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A circle criterion observer for estimating the unmeasured membrane potential of neuronal populations

Michelle Chong, Romain Postoyan, Dragan Nešić, Levin Kuhlmann, Andrea Varsavsky

Abstract—A circle criterion observer is designed for estimating the unmeasured membrane potential of neuronal populations using the electroencephalogram (EEG) from a class of parameterised models that replicates patterns seen on the EEG. Compared to existing similar designs, we provide a less conservative linear matrix inequality (LMI) condition that is shown to be fulfilled for the neural models we consider. The designed observer is robust towards disturbances in the input and measurement, as well as model uncertainty. We show that the observer can be designed for a model that reproduces alpha rhythms in the EEG as an illustrative example.

I. INTRODUCTION

The problem of reconstructing the unmeasured activities of the brain has been central to the pursuit in better understanding it, especially for the development of diagnosis and treatment strategies for neural disorders. The physical quantities of interest for this endeavour depend on the level of detail one is interested in. At the *macroscopic* level, where the activity of *neuronal populations* is concerned, the physical variable of interest is the mean membrane potential of each population.

A straightforward solution to this problem is to directly measure each population’s mean membrane potential. However, measuring this quantity is difficult to perform, especially in humans. A widely used diagnostic measurement method is the electroencephalogram (EEG), which in the cortex, is predominantly a measurement proportional to the mean membrane potential of the pyramidal neurons [1]. Using the EEG measurement, we aim to estimate the mean membrane potential of the neuronal populations of a given cortical region, namely the mean membrane potential of the pyramidal neurons, the excitatory and inhibitory populations. For this purpose, we design nonlinear deterministic observers based on a known model of the brain region of interest.

In this paper, we consider the model by Jansen and Rit [2] which is given as a set of ordinary differential equations that describes the generation of alpha rhythms in the scalp EEG, specifically the visual pathway when the brain is in an idle state. Our results also apply to other neural models such as the model by Stam et. al. [3] which replicates alpha

rhythms in scalp EEG and the model by Wendling et. al. [4] which describes epileptic activity in the hippocampus seen in intracranial EEG.

The task of designing observers for estimating brain activity from EEG measurement has traditionally been performed under the stochastic framework [5], [6]. These pursuits employ variants of the stochastic Kalman filter, which produce satisfactory results only under some conditions [7]. Three main drawbacks plague the robustness of this approach. Firstly, the filter is required to be initialised close to the true initial condition, that is often unknown. Besides, the convergence of the estimates to the true states is not guaranteed for every trajectory. Furthermore, the input to the system is required to be a Gaussian signal to derive a computationally tractable solution to the filtering problem. This is not warranted for all models considered. These issues can be circumvented using deterministic nonlinear observers.

For the neural models considered, preliminary results showing that an estimator can be designed for the model by Wendling et. al. [4] is presented in [8]. One drawback concerning the estimator is the inability to control the convergence speed of the error dynamics. Hence, other designs need to be explored.

In this paper, we propose to apply circle criterion observers which have originally been developed by Arcaç and Kokotović in [9] and extended in [10], [11], [12]. Unfortunately, none of these designs apply to the neural models studied because the required LMI condition is not satisfied. Hence, we combine the results for monotonic and globally Lipschitz nonlinearities [10], [12], as well as introduce a multiplier [11] to obtain a less restrictive linear matrix inequality (LMI) that is feasible for the neural models considered.

The paper is structured as follows. We present a simple extension of the circle criterion observer [10], [11], [12] in Section II. Next, we analyse the robustness of the observer in Section III. We then apply the designed observer to a model that replicates alpha rhythms [2] in Section IV and explain how it can be used for the models in [3], [4]. Lastly, we illustrate our results with simulations in Section V.

Notation

- The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is denoted (a, b) , for all $a, b \in \mathbb{R}$.
- The block diagonal matrix with square matrices $A_i \in \mathbb{R}^{n \times n}$ is denoted as $\text{diag}(A_1, \dots, A_n)$.
- The set \mathcal{L}_∞ denotes the set of functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$, for some $n \in \mathbb{Z}$, such that for any $0 \leq t_1 \leq t_2 < \infty$ there exists $r \geq 0$ so that $\|f\|_{[t_1, t_2]} :=$

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ess sup $_{\tau \in [t_1, t_2]} |f(\tau)| \leq r$.

- For a positive definite and symmetric matrix P , the largest (smallest) eigenvalue of P is denoted $\lambda_{max}(P)$ ($\lambda_{min}(P)$).
- A signal $\eta \in \mathbb{R}$ that is drawn from a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}$ is denoted $\eta \sim \mathcal{N}(\mu, \sigma^2)$.

