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GLOBAL WELL-POSEDNESS AND LIMIT BEHAVIOR FOR A HIGHER-ORDER BENJAMIN-ONO EQUATION

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Abstract. In this paper, we prove that the Cauchy problem associated to the following higher-order Benjamin-Ono equation

$$\partial_t v - b\mathcal{H}\partial_x^2 v - a\epsilon\partial_x^3 v = cv\partial_x v - d\epsilon\partial_x(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)), \tag{0.1}$$

is globally well-posed in the energy space $H^1(\mathbb{R})$. Moreover, we study the limit behavior when the small positive parameter ϵ tends to zero and show that, under a condition on the coefficients a, b, c and d, the solution v_ϵ to (0.1) converges to the corresponding solution of the Benjamin-Ono equation.

1. Introduction

Considered here is the following higher-order Benjamin-Ono equation

$$\partial_t v - b\mathcal{H}\partial_x^2 v - a\epsilon\partial_x^3 v = cv\partial_x v - d\epsilon\partial_x(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)), \tag{1.1}$$

where x, $t \in \mathbb{R}$, v is a real-valued function, a, b, c and d are positive constants, $\epsilon > 0$ is a small positive parameter and $\mathcal{H}$ is the Hilbert transform, defined on the line by

$$\mathcal{H}f(x) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy. \tag{1.2}$$

The equation above corresponds to a second order approximation of the unidirectional evolution of weakly nonlinear dispersive internal long waves at the interface of a two-layer system of fluids, the lower one being infinitely deep. It was derived by Craig, Guyenne and Kalisch (see equation (5.38) in [5]), using a Hamiltonian perturbation theory. Here, v represents the dislocation of the interface around its position of equilibrium, the coefficients a, b, c and d are respectively given by

$$a = \frac{h_1^2}{2}\Big(\frac{\rho^2}{\rho_1^2} - \frac{1}{3}\Big)\sqrt{\frac{gh_1(\rho-\rho_1)}{\rho_1}}, \qquad b = \frac{\rho h_1^2}{2\rho_1^2}\sqrt{\frac{g\rho_1(\rho-\rho_1)}{h_1}}, \tag{1.3}$$

$$c = \frac{3\sqrt{2}}{4\rho_1}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}} \quad \text{and} \quad d = \frac{\sqrt{2}\rho h_1}{2\rho_1^2}\sqrt[4]{\frac{g\rho_1(\rho-\rho_1)}{h_1}}, \tag{1.4}$$

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where h_1 represents the depth of the upper layer when the fluid is at rest, ρ_1 is the density of the upper fluid and ρ is the density of the lower fluid. Moreover, the system is assumed to be in a stable configuration, which is to say that $\rho > \rho_1$, so that the coefficients a, b, c and d are positive.

It is worth noting that the equation obtained at the first order approximation of the above physical model is the well-known Benjamin-Ono equation

$$\partial_t v - b\mathcal{H}\partial_x^2 v = cv\partial_x v, \tag{1.5}$$

and therefore equation (1.1) can be seen as an higher-order perturbation of equation (1.5). Moreover, the quantities

$$M(v) = \int_{\mathbb{R}} v^2 dx \tag{1.6}$$

and

$$H(v) = \int_{\mathbb{R}} \Big(a\epsilon(\partial_x v)^2 - bv\mathcal{H}\partial_x v - \frac{c}{3}v^3 + d\epsilon v^2 \mathcal{H}\partial_x v \Big)\, dx \tag{1.7}$$

are conserved by the flow associated to (1.1).

The initial value problem (IVP) associated to the Benjamin-Ono equation on the line has been extensively studied in the recent years and has been proved to be globally well-posed in $L^2(\mathbb{R})$ by Ionescu and Kenig in [9] (see [23] for another proof and [1, 4, 11, 12, 17, 26, 27] for former results). The IVP associated to (1.1) presents the same mathematical difficulties as for the Benjamin-Ono equation. Indeed, it has been shown in [25] that the flow map data-solution cannot be C^2 in any L^2-based Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$, by using the same counter-example as for the Benjamin-Ono equation in [24]. On the other hand, the Cauchy problem associated to (1.1) was proved in [18] to be locally well-posed in $H^s(\mathbb{R})$, for $s \geq 2$ (and also in weighted Sobolev spaces $H^k(\mathbb{R}) \cap L^2(\mathbb{R}; x^2 dx)$, for $k \in \mathbb{Z}_+$, $k \geq 2$). However, there are no conserved quantities at the H^2 level and thus it is not known wether these local solutions extend globally in time or not. Therefore, as commented in [18], the question of the local well-posedness in $H^1(\mathbb{R})$, which would directly imply global well-posedness by using (1.6) and (1.7), arises naturally.

The first aim of this paper is to give a positive answer to this issue. The result states as follows.

Theorem 1.1. *Fix $\epsilon > 0$ and let $s \geq 1$ be given. Then, for all $v_0 \in H^s(\mathbb{R})$ and all $T > 0$, there exists a unique solution v to equation* (1.1) *in the space*

$$C([0,T]; H^s(\mathbb{R})) \cap L^4_T W^{s,4}_x \cap L^2_x L^\infty_T \cap X^{s-2\theta,\theta}_{\epsilon,T}, \quad \text{for all } 0 \leq \theta \leq 1. \tag{1.8}$$

satisfying

$$v(\cdot, 0) = v_0 \tag{1.9}$$

and

$$w = \partial_x P_{+hi}\big(e^{-iF[v]}\big) \in X^{s,\frac{1}{2},1}_{\epsilon,T}, \tag{1.10}$$

where $F[v]$ is a spatial primitive of v defined in Section 3.

Moreover, $v \in C_b(\mathbb{R}; H^1(\mathbb{R}))$ and the flow map data-solution $S_\epsilon : v_0 \mapsto v$ is continuous from $H^s(\mathbb{R})$ into $C([0,T]; H^s(\mathbb{R}))$.

Note that above $H^s(\mathbb{R})$ denotes the space of all real-valued functions with the usual norm, while $X^{s,b}_{\epsilon,T}$ and $X^{s,b,q}_{\epsilon,T}$ are Bourgain spaces defined in Subsection 2.2.

Since it follows from the result of ill-posedness in [25] that the Cauchy problem associated to (1.1) cannot be solved by using a fixed point theorem on the integral equation, we use a compactness argument based on the smooth solutions obtained in [18]. To derive *a priori* estimates at the H^1 level, we introduce a gauge transformation which weakens the high-low frequency interactions in the nonlinearity of (1.1), as it was done by Tao in [27] for the Benjamin-Ono equation. Note that the same kind of gauge transformation was already introduced in [18] to obtain the solutions in $H^2(\mathbb{R})$. However, to lower the regularity till $H^1(\mathbb{R})$, we will need to combine this transformation with the use of Bourgain's spaces (as it was already done in [4, 9, 23] for BO). More precisely, we need to work in a Besov version of Bourgain's spaces (introcuded in [29] in the context of waves maps). Indeed, on one hand we have to work in Bourgain' spaces of conormal regularity 1/2 to establish the main bilinear estimate (see Proposition 4.2 below). On the other hand, to control some remaining terms appearing in the transformation, we need the full Kato smoothing effect for functions that are localized in space frequencies (see Proposition 4.4). The rest of the proof follows closely the one in [23] for the Benjamin-Ono equation (see also [20]).

In the second part of this article, we investigate the limit behavior of the solutions v_ϵ to (1.1), obtained in Theorem 1.1, as ϵ tends to zero. First, it is interesting to observe that a direct argument based on compactness methods (see for example [22] in the case of the Benjamin-Ono-Burgers equation) does not seem to work. Indeed, the leading terms in the energy H, which is to say $a\epsilon(\partial_x v)^2$ and $bv\mathcal{H}v$, have opposite signs, so that (1.6) and (1.7) do not provide *a priori* bounds, uniformly in ϵ, on $\epsilon\|v_\epsilon\|^2_{H^1} + \|v_\epsilon\|^2_{H^{\frac{1}{2}}}$. Therefore, the problem of studying the limit of v_ϵ, as ϵ goes to zero, turns out to be far from trivial.

Nevertheless, we are able to prove the convergence of solutions of (1.1) toward a solution of the Benjamin-Ono equation in the special case where the ratio of the densities is equal to $\sqrt{3}$.

Theorem 1.2. *Assume that* $\frac{3ac}{4d} = b \quad \Leftrightarrow \quad \rho^2 = 3\rho_1^2$. *Let* $v_0 \in H^1(\mathbb{R})$ *and for any* $\varepsilon > 0$ *denote by* $S_\varepsilon(t)v_0 \in C(\mathbb{R}; H^1(\mathbb{R}))$ *the solution to* (1.1) *emanating from* v_0. *Then for any* $T > 0$ *it holds*

$$\|S_\varepsilon(t)v_0 - S(t)v_0\|_{L^\infty(0,T;H^1(\mathbb{R}))} \longrightarrow 0 \text{ as } \varepsilon \to 0 \tag{1.11}$$

where $S(t)v_0$ *is the solution to the Benjamin-Ono equation emanating from* v_0.

In the case where $\frac{\rho}{\rho_1} = \sqrt{3}$, the spatial primitive chosen to perform the gauge transformation for equation (1.1) corresponds to the one chosen for the Benjamin-Ono equation. Then, we can show that the Cauchy problem associated to (1.1) is uniformly in ϵ well-posed in $H^1(\mathbb{R})$, which will in a classical way (see for example [7]) lead to Theorem 1.2. The main difficulty here arises from the fact that the dispersive linear terms $\epsilon\partial_x^3$ and $\mathcal{H}\partial_x^2$ compete together as in the Benjamin equation (see the introduction in [2]). Therefore, we are only allowed to use the dispersive smoothing effects associated to (1.1) in some well behaved regions in spatial frequency and we need to refine the bilinear estimates obtained in the proof of Theorem 1.1 in the other regions.

