1. If $f_1(\mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) \leq q$, then

$$\mathcal{RS}_1^q\Big((\rho^l, v^l), (\rho^r, v^r)\Big)(x) = \mathcal{RS}\Big((\rho^l, v^l), (\rho^r, v^r)\Big)(x) \tag{2.8}$$

   for every $x \in \mathbb{R}$.

2. If $f_1(\mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) > q$, then

$$\mathcal{RS}_1^q\Big((\rho^l, v^l), (\rho^r, v^r)\Big)(x) = \begin{cases} \mathcal{RS}\left((\rho^l, v^l), (\hat{\rho}, \hat{v})\right)(x), & \text{if } x < 0, \\ \mathcal{RS}\left((\check{\rho}_1, \check{v}_1), (\rho^r, v^r)\right)(x), & \text{if } x > 0. \end{cases} \tag{2.9}$$

Proposition 2.1 *The Riemann solver $\mathcal{RS}_1^q$ satisfies*

$$f_1(\mathcal{RS}_1^q\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) \leq q$$

for every (ρ^l, v^l) and (ρ^r, v^r).

The proof follows directly from the construction of the Riemann solver $\mathcal{RS}_1^q$.

Remark 1 *The Riemann solver $\mathcal{RS}_1^q$ is determined by imposing the conservation of both quantities ρ and y at $x = 0$ and by respecting the constraint condition (1.2); see also [14] for an example of a Riemann solver at a node, which conserves both ρ and y.*

In the following, we denote by $w(\mathcal{RS}_1^q\left((\rho^l, v^l), (\rho^r, v^r)\right)(x))$ the w component of the Riemann solver $\mathcal{RS}_1^q\left((\rho^l, v^l), (\rho^r, v^r)\right)(x)$.

Proposition 2.2 *The Riemann solver $\mathcal{RS}_1^q$ satisfies the maximum principle on the second Riemann invariant $w = v + p(\rho)$, i.e.*

$$\min\left\{w^l, w^r\right\} \leq w(\mathcal{RS}_1^q\left((\rho^l, v^l), (\rho^r, v^r)\right)(x)) \leq \max\left\{w^l, w^r\right\}, \quad \forall x \in \mathbb{R}.$$

The property easy follows from the maximum principle satisfied by the classical Riemann solvers $\mathcal{RS}\left((\rho^l, v^l), (\hat{\rho}, \hat{v})\right)$ for $x < 0$ and $\mathcal{RS}\left((\check{\rho}_1, \check{v}_1), (\rho^r, v^r)\right)$ for $x > 0$, and by the fact that $\hat{w} = \check{w}_1 = w^l$.

2.2 The constrained Riemann solver $\mathcal{RS}_2^q$

In this part we describe the Riemann solver $\mathcal{RS}_2^q$, which conserves only the car density ρ at $x = 0$.

Fix $(\rho^l, v^l) \in \mathcal{D}$, $(\rho^r, v^r) \in \mathcal{D}$. The Riemann solver $\mathcal{RS}_2^q$ is defined as follows.

1. If $f_1(\mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) \leq q$, then we put

$$\mathcal{RS}_2^q\left((\rho^l, v^l), (\rho^r, v^r)\right)(x) = \mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(x) \tag{2.10}$$

   for every $x \in \mathbb{R}$.

2. If $f_1(\mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) > q$, then

$$\mathcal{RS}_2^q\left((\rho^l, v^l), (\rho^r, v^r)\right)(x) = \begin{cases} \mathcal{RS}\left((\rho^l, v^l), (\hat{\rho}, \hat{v})\right)(x), & \text{if } x < 0, \\ \mathcal{RS}\left((\check{\rho}_2, \check{v}_2), (\rho^r, v^r)\right)(x), & \text{if } x > 0. \end{cases} \tag{2.11}$$

Proposition 2.3 *The Riemann solver $\mathcal{RS}_2^q$ satisfies*

$$f_1(\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))(0)) \leq q$$

for every (ρ^l, v^l) and (ρ^r, v^r).

The proof follows directly from the construction of the Riemann solver $\mathcal{RS}_2^q$.

Remark 2 *The Riemann solver $\mathcal{RS}_2^q$ conserves only the density at $x = 0$; therefore it is in the same spirit of Riemann solvers introduced for traffic at junctions in [11].*

3 Invariant domains for $\mathcal{RS}_1^q$ and $\mathcal{RS}_2^q$

In this section, we want to describe the invariant regions for the Aw-Rascle system with constraints. First, we recall that, for every $0 < v_1 < v_2$, $0 < w_1 < w_2$ and $v_2 < w_2$, the set

$$\mathcal{D}_{v_1,v_2,w_1,w_2} = \{(\rho, v) \in \mathcal{D} : v_1 \le v \le v_2,\ w_1 \le v + p(\rho) \le w_2\} \tag{3.1}$$

is invariant for (1.1); see Figure 3.1 and [16]. The hypothesis $v_2 < w_2$ implies that the Riemann invariants $w = w_2$ and $z = v_2$ intersect in $\mathcal{D}$ at a point different from the origin. For a given $q > 0$, we define the function of class $C^2(]0, +\infty[)$

$$\begin{array}{rccc} h_q : &]0, +\infty[& \longrightarrow & \mathbb{R} \\ & v & \longmapsto & p\left(\frac{q}{v}\right) + v, \end{array} \tag{3.2}$$

which gives the value of the Riemann invariant w of the point $(\tilde{\rho}, v) \in \mathcal{D}$ such that $\tilde{\rho} v = q$. Indeed we have that $h_q(v) = w$ if and only if $w = v + p(\rho)$ with $\rho v = q$.

Lemma 3.1 *Fix $q > 0$ and assume (1.3). There exists $\bar{v} = \bar{v}(q) > 0$ such that the function $h_q(v)$ is strictly decreasing in $]0, \bar{v}[$ and strictly increasing in $]\bar{v}, +\infty[$.*

Proof. We have

$$h_q''(v) = \frac{q}{v^3}\left[2p'\left(\frac{q}{v}\right) + \frac{q}{v}p''\left(\frac{q}{v}\right)\right]$$

and so, by (1.3), we deduce that $h_q''(v) > 0$ for every $v > 0$; this means that $h_q'(v)$ is a strictly increasing function. Note also that (1.3) implies that

$$\lim_{\rho \to +\infty} p(\rho) = +\infty. \tag{3.3}$$

Indeed, if (3.3) does not hold, then there exists $M > 0$ such that $p(\rho) \le M$ and so $\rho p(\rho) \le M\rho$ for every $\rho > 0$. This is not possible since the map $\rho \mapsto \rho p(\rho)$ is strictly convex. This implies that

$$\lim_{v \to 0^+} h_q(v) = +\infty \quad \text{and} \quad \lim_{v \to +\infty} h_q'(v) = 1;$$

hence, since h_q' is a strictly increasing function, there exists a unique $\bar{v} > 0$ such that $h_q'(\bar{v}) = 0$. Therefore h_q is strictly decreasing in $]0, \bar{v}[$ and strictly increasing in $]\bar{v}, +\infty[$. □