II. AN EXTENSION OF THE CIRCLE CRITERION OBSERVER

We consider the following class of nonlinear systems:

$$\begin{aligned}\dot{x} &= Ax + G\gamma(Hx) + \sigma(y, u) \\ y &= Cx,\end{aligned}\quad (1)$$

where the state vector is $x \in \mathbb{R}^n$, the input is $u \in \mathbb{R}^r$, the measurement is $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{p \times n}$, $G \in \mathbb{R}^{n \times m}$, $H \in \mathbb{R}^{q \times n}$, $\gamma = (\gamma_1, \dots, \gamma_m) : \mathbb{R}^q \rightarrow \mathbb{R}^m$ and $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{R}^p \times \mathbb{R}^r \rightarrow \mathbb{R}^n$. Suppose γ is both globally Lipschitz and monotonically increasing as follows:

Assumption 1: For any $i \in \{1, \dots, m\}$, there exists constants $0 \leq a_i \leq b_i < \infty$, so that the following holds:

$$a_i \leq \frac{\gamma_i(v_i) - \gamma_i(w_i)}{v_i - w_i} \leq b_i, \quad \forall v_i, w_i \in \mathbb{R} \text{ with } v_i \neq w_i.$$

Assumption 1 is an extension of the slope restriction condition from [10, Equation 1] to vector nonlinearity $\gamma = (\gamma_1, \dots, \gamma_m)$. Constant b_i is the Lipschitz constant of γ_i .

We consider the following type of observer originally proposed in [9]:

$$\dot{\hat{x}} = A\hat{x} + G\gamma(H\hat{x} + K(C\hat{x} - y)) + L(C\hat{x} - y) + \sigma(y, u), \quad (2)$$

where \hat{x} is the state estimate and $K \in \mathbb{R}^{q \times p}$, $L \in \mathbb{R}^{n \times p}$ are the observer matrices to be designed.

Denoting the observation error as $e := x - \hat{x}$ and $\eta := v - w$ where $v := Hx$ and $w := H\hat{x} + K(C\hat{x} - y)$, the observation error system from (1) and (2) is:

$$\dot{e} = (A + LC)e + G(\gamma(v) - \gamma(w)).$$

Note that from Assumption 1, we know that for any $i \in \{1, \dots, m\}$, there exists a time-varying gain $\delta_i(t)$ taking values in the interval $[0, b_i]$ so that:

$$\gamma_i(v_i(t)) - \gamma_i(w_i(t)) = \delta_i(t)(v_i(t) - w_i(t)), \quad \forall v_i, w_i \in \mathbb{R}. \quad (3)$$

By (3), we obtain the observation error system as

$$\dot{e} = (A + LC)e + G\delta(t)\eta, \quad (4)$$

where $\delta(t) = \text{diag}(\delta_1(t), \dots, \delta_m(t))$.

We show in Theorem 1 that the origin of the observation error system (4) is globally exponentially stable (GES) under certain conditions.

Theorem 1: Suppose $x(t)$ exists for all $t \geq 0$. Under Assumption 1, if there exist a matrix $P = P^T > 0$, a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ with strictly positive components and a constant $\nu > 0$ such that:

$$\begin{bmatrix} (A + LC)^T P + P(A + LC) + \nu I & PG + (H + KC)^T \Lambda \\ G^T P + \Lambda(H + KC) & -2\Lambda \text{diag}\left(\frac{1}{b_1}, \dots, \frac{1}{b_m}\right) \end{bmatrix} \leq 0, \quad (5)$$

then there exists $k, \beta > 0$ such that the following holds:

$$|e(t)| \leq k \exp(-\beta t) |e(0)|, \quad \forall t \geq 0, \forall e(0) \in \mathbb{R}^n. \quad (6)$$

Proof: Let $V(e) = e^T P e$, its derivative along the solutions to (4) is:

$$\begin{aligned}\dot{V} &= e^T (P(A + LC) + (A + LC)^T P) e + 2e^T P G \delta(t) \eta \\ &= \begin{bmatrix} e \\ \delta(t)\eta \end{bmatrix}^T \begin{bmatrix} (A + LC)^T P + P(A + LC) & PG \\ G^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ \delta(t)\eta \end{bmatrix}.\end{aligned}$$

According to (5),

$$\begin{aligned}\dot{V} &\leq \begin{bmatrix} e \\ \delta(t)\eta \end{bmatrix}^T \begin{bmatrix} -\nu I & -(H + KC)^T \Lambda \\ -\Lambda(H + KC) & 2\Lambda \text{diag}\left(\frac{1}{b_1}, \dots, \frac{1}{b_m}\right) \end{bmatrix} \begin{bmatrix} e \\ \delta(t)\eta \end{bmatrix} \\ &= -\nu e^T e - 2\eta^T \delta(t) \Lambda (H + KC) e \\ &\quad + 2\eta^T \delta(t) \Lambda \text{diag}\left(\frac{1}{b_1}, \dots, \frac{1}{b_m}\right) \delta(t) \eta.\end{aligned}$$

Recall that $\eta = v - w = (H + KC)e$ and since Λ and δ are diagonal, $\Lambda = \Lambda^T$, $\delta(t) = \delta(t)^T$ and $\Lambda \delta(t) = \delta(t) \Lambda$,

$$\begin{aligned}\dot{V} &\leq -\nu e^T e - 2\eta^T \delta(t) \Lambda \eta + 2\eta^T \delta(t) \Lambda \text{diag}\left(\frac{1}{b_1}, \dots, \frac{1}{b_m}\right) \delta(t) \eta \\ &= -\nu e^T e - 2\eta^T \left(\delta(t) \Lambda - \delta(t) \Lambda \text{diag}\left(\frac{1}{b_1}, \dots, \frac{1}{b_m}\right) \delta(t) \right) \eta.\end{aligned}$$