It would be interesting to derive a class of higher-order equation for internal long waves from the first order one derived by Bona, Lannes and Saut in [3]. Among those equations, which would be formally equivalent to (1.1), one might find some with better behaved linear parts, which would avoid to deal with those technical difficulties.

Finally, we observe that the techniques introduced here would likely lead to similar results for the following intermediate long wave equation

$$\partial_t u - b\mathcal{F}_h\partial_x^2 u + (a_1\mathcal{F}_h^2 + a_2)\epsilon\partial_x^3 u = cu\partial_x u - d\epsilon\partial_x(u\mathcal{F}_h\partial_x u + \mathcal{F}_h(u\partial_x u)) \tag{1.12}$$

where $\mathcal{F}_h$ is the Fourier multiplier $-i\coth(h\xi)$, u is a real-valued solution, and a_1, a_2, b, c, d and h are positive constants, and which was also derived in [5]. Note that the same ill-posedness results as for equation (1.1) also hold for this equation (see [25]).

The paper is organized as follows: in the next section, we introduce the notations, define the functions spaces and recall some classical estimates. Sections 3 and 4 are devoted the key nonlinear estimates, which are used in Section 5 to prove Theorem 1.1. Finally, in Section 6, we prove Theorem 1.2.

2. Notations, function spaces and preliminary estimates

2.1. **Notation.** For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, α_+, respectively α_-, will denote a number slightly greater, respectively lesser, than α.

For $u = u(x,t) \in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}u = \widehat{u}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = (u)^{\wedge_x}$, respectively $\mathcal{F}_t u = (u)^{\wedge_t}$, will denote its Fourier transform in space, respectively in time. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J_x^s and D_x^s, by

$$J_x^s u = \mathcal{F}_x^{-1}\big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_x u\big) \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}\big(|\xi|^s\mathcal{F}_x u\big).$$

Throughout the paper, we fix a smooth cutoff function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \leq \eta \leq 1, \quad \eta_{|_{[-1,1]}} = 1 \quad \text{and} \quad \text{supp}(\eta) \subset [-2,2].$$

Then if A is a positive number, $P_{\lesssim A}$ denote the Fourier multiplier whose symbol is given by $\eta(\frac{\cdot}{cA})$ and $P_{\gtrsim A}$ is defined by $P_{\gtrsim A} = 1 - P_{\lesssim A}$. For $l \in \mathbb{Z}_+$, we define

$$\phi(\xi) := \eta(\xi) - \eta(2\xi), \quad \phi_{2^l}(\xi) := \phi(2^{-l}\xi),$$

and

$$\psi_{2^l}(\xi,\tau) = \phi_{2^l}(\tau - b|\xi|\xi + a\epsilon\xi^3).$$

By convention, we also denote

$$\phi_0(\xi) := \eta(2\xi), \quad \text{and} \quad \psi_0(\xi) := (\xi,\tau) = \phi_0(2(\tau - b|\xi|\xi + a\epsilon\xi^3)),$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic with N, L, K or $M \geq 0$, *i.e.*, these variables range over numbers of the form $\{2^n : n \in \mathbb{Z}_+\} \cup \{0\}$. Then, we have that

$$\sum_N \phi_N(\xi) = 1, \quad \text{supp}\,(\phi_N) \subset \{\frac{N}{2} \leq |\xi| \leq 2N\},\ N \geq 1, \quad \text{and} \quad \text{supp}\,(\phi_0) \subset \{|\xi| \leq 1\}.$$

Let us define the Littlewood-Paley multipliers by

$$P_N u = \mathcal{F}_x^{-1}\big(\phi_N\mathcal{F}_x u\big), \quad Q_L u = \mathcal{F}^{-1}\big(\psi_L\mathcal{F}u\big),$$

and $P_{\geq N} := \sum_{K\geq N} P_K$. Moreover, we also define the operators P_{hi}, P_{HI}, P_{lo} and P_{LO} by

$$P_{hi} = \sum_{N\geq 2} P_N, \quad P_{HI} = \sum_{N\geq 2^4} P_N, \quad P_{lo} = 1 - P_{hi}, \quad \text{and} \quad P_{LO} = 1 - P_{HI}.$$

Let P_+ and P_- denote the projection on respectively the positive and the negative Fourier frequencies. Then

$$P_\pm u = \mathcal{F}_x^{-1}\big(\chi_{\mathbb{R}_\pm}\mathcal{F}_x u\big),$$

and we also denote $P_{\pm hi} = P_\pm P_{hi}$, $P_{\pm HI} = P_\pm P_{HI}$, $P_{\pm lo} = P_\pm P_{lo}$, $P_{\pm LO} = P_\pm P_{LO}$ and $P_{\pm N} = P_\pm P_N$. Observe that P_{hi}, P_{HI}, P_{lo}, P_{LO}, P_N and $P_{\pm N}$ are bounded (uniformly in N) operators on $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$, while $P_\pm$ are only bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. We also note that

$$\mathcal{H} = -iP_+ + iP_-.$$

Finally, we denote by $V_\epsilon(t) = e^{t(b\mathcal{H}\partial_x^2 + a\epsilon\partial_x^3)}$ the free group associated with the linearized part of equation (1.1), which is to say,

$$\mathcal{F}_x\big(V_\epsilon(t)f\big)(\xi) = e^{it(b|\xi|\xi - a\epsilon\xi^3)}\mathcal{F}_x f(\xi). \tag{2.1}$$

2.2. **Function spaces.** For $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$, and for $s \in \mathbb{R}$, the real-valued Sobolev spaces $H^s(\mathbb{R})$ and $W^{s,p}(\mathbb{R})$ denote the spaces of all real-valued functions with the usual norms

$$\|\phi\|_{H^s} = \|J_x^s\phi\|_{L^2} \quad \text{and} \quad \|\phi\|_{W^{s,p}} = \|J_x^s\phi\|_{L^p}.$$

If $f = f(x,t)$ is a function defined for $x \in \mathbb{R}$ and t in the time interval $[0,T]$, with $T > 0$, if B is one of the spaces defined above, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, we will define the mixed space-time spaces $L_T^pB_x$, $L_t^pB_x$, $L_x^qL_T^p$ by the norms

$$\|f\|_{L_T^pB_x} = \Big(\int_0^T \|f(\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}} \quad \|f\|_{L_t^pB_x} = \Big(\int_{\mathbb{R}} \|f(\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}},$$

and

$$\|f\|_{L_x^qL_T^p} = \left(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^p dt\Big)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}.$$

Moreover, if $s \in \mathbb{R}$, $1 \leq q \leq \infty$ and X denotes one of the mixed space-time spaces defined above, we define its dyadic version $\mathcal{B}^{s,q}(X)$ as

$$\|f\|_{\mathcal{B}^{s,q}(X)} = \left(\sum_N \langle N\rangle^{sq}\|P_N f\|_X^q\right)^{\frac{1}{q}}.$$

In the special case $(s,q) = (0,2)$, the space $\mathcal{B}^{s,q}(X)$ will be simply denoted by $\widetilde{X}$.

For s, $b \in \mathbb{R}$, we introduce the Bourgain spaces $X_\epsilon^{s,b}$ related to the linear part of (1.1) as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X_\epsilon^{s,b}} := \left(\int_{\mathbb{R}^2} \langle\tau - b|\xi|\xi + a\epsilon\xi^3\rangle^{2b}\langle\xi\rangle^{2s}|\widehat{v}(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}, \tag{2.2}$$

where $\langle x\rangle := 1+|x|$. We will also use a dyadic version of those spaces introduced in [29] in the context of wave maps. For s, $b\in\mathbb{R}$, $1\le q\le\infty$, $X_\epsilon^{s,b,q}$ will denote the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X_\epsilon^{s,b,q}} := \left(\sum_N \Big(\sum_L \langle N\rangle^{sq}\langle L\rangle^{bq}\|P_N Q_L v\|_{L^2_{x,t}}^q\Big)^{\frac{2}{q}}\right)^{\frac12}. \tag{2.3}$$

Moreover, we define a localized (in time) version of these spaces. Let $T>0$ be a positive time and $Y=X_\epsilon^{s,b}$ or $Y=X_\epsilon^{s,b,q}$. Then, if $v:\mathbb{R}\times[0,T]\to\mathbb{C}$, we have that

$$\|v\|_{Y_T} := \inf\{\|\tilde v\|_Y \mid \tilde v:\mathbb{R}\times\mathbb{R}\to\mathbb{C},\ \tilde v|_{\mathbb{R}\times[0,T]}=v\}.$$

When $\epsilon=1$, we will denote $X^{s,b}=X_1^{s,b}$, $X_T^{s,b}=X_{1,T}^{s,b}$, $X^{s,b,q}=X_1^{s,b,q}$ and $X_T^{s,b,q}=X_{1,T}^{s,b,q}$.

Finally we list some useful properties of the Bourgain spaces defined above.