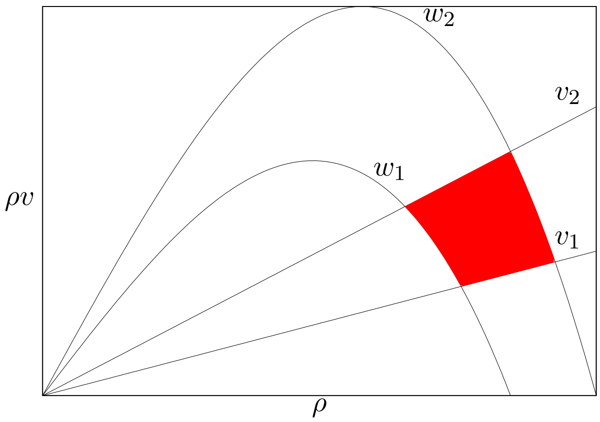

Figure 2: The invariant domain $\mathcal{D}_{v_1,v_2,w_1,w_2}$.

Proposition 3.1 *Fix $0 < v_1 < v_2$, $0 < w_1 < w_2$, $v_2 < w_2$ and $q > 0$. If $h_q(v) \geq w_2$ for every $v \in [v_1, v_2]$, then $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for both the Riemann solvers $\mathcal{RS}_1^q$ and $\mathcal{RS}_2^q$.*

PROOF. The hypothesis $h_q(v) \geq w_2$ for every $v \in [v_1, v_2]$ implies that

$$\sup\left\{f_1(\rho, v) : (\rho, v) \in \mathcal{D}_{v_1,v_2,w_1,w_2}, v \in [v_1, v_2], v + p(\rho) = w_2\right\} \leq q$$

and so

$$\sup\left\{f_1(\rho, v) : (\rho, v) \in \mathcal{D}_{v_1,v_2,w_1,w_2}\right\} \leq q.$$

Therefore the Riemann solvers $\mathcal{RS}_1^q$ and $\mathcal{RS}_2^q$ in the domain $\mathcal{D}_{v_1,v_2,w_1,w_2}$ coincide with $\mathcal{RS}$. $\square$

3.1 The Riemann solver $\mathcal{RS}_1^q$

The next proposition describes the invariant domains for $\mathcal{RS}_1^q$.

Proposition 3.2 *Fix $0 < v_1 < v_2$, $0 < w_1 < w_2$, $v_2 < w_2$ and $q > 0$. Assume (1.3) and that there exists $\bar{v} \in [v_1, v_2]$ such that $h_q(\bar{v}) < w_2$. The set $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for the Riemann solver $\mathcal{RS}_1^q$, if and only if*

$$h_q(v_1) \geq w_2 \quad \textit{and} \quad h_q(v_2) \geq w_2; \tag{3.4}$$

see Figure 3.2.

PROOF. Clearly, if condition (3.4) holds, then the set $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for $\mathcal{RS}_1^q$, since both $(\hat{\rho}, \hat{v})$ and $(\check{\rho}_1, \check{v}_1)$ belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$ for every possible choice of initial conditions in $\mathcal{D}_{v_1,v_2,w_1,w_2}$.

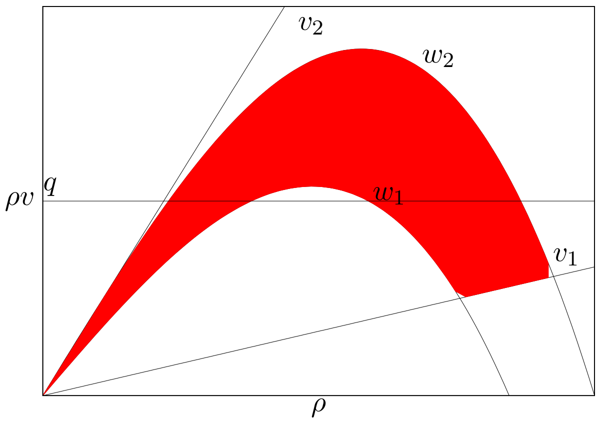

Figure 3: The invariant domain $\mathcal{D}_{v_1,v_2,w_1,w_2}$ for the Riemann solver $\mathcal{RS}_1^q$.

Assume now that $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for $\mathcal{RS}_1^q$.
If $h_q(v_1) < w_2$, then denote with $(\rho^l, v^l) = (\rho^r, v^r) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ the solution to the system

$$\begin{cases} v^l + p(\rho^l) = w_2, \\ v^l = v_1. \end{cases}$$

By hypotheses, we deduce that $\rho^l v^l > q$ and so the trace of the Riemann solver $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r))$ at the point $0-$ is given by $(\hat{\rho}, \hat{v})$, which does not belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$, since $h_q(v_1) < w_2$. This argument shows that $h_q(v_1) \geq w_2$.
If $h_q(v_2) < w_2$, then denote with $(\rho^l, v^l) = (\rho^r, v^r) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ the solution to the system

$$\begin{cases} v^l + p(\rho^l) = w_2, \\ v^l = v_2. \end{cases}$$

By hypotheses, we deduce that $\rho^l v^l > q$ and so the trace of the Riemann solver $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r))$ at the point $0+$ is given by $(\check{\rho}_1, \check{v}_1)$, which does not belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$, since $h_q(v_2) < w_2$. This argument shows that $h_q(v_2) \geq w_2$. This completes the proof. $\square$

3.2 The Riemann solver $\mathcal{RS}_2^q$

In this part, we describe the invariant domains for $\mathcal{RS}_2^q$. First let us introduce the following necessary conditions.

Lemma 3.2 *Fix $0 < v_1 < v_2$, $0 < w_1 < w_2$, $v_2 < w_2$ and $q > 0$. Assume (1.3) and that there exists $\bar{v} \in [v_1, v_2]$ such that $h_q(\bar{v}) < w_2$. If the set $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for the Riemann solver $\mathcal{RS}_2^q$, then $h_q(v_1) \geq w_2$.*

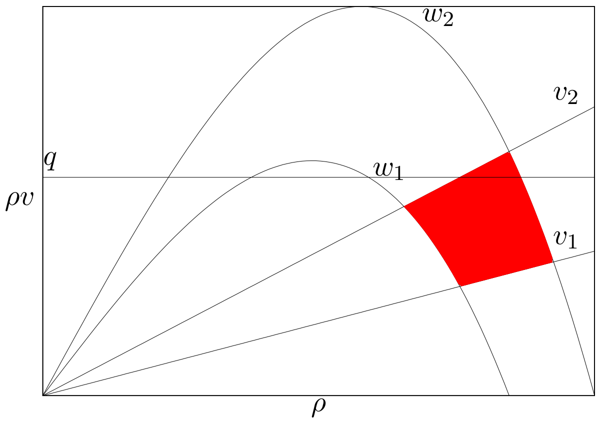

Figure 4: The invariant domain $\mathcal{D}_{v_1,v_2,w_1,w_2}$ for the Riemann solver $\mathcal{RS}_2^q$.