If we examine the last term component-wise, we obtain:

$$\delta_i(t) \lambda_i - \delta_i(t) \lambda_i b_i^{-1} \delta_i(t) = \delta_i(t) \lambda_i (1 - b_i^{-1} \delta_i(t)).$$

From Assumption 1 and (3), we know that $1 - b_i^{-1} \delta_i(t) > 0$, hence,

$$\left(\delta(t) \Lambda - \delta(t) \Lambda \text{diag}\left(\frac{1}{b_1}, \dots, \frac{1}{b_m}\right) \delta(t) \right) \geq 0.$$

We obtain $\dot{V} \leq -\nu e^T e$. By [13, Theorem 4.10], we deduce that (6) holds. \blacksquare

Theorem 1 states that an observer of the form (2) can be designed for system (1) if the observer matrices K and L can be found such that LMI (5) is satisfied. As explained in [9] and in Section IV-B, inequality (5) can be treated as an LMI in $P, PL, \Lambda, \Lambda K$ and ν . Hence, widely available software packages such as the LMI Lab in MATLAB can be used to solve (5).

Existing circle criterion observer results [10], [11], [12] yield LMIs that are infeasible for the neural models we consider in Section IV. Therefore, we tailored the results of [11] to the case where vector nonlinearities γ are not only monotonically increasing but also globally Lipschitz. In that way, the LMI condition (5) differs from the LMI in (17) of [11], where element (2, 2) in (5) is non-zero which makes it less restrictive. The LMI condition (5) also differs from (13) of [12], where the presence of multiplier Λ in elements (1, 2), (2, 1) and (2, 2) of (5), gives us more flexibility. Hence, we are able to design circle criterion observers for the neural models of interest.

III. ROBUSTNESS ANALYSIS

The robustness of the observer towards uncertainties and disturbances is important for its practicality in a realistic setting. Therefore, we introduce ϵ_y for measurement noise, ϵ_u for input uncertainty and μ to characterise modelling uncertainty. Instead of (1) and (2), our model and observer are now restated as follows:

$$\begin{aligned} \dot{x} &= Ax + G\gamma(Hx) + \sigma(y, u) + \mu \\ y &= Cx, \end{aligned} \quad (7)$$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + G\gamma\left(H\hat{x} + K(C\hat{x} - (y + \epsilon_y))\right) \\ &\quad + L(C\hat{x} - (y + \epsilon_y)) + \sigma(y + \epsilon_y, u + \epsilon_u). \end{aligned} \quad (8)$$

We make the following additional assumption:

Assumption 2: Each component $\sigma_i : \mathbb{R}^p \times \mathbb{R}^r \rightarrow \mathbb{R}$ for $i \in \{1, \dots, n\}$ is globally Lipschitz with constant \tilde{b}_i .

Note that due to Assumption 1-2 and assuming that $\mu \in \mathcal{L}_\infty$, the solution $x(t)$ exists for all time $t \geq 0$ [13, Theorem 3.2]. The following theorem shows that the observation error e from (7) and (8) is input-to-state stable (ISS) [14] with respect to disturbances ϵ_y , ϵ_u and uncertainty μ .

Theorem 2: Consider the perturbed model (7) and observer (8). If Assumptions 1 - 2 hold and LMI (5) is feasible, then there exist \tilde{k} , $\tilde{\beta}$, γ_u , γ_y , $\gamma_\mu > 0$ such that (9) holds for all $t \geq 0$, $e(0) \in \mathbb{R}^n$ and for all ϵ_u , ϵ_y , $\mu \in \mathcal{L}_\infty$,

$$\begin{aligned} |e(t)| &\leq \tilde{k} \exp(-\tilde{\beta}t) |e(0)| + \gamma_u \|\epsilon_u\|_{[0,t]} \\ &\quad + \gamma_y \|\epsilon_y\|_{[0,t]} + \gamma_\mu \|\mu\|_{[0,t]}. \end{aligned} \quad (9)$$

Proof: From (7) and (8), the observation error system is as follows:

$$\begin{aligned} \dot{e} &= (A + LC)e + L\epsilon_y + \sigma(y, u) - \sigma(y + \epsilon_y, u + \epsilon_u) \\ &\quad + G\gamma(Hx) - G\gamma\left(H\hat{x} + K(C\hat{x} - (y + \epsilon_y))\right) + \mu. \end{aligned} \quad (10)$$

By introducing $\pm G\gamma\left(H\hat{x} + K(C\hat{x} - y)\right)$ to (10), we obtain an observation error system that can be considered as the nominal system (4) perturbed by a term that depends on ϵ_y , ϵ_u and μ as shown below:

$$\begin{aligned} \dot{e} &= \underbrace{(A + LC)e + G(\gamma(Hx) - \gamma(H\hat{x} + K(C\hat{x} - y)))}_{\text{nominal system (4)}} \\ &\quad + \underbrace{\Psi_\epsilon(x, \hat{x}, \epsilon_y, \epsilon_u, \mu)}_{\text{perturbation terms}}, \end{aligned} \quad (11)$$

where $\Psi_\epsilon = G\gamma(H\hat{x} + K(C\hat{x} - y)) - G\gamma\left(H\hat{x} + K(C\hat{x} - (y + \epsilon_y))\right) + L\epsilon_y + \sigma(y, u) - \sigma(y + \epsilon_y, u + \epsilon_u) + \mu$.