Proposition 2.1. *Fix $\delta>0$, $s\in\mathbb{R}$ and $\epsilon>0$. Then it holds that*

$$\|v\|_{X_\epsilon^{s,\frac12}} \lesssim \|v\|_{X_\epsilon^{s,\frac12,1}} \lesssim \|v\|_{X_\epsilon^{s,\frac12+\delta}}, \tag{2.4}$$

$$\|v\|_{L^\infty_t H^s_x} \lesssim \|\widehat{J^s_x v}\|_{L^2_\xi L^1_\tau} \lesssim \|v\|_{X_\epsilon^{s,\frac12,1}}, \tag{2.5}$$

and

$$\|f\|_{X_\epsilon^{s,-\frac12+\delta}} \lesssim \|f\|_{L^{1+\delta'}_t H^s_x}, \tag{2.6}$$

for $\delta'>0$ satisfying $1+\delta'=\frac{1}{1-\delta}$. In other words, the injections

$$X_\epsilon^{s,\frac12+\delta}\hookrightarrow X_\epsilon^{s,\frac12,1}\hookrightarrow X_\epsilon^{s,\frac12},\quad X_\epsilon^{s,\frac12,1}\hookrightarrow L^\infty_t H^s_x,$$

and

$$L^{1+\delta'}_t H^s_x \hookrightarrow X_\epsilon^{s,-\frac12+\delta}$$

are continuous.

2.3. **Linear estimates.** First, we recall some linear estimates in Bourgain's spaces which will be needed later (see for instance [29]).

Lemma 2.2 (Homogeneous linear estimate)**.** *Let $s\in\mathbb{R}$ and $\epsilon>0$. Then*

$$\|\eta(t)V_\epsilon(t)\phi\|_{X_\epsilon^{s,\frac12,1}} \lesssim \|\phi\|_{H^s}. \tag{2.7}$$

Lemma 2.3 (Non-homogeneous linear estimate)**.** *Let $s\in\mathbb{R}$ and $\epsilon>0$. Then, it holds that*

$$\Big\|\eta(t)\int_0^t V_\epsilon(t-t')g(t')dt'\Big\|_{X_\epsilon^{s,\frac12,1}} \lesssim \|g\|_{X_\epsilon^{s,-\frac12,1}}. \tag{2.8}$$

Next, we derive local and global smoothing effects associated to the group $\{V_\epsilon(t)\}$, for the KdV scaling, in the context of Bourgain's spaces. We begin with the Strichartz estimates.

Lemma 2.4. *For all $0<\epsilon<1$, $T>0$ and $0\le\theta\le1$, we have that*

$$\|v\|_{L^{p_\theta}_{x,t}} \lesssim \|v\|_{\widetilde{L^{p_\theta}_{x,t}}} \lesssim \epsilon^{-\frac{\theta}{8}}\|v\|_{X_\epsilon^{0,\frac{\theta}{2}+}}, \tag{2.9}$$

and

$$\|v\|_{L^{p_\theta}_{x,T}} \lesssim \epsilon^{-\frac{\theta}{8}} \|v\|_{X^{0,\frac{\theta}{2}+}_{\epsilon,T}}, \tag{2.10}$$

where $\frac{1}{p_\theta} = \frac{\theta}{8} + \frac{1-\theta}{2}$.

Proof. First, we observe, arguing as in Lemma 2.1 in [18], that w is a solution to the linear equation

$$\partial_t w - a\epsilon \partial_x^3 w \pm ib\partial_x^2 w = 0, \tag{2.11}$$

if and only if

$$u(x,t) = e^{\pm i \frac{2b^3}{27a^2\epsilon^2} t} e^{\mp i \frac{b}{3a\epsilon} x} w(x - \frac{b^2 t}{3a\epsilon}, t) \tag{2.12}$$

is a solution to

$$\partial_t u - a\epsilon \partial_x^3 u = 0. \tag{2.13}$$

Let us denote by $\{W_\epsilon^\pm(t)\}$ and $\{U_\epsilon(t)\}$ the groups associated to (2.11) and (2.13). Since $U_\epsilon(t) = U_1(\epsilon t)$, we deduce from the classical Strichartz estimate for the KdV equation (cf. for example [19], chapter 4) that

$$\|U_\epsilon(t)\phi\|_{L^8_{x,t}} \lesssim \epsilon^{-\frac{1}{8}} \|\phi\|_{L^2}. \tag{2.14}$$

Then, it follows gathering (2.11)–(2.14) with the identity

$$V_\epsilon(t) = W_\epsilon^+(t)P_+ + W_\epsilon^-(t)P_-, \tag{2.15}$$

that

$$\|V_\epsilon(t)\phi\|_{L^8_{x,t}} \lesssim \epsilon^{-\frac{1}{8}} \|\phi\|_{L^2}. \tag{2.16}$$

Next, we use Lemma 3.3 in [6] to rewrite estimate (2.16) in the context of Bourgain's spaces. We get that

$$\|v\|_{L^8_{x,t}} \lesssim \epsilon^{-\frac{1}{8}} \|v\|_{X^{0,\frac{1}{2}+}}. \tag{2.17}$$

Therefore, we deduce by using Stein's theorem to interpolate estimate (2.17) with Plancherel's identity $\|v\|_{L^2_{x,t}} = \|v\|_{X^{0,0}_\epsilon}$, that

$$\|v\|_{L^{p_\theta}_{x,t}} \lesssim \epsilon^{-\frac{\theta}{8}} \|v\|_{X^{0,\frac{\theta}{2}+}_\epsilon}. \tag{2.18}$$

Finally, estimate (2.9) follows directly by applying estimate (2.18) to each dyadic block of $\|v\|_{\widetilde{L^{p_\theta}_{x,t}}}$. □

Next, we turn to the local Kato type smoothing effect.

Lemma 2.5. *Let* $0 < \epsilon \leq 1$ *and* $T > 0$ *and* $N \gtrsim \frac{1}{\epsilon}$. *Then, it holds that*

$$\|\partial_x P_N v\|_{L^\infty_x L^2_t} \lesssim \epsilon^{-\frac{1}{2}} \|P_N v\|_{X^{0,\frac{1}{2},1}_\epsilon}, \tag{2.19}$$

and

$$\|\partial_x v\|_{\widetilde{L^\infty_x L^2_T}} \lesssim T^{\frac{1}{2}} \epsilon^{-\frac{3}{2}+} \|v\|_{X^{0,\frac{1}{2},1}_\epsilon}. \tag{2.20}$$

Proof. Since $N \gtrsim \frac{1}{\epsilon}$, we obtain applying estimate (4.3) in Theorem 4.1 of [13] that

$$\begin{aligned}\|\partial_x V_\epsilon(t)P_N v_0\|_{L^\infty_x L^2_t} &\lesssim \Big(\int_{|\xi|\gtrsim\frac{1}{\epsilon}} \frac{|\xi|^2}{|2b\xi-3a\epsilon\xi^2|}|\big(P_N v_0\big)^\wedge(\xi)|^2 d\xi\Big)^{\frac12}\\ &\lesssim \epsilon^{-\frac12}\|P_N v_0\|_{L^2_x}.\end{aligned} \tag{2.21}$$

Moreover, by applying the Fourier inverse formula, it follows that

$$\partial_x P_N v(x,t) = \int_{\mathbb{R}} \partial_x V_\epsilon(t)\big(V_\epsilon(-\cdot)P_N v\big)^{\wedge_t}(x,\tau)e^{it\tau}d\tau.$$

Therefore, Minkowski's inequality, estimate (2.21), Plancherel's identity and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}\|\partial_x P_N v\|_{L^\infty_x L^2_t} &\lesssim \int_{\mathbb{R}} \big\|\big(V_\epsilon(-\cdot)P_N v\big)^\wedge(\cdot,\tau)\big\|_{L^2_\xi}d\tau\\ &\lesssim \sum_L \langle L\rangle^{\frac12}\big\|\phi_L(\tau)\big(V_\epsilon(-\cdot)P_N v\big)^\wedge\big\|_{L^2_{\xi,\tau}},\end{aligned} \tag{2.22}$$

which leads to estimate (2.19) since

$$\big(V_\epsilon(-\cdot)P_N v\big)^\wedge(\xi,\tau) = \big(P_N v\big)^\wedge(\xi,\tau+b|\xi|\xi-a\epsilon\xi^3).$$

On the other hand, if $N \lesssim \frac{1}{\epsilon}$, we deduce from the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, whenever $s>\frac12$, that

$$\begin{aligned}\|\partial_x V_\epsilon(t)P_N v_0\|_{L^\infty_x L^2_T} &\lesssim T^{\frac12}\|\partial_x V_\epsilon(t)P_N v_0\|_{L^\infty_{x,T}}\\ &\lesssim T^{\frac12}\|\partial_x V_\epsilon(t)P_N v_0\|_{L^\infty_T H^s_x}\\ &\lesssim T^{\frac12}\epsilon^{-1}(1+\epsilon^{-s})\|P_N v_0\|_{L^2_x}.\end{aligned}$$

Therefore, we deduce arguing as above that

$$\|\partial_x P_N v\|_{L^\infty_x L^2_T} \lesssim T^{\frac12}\epsilon^{-1}(1+\epsilon^{-s})\|P_N v\|_{X^{0,\frac12,1}_{\epsilon,T}}, \tag{2.23}$$

whenever $N \lesssim \frac{1}{\epsilon}$.

Estimate (2.20) follows gathering estimates (2.19) and (2.23) and by squaring and summing over N. □

Finally, we derive the maximal function estimate.