Proof. Assume by contradiction that $h_q(v_1) < w_2$. Denote $(\rho^l, v^l) = (\rho^r, v^r) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ the solution to the system

$$\begin{cases} v^l + p(\rho^l) = w_2, \\ v^l = \bar{v}. \end{cases}$$

By hypotheses, we deduce that $\rho^l v^l > q$ and so the trace of the Riemann solver $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$ at the point $0-$ is given by $(\hat{\rho}, \hat{v})$, which does not belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$, since $h_q(v_1) < w_2$. $\square$

Lemma 3.3 *Fix $0 < v_1 < v_2$, $0 < w_1 < w_2$, $v_2 < w_2$ and $q > 0$. Assume (1.3) and that there exists $\bar{v} \in [v_1, v_2]$ such that $h_q(\bar{v}) < w_2$. If the set $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for the Riemann solver $\mathcal{RS}_2^q$, then $h_q(v) \geq w_1$ for every $v \in [v_1, v_2]$.*

Proof. Assume by contradiction that $h_q(\tilde{v}) < w_1$ for some $\tilde{v} \in [v_1, v_2]$. Denote $(\rho^l, v^l) = (\rho^r, v^r) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ the solution to the system

$$\begin{cases} v^l + p(\rho^l) = w_2, \\ v^l = \tilde{v}. \end{cases}$$

By hypotheses, we deduce that $\rho^l v^l > q$ and so the trace of the Riemann solver $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$ at the point $0+$ is given by $(\check{\rho}_2, \check{v}_2)$, which does not belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$, since $\check{v}_2 = \tilde{v}$ and $h_q(\tilde{v}) < w_1$. $\square$

We have the following proposition about necessary and sufficient conditions for a domain to be invariant for $\mathcal{RS}_2^q$.

Proposition 3.3 *Fix $0 < v_1 < v_2$, $0 < w_1 < w_2$, $v_2 < w_2$ and $q > 0$. Assume (1.3) and that there exists $\bar{v} \in [v_1, v_2]$ such that $h_q(\bar{v}) < w_2$. The set $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for the Riemann solver $\mathcal{RS}_2^q$ (see Figure 3.3) if and only if*

$$h_q(v_1) \geq w_2 \quad \text{and} \quad h_q(v) \geq w_1 \quad \forall v \in [v_1, v_2]. \tag{3.5}$$

Proof. By Lemma 3.2 and Lemma 3.3, we need to prove that condition (3.5) is sufficient in order $\mathcal{D}_{v_1,v_2,w_1,w_2}$ be invariant for the Riemann solver $\mathcal{RS}_2^q$. Thus we assume that condition (3.5) holds.

Since $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for (1.1), it is sufficient to prove that the left and the right traces at $x = 0$ for $\mathcal{RS}_2^q$ belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$. So fix (ρ^l, v^l) and (ρ^r, v^r) in $\mathcal{D}_{v_1,v_2,w_1,w_2}$. If $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$ produces the classical solution, then we conclude. Assume therefore that $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$ does not produce the classical solution and denote with $(\hat{\rho}, \hat{v})$ and $(\check{\rho}_2, \check{v}_2)$ the left and right traces at $x = 0$ for $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r))$.
If $(\hat{\rho}, \hat{v}) \notin \mathcal{D}_{v_1,v_2,w_1,w_2}$, then every point (ρ, v) of the Lax curve of the first family through (ρ^l, v^l) contained in $\mathcal{D}_{v_1,v_2,w_1,w_2}$ has the property that $\rho v \leq q$ and so the Riemann solver gives the classical solution, since waves of the second family have strictly positive speed. This permits to prove that $(\hat{\rho}, \hat{v}) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$.
If $(\check{\rho}_2, \check{v}_2) \notin \mathcal{D}_{v_1,v_2,w_1,w_2}$, then every point (ρ, v) of the Lax curve of the second family through (ρ^r, v^r) contained in $\mathcal{D}_{v_1,v_2,w_1,w_2}$ has the property that $\rho v \leq q$ and so the Riemann solver gives the classical solution. In fact, if $v^l > v^r$, then a shock wave of the first family with strictly negative speed appears, if $v^l = v^r$, then no wave of the first family appears, whereas if $v^l < v^r$, then all the states (ρ, v) of the rarefaction wave have flux ρv less than or equal to q. This permits to prove that $(\check{\rho}_2, \check{v}_2) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$.

The proof is thus completed. □

4 Total variation estimates for $\mathcal{RS}_1^q$ and $\mathcal{RS}_2^q$

In this section we make a comparison between the two Riemann solvers $\mathcal{RS}_1^q$ and $\mathcal{RS}_2^q$ in terms of the changes in the total variation of various quantities.

Fix $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$. We denote with $\tilde{\rho}_1$ and $\tilde{\rho}_2$ respectively the ρ-component of $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r))$ and of $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$. Moreover we denote with $\tilde{v}_1$, $\tilde{v}_2$ respectively the v-component of $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r))$ and of $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$. Finally, we put $\tilde{y}_1 = \tilde{\rho}_1(\tilde{v}_1 + p(\tilde{\rho}_1))$, $\tilde{y}_2 = \tilde{\rho}_2(\tilde{v}_2 + p(\tilde{\rho}_2))$, $\tilde{w}_1 = \tilde{v}_1 + p(\tilde{\rho}_1)$, $\tilde{w}_2 = \tilde{v}_2 + p(\tilde{\rho}_2)$.

In order to facilitate the reading of the following calculation, we refer to Figures 4.1 and 4.2.

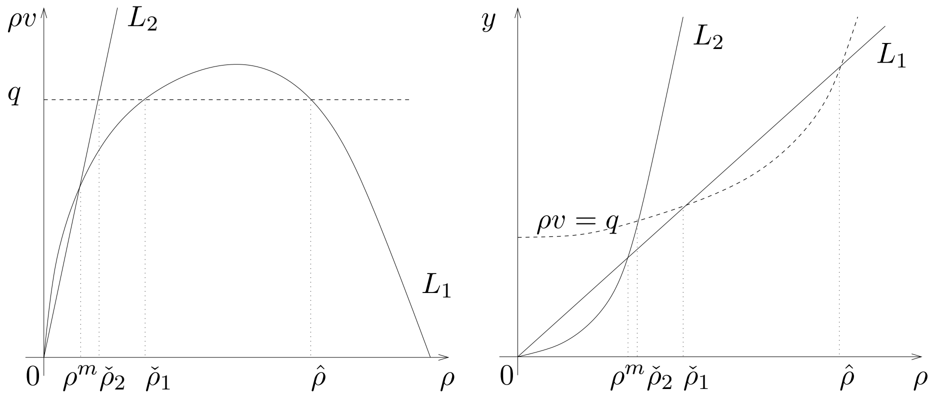

Figure 5: Notations used in the paper: case $\check\rho_1 > \check\rho_2$.