Let $V(e) = e^T P e$, by following the same arguments as in the proof of Theorem 1, along solutions to (11), we obtain

$$\dot{V} \leq -\nu e^T e + 2e^T P \Psi_\epsilon. \quad (12)$$

The perturbation Ψ_ϵ can be bounded as follows:

$$|\Psi_\epsilon| \leq \sigma_y |\epsilon_y| + \sigma_u |\epsilon_u| + |\mu|, \quad (13)$$

where $\sigma_y = |G| |\text{diag}(b_1, \dots, b_m)| |K| + |L| + |\text{diag}(\tilde{b}_1, \dots, \tilde{b}_n)|$ and $\sigma_u = |\text{diag}(\tilde{b}_1, \dots, \tilde{b}_n)|$.

From (12) and the bound on the perturbation terms (13):

$$\dot{V} \leq -\nu |e|^2 + 2|e| |P| (\sigma_y |\epsilon_y| + \sigma_u |\epsilon_u| + |\mu|).$$

Consequently, if $|e| > \frac{4|P|}{\nu} (\sigma_y |\epsilon_y| + \sigma_u |\epsilon_u| + |\mu|)$, then $\dot{V} \leq -\frac{\nu}{2} |e|^2$. By [13, Theorem 4.19], (9) is satisfied with $\gamma_y = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \frac{4|P|}{\nu} \sigma_y$, $\gamma_u = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \frac{4|P|}{\nu} \sigma_u$ and $\gamma_\mu = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \frac{4|P|}{\nu}$. ■

The ISS property guarantees that the observation error converges to a neighbourhood of the origin whose size depends on the norms of input uncertainty ϵ_u , modelling uncertainty μ and disturbance in the measurement ϵ_y . Note that we recover the global exponential property of the observation error system in Theorem 1 when all the uncertainties and disturbances are set to 0.

IV. APPLICATION TO A NEURAL MODEL

A. Description of the model

We consider the neural model developed in [2] that describes the interconnection between populations of neurons to replicate alpha rhythms in the EEG. This model can be written as follows:

$$\begin{aligned} \dot{x} &= Ax + G(\theta)\gamma(Hx) + \tilde{\sigma}(y, u, \theta) \\ y &= Cx, \end{aligned} \quad (14)$$

where the state vector is $x = (x_1, z_1, x_2, z_2, x_3, z_3, x_4, z_4)$, where x_i are the membrane potential contribution from one neuronal population to another and z_i is the time derivative of x_i for $i \in \{1, \dots, 4\}$. The input u is the afferent influence from nearby populations and is identified in [2] to be uniformly distributed between 120 and 320mV. The output of the model y is the EEG measurement available to the observer. The model is parameterised by $\theta = (\theta_A, \theta_B, C_1, C_2, C_3, C_4)$, where the different values of θ produce different EEG patterns corresponding to various brain activity, e.g. alpha rhythms. As identified in [2, Section 2.3], the parameters reside in a compact set:

$$\Theta = [\theta_{A_{\min}}, \theta_{A_{\max}}] \times [\theta_{B_{\min}}, \theta_{B_{\max}}] \times [C_{1_{\min}}, C_{1_{\max}}] \times [C_{2_{\min}}, C_{2_{\max}}] \times [C_{3_{\min}}, C_{3_{\max}}] \times [C_{4_{\min}}, C_{4_{\max}}].$$

All values of the constants are non-negative and their physiological meaning can be found in [2, Section 2.3].

When the parameter θ is known and constant, we see that (14) is of the form (1) with:

- $A = \text{diag}(A_1, \dots, A_4)$ where $A_i = \begin{bmatrix} 0 & 1 \\ -k_i^2 & -2k_i \end{bmatrix}$, $k_1 = k_3 = k_4 = a$ and $k_2 = b$, where $a, b > 0$.
- $G(\theta) = (0, \theta_A a C_2, 0, \theta_B b C_4, 0, 0, 0, 0)$.

$$\gamma(s) = \frac{\alpha}{1 + \exp(-r(s - V_0))}, \forall s \in \mathbb{R}. \quad (15)$$

where $\alpha, r > 0$. Function γ is bounded by α and satisfies Assumption 1 with $a_1 = 0$, $b_1 = \rho$, where $\rho = \frac{1}{4}\alpha r$ is its Lipschitz constant.