Lemma 2.6. *Let $s>\frac34$, $0<\epsilon\le 1$, and $T>0$ be such that $0<\epsilon T\le 1$. Then, we have that*

$$\|v\|_{\widetilde{L^2_x L^\infty_T}} \lesssim \epsilon^{-s}\|v\|_{X^{s,\frac12,1}_{\epsilon,T}}. \tag{2.24}$$

Proof. The L^2_x-maximal function for the KdV group $\{U_1(t)\}$ derived in Theorem 2.7 of [15] implies that

$$\Big(\int_{\mathbb{R}}\sup_{|t|\le 1}|U_1(t)u_0(x)|^2dx\Big)^{\frac12} \lesssim \|u_0\|_{H^s}, \tag{2.25}$$

if $s>\frac34$. Then, a scaling argument and estimate (2.25) yield

$$\begin{aligned}\|U_\epsilon(t)u_0\|_{L^2_x L^\infty_T} &= \|U_1(\epsilon t)u_0\|_{L^2_x L^\infty_T}\\ &= \Big(\int_{\mathbb{R}}\sup_{|s|\le\epsilon T}|U_1(s)u_0(x)|^2dx\Big)^{\frac12} \lesssim \|u_0\|^2_{H^s},\end{aligned} \tag{2.26}$$

since $s > \frac{3}{4}$ and $0 < \epsilon T \leq 1$.

Thus, if w and u are the solutions associated to (2.11) and (2.13) with respective initial data w_0 and u_0, it follows from (2.12) and (2.26) that

$$\|w\|_{L^2_x L^\infty_T} = \|u\|_{L^2_x L^\infty_T} \lesssim \|u_0\|_{H^s_x} \lesssim \epsilon^{-s}\|w_0\|_{H^s_x}. \tag{2.27}$$

Therefore, we conclude gathering (2.15) and (2.27) that

$$\|V_\epsilon(t)v_0\|_{L^2_x L^\infty_T} \lesssim \epsilon^{-s}\|v_0\|_{H^s_x}, \tag{2.28}$$

whenever $s > \frac{3}{4}$ and ϵ, T satisfying $0 < \epsilon T \leq 1$. This implies estimate arguing as in (2.22) that

$$\|P_N v\|_{L^2_x L^\infty_T} \lesssim \epsilon^{-s}\langle N\rangle^s \|P_N v\|_{X^{0,\frac{1}{2},1}_{\epsilon,T}}.$$

for any $N \geq 0$ and $s > \frac{3}{4}$, which leads to (2.24) by squaring and summing over N. □

2.4. **Fractional Leibniz's rules.** First we state the classical fractional Leibniz rule estimate derived by Kenig, Ponce and Vega (See Theorems A.8 and A.12 in [15]).

Proposition 2.7. *Let $0 < \alpha < 1$, p, p_1, $p_2 \in (1, +\infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and α_1, $\alpha_2 \in [0, \alpha]$ with $\alpha = \alpha_1 + \alpha_2$. Then,*

$$\left\|D^\alpha_x(fg) - fD^\alpha_x g - gD^\alpha_x f\right\|_{L^p} \lesssim \|D^{\alpha_1}_x g\|_{L^{p_1}}\|D^{\alpha_2}_x f\|_{L^{p_2}}. \tag{2.29}$$

Moreover, for $\alpha_1 = 0$, the value $p_1 = +\infty$ is allowed.

The next estimate is a frequency localized version of estimate (2.29), proved in [20], in the same spirit as Lemma 3.2 in [27].

Lemma 2.8. *Let $\alpha \geq 0$ and $1 < q < \infty$. Then,*

$$\left\|D^\alpha_x P_+\big(fP_-\partial_x g\big)\right\|_{L^q} \lesssim \|D^{\alpha_1}_x f\|_{L^{q_1}}\|D^{\alpha_2}_x g\|_{L^{q_2}}, \tag{2.30}$$

with $1 < q_i < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ and $\alpha_1 \geq \alpha$, $\alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1 + \alpha$.

We also state an estimate to handle the multiplication by a term on the form $e^{\pm\frac{i}{2}F}$, where F is a real-valued function, in fractional Sobolev spaces.

Lemma 2.9. *Let $2 \leq q < \infty$ and $1 \leq s \leq \frac{3}{2}$. Consider F and F_1 two real-valued functions such that $v = \partial_x F$ and $v_1 = \partial_x F_1$ belong to $L^2(\mathbb{R})$. Then, it holds that*

$$\|J^s_x\big(e^{\pm iF}g\big)\|_{L^q} \lesssim (1 + \|v\|^2_{H^1})\|J^s_x g\|_{L^q}. \tag{2.31}$$

Remark 2.10. The proof follows the lines of Lemma 2.7 in [23] (see also [20, 21]). A version of Lemma 2.9 could also be stated for $s > \frac{3}{2}$.

3. The gauge transformation

The gauge transform we will use is the one introduced by Tao in [27]. First we define an antiderivative $F = F[v]$ of v. We determine F on the time axis $x = 0$ by solving the ODE

$$\begin{cases} \partial_t F(0,t) = \big(b\mathcal{H}v_x + a\epsilon v_{xx} + \frac{c}{2}A^{-1}v^2 - \frac{3}{2}a\epsilon(v\mathcal{H}v_x + \mathcal{H}(vv_x))\big)(0,t), \\ F(0,0) = 0, \end{cases}$$

Then we extend F on the whole plan by setting

$$\partial_x F = Av, \quad \text{where} \quad A = \frac{2d}{3a}.$$

Clearly, it holds

$$\partial_x\big(\partial_t F - b\mathcal{H}\partial_x^2 F - a\epsilon\partial_x^3 F\big) = \partial_x\big(\frac{c}{2}A^{-1}F_x^2 - \frac{3}{2}a\epsilon(F_x\mathcal{H}F_{xx} + \mathcal{H}(F_xF_{xx}))\big).$$

and, according to the choice of F on the time axis, it satisfies the equation

$$\partial_t F - b\mathcal{H}\partial_x^2 F - a\epsilon\partial_x^3 F = \frac{c}{2}A^{-1}F_x^2 - \frac{3}{2}a\epsilon(F_x\mathcal{H}F_{xx} + \mathcal{H}(F_xF_{xx})). \tag{3.1}$$

Now, we perform the following nonlinear transformation

$$W = P_{+hi}(e^{iF}) \quad \text{and} \quad w = W_x = iAP_{+hi}(e^{iF}v). \tag{3.2}$$

First, using the identity $\mathcal{H}P_+ = -iP_+$, we compute

$$\begin{aligned}&\partial_t W + ib\partial_x^2 W - a\epsilon\partial_x^3 W\\ &\quad = iP_{+hi}\big(e^{iF}(\partial_t F + ib\partial_x^2 F - a\epsilon\partial_x^3 F - bF_x^2 - 3ia\epsilon F_xF_{xx} - a\epsilon F_x^3)\big).\end{aligned}$$

Then using (3.1) and the identity $\mathcal{H} + i = 2iP_-$ it follows that

$$\begin{aligned}&\partial_t W + ib\partial_x^2 W - a\epsilon\partial_x^3 W\\ &\quad = P_{+hi}\big(e^{iF}(i(\frac{c}{2}A^{-1} - b)F_x^2 - ia\epsilon F_x^3 - 2bP_-F_{xx} + 3a\epsilon F_xP_-F_{xx} + 3a\epsilon P_-(F_xF_{xx}))\big)\\ &\quad = P_{+hi}(e^{iF}\big(\alpha_1 v^2 + \alpha_2\epsilon v^3)\big) + \alpha_3 P_{+hi}(WP_-v_x) + \alpha_3 P_{+hi}(P_{lo}(e^{iF})P_-v_x)\\ &\qquad + \alpha_4\epsilon P_{+hi}(wP_-v_x) + \alpha_5\epsilon P_{+hi}(P_{lo}(e^{iF}v)P_-v_x) + \alpha_6\epsilon P_{+hi}(WP_-(vv_x))\\ &\qquad + \alpha_6\epsilon P_{+hi}(P_{lo}(e^{iF})P_-(vv_x)),\end{aligned}$$

where $\alpha_j,\ j = 1\cdots 6$ are complex constants depending on a, b, c and d.

Remark 3.1. We observe from the definition of the coefficients a, b, c and d in (1.3) and (1.4) that

$$\alpha_1 = 0 \quad \Leftrightarrow \quad \frac{3ac}{4d} = b \quad \Leftrightarrow \quad \rho^2 = 3\rho_1^2. \tag{3.3}$$

In the following, we will fix $\alpha_1 = \cdots \alpha_6 = 1$ for sake of simplicity. Therefore, we deduce by differentiating the above equation that w is a solution to

$$\begin{aligned}\partial_t w + ib\partial_x^2 w - a\epsilon\partial_x^3 w &= \partial_x P_{+hi}(e^{iF}(v^2 + \epsilon v^3))\\ &\quad + \partial_x P_{+hi}(WP_-v_x) + \partial_x P_{+hi}(P_{lo}(e^{iF})P_-v_x)\\ &\quad + \epsilon\partial_x P_{+hi}(wP_-v_x) + \epsilon\partial_x P_{+hi}(P_{lo}(e^{iF}v)P_-v_x)\\ &\quad + \epsilon\partial_x P_{+hi}(WP_-(vv_x)) + \epsilon\partial_x P_{+hi}(P_{lo}(e^{iF})P_-(vv_x))\\ &\quad := N(e^{iF}, v, W, w).\end{aligned} \tag{3.4}$$

On the other hand, we can recover v as a function of w by writing

$$iAv = e^{-iF}\partial_x(e^{iF}) = e^{-iF}w + e^{-iF}\partial_x P_{lo}(e^{iF}) + e^{-iF}\partial_x P_{-hi}(e^{iF}), \tag{3.5}$$

so that it follows from the frequency localization

$$\begin{aligned}iAP_{+HI}v &= P_{+HI}(e^{-iF}w) + P_{+HI}(P_{+hi}e^{-iF}\partial_x P_{lo}(e^{iF}))\\ &\quad + P_{+HI}(P_{+HI}e^{-iF}\partial_x P_{-hi}(e^{iF})).\end{aligned} \tag{3.6}$$

Then, we have the following *a priori* estimates on v in terms of w.