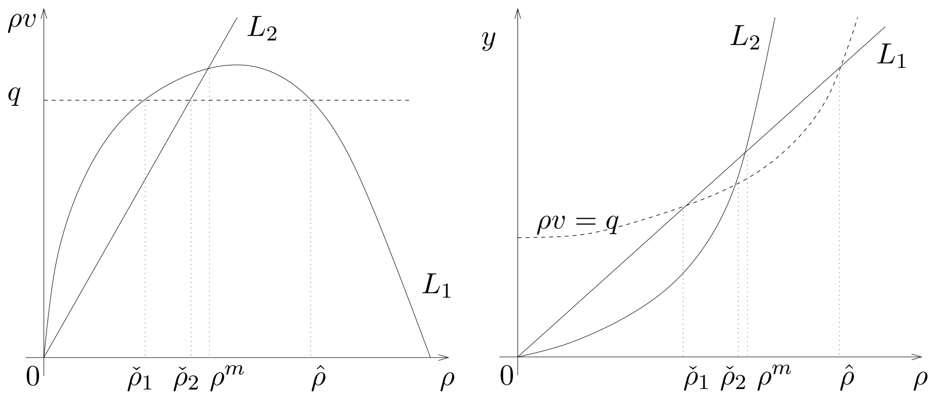

Figure 6: Notations used in the paper: case $\check\rho_1 < \check\rho_2$.

4.1 Total variation of the density ρ

This subsection deals with Tot.Var.$(\tilde\rho_1)$ and Tot.Var.$(\tilde\rho_2)$. The following proposition holds.

Proposition 4.1 *For every initial conditions* $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$, *we have that*

$$\textit{Tot. Var.}(\tilde\rho_1) \geq \textit{Tot. Var.}(\tilde\rho_2). \tag{4.1}$$

Proof. If $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) = \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$, then Tot.Var.$(\tilde\rho_1) =$ Tot.Var.$(\tilde\rho_2)$. Therefore, we assume that

$$\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) \neq \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r)).$$

In this case we have that $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(0)) > q$ and so, by construction of $\mathcal{RS}^1$ and $\mathcal{RS}^2$, we deduce that $\tilde\rho_1(x) = \tilde\rho_2(x)$ for a.e. $x < 0$. Moreover, for $x > 0$, Tot.Var. $\left(\tilde\rho_{2|_{]0,+\infty[}}\right) = |\check\rho_2 - \rho^r|$, since the states $(\check\rho_2, \check v_2)$

and (ρ^r, v^r) are connected by a contact discontinuity wave of the second family. Hence

$$\begin{aligned}\text{Tot.Var.}(\tilde{\rho}_2) &= \text{Tot.Var.}\left(\tilde{\rho}_{2|]-\infty,0[}\right) + |\hat{\rho} - \check{\rho}_2| + |\check{\rho}_2 - \rho^r| \\ &= \left|\rho^l - \hat{\rho}\right| + |\hat{\rho} - \check{\rho}_2| + |\check{\rho}_2 - \rho^r| .\end{aligned}$$

First consider the case $v^r = L_1(\rho^r; \rho^l, v^l)$, so that $(\check{\rho}_1, \check{v}_1)$ and (ρ^r, v^r) can be connected by a wave of the first family. We get that

$$\begin{aligned}\text{Tot.Var.}(\tilde{\rho}_1) &= \text{Tot.Var.}\left(\tilde{\rho}_{1|]-\infty,0[}\right) + |\hat{\rho} - \check{\rho}_1| + |\check{\rho}_1 - \rho^r| \\ &= \left|\rho^l - \hat{\rho}\right| + |\hat{\rho} - \check{\rho}_1| + |\check{\rho}_1 - \rho^r| .\end{aligned}$$

If $\rho^r \le \check{\rho}_1$, then $\check{\rho}_2 \in]\rho^r, \check{\rho}_1]$ and we obtain $\text{Tot.Var.}(\tilde{\rho}_1) = \text{Tot.Var.}(\tilde{\rho}_2)$.
If $\rho^r > \check{\rho}_1$, then $\check{\rho}_1 < \check{\rho}_2 < \rho^r \le \hat{\rho}$ and so $\text{Tot.Var.}(\tilde{\rho}_1) - \text{Tot.Var.}(\tilde{\rho}_2) = 2(\check{\rho}_2 - \check{\rho}_1) > 0$.
Consider now the case $v^r \neq L_1(\rho^r; \rho^l, v^l)$. We have that

$$\text{Tot.Var.}(\tilde{\rho}_1) = \left|\rho^l - \hat{\rho}\right| + |\hat{\rho} - \check{\rho}_1| + |\check{\rho}_1 - \rho^m| + |\rho^m - \rho^r| .$$

If $\check{\rho}_2 \le \check{\rho}_1$, then we get that $\rho^m \le \check{\rho}_2$ and

$$\text{Tot.Var.}(\tilde{\rho}_1) - \text{Tot.Var.}(\tilde{\rho}_2) = \check{\rho}_2 - \rho^m + |\rho^m - \rho^r| - |\check{\rho}_2 - \rho^r| \ge 0$$

by the triangular inequality. If $\check{\rho}_2 > \check{\rho}_1$, then we get that $\rho^m > \check{\rho}_2$ and

$$\begin{aligned}\text{Tot.Var.}(\tilde{\rho}_1) - \text{Tot.Var.}(\tilde{\rho}_2) &= \check{\rho}_2 + \rho^m - 2\check{\rho}_1 + |\rho^m - \rho^r| - |\check{\rho}_2 - \rho^r| \\ &\ge 2(\check{\rho}_2 - \check{\rho}_1) > 0\end{aligned}$$

by the triangular inequality. This completes the proof. □

4.2 Total variation of the velocity v (i.e. the first Riemann invariant)

This subsection deals with the total variation of the velocity, i.e. of the first Riemann invariant z.

Proposition 4.2 *For every initial conditions $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$, we have that*

$$\textit{Tot. Var.}(\tilde{v}_1) \ge \textit{Tot. Var.}(\tilde{v}_2). \tag{4.2}$$

PROOF. If $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) = \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$, then the thesis clearly holds. Therefore we assume that

$$\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) \neq \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r)).$$

In this situation we have that $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(0)) > q$ and so, by construction of $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r))$ and $\mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$, we deduce that $\tilde{v}_1(x) = \tilde{v}_2(x)$ for a.e. $x < 0$. It is clear that

$$\begin{aligned} \text{Tot.Var.}(\tilde{v}_2) &= \left| v^l - \hat{v} \right| + |\hat{v} - \check{v}_2| + |\check{v}_2 - v^r| \\ &= \left| v^l - \hat{v} \right| + |\hat{v} - v^r| \end{aligned}$$

since $\check{v}_2 = L_2(\check{\rho}_2; \rho^r, v^r) = v^r$.