- $H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

- $\tilde{\sigma}(y, u, \theta) = (0, \theta_A a u, 0, 0, 0, \theta_A a C_3 \gamma(y), 0, \theta_A a C_1 \gamma(y))$ where γ is defined by (15). This satisfies Assumption 2 with $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{b}_4 = \tilde{b}_5 = \tilde{b}_7 = 0$, $\tilde{b}_6 = \theta_A a C_3 \alpha$ and $\tilde{b}_8 = \theta_A a C_1 \alpha$.
- $C = [1 \ 0 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0]$.

B. Observer design with uncertainties and disturbances

Under practical conditions, the parameters θ_A , θ_B , C_1 , C_2 , C_3 and C_4 are not known and are time-varying. To characterise the uncertainty in the parameters, we introduce ϵ_θ so that the true parameter of the system (14) is $\theta + \epsilon_\theta$, where θ is fixed and known, ϵ_θ is bounded and unknown. Therefore, the model (14) becomes:

$$\begin{aligned} \dot{x} &= Ax + G(\theta + \epsilon_\theta) \gamma(Hx) + \tilde{\sigma}(y, u, \theta + \epsilon_\theta), \\ y &= Cx. \end{aligned} \quad (16)$$

Moreover, the EEG measurement y is not noise-free and the input u from afferent populations is not quantifiable in practice. As performed in Section III, we introduce ϵ_y to characterise measurement noise such that the EEG measurement available to the observer is now $y + \epsilon_y$ and the input u is allowed to be uncertain by introducing ϵ_u such that the input available to the observer is $u + \epsilon_u$.

We consider the observer in the form of (8) as follows:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + G(\theta) \gamma(H\hat{x} + K(\theta)(C\hat{x} - (y + \epsilon_y))) \\ &\quad + \sigma(y + \epsilon_y, u + \epsilon_u, \theta) + L(\theta)(C\hat{x} - (y + \epsilon_y)), \end{aligned} \quad (17)$$

where $K(\theta)$ and $L(\theta)$ are obtained by solving LMI (5) where G is dependent on θ . The overall system (16) and (17) is depicted in Figure 1.

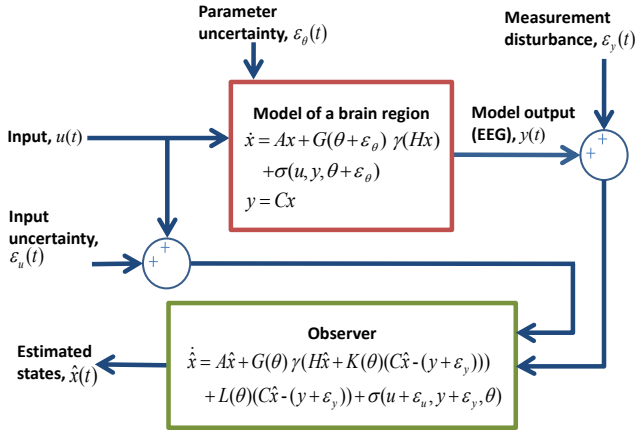


Fig. 1. Perturbed systems.

Note that as the LMI (5) is dependent on $G(\theta)$, verifying that (5) is satisfied for $\theta \in \Theta$ will require infinite number of LMIs to be checked. To avoid this, we proceed as follows. We note that $G(\theta)$ is:

$$G(\theta) = (0, \theta_1 a, 0, \theta_2 b, 0, 0, 0, 0), \quad (18)$$

where $\tilde{\theta} := (\theta_1, \theta_2) = (\theta_A C_2, \theta_B C_4)$. Henceforth, we denote $G(\theta)$ as $G(\tilde{\theta})$ with slight abuse of notation. The

tuple of parameters (θ_1, θ_2) takes values in the convex set $\tilde{\Theta} := [\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \bar{\theta}_2]$, where $\bar{\theta}_1 = \theta_{A_{max}} C_{2_{max}}$, $\underline{\theta}_1 = \theta_{A_{min}} C_{2_{min}}$, $\bar{\theta}_2 = \theta_{B_{max}} C_{4_{max}}$ and $\underline{\theta}_2 = \theta_{B_{min}} C_{4_{min}}$. The vertices of $\tilde{\Theta}$ are then:

$$\begin{cases} \theta_1^* = (\underline{\theta}_1, \underline{\theta}_2) \\ \theta_2^* = (\underline{\theta}_1, \bar{\theta}_2) \\ \theta_3^* = (\bar{\theta}_1, \underline{\theta}_2) \\ \theta_4^* = (\bar{\theta}_1, \bar{\theta}_2). \end{cases} \quad (19)$$