Proposition 3.2. *Let $s \geq 1$, $0 < T \leq 1$, $0 \leq \theta \leq 1$, $0 < \epsilon \leq 1$ and v be a solution to (1.1) in the time interval $[0,T]$. Then, it holds that*

$$\|v\|_{X^{s-2\theta,\theta}_{\epsilon,T}} \lesssim \|v\|_{L^\infty_T H^s_x} + \|v\|^2_{L^\infty_T H^s_x} + \epsilon \|J^s_x v\|^2_{L^4_{T,x}}. \tag{3.7}$$

Moreover, if $1 \leq s \leq \frac{3}{2}$, it holds that

$$\|J^s_x v\|_{L^\infty_T L^2_x} \lesssim \|v_0\|_{H^1} + \big(1 + \|v\|^2_{L^\infty_T H^1_x}\big)\big(\|w\|_{X^{s,\frac{1}{2},1}_{\epsilon,T}} + \|v\|^2_{L^\infty_T H^1_x}\big), \tag{3.8}$$

$$\|J^s_x v\|_{L^4_{x,t}} \lesssim \|v_0\|_{H^1} + \big(1 + \|v\|^2_{L^\infty_T H^1_x}\big)\big(\epsilon^{-\frac{1}{12}}\|w\|_{X^{s,\frac{1}{3}+}_{\epsilon,T}} + \|v\|^2_{L^\infty_T H^1_x}\big), \tag{3.9}$$

$$\|v\|_{L^2_x L^\infty_T} \lesssim \varepsilon^{-1}\Big(\|v_0\|_{H^1} + \|v\|_{L^\infty_T H^1_x}(\|w\|_{X^{1,1/2,1}_{\epsilon,T}} + \|v\|_{L^\infty_T H^1_x} + \|v\|_{L^2_x L^\infty_T})\Big), \tag{3.10}$$

and

$$\begin{aligned}\|J^s_x \partial_x v\|_{\widetilde{L^\infty_x L^2_T}} \lesssim \Big(&\epsilon^{-\frac{3}{2}+}\|v_0\|_{H^s} + \epsilon^{-\frac{3}{2}+}\|w\|_{X^{s,\frac{1}{2},1}_{\epsilon,T}} \\ &+ \|v\|_{L^\infty_T H^1_x}\big(\|w\|_{X^{1,\frac{1}{2},1}_{\epsilon,T}} + \|J^s_x \partial_x v\|_{\widetilde{L^\infty_x L^2_T}} + \|v\|_{L^\infty_T H^1_x}\big)\Big).\end{aligned} \tag{3.11}$$

Remark 3.3. It is worth noticing that estimates (3.8) and (3.9) could be rewritten in a convenient form for $s > \frac{3}{2}$.

Proof. (3.8) and (3.9) follow from (3.6) for the high frequencies and from (1.1) for the low frequencies (see for instance [23]). To prove (3.7) we proceed as in [23], noticing that according to (1.1),

$$\|\partial_t(V_\varepsilon(-t)u(t))\|_{L^2_T H^{s-2}_x} \lesssim \|J^s_x u\|^2_{L^4_{Tx}}.$$

To prove estimate (3.10), we also split v between its high and low Fourier modes

$$\|v\|_{\widetilde{L^2_x L^\infty_T}} \lesssim \|P_{LO} v\|_{\widetilde{L^2_x L^\infty_T}} + \|P_{HI} v\|_{\widetilde{L^2_x L^\infty_T}}. \tag{3.12}$$

The low frequency term on the right-hand side of (3.12) can be treated by using (1.1) and the maximal function estimate (2.28) to get

$$\|P_{LO} v\|_{\widetilde{L^2_x L^\infty_T}} \lesssim \epsilon^{-(\frac{3}{4}+)}\left(\|v_0\|_{H^1} + \|v\|^2_{L^\infty_T H^1_x}\right). \tag{3.13}$$

To treat the high frequency term we use that v is real-valued to first notice that

$$\|P_{HI} v\|_{L^2_x L^\infty_T} \lesssim 2\|P_{+HI} v\|_{L^2_x L^\infty_T}$$

so that we are reduced to estimate each terms on the right-hand side of (3.6). Now the problem is that P_{+HI} is not continuous in $L^2_x L^\infty_T$. We will overcome this difficulty by noticing that $P_{+HI} = \sum_{k \geq 4} P_{+2^k}$ and that the family of operators P_{+2k} is bounded in $L^2_x L^\infty_T$. To treat the first term of the right-hand side of (3.6) we first notice that for $k \geq 4$,

$$P_{+2^k}(e^{-iF} w) = P_{+2^k}\Big(\sum_{j \geq k-3} P_{2^j} w P_{\leq 2^{j-1}}(e^{-iF})\Big) + P_{+2^k}\Big(\sum_{j \geq k-3} P_{\leq 2^j} w P_{2^j}(e^{-iF})\Big) \tag{3.14}$$

so that

$$\begin{aligned}\|P_{+HI}(e^{-iF}w)\|_{L^2_xL^\infty_T} &\lesssim \sum_{k\geq 4}\|P_{+2^k}(e^{-iF}w)\|_{L^2_xL^\infty_T}\\ &\lesssim \sum_{k\geq 4}\Big[\sum_{j\geq k-3}\|P_{2^j}w\|_{L^2_xL^\infty_T}\\ &\quad+\sum_{j\geq k-3}\|P_{2^j}(e^{-iF})\|_{L^\infty_{T,x}}\|P_{\leq 2^j}w\|_{L^2_xL^\infty_T}\Big].\end{aligned}$$

But on one hand, for $s\geq 1$ we deduce from (2.24) and Bernstein inequalities that for $\alpha\in]0,1/4[$,

$$\begin{aligned}\sum_{k\geq 4}\sum_{j\geq k-3}\|P_{+2^j}w\|_{L^2_xL^\infty_T} &\lesssim \sum_{k\geq 4}\sum_{j\geq k-3}2^{-\alpha j}\|D^\alpha_xP_{+2^j}w\|_{L^2_xL^\infty_T}\\ &\lesssim \sup_{k\geq 3}\|D^\alpha_xP_{2^k}w\|_{L^2_xL^\infty_T}\lesssim \varepsilon^{-1}\|w\|_{X^{1,1/2,1}_T}\end{aligned}$$

and on the other hand,

$$\begin{aligned}\sum_{k\geq 4}\sum_{j\geq k-3}\|P_{2^j}(e^{-iF})\|_{L^\infty_{Tx}}\|P_{\leq 2^j}w\|_{L^2_xL^\infty_T} &\lesssim \sum_{k\geq 4}\sum_{j\geq k-3}2^{-j}\|v\|_{L^\infty_{Tx}}\|P_{\leq 2^j}w\|_{L^2_xL^\infty_T}\\ &\lesssim \|v\|_{L^\infty_TH^1_x}\sup_{k\geq 1}\|P_{2^k}w\|_{L^2_xL^\infty_T}\\ &\lesssim \varepsilon^{-1}\|w\|_{X^{1,1/2,1}_T}\|v\|_{L^\infty_TH^1_x},\end{aligned}$$

which completes the estimate of the term $P_{+HI}(e^{-iF}w)$. To treat the second term and third terms of the right-hand side of (3.6) we proceed as above, using the frequency localization due to the projections, to obtain

$$\begin{aligned}\big\|P_{+HI}\big(e^{-iF}P_{-hi}(e^{iF}v)\big)\big\|_{L^2_xL^\infty_T} &\lesssim \sum_{N\geq 2^4}\sum_{K\geq N/4}\|P_Ke^{iF}\|_{L^\infty_{x,T}}\|P_{\leq K}P_{-hi}(e^{iF}v)\|_{L^2_xL^\infty_T}\Big)\\ &\lesssim \sum_{N\geq 2^4}\Big(\sum_{K\geq N/4}K^{-1}\|v\|_{L^\infty_{x,T}}\sum_{2\leq Q\leq K}\|P_{+Q}v\|_{L^2_xL^\infty_T}\Big)\\ &\lesssim \|v\|_{L^\infty_TH^1_x}\|v\|_{L^2_xL^\infty_T}.\end{aligned}\tag{3.15}$$

which completes the proof of (3.10).

Finally, to prove (3.11) we proceed similarly. First we use (1.1) and Sobolev inequality to get

$$\|P_{LO}v\|_{\widetilde{L^\infty_xL^2_T}}\lesssim\Big(\|v_0\|_{H^1}+\|v\|^2_{L^\infty_TH^1_x}\Big).$$

Second, from (3.14) we deduce that for any integer $k\geq 4$,

$$\begin{aligned}&\|J^s_x\partial_xP_{+2^k}(e^{-iF}w)\|_{L^\infty_xL^2_T}\\ &\quad\lesssim 2^{k(s+1)}\Big(\sum_{j\geq k-3}\|P_{2^j}w\|_{L^\infty_xL^2_T}+\sum_{j\geq k-3}\|P_{\leq 2^j}w\|_{L^\infty_{T,x}}\|P_{2^j}e^{-iF}\|_{L^\infty_xL^2_T}\Big)\\ &\quad\lesssim \sum_{j\geq k-3}2^{(k-j)(s+1)}\|J^s_x\partial_xP_{2^j}w\|_{L^\infty_xL^2_T}+\|w\|_{L^2_TH^1_x}\sum_{j\geq k-3}2^{-j}\|J^s_x\partial_xP_{2^j}v\|_{L^\infty_xL^2_T}.\end{aligned}$$