If $v^r = L_1(\rho^r; \rho^l, v^l)$, then

$$\text{Tot.Var.}(\tilde{v}_1) = \left| v^l - \hat{v} \right| + |\hat{v} - \check{v}_1| + |\check{v}_1 - v^r|$$

and so, by the triangular inequality, we deduce $\text{Tot.Var.}(\tilde{v}_1) \geq \text{Tot.Var.}(\tilde{v}_2)$.

If $v^r \neq L_1(\rho^r; \rho^l, v^l)$, then

$$\begin{aligned} \text{Tot.Var.}(\tilde{v}_1) &= \left| v^l - \hat{v} \right| + |\hat{v} - \check{v}_1| + |\check{v}_1 - v^m| + |v^m - v^r| \\ &= \left| v^l - \hat{v} \right| + |\hat{v} - \check{v}_1| + |\check{v}_1 - v^r| \end{aligned}$$

since $v^m = v^r$ by (2.7). Again, $\text{Tot.Var.}(\tilde{v}_1) \geq \text{Tot.Var.}(\tilde{v}_2)$ by the triangular inequality.

The proof is so finished. □

4.3 Total variation of the generalized momentum y

This subsection deals with the total variation of the generalized momentum $y = \rho(v + p(\rho))$.

Proposition 4.3 *Assume that hypothesis (1.3) holds. For every initial conditions $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$, we have that*

$$\textit{Tot.Var.}(\tilde{y}_1) \geq \textit{Tot.Var.}(\tilde{y}_2). \tag{4.3}$$

PROOF. If $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) = \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$, then $\text{Tot.Var.}(\tilde{y}_1) = \text{Tot.Var.}(\tilde{y}_2)$. Therefore we assume that

$$\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) \neq \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r)).$$

In this situation we have that $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(0)) > q$ and so $\tilde{y}_1(x) = \tilde{y}_2(x)$ for a.e. $x < 0$. We define $\hat{y} = \hat{\rho}(\hat{v} + p(\hat{\rho}))$, $y^l = \rho^l(v^l + p(\rho^l))$, $y^r = \rho^r(v^r + p(\rho^r))$, $\check{y}_1 = \check{\rho}_1(\check{v}_1 + p(\check{\rho}_1))$, $\check{y}_2 = \check{\rho}_2(\check{v}_2 + p(\check{\rho}_2))$. Note that $\hat{y} \geq \max\{\check{y}_1, \check{y}_2\}$.

We have that

$$\text{Tot.Var.}(\tilde{y}_2) = \left| y^l - \hat{y} \right| + |\hat{y} - \check{y}_2| + |\check{y}_2 - y^r|.$$

Consider first the case $v^r = L_1(\rho^r; \rho^l, v^l)$, which implies that

$$\text{Tot.Var.}(\tilde{y}_1) = \left| y^l - \hat{y} \right| + |\hat{y} - \check{y}_1| + |\check{y}_1 - y^r| \, .$$

If $\rho^r \le \check{\rho}_1$, then, by (1.3), we easily get that $y^r \le \check{y}_2 \le \check{y}_1$ and consequently $\text{Tot.Var.}(\tilde{y}_1) = \text{Tot.Var.}(\tilde{y}_2)$.
If $\rho^r > \check{\rho}_1$, then, by (1.3), $y^r > \check{y}_2 > \check{y}_1$ and so $\text{Tot.Var.}(\tilde{y}_1) - \text{Tot.Var.}(\tilde{y}_2) = 2(\check{y}_2 - \check{y}_1)$.

Consider now the case $v^r \neq L_1(\rho^r; \rho^l, v^l)$. We have that

$$\text{Tot.Var.}(\tilde{y}_1) = \left| y^l - \hat{y} \right| + |\hat{y} - \check{y}_1| + |\check{y}_1 - y^m| + |y^m - y^r| \, ,$$

where $y^m = \rho^m(v^m + p(\rho^m))$.
If $\check{\rho}_1 \le \check{\rho}_2$, then, by (1.3), we deduce that $\check{y}_1 \le \check{y}_2 \le y^m$ and so

$$\begin{aligned}
\text{Tot.Var.}(\tilde{y}_2) &= \left| y^l - \hat{y} \right| + (\hat{y} - \check{y}_2) + |\check{y}_2 - y^r| \\
&\le \left| y^l - \hat{y} \right| + (\hat{y} - \check{y}_1) + |\check{y}_2 - y^m| + |y^m - y^r| \\
&\le \left| y^l - \hat{y} \right| + (\hat{y} - \check{y}_1) + |\check{y}_1 - y^m| + |y^m - y^r| = \text{Tot.Var.}(\tilde{y}_1).
\end{aligned}$$

If $\check{\rho}_1 > \check{\rho}_2$, then, by (1.3), we deduce that $\check{y}_1 > \check{y}_2 > y^m$ and so

$$\begin{aligned}
\text{Tot.Var.}(\tilde{y}_2) &= \left| y^l - \hat{y} \right| + (\hat{y} - \check{y}_2) + |\check{y}_2 - y^r| \\
&\le \left| y^l - \hat{y} \right| + (\hat{y} - \check{y}_2) + |\check{y}_2 - y^m| + |y^m - y^r| \\
&= \left| y^l - \hat{y} \right| + (\hat{y} - y^m) + |y^m - y^r| = \text{Tot.Var.}(\tilde{y}_1).
\end{aligned}$$

The proof is completed. □

4.4 Total variation of the second Riemann invariant w

This subsection deals with the total variation of the second Riemann coordinate $w = v + p(\rho)$.

Proposition 4.4 *For every initial conditions $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}$, we have that*

$$\textit{Tot.Var.}(\tilde{w}_1) \le \textit{Tot.Var.}(\tilde{w}_2). \tag{4.4}$$

Proof. If $\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) = \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r))$, then $\text{Tot.Var.}(\tilde{z}_1) = \text{Tot.Var.}(\tilde{z}_2)$. Therefore we assume that

$$\mathcal{RS}_1^q((\rho^l, v^l), (\rho^r, v^r)) \neq \mathcal{RS}_2^q((\rho^l, v^l), (\rho^r, v^r)).$$

In this situation we have that $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(0)) > q$ and so $\tilde{w}_1(x) = \tilde{w}_2(x)$ for a.e. $x < 0$. We define $\hat{w} = \hat{v}+p(\hat{\rho})$, $w^l = v^l+p(\rho^l)$, $w^r = v^r+p(\rho^r)$, $\check{w}_1 = \check{v}_1 + p(\check{\rho}_1)$, $\check{w}_2 = \check{v}_2 + p(\check{\rho}_2)$. Note that $w^l = \hat{w} = \check{w}_1$.
We have that

$$\text{Tot.Var.}(\tilde{w}_2) = \left| w^l - \hat{w} \right| + |\hat{w} - \check{w}_2| + |\check{w}_2 - w^r| .$$

Consider first the case $v^r = L_1(\rho^r; \rho^l, v^l)$, which implies that

$$\text{Tot.Var.}(\tilde{w}_1) = \left| w^l - \hat{w} \right| \leq \text{Tot.Var.}(\tilde{w}_2).$$

Consider now the case $v^r \neq L_1(\rho^r; \rho^l, v^l)$. In this case we have that

$$\text{Tot.Var.}(\tilde{w}_1) = \left| w^l - \hat{w} \right| + |w^m - w^r| ,$$

where $w^m = v^m + p(\rho^m)$. Since $w^m = \hat{w}$, we conclude by the triangular inequality.