We now show that if LMI (5) is solvable for each $\theta_i^* \in \tilde{\Theta}$, $i \in \{1, \dots, 4\}$ with the same positive definite matrix $P = P^T$, then it is solvable for any $\theta \in \tilde{\Theta}$. In that way, only four LMIs need to be verified to ensure the feasibility of (5). Due to the convexity of $\tilde{\Theta}$, any point $\tilde{\theta} = (\theta_1, \theta_2) \in \tilde{\Theta}$ can be written in the form of:

$$\tilde{\theta} = \sum_{i=1}^4 \lambda_i \theta_i^* \quad (20)$$

where $\lambda_i \geq 0$, $i \in \{1, \dots, 4\}$ and $\sum_{i=1}^4 \lambda_i = 1$. From (18), $G(\tilde{\theta})$ is linear in $\tilde{\theta}$, hence:

$$G(\tilde{\theta}) = \sum_{i=1}^4 \lambda_i G(\theta_i^*). \quad (21)$$

Now, we convert the matrix inequality of Theorem 1 into the following LMI as explained in [9]:

$$\begin{aligned} \mathcal{M}_{\tilde{\theta}}(P, R, S, \Lambda, \nu) = \\ \left[\begin{array}{cc} A^T P + P A + R C + C^T R^T + \nu I & P G(\theta) + H^T \Lambda + C^T S^T \\ G(\theta)^T P + \Lambda H + S C & -2\Lambda \text{diag}(b_1^{-1}, \dots, b_m^{-1}) \end{array} \right] \leq 0 \end{aligned} \quad (22)$$

where $R = PL$ and $S = \Lambda K$. Suppose that there exist positive definite matrix $P = P^T$, matrices R_i and S_i , positive definite diagonal matrix Λ_i and strictly positive constant ν_i for $i \in \{1, \dots, 4\}$ such that the following holds:

$$\mathcal{M}_{\theta_i^*}(P, R_i, S_i, \Lambda_i, \nu_i) \leq 0, \quad (23)$$

for each vertex θ_i^* defined in (19). Consider $\tilde{\theta} \in \tilde{\Theta}$, we know that there exist λ_i such that (20) holds for all $\lambda_i \geq 0$ and $\sum_{i=1}^4 \lambda_i = 1$. Due to the linearity of $\mathcal{M}_{\theta}(P, R, S, \Lambda, \nu)$ in its arguments and (21), the following is satisfied:

$$\mathcal{M}_{\tilde{\theta}}(P, R, S, \Lambda, \nu) = \sum_{i=1}^4 \lambda_i \mathcal{M}_{\theta_i^*}(P, R_i, S_i, \Lambda_i, \nu_i), \quad (24)$$

where $R = \sum_{i=1}^4 \lambda_i R_i$, $S = \sum_{i=1}^4 \lambda_i S_i$, $\Lambda = \sum_{i=1}^4 \lambda_i \Lambda_i$ is a positive definite diagonal matrix and $\nu = \sum_{i=1}^4 \lambda_i \nu_i > 0$. Consequently, the feasibility of all the LMIs (23) for all $i \in \{1, \dots, 4\}$ with the same P implies that the LMI (5) (equivalently (22)) is satisfied for all $\theta \in \tilde{\Theta}$. Note that the same P that satisfies LMI (22) also has to work for all LMIs (23) for $i \in \{1, \dots, 4\}$, i.e. we have a simultaneous Lyapunov function for all vertices [15, Section 10.1.3]. We have computationally verified that the LMIs (23) are satisfied for the numerical values provided in [2, Section 2.3]. Therefore, we are able to state the following result which guarantees the existence of observer matrices $K(\theta)$ and $L(\theta)$ for all $\theta \in \tilde{\Theta}$ such that the state estimation error system e

is ISS with respect to the input and parameter uncertainties $\epsilon_u, \epsilon_\theta$ as well as measurement noise ϵ_y .

Proposition 1: Consider the perturbed model (16) and observer (17). For any $\theta \in \Theta$ (where Θ is defined in Section IV-A), there exist observer matrices $L(\theta)$ and $K(\theta)$ that satisfy LMI (5) such that the observation error system e satisfies for all $t \geq 0$, $e(0) \in \mathbb{R}^8$ and for all $\epsilon_u, \epsilon_y, \epsilon_\theta \in \mathcal{L}_\infty$,

$$|e(t)| \leq \bar{k} \exp(-\bar{\beta}t) |e(0)| + \bar{\gamma}_y \|\epsilon_y\|_{[0,t]} + \bar{\gamma}_u \|\epsilon_u\|_{[0,t]} + \bar{\gamma}_\theta \|\epsilon_\theta\|_{[0,t]}, \quad (25)$$

where $\bar{k}, \bar{\beta}, \bar{\gamma}_y, \bar{\gamma}_u, \bar{\gamma}_\theta \geq 0$.