Therefore, noticing that the first term on the right-hand side of the above inequality is a discrete convolutions between $\{2^{q(s+1)}\}_{q\leq 1}\in l^1(\mathbb{Z})$ and $\{\|J^s_x\partial_xP_{2^j}w\|_{L^\infty_xL^2_T}\}_{j\in\mathbb{Z}_+}$,

we deduce from Young's inequality that

$$\begin{aligned}\|J_x^s\partial_x P_{+HI}(e^{-iF}w)\|_{\widetilde{L_x^\infty L_T^2}} &\lesssim \|J_x^s\partial_x w\|_{\widetilde{L_x^\infty L_T^2}} + \|w\|_{L_T^2H_x^1}\sup_{j\geq 3}\|J_x^s\partial_x v\|_{L_x^\infty L_T^2}\\ &\lesssim \epsilon^{-\frac{3}{2}+}\|w\|_{X_{\epsilon,T}^{s,1/2,1}} + \|w\|_{X_{\epsilon,T}^{1,1/2,1}}\|J_x^s\partial_x v\|_{\widetilde{L_x^\infty L_T^2}},\end{aligned}$$

where we made use of (2.20) in the last step. Third, we proceed similarly to estimate the second and third term of the right-hand side of (3.6) by

$$\begin{aligned}&\big\|J_x^s\partial_x P_{+HI}\big(e^{-iF}P_{lo}(e^{iF}v)\big)\big\|_{\widetilde{L_x^\infty L_T^2}} + \big\|J_x^s\partial_x P_{+HI}\big(e^{-iF}P_{-hi}(e^{iF}v)\big)\big\|_{\widetilde{L_x^\infty L_T^2}}\\ &\qquad\lesssim \|v\|_{L_T^\infty H_x^1}\sup_{j\geq 3}\|J_x^s\partial_x P_{2^j}v\|_{L_x^\infty L_T^2}.\end{aligned}$$

This completes the proof of (3.11) and of the proposition. □

4. Bilinear estimates

In this section, we fix $\epsilon = 1$. The aim of this section is to derive an estimate on $\|w\|_{X_T^{s,\frac{1}{2},1}}$.

Proposition 4.1. *Let $0 < T \leq 1$, $1 \leq s \leq \frac{3}{2}$, v be a solution to (1.1) on the time interval $[0,T]$ and w defined in (3.2). Then it holds that*

(4.1)
$$\begin{aligned}&\|w\|_{X_T^{s,\frac{1}{2},1}} \lesssim \big(1+\|v_0\|_{H^1}^2\big)\|v_0\|_{H^s}\\ &+ p\big(\|w\|_{X^{s,\frac{1}{2},1}}, \|v\|_{L_T^\infty H_x^1}, \|v\|_{L_T^4W_x^{1,4}}, \|v\|_{L_x^2L_T^\infty}, \|J_x^s\partial_x v\|_{\widetilde{L_x^\infty L_T^2}}, \sup_{0\leq\theta\leq 1}\|v\|_{X_T^{1-2\theta,\theta}}\big),\end{aligned}$$

where p is a polynomial function at least quadratic in its arguments.

The main tools to prove Proposition 4.1 are the following crucial bilinear estimates.

Proposition 4.2. *For any $s \geq 1$, we have that*

$$\|\partial_x P_{+hi}\big(wP_-\partial_x v\big)\|_{X^{s,-\frac{1}{2},1}} \lesssim \|w\|_{X^{s,\frac{1}{2},1}}\sup_{0\leq\theta\leq 1}\|v\|_{X^{1-2\theta,\theta}}. \tag{4.2}$$

Proof. We only prove estimate (4.2) in the case $s = 1$, since the case $s > 1$ follows by similar arguments due to the frequency localization on the functions w and v. By duality, estimate (4.2) is equivalent to

$$\big|I\big| \lesssim \Big(\sum_N \sup_L \|h_{N,L}\|_{L_{\xi,\tau}^2}^2\Big)^{\frac{1}{2}}\|w\|_{X^{1,\frac{1}{2},1}}\sup_{0\leq\theta\leq 1}\|v\|_{X^{1-2\theta,\theta}}, \tag{4.3}$$

where

$$I = \sum_{N,L}\langle N\rangle\langle L\rangle^{-\frac{1}{2}}\int_{\mathcal{D}}\xi h_{N,L}(\xi,\tau)\phi_N(\xi)\psi_L(\xi,\tau)\widehat{w}(\xi_1,\tau_1)\xi_2\widehat{v}(\xi_2,\tau_2)d\nu, \tag{4.4}$$

$$d\nu = d\xi d\xi_1 d\tau d\tau_1, \quad \xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \tag{4.5}$$

$$\sigma = \tau - |\xi|\xi + \xi^3, \quad \sigma_i = \tau_i - \xi_i|\xi_i| + \xi_i^3,\ i = 1,2, \tag{4.6}$$

and

$$\mathcal{D} = \big\{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 \mid \xi \geq 1,\ \xi_1 \geq 1 \text{ and } \xi_2 \leq 0\big\}. \tag{4.7}$$

Observe that we always have in $\mathcal{D}$ that

$$\xi_1 \geq \xi \geq 1 \quad \text{and} \quad \xi_1 \geq |\xi_2|. \tag{4.8}$$

Then, we obtain by performing dyadic decompositions in ξ_1, ξ_2, σ_1 and σ_2 that

$$\begin{aligned} I = \sum_{N,N,N_2} \sum_{L,L_1,L_2} \langle N\rangle \langle L\rangle^{-\frac{1}{2}} \int_{\mathcal{D}} \xi h_{N,L}(\xi,\tau)\phi_N(\xi)\psi_L(\xi,\tau) \\ \times (P_{N_1}Q_{L_1}w)^\wedge(\xi_1,\tau_1)\xi_2(P_{N_2}Q_{L_2}v)^\wedge(\xi_2,\tau_2)d\nu. \end{aligned} \tag{4.9}$$

Due to the second identity in (4.5) and (4.8), we can always assume that one of the following cases holds:

(1) high-low interaction: $N_1 \sim N$ and $N_2 \leq N_1$
(2) high-high interaction: $N_1 \sim N_2$ and $N \leq N_1$.

Moreover, the resonance identity

$$\sigma - \sigma_1 - \sigma_2 = 3\xi\xi_2(\xi_1 - \frac{2}{3}) \tag{4.10}$$

holds in $\mathcal{D}$, so that for fixed N, N_1 and N_2, we can always assume that

$$L_{max} \sim \max\{L_{med}, NN_1N_2\}, \tag{4.11}$$

where L_{max}, L_{med} and L_{min} denote respectively the maximum, median and minimum of L, L_1 and L_2.

To estimate I, we will divide the sum in (4.9) depending on the high-low or high-high interactions regime and on wether $L_{\max} = L,\ L_1$ or L_2.

Case high-low interaction and $L_{\max} = L_2$. In this case, we can estimate I as

$$\begin{aligned} |I| \lesssim \sum_{N_1,N_2\leq N_1} \sum_{L_2,L\leq L_2} \frac{\langle N_1\rangle\langle L\rangle^{-\delta}}{\langle L_2\rangle} \int_{\mathcal{D}} \frac{|\xi\xi_2|\langle\xi_2\rangle}{\langle\xi_1\rangle} \frac{|h_{N_1,L}(\xi,\tau)|}{\langle\sigma\rangle^{\frac{1}{2}-\delta}} \phi_{N_1}(\xi)\psi_L(\xi,\tau) \\ \times |(P_{N_1}J_x^1 w)^\wedge(\xi_1,\tau_1)| \frac{\langle\sigma_2\rangle}{\langle\xi_2\rangle} |(P_{N_2}Q_{L_2}v)^\wedge(\xi_2,\tau_2)| d\nu. \end{aligned}$$

Since $L_2 = L_{max}$, we deduce from (4.10) that $L_2 \geq N_1^2N_2$. Therefore, $L_2 \sim 2^kN_1^2N_2$ for $k \in \mathbb{Z}_+$, so that, we obtain by using Plancherel's identity and Hölder's inequality

$$\begin{aligned} |I| &\lesssim \sum_{N_1,N_2\leq N_1} \frac{N_2}{N_1} \sup_L \Big\| \Big(\frac{|h_{N_1,L}|}{\langle\sigma\rangle^{\frac{1}{2}-\delta}}\Big)^\vee \Big\|_{L^4_{x,t}} \Big\| \big(|(P_{N_1}J_x^1 w)^\wedge|\big)^\vee \Big\|_{L^4_{x,t}} \\ &\qquad \times \sum_{k\in\mathbb{Z}_+} 2^{-k} \Big\| \frac{\langle\sigma\rangle}{\langle\xi\rangle} (P_{N_2}Q_{2^kN_1^2N_2}v)^\wedge \Big\|_{L^2_{\xi,\tau}}. \\ &\lesssim \sum_{N_1} \sup_L \|h_{N_1,L}\|_{L^2_{\xi,\tau}} \Big\| \big(|(P_{N_1}J_x^1 w)^\wedge|\big)^\vee \Big\|_{L^4_{x,t}} \|v\|_{X^{-1,1}} \\ &\lesssim \Big(\sum_{N_1} \sup_L \|h_{N_1,L}\|^2_{L^2_{\xi,\tau}}\Big)^{\frac{1}{2}} \Big(\sum_{N_1} \Big\| \big(|(P_{N_1}J_x^1 w)^\wedge|\big)^\vee \Big\|^2_{L^4_{x,t}}\Big)^{\frac{1}{2}} \|v\|_{X^{-1,1}}, \end{aligned}$$

which, combined to estimates (2.9) and (2.4) leads to estimate (4.3) in this case. Note that we have used here any $0 < \delta < \frac{1}{6}-$, since $X^{0,\frac{1}{3}+} \hookrightarrow L^4_{x,t}$ due to estimate (2.9).