The proof is so finished. □

5 Numerical schemes

This section is devoted to the construction of finite volume numerical schemes to capture the solutions corresponding to $\mathcal{RS}_1^q$ and $\mathcal{RS}_2^q$.

Let Δx and Δt be two constant increments for space and time discretization. We then define the mesh interfaces $x_{j+1/2} = j\Delta x$ (so that $x_{1/2} = 0$ corresponds to the constraint location) and the cell centers $x_j = (j-1/2)\Delta x$ for $j \in \mathbb{Z}$, the intermediate times $t^n = n\Delta t$ for $n \in \mathbb{N}$, and at each time t^n we denote $\mathbf{u}_j^n$ an approximate mean value of the solution of (1.1), (1.2) on the interval $\mathcal{C}_j = [x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$. In other words, a piecewise constant approximation $x \to \mathbf{u}(t^n, x)$ of the conserved variables $\mathbf{u} = (\rho, y)$ is given by

$$\mathbf{u}(t^n, x) = \mathbf{u}_j^n \text{ for all } x \in \mathcal{C}_j, \ j \in \mathbb{Z}, \ n \in \mathbb{N}.$$

When $n = 0$, we set

$$\mathbf{u}_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x) dx, \text{ for all } j \in \mathbb{Z}, \tag{5.1}$$

where $\mathbf{u}_0 = (\rho_0, y_0) \in \mathcal{D}$ is a given initial data (we will restrict the study to Riemann-type initial data).

Given a sequence $(\mathbf{u}_j^n)_{j\in\mathbb{Z}}$ at time t^n, we concentrate now on the computation of an approximate solution at the next time level t^{n+1}.

We will concentrate on Godunov scheme and show how to adapt it in order to match the constraint condition (1.2) at $x = 0$. We recall that, as pointed out in [5], classical conservative schemes (like Godunov method) may generate important non-physical oscillations near contact discontinuities. For this reason we will restrict to Riemann data lying on the same second Riemann invariant, i.e. we take $v^l + p(\rho^l) = v^r + p(\rho^r)$. More general cases can be treated for example combining the techniques presented here with the Transport-Equilibrium scheme described in [5]. Note that, in any case, a contact discontinuity appears when applying the Riemann solver $\mathcal{RS}_2^q$.

For sake of completeness, we recall that classical Godunov scheme writes

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{f}_{j+1/2}^n - \mathbf{f}_{j-1/2}^n) \text{ for all } j \in \mathbb{Z}, \tag{5.2}$$

where the numerical fluxes are given by

$$\mathbf{f}_{j+1/2}^n = \mathbf{f}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) = f(\mathcal{RS}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n)(0)) \text{ for all } j \in \mathbb{Z}, \tag{5.3}$$

and the usual CFL condition

$$\frac{\Delta t}{\Delta x} \max_{j\in\mathbb{Z}}\{|\lambda_i(\mathbf{u}_j^n)|,\ i = 1,2\} \leq \frac{1}{2} \tag{5.4}$$

holds. In the following sections we describe how to modify the definition of the numerical flux (5.3) for $j = 0$. The simulations have been performed taking $p(\rho) = \rho$ and $\Delta x = 0.002$.

5.1 The Constrained Godunov scheme for $\mathcal{RS}_1^q$

We follow the idea introduced in [1] for the scalar case. We redefine the numerical flux at the interface $x_{1/2} = 0$ to take into account the imposed constraint (1.2). We denote by $\mathbf{f}_{1,j+1/2}^n$, $\mathbf{f}_{2,j+1/2}^n$ the components of the classical Godunov flux:

$$\mathbf{f}_{j+1/2}^n = \begin{pmatrix} \mathbf{f}_{1,j+1/2}^n \\ \mathbf{f}_{2,j+1/2}^n \end{pmatrix}.$$

For $j = 0$, we replace it by $\hat{\mathbf{f}}_{1/2}^n$, where

$$\begin{aligned} \hat{\mathbf{f}}_{1,1/2}^n &= \min\left\{\mathbf{f}_{1,1/2}^n\ ,\ q\right\}, \\ \hat{\mathbf{f}}_{2,1/2}^n &= \hat{\mathbf{f}}_{1,1/2}^n \frac{\mathbf{f}_{2,1/2}^n}{\mathbf{f}_{1,1/2}^n} = \min\left\{\mathbf{f}_{2,1/2}^n\ ,\ q\,\frac{\mathbf{f}_{2,1/2}^n}{\mathbf{f}_{1,1/2}^n}\right\}. \end{aligned} \tag{5.5}$$

We stress that the above construction preserves conservation, in agreement with the conservative character of $\mathcal{RS}_1^q$.

Theorem 5.1 (Maximum principle) *Under the CFL restriction* (5.4), *the finite volume numerical scheme defined by* (5.2), (5.3) *and* (5.5) *satisfies the maximum principle property*

$$\inf_{l\in\mathbb{Z}} \left(v_l^0 + p(\rho_l^0)\right) \leq v_j^n + p(\rho_j^n) \leq \max_{l\in\mathbb{Z}} \left(v_l^0 + p(\rho_l^0)\right)$$

for all $j \in \mathbb{Z}$ and all $n \in \mathbb{N}$, where $\mathbf{u}_j^n = \begin{pmatrix} \rho_j^n \\ y_j^n \end{pmatrix}$ *and* $v_j^n + p(\rho_j^n) = \frac{y_j^n}{\rho_j^n} = w_j^n$.

Proof. We observe first that the above maximum principle property on the second Riemann invariant is satisfied by the classical Godunov scheme (5.2), (5.3) (see for example [5, Remark 3.1 (ii)] for a detailed computation). Thus we only need to check what happens for $j = 0, 1$.