Proof: Let $\theta \in \Theta$, the perturbed model (16) has solutions for all $t \geq 0$ as γ and $\tilde{\sigma}$ are globally Lipschitz, the input u and the uncertainty ϵ_θ are in \mathcal{L}_∞ [13, Theorem 3.2]. We can write the perturbed model (16) in the form of (7) with $\mu = G(\theta + \epsilon_\theta)\gamma(Hx) - G(\theta)\gamma(Hx) + \tilde{\sigma}(y, u, \theta + \epsilon_\theta) - \tilde{\sigma}(y, u, \theta)$. As Assumptions 1-2 are satisfied, we apply Theorem 2 and obtain the following:

$$|e(t)| \leq \bar{k} \exp(-\bar{\beta}t) |e(0)| + \bar{\gamma}_y \|\epsilon_y\|_{[0,t]} + \bar{\gamma}_u \|\epsilon_u\|_{[0,t]} + \gamma_\mu \|\mu\|_{[0,t]}. \quad (26)$$

Noting that the uncertainty is $\mu = G(\theta + \epsilon_\theta)\gamma(Hx) - G(\theta)\gamma(Hx) + \tilde{\sigma}(y, u, \theta + \epsilon_\theta) - \tilde{\sigma}(y, u, \theta)$, we use the fact that γ and $\tilde{\sigma}$ are globally Lipschitz and bounded to show that μ satisfies the following:

$$|\mu| \leq \sigma_\theta |\epsilon_\theta|, \quad (27)$$

where $\sigma_\theta = |(aC_2\alpha, bC_4\alpha)| + |(a\|u\|_{[0,\infty]}, aC_3\alpha, aC_1\alpha)|$, recalling that $\|u\|_{[0,\infty]} \leq 320\text{mV}$ [2, Section 3.1]. We deduce from (26) and (27) that (25) is satisfied with $\bar{\gamma}_\theta = \sigma_\theta \gamma_\mu$. ■

C. Discussion

Due to space constraints, we focus on presenting results for the model by Jansen and Rit [2]. However, similar results can be derived for the models by Stam et. al. [3] and Wendling et. al. [4] describing alpha rhythms and epileptic activity in the hippocampus respectively. Indeed, these models can be written in the form of (16) (where parameters θ are constant and known) and we have verified that the conditions of Theorem 2 and Proposition 1 are met.

V. SIMULATION RESULTS

We show the simulation results for the model (16) and observer (8). The simulations are performed under two scenarios: (1) without disturbances and uncertainties, (2) under the more practical condition with disturbances and uncertainties.

We initialise the model at $x(0) = (6, 6, 6, 6, 6, 6, 6, 6)$ and the observer at $\hat{x}(0) = (0, 0, 0, 0, 0, 0, 0, 0)$. The input to the model u as described in [2] is uniformly distributed between 120 and 320mV. The parameters were chosen to correspond to alpha-like activity $\theta = (\theta_A, \theta_B, C_1, C_2, C_3, C_4) = (3.25, 22, 135, 108, 33.75, 33.75)$ as identified in [2]. All other constants used in simulation are as described in [2, Section 3.1].

By solving the LMI (5), we obtain the observer matrices $L = 10^4 \times$

$(0.0053, -2.2306, 0.0077, 5.3849, 0.0032, -0.1266, -0.0017, 0.0514)$ and $K = (-0.0586, -0.1422)$.

A. Scenario 1: the ideal condition without disturbance and uncertainties

In the ideal scenario, Theorem 1 states that the observer provides estimates that converge to the true states in exponential time. As illustrated in Figure 2, at $t = 0.5\text{s}$, the absolute observation error is less than 0.01%.

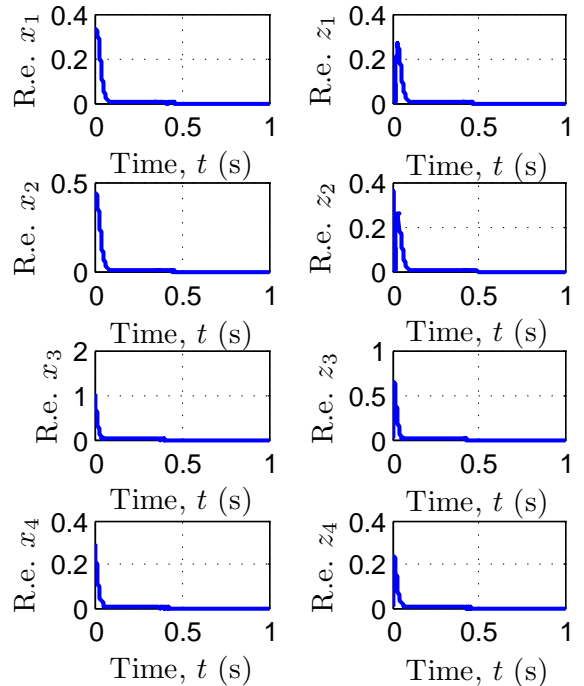


Fig. 2. Scenario 1: Absolute observation error relative to the amplitude of the signal, $\frac{|e_i|}{|\max(x_i) - \min(x_i)|}$, for $i \in \{1, \dots, 4\}$.

B. Scenario 2: robustness of the observer towards uncertainty in parameters, input and measurement

We now introduce parameter uncertainty that is 10% of the actual values $\epsilon_\theta = 0.1 \times (\theta_A, \theta_B, C_1, C_2, C_3, C_4) = (0.33, 2.2, 13.5, 10.8, 3.375, 3.375)$, input uncertainty $\epsilon_u \sim \mathcal{N}(0, 10^2)$ and measurement noise $\epsilon_y \sim \mathcal{N}(0, 0.7^2)$, as shown in our perturbed model (16) and observer (8). Our simulation results (Figures 3 - 5) confirm the results stated in Proposition 1, i.e. the norm of the observation error is small with small \mathcal{L}_∞ norm of the uncertainties. The absolute observation error relative to the amplitude of the signal as shown in Figure 3 remains less than 25% for all states. Figures 4 and 5 show that all the estimated states do converge to a neighbourhood of the true states.