Case high-high interaction and $L_{\max} = L_2$. This case works exactly as in the precedent case, summing in $N \leq N_1$ instead of $N_2 \leq N$.

Case high-low interaction and $L_{\max} = L_1$. We obtain from the frequency localization properties that

$$|I| \lesssim \sum_{N_1, N_2 \leq N_1} \sum_{L_1, L \leq L_1} \frac{\langle N_1 \rangle \langle L \rangle^{-\delta} \langle L_1 \rangle^{\frac{1}{2}}}{\langle L_1 \rangle^{\frac{1}{2}}} \int_{\mathcal{D}} |\xi \xi_2^{\frac{1}{2}-\delta_2}| \frac{|h_{N_1,L}(\xi,\tau)|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \phi_{N_1}(\xi) \psi_L(\xi,\tau)$$
$$\times |(P_{N_1} Q_{L_1} w)^\wedge(\xi_1,\tau_1)| |(P_{N_2} D_x^{\frac{1}{2}+\delta_2} v)^\wedge(\xi_2,\tau_2)| d\nu,$$

in this case. Therefore, it follows from (4.10), estimate (2.9), Plancherel's identity and Hölder's inequality that

(4.12)
$$\begin{aligned}
|I| &\lesssim \sum_{N_1, N_2 \leq N_1} N_2^{-\delta_2} \sup_L \Big\| \Big(\frac{|h_{N_1,L}|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \Big)^\vee \Big\|_{L^6_{x,t}} \\
&\quad \times \sum_{L_1} \langle N_1 \rangle \langle L_1 \rangle^{\frac{1}{2}} \|P_{N_1} Q_{L_1} w\|_{L^2_{x,t}} \Big\| \big(|(P_{N_2} D_x^{\frac{1}{2}+\delta_2} v)^\wedge| \big)^\vee \Big\|_{L^3_{x,t}} \\
&\lesssim \Big(\sum_{N_1} \sup_L \Big\| \Big(\frac{|h_{N_1,L}|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \Big)^\vee \Big\|^2_{L^6_{x,t}} \Big)^{\frac{1}{2}} \|w\|_{X^{1,\frac{1}{2},1}} \Big(\sum_{N_2} \Big\| \big(|(P_{N_2} D_x^{\frac{1}{2}+\delta_2} v)^\wedge| \big)^\vee \Big\|^2_{L^3_{x,t}} \Big)^{\frac{1}{2}},
\end{aligned}$$

where δ and δ_2 are two small positive numbers. Now, we have from (2.9) that the injection $X^{0,\frac{2}{9}+} \hookrightarrow L^3_{x,t}$ is continuous. Then, we can choose δ_2 postive small enough such that

$$\Big(\sum_{N_2} \Big\| \big(|(P_{N_2} D_x^{\frac{1}{2}+\delta_2} v)^\wedge| \big)^\vee \Big\|^2_{L^3_{x,t}} \Big)^{\frac{1}{2}} \lesssim \|v\|_{X^{\frac{1}{2}+\delta_2, \frac{1}{4}-\frac{\delta_2}{2}}} \lesssim \sup_{0 \leq \theta \leq 1} \|v\|_{X^{1-2\theta,\theta}}. \tag{4.13}$$

Estimate (2.9) also implies $X^{0,\frac{4}{9}+} \hookrightarrow L^6_{x,t}$, so that

$$\Big(\sum_{N_1} \sup_L \Big\| \Big(\frac{|h_{N_1,L}|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \Big)^\vee \Big\|^2_{L^6_{x,t}} \Big)^{\frac{1}{2}} \lesssim \Big(\sum_{N_1} \sup_L \|h_{N_1,L}\|^2_{L^2_{\xi,\tau}} \Big)^{\frac{1}{2}}, \tag{4.14}$$

for $0 < \delta$ and small enough. We deduce estimate (4.3) in this case gathering (4.12)–(4.14).

Case high-high interaction and $L_{\max} = L_1$. We proceed exactly as in the precedent case. I can be estimate as

$$\begin{aligned}
|I| &\lesssim \sum_{N_1, N \leq N_1} \sum_{L_1, L \leq L_1} \frac{\langle N \rangle \langle L \rangle^{-\delta} \langle L_1 \rangle^{\frac{1}{2}}}{\langle L_1 \rangle^{\frac{1}{2}}} \int_{\mathcal{D}} |\xi \xi_2^{\frac{1}{2}}| \frac{|h_{N,L}(\xi,\tau)|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \phi_N(\xi) \psi_L(\xi,\tau) \\
&\qquad \times |(P_{N_1} Q_{L_1} w)^\wedge(\xi_1,\tau_1)| |(P_{N_1} D_x^{\frac{1}{2}} v)^\wedge(\xi_2,\tau_2)| d\nu \\
&\lesssim \sum_{N_1, N \leq N_1} \Big(\frac{N}{N_1} \Big)^{\frac{1}{2}} \sup_L \Big\| \Big(\frac{|h_{N,L}|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \Big)^\vee \Big\|_{L^6_{x,t}} \\
&\qquad \times \sum_{L_1} \langle N_1 \rangle \langle L_1 \rangle^{\frac{1}{2}} \|P_{N_1} Q_{L_1} w\|_{L^2_{x,t}} \Big\| \big(|(P_{N_1} D_x^{\frac{1}{2}} v)^\wedge| \big)^\vee \Big\|_{L^3_{x,t}} \\
&\lesssim \Big(\sum_{N} \sup_L \Big\| \Big(\frac{|h_{N,L}|}{\langle \sigma \rangle^{\frac{1}{2}-\delta}} \Big)^\vee \Big\|^2_{L^6_{x,t}} \Big)^{\frac{1}{2}} \|w\|_{X^{1,\frac{1}{2}}} \Big(\sum_{N_1} \Big\| \big(|(P_{N_1} D_x^{\frac{1}{2}+\delta_2} v)^\wedge| \big)^\vee \Big\|^2_{L^3_{x,t}} \Big)^{\frac{1}{2}},
\end{aligned}$$

which yields estimate (4.3) in this case, recalling (4.13) and (4.14).

Case high-low interaction and $L_{\max} = L$. From (4.11), we can always assume that $L \sim (NN_1N_2)^{\frac{1}{2}}$ in this case, since when $L \sim L_{med}$ we are in one of the precedent cases. Therefore, for fixed N, N_1 and N_2, we only have a finite number of terms in the sum in L appearing on (4.9), so that the following estimate holds in this case,

$$\begin{aligned}
|I| &\lesssim \sum_{N_1,N_2\leq N_1} \frac{\langle N_1\rangle}{(N_1^2N_2)^{\frac{1}{2}}} \int_{\mathcal{D}} |\xi\xi_2^{\frac{1}{2}-\delta_2}||h_{N_1,L}(\xi,\tau)|\phi_{N_1}(\xi)\psi_L(\xi,\tau) \\
&\qquad\qquad \times |(P_{N_1}w)^\wedge(\xi_1,\tau_1)||(P_{N_2}D_x^{\frac{1}{2}+\delta_2}v)^\wedge(\xi_2,\tau_2)|d\nu \\
&\lesssim \sum_{N_1,N_2\leq N_1} N_2^{-\delta_2} \sup_L \|h_{N_1,L}\|_{L^2_{\xi,\tau}} \|\big(|(P_{N_1}J_x^1w)^\wedge|\big)^\vee\|_{L^6_{x,t}} \|\big(|(P_{N_2}D_x^{\frac{1}{2}+\delta_2}v)^\wedge|\big)^\vee\|_{L^3_{x,t}} \\
&\lesssim \Big(\sum_{N_1} \sup_L \|h_{N_1,L}\|^2_{L^2_{\xi,\tau}}\Big)^{\frac{1}{2}} \|w\|_{X^{1,\frac{1}{2},1}} \|v\|_{X^{\frac{1}{2}+\delta_2,\frac{1}{4}-\frac{\delta_2}{2}}}.
\end{aligned}$$

This proves estimate (4.3) in this case.

Case high-high interaction and $L_{\max} = L$. This case can be treated combining the ideas used for the *high-high interaction and* $L_{\max} = L_1$ *case* and the *high-low interaction and* $L_{\max} = L$ case. □

Proposition 4.3. *Let* $0 < T \leq 1$, $s \geq 1$, $v \in L^\infty(\mathbb{R}; L^2(\mathbb{R})) \cap L^4(\mathbb{R}; W^{1,4}(\mathbb{R}))$ *and* $w \in X^{1,\frac{1}{2},1}$ *supported in the time interval* $[0, 2T]$. *Then, it holds that*

$$\|\partial_x P_{+hi}\big(WP_-v_x\big)\|_{X^{s,-\frac{1}{2},1}} \lesssim \|w\|_{X^{s,\frac{1}{2},1}} \|\partial_x v\|_{L^4_{x,t}}, \tag{4.15}$$

$$\|\partial_x P_{+hi}\big(WP_-\partial_x(v^2)\big)\|_{X^{s,-\frac{1}{2},1}} \lesssim \|w\|_{X^{s,\frac{1}{2},1}} \|\partial_x v\|_{L^4_{x,t}} \|v\|_{L^\infty_t H^1_x}, \tag{4.16}$$

and

$$\begin{aligned}
&\|\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-v_x\big)\|_{X^{s,-\frac{1}{2},1}} + \|\partial_x P_{+hi}\big(P_{lo}(e^{iF}v)P_-v_x\big)\|_{X^{s,-\frac{1}{2},1}} \\
&\qquad + \|\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-\partial_x(v^2)\big)\|_{X^{s,-\frac{1}{2},1}} \lesssim \|v\|^2_{L^4_{x,t}}.
\end{aligned} \tag{4.17}$$