If $\hat{\mathbf{f}}_{1,1/2}^n = \mathbf{f}_{1,1/2}^n$, then also $\hat{\mathbf{f}}_{2,1/2}^n = \mathbf{f}_{2,1/2}^n$ and the scheme reduces to the classical Godunov scheme. Therefore we assume that $\hat{\mathbf{f}}_{1,1/2}^n = q < \mathbf{f}_{1,1/2}^n$ and $\hat{\mathbf{f}}_{2,1/2}^n = q\,\mathbf{f}_{2,1/2}^n / \mathbf{f}_{1,1/2}^n < \mathbf{f}_{2,1/2}^n$. In this case, recalling the construction of $\mathcal{RS}_1^q$, it is easy to see that

$$\begin{aligned} \hat{\mathbf{f}}_{1,1/2}^n &= \mathbf{f}_1(\mathbf{u}_0^n, \hat{\mathbf{u}}) = \mathbf{f}_1(\check{\mathbf{u}}_1, \mathbf{u}_1^n)\,, \\ \hat{\mathbf{f}}_{2,1/2}^n &= q\, w(\mathcal{RS}\,(\mathbf{u}_0^n, \mathbf{u}_1^n)\,(0)) = \mathbf{f}_2(\mathbf{u}_0^n, \hat{\mathbf{u}}) = \mathbf{f}_2(\check{\mathbf{u}}_1, \mathbf{u}_1^n)\,, \end{aligned}$$

where $\hat{\mathbf{u}}$ and $\check{\mathbf{u}}_1$ are, respectively, the left and right traces at $x = 0$ of $\mathcal{RS}_1^q(\mathbf{u}_0^n, \mathbf{u}_1^n)$. In fact, since $\mathcal{RS}\,(\mathbf{u}_0^n, \hat{\mathbf{u}})$ counts only waves of negative speed, we have that

$$\mathbf{f}(\mathbf{u}_0^n, \hat{\mathbf{u}}) = f\,(\mathcal{RS}\,(\mathbf{u}_0^n, \hat{\mathbf{u}})\,(0)) = f(\hat{\mathbf{u}}) = \begin{pmatrix} q \\ q w_0^n \end{pmatrix}.$$

On the other side, since $\mathcal{RS}\,(\check{\mathbf{u}}_1, \mathbf{u}_1^n)$ counts only waves of positive speed, we have that

$$\mathbf{f}(\check{\mathbf{u}}_1, \mathbf{u}_1^n) = f\,(\mathcal{RS}\,(\check{\mathbf{u}}_1, \mathbf{u}_1^n)\,(0)) = f(\check{\mathbf{u}}_1) = \begin{pmatrix} q \\ q w_0^n \end{pmatrix}.$$

Hence the following bounds hold for $j = 0, 1$:

$$\inf_{l=-1,\ldots,2} \{w_l^n, \hat{w}, \check{w}_1\} \le w_j^n \le \max_{l=-1,\ldots,2} \{w_l^n, \hat{w}, \check{w}_1\}\,,$$

and we conclude observing that $\hat{w} = \check{w}_1 = w_0^n$. □

We have tested our method on Riemann data lying on the same 1-Riemann invariant, in order to avoid spurious oscillations due to the presence of contact discontinuities. More general data can be dealt with using the technique presented in [5]. Figures 5.1, 5.2, shows that the numerical solutions are in good agreement with exact solutions. In particular, our scheme perfectly captures the nonclassical shock at $x = 0$.

5.2 The Constrained Godunov scheme for $\mathcal{RS}_2^q$

The Constrained Riemann Solver $\mathcal{RS}_2^q$ is not globally conservative at the point $x = 0$ (by definition, conservation holds only for the first equation

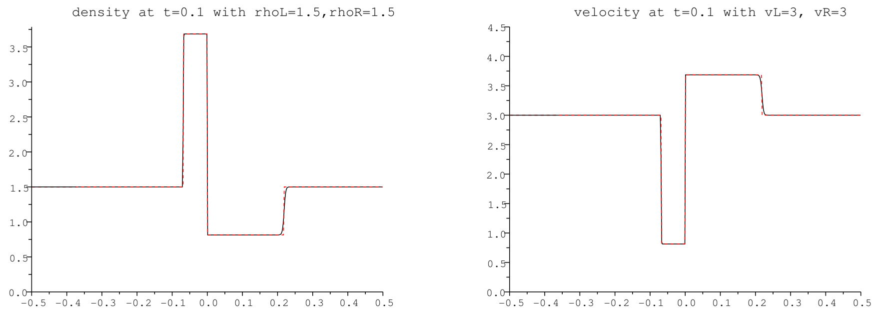

Figure 7: **Test 1a** : Solution of the constrained Riemann solver $\mathcal{RS}_1^q$ with data $\rho^l = \rho^r = 1.5$, $v^l = v^r = 3$ and $q = 3$: exact solution (dashed line), numerical approximation (continuous line).

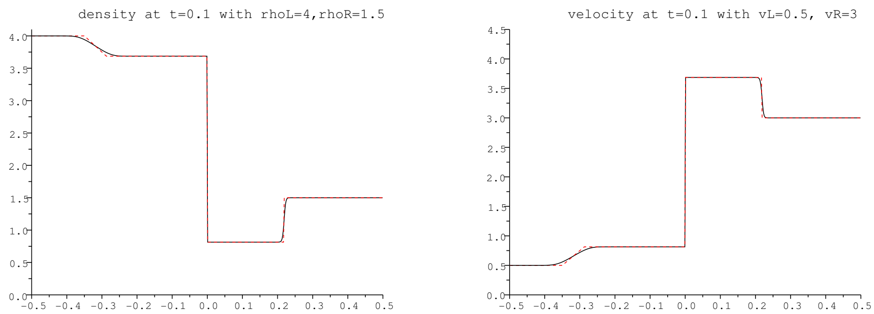

Figure 8: **Test 1b** : Solution of the constrained Riemann solver $\mathcal{RS}_1^q$ with data $\rho^l = 4$, $\rho^r = 1.5$, $v^l = 0.5$, $v^r = 3$ and $q = 3$: exact solution (dashed line), numerical approximation (continuous line).

in (1.1) and therefore only car density ρ is conserved). As a consequence, we look for a non-conservative numerical scheme, i.e. we define two numerical fluxes $\tilde{\mathbf{f}}^{n,-}_{1/2} \neq \tilde{\mathbf{f}}^{n,+}_{1/2}$ such that

$$\begin{aligned} \mathbf{u}^{n+1}_0 &= \mathbf{u}^n_0 - \frac{\Delta t}{\Delta x}(\tilde{\mathbf{f}}^{n,-}_{1/2} - \mathbf{f}^n_{-1/2})\,, && (5.6)\\ \mathbf{u}^{n+1}_1 &= \mathbf{u}^n_1 - \frac{\Delta t}{\Delta x}(\mathbf{f}^n_{3/2} - \tilde{\mathbf{f}}^{n,+}_{1/2})\,. && (5.7) \end{aligned}$$

We set

$$\tilde{\mathbf{f}}^{n,-}_{1,1/2} = \tilde{\mathbf{f}}^{n,+}_{1,1/2} = \min\left\{\mathbf{f}^n_{1,1/2}, q\right\}\,,$$

and

$$\tilde{\mathbf{f}}^{n,-}_{2,1/2} = \tilde{\mathbf{f}}^n_{1,1/2}\,\frac{\mathbf{f}^n_{2,1/2}}{\mathbf{f}^n_{1,1/2}}$$