VI. CONCLUSIONS AND FUTURE WORKS

We have proposed a circle criterion observer by combining existing techniques from [11], [12] to achieve a new less restrictive LMI condition. As a result of this extension, a circle criterion observer can be designed for a class of neural models that includes the model by Jansen and Rit

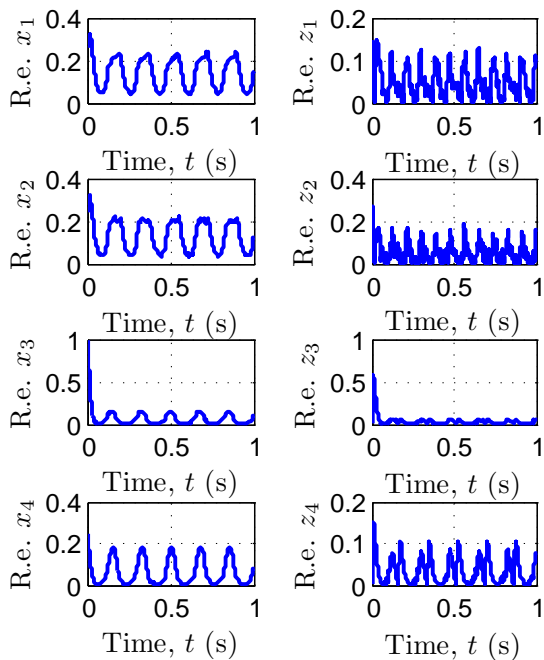


Fig. 3. Scenario 2: Absolute state estimation error relative to the amplitude of the signal $\frac{|e_i|}{|\max(x_i) - \min(x_i)|}$ for $i \in \{1, \dots, 4\}$.

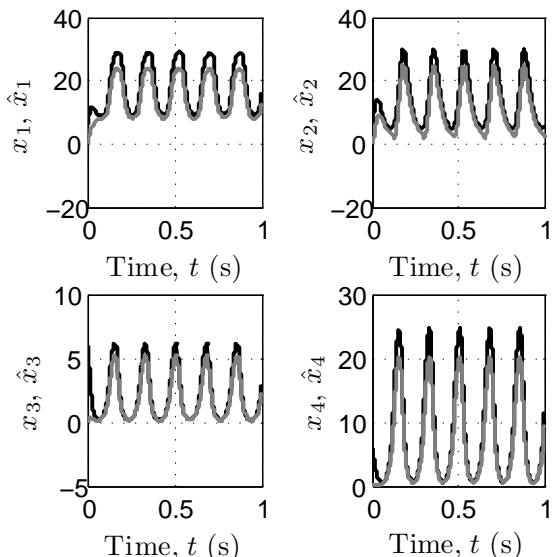


Fig. 4. Scenario 2: Estimated membrane potential contribution \hat{x}_i (grey line) and the true membrane potential contribution x_i (black line), for $i \in \{1, \dots, 4\}$.

[2], Stam et. al. [3] and Wendling et. al. [4]. We have shown its robustness towards uncertainties and disturbances, in the sense that small perturbations result in small observation error. We then applied the observer on the model by Jansen and Rit [2] to estimate the mean membrane potential of neuronal populations from the EEG measurement.

The model considered in this study is parameterised, where regions of the parameter space were identified to correspond to different brain activities of interest. Future work would concentrate on building an adaptive observer that serves to achieve observer-based parameter estimation

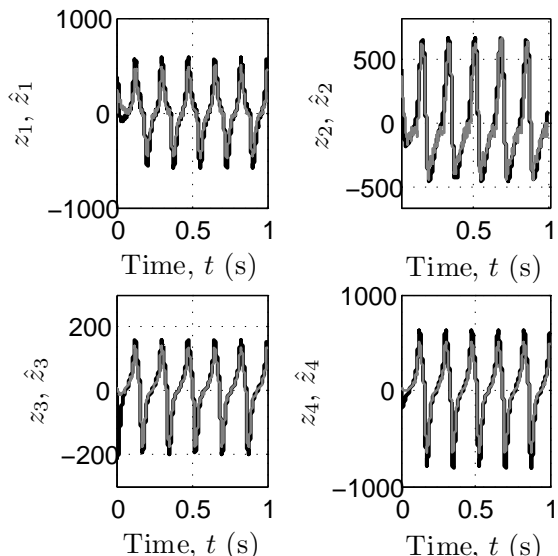


Fig. 5. Scenario 2: Estimated rate of change of the membrane potential contribution \hat{z}_i (grey line) and the true rate of change of the membrane potential contribution z_i (black line), for $i \in \{1, \dots, 4\}$.

to estimate both variables and parameters.

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