In order to capture the right trace at $x = 0$, we could envisage using a *ghost cell* type method (introduced in [10], see also [18] and references therein for other applications), computing the ghost value $\check{\mathbf{u}}^n_1$ corresponding to $\mathbf{u}^n_1$, whose (ρ, v) components are given by

$$\check{\rho}^n_1 = q/v^n_1\,, \quad \check{v}^n_1 = v^n_1\,,$$

where $v^n_1 = \dfrac{y^n_1}{\rho^n_1} - p(\rho^n_1)$. This is obtained using the following flux

$$\tilde{\mathbf{f}}^{n,+}_{2,1/2} = q\left(v^n_1 + p(q/v^n_1)\right)\,,$$

whenever $\mathbf{f}^n_{1,1/2} < q$. Unfortunately, due to the convexity assumption (1.3) on the function $\rho \mapsto \rho p(\rho)$, the velocity component is overestimated during the projection step of Godunov scheme in (5.7) (see [5]). Therefore, the right trace cannot be captured properly: the velocity component is overestimated and the density is underestimated, see Figures 5.3, 5.5. In fact, at each time-step, we have $\check{v}^{n+1}_1 = v^{n+1}_1 \geq v^n_1$ and $\check{\rho}^{n+1}_1 \leq \check{\rho}^n_1$, where the inequality is strict generally speaking.

In order to overcome this difficulty, we propose to simply keep the value of the velocity component fixed for $j = 1$, i.e. to replace the value obtained by (5.7) with $\tilde{\mathbf{f}}^{n,+}_{2,1/2} = \tilde{\mathbf{f}}^{n,-}_{2,1/2}$ by $v^{n+1}_1 = v^n_1$, and then updating the conservative component as

$$y^{n+1}_1 = \rho^{n+1}_1\left(v^n_1 + p(\rho^{n+1}_1)\right)\,,$$

whenever $\mathbf{f}^n_{1,1/2} < q$. This allows to capture precisely the right trace of the discontinuity at $x = 0$, as shown by numerical simulation in Figures 5.3, 5.5. Only, a small amplitude oscillation traveling at speed $v = v^r$ is produced, see Figures 5.4, 5.6.

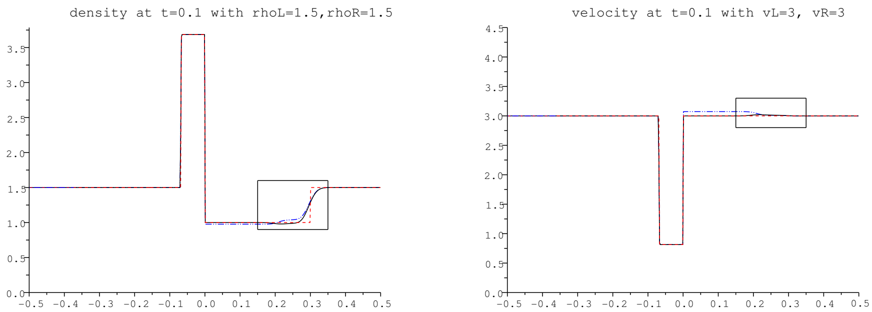

Figure 9: **Test 2a**: Solution of the constrained Riemann solver $\mathcal{RS}_2^q$ with data $\rho^l = \rho^r = 1.5$, $v^l = v^r = 3$ and $q = 3$: exact solution (dashed line), ghost cell method (dash-dotted line), our method (continuous line). The rectangles select the zoomed areas plotted in Figure 5.4.

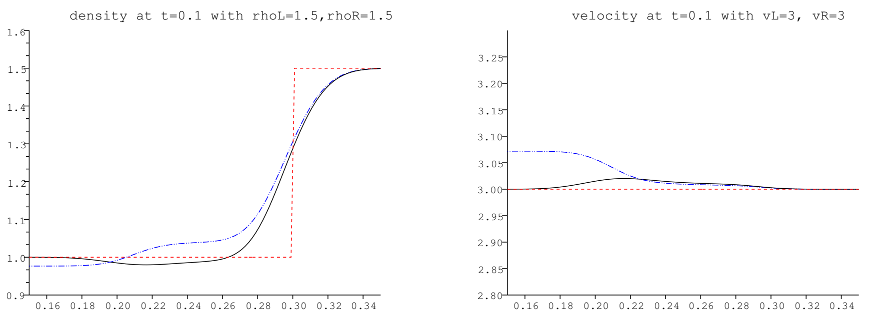

Figure 10: **Test 2a**: Detailed view of a part of the computational domain of Figure 5.3.

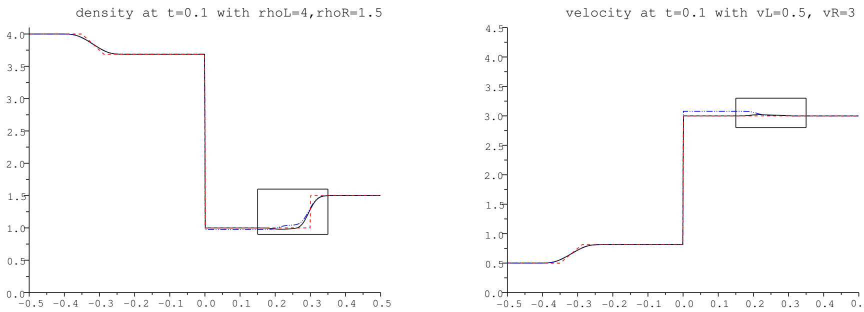

Figure 11: **Test 2b**: Solution of the constrained Riemann solver $\mathcal{RS}_2^q$ with data $\rho^l = 4$, $\rho^r = 1.5$, $v^l = 0.5$, $v^r = 3$ and $q = 3$: exact solution (dashed line), ghost cell method (dash-dotted line), our method (continuous line). The rectangles select the zoomed areas plotted in Figure 5.6.

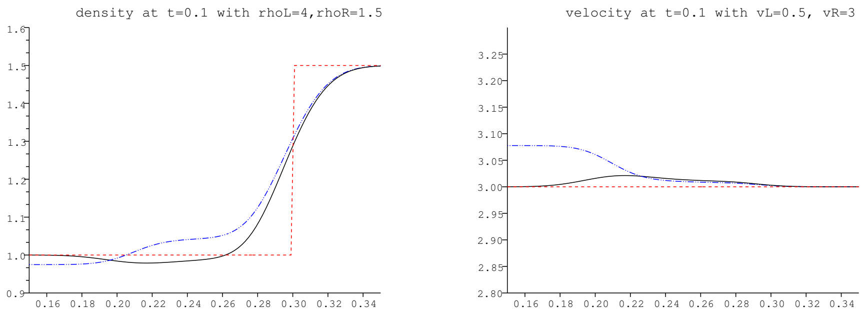

Figure 12: **Test 2b**: Detailed view of a part of the computational domain of Figure 5.5.

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