Proof. We begin with the proof of estimate (4.15). We deduce from the Cauchy-Schwarz inequality and estimate (2.6) that

$$\begin{aligned}
\|\partial_x P_{+hi}\big(WP_-v_x\big)\|_{X^{s,-\frac{1}{2},1}} &\lesssim \|\partial_x P_{+hi}\big(WP_-v_x\big)\|_{L^{1+}_t H^s_x} \\
&\lesssim \|\partial_x P_{+hi}\big(WP_-v_x\big)\|_{L^{1+}_t L^2_x} + \|D_x^s\partial_x P_{+hi}\big(WP_-v_x\big)\|_{L^{1+}_t L^2_x}.
\end{aligned}$$

Thus, it follows applying estimate (2.30) and Hölder's inequality in time that

$$\|\partial_x P_{+hi}\big(WP_-v_x\big)\|_{X^{s,-\frac{1}{2},1}} \lesssim T^{\frac{1}{2}-}\big(\|w\|_{L^4_{x,t}} + \|D_x^s w\|_{L^4_{x,t}}\big)\|v_x\|_{L^4_{x,t}},$$

which proves estimate (4.15) since $X^{0,\frac{1}{2},1} \hookrightarrow X^{0,\frac{1}{3}+} \hookrightarrow L^4_{x,t}$ by combining (2.4) and (2.9). Similar arguments combined to the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ imply estimate (4.16).

Finally, we turn to the proof of estimate (4.17). We will only bound the first term on the left-hand side, since the other ones can be treated exactly by the same

way. The Cauchy-Schwarz inequality and estimate (2.6) imply that

$$\begin{aligned}&\|\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-v_x\big)\|_{X^{s,-\frac{1}{2},1}}\\&\quad\lesssim \|\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-v_x\big)\|_{L^{1+}_tL^2_x}+\|D^s_x\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-v_x\big)\|_{L^{1+}_tL^2_x}.\end{aligned}$$

On the other hand, we get from the frequency localization that

$$\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-v_x\big)=\partial_x P_{+LO}\big(P_{lo}(e^{iF})P_{-LO}v_x\big).$$

Therefore, we deduce from Bernstein's inequalities and estimate (2.30) that

$$\|\partial_x P_{+hi}\big(P_{lo}(e^{iF})P_-v_x\big)\|_{X^{s,-\frac{1}{2},1}}\lesssim T^{\frac{1}{2}-}\|\partial_x e^{iF}\|_{L^4_{x,t}}\|v\|_{L^4_{x,t}}\lesssim \|v\|^2_{L^4_{x,t}},$$

recalling that $\partial_x F=Av$. This concludes the proof of Proposition 4.3. □

Proposition 4.4. *Let $1\le s\le \frac{3}{2}$, $0\le T\le 1$, v a solution to (1.1) which belongs to $L^\infty(\mathbb{R};L^2(\mathbb{R}))\cap L^2(\mathbb{R};L^\infty[0,T])$, such that $J^s_x\partial_x v\in \widetilde{L^\infty_x L^2_T}$ and supported in the time interval $[0,2T]$. Then, it holds that*

(4.18)
$$\|\partial_x P_{+hi}\big(e^{iF}v^2\big)\|_{X^{s,-\frac{1}{2},1}}\lesssim p_1(\|v\|_{L^\infty_T H^1_x})\|v\|_{L^\infty_T H^1_x}\|v\|_{L^\infty_T H^s_x}+\|v\|_{L^2_xL^\infty_T}\|J^s_x\partial_x v\|_{\widetilde{L^\infty_x L^2_T}}$$

and

(4.19)
$$\|\partial_x P_{+hi}\big(e^{iF}v^3\big)\|_{X^{s,-\frac{1}{2},1}}\lesssim p_2(\|v\|_{L^\infty_T H^1_x})\|v\|_{L^\infty_T H^1_x}\Big(\|v\|_{L^\infty_T H^s_x}+\|v\|_{L^2_xL^\infty_T}\|J^s_x\partial_x v\|_{\widetilde{L^\infty_x L^2_T}}\Big),$$

where p_1 and p_2 are polynomial functions.

Proof. First we notice that according to (2.31)

$$\begin{aligned}\|\partial_x P_{+hi}\big(e^{iF}v^2\big)\|_{X^{s,-\frac{1}{2},1}}&\lesssim \|\partial_x P_{+hi}\big(e^{iF}v^2\big)\|_{L^{1+}_TH^s}\\&\lesssim \|e^{iF}v^3\|_{L^{1+}_TH^s_x}+\|e^{iF}\partial_x(v^2)\big)\|_{L^{1+}_TH^s_x}\\&\lesssim (1+\|v\|^2_{L^\infty_TH^1_x})(\|v\|^2_{L^\infty_TH^1_x}\|v\|_{L^\infty_TH^s_x}+\|\partial_x(v^2)\|_{L^2_TH^s_x}).\end{aligned}$$

It thus remains to control $\|P_{HI}\partial_x(v^2)\|_{L^2_TH^s_x}$ since the low frequency part of the contribution of $\partial_x(v^2)$ is easily estimated by $\|v\|^2_{L^\infty_TH^1_x}$. To this aim we use the same decomposition as in (3.14) and write for $k\ge 4$,

$$P_{2^k}(v^2)=P_{2^k}\Big(\sum_{j\ge k-3}P_{2^j}vP_{\le 2^{j-1}}v+\sum_{j\ge k-3}P_{2^j}vP_{\le 2^j}v\Big).$$

Hence,

$$\begin{aligned}\|P_{2^k}\partial_x(v^2)\|_{L^2_TH^s_x}&\lesssim 2^{k(s+1)}\sum_{j\ge k-2}2^{-j(s+1)}\|P_{2^j}J^s_xv_x\|_{L^\infty_xL^2_T}\|v\|_{L^2_xL^\infty_T}\\&\lesssim \|v\|_{L^2_xL^\infty_T}\sum_{j\ge k-3}2^{(k-j)(s+1)}\|P_{2^j}J^s_x\partial_x v\|_{L^\infty_xL^2_T},\end{aligned}$$

and by Young's inequality we obtain that

$$\|P_{HI}\partial_x(v^2)\|^2_{L^2_TH^s_x}\lesssim\sum_{k\ge 4}\|P_{2^k}\partial_x(v^2)\|^2_{L^2_TH^s_x}\lesssim \|v\|^2_{L^2_xL^\infty_T}\|J^s_xv_x\|^2_{\widetilde{L^\infty_xL^2_T}}.$$

This completes the proof of (4.18). The proof of (4.19) follows a similar way and will thus be omitted. □

We are now in position to give the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $s \geq 1$, $0 < T \leq 1$, and $\tilde{v}$ and $\tilde{w}$ be extensions of v and w such that $\|\tilde{v}\|_{X^{1-2\theta,\theta}} \leq 2\|v\|_{X_T^{1-2\theta,\theta}}$ for all $0 \leq \theta \leq 1$ and $\|\tilde{w}\|_{X^{1,\frac{1}{2},1}} \leq 2\|w\|_{X_T^{1,\frac{1}{2},1}}$. By the Duhamel principle, the integral formulation associated to (3.4) writes

$$w(t) = \eta(t)w(0) + \eta(t)\int_0^t N(\eta_T e^{iF}, \eta_T v, \eta_T W, \eta_T w)(\tau)d\tau,$$

where $N(e^{iF}, v, W, w)$ is defined in (3.4), for $0 < t \leq T \leq 1$. Therefore, we deduce gathering estimates (2.7), (2.8), (4.2) and (4.15)–(4.19) that

$$\begin{aligned}\|w\|_{X^{s,\frac{1}{2},1}} &\lesssim \|w(0)\|_{H^s} \\ &\quad + p\big(\|w\|_{X^{s,\frac{1}{2},1}}, \|v\|_{L^\infty_T H^1_x}, \|v\|_{L^4_T W^{1,4}_x}, \|v\|_{L^2_x L^\infty_T}, \|J^s_x v_x\|_{\widetilde{L^\infty_x L^2_T}}, \sup_{0\leq\theta\leq 1}\|v\|_{X_T^{1-2\theta,\theta}}\big),\end{aligned}$$

where p is polynomial at least quadratic in its arguments. This concludes the proof of estimate (4.1), since

$$\|w(0)\|_{H^s} \lesssim \|J^s_x\big(e^{-iF[v_0]}v_0\big)\|_{L^2} \lesssim \big(1 + \|v_0\|^2_{H^1}\big)\|v_0\|_{H^s},$$

due to Lemma 2.9. □

5. Proof of Theorem 1.1

Without loss of generality, we will fix $\epsilon = 1$ in this section. First observe that, unlike to the Benjamin-Ono equation, equation (1.1) is not invariant under scaling. However, if v is a solution to the equation (1.1) on the time interval $[0, T]$ with initial data v_0, then for every $0 < \lambda < \infty$, $v_\lambda(x,t) = \lambda v(\lambda x, \lambda^3 t)$ is a solution to

$$\partial_t v - b\lambda\mathcal{H}\partial_x^2 v - a\partial_x^3 v = c\lambda v\partial_x v - d\partial_x(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)), \tag{5.1}$$

on the time interval $[0, \lambda^{-3}T]$. Then, since

$$\|v_\lambda(\cdot,0)\|_{H^1} \sim \lambda^{\frac{1}{2}}\|v_0\|_{L^2} + \lambda^{\frac{3}{2}}\|\partial_x v_0\|_{L^2} \lesssim \lambda^{\frac{1}{2}}\|v_0\|_{H^1},$$

we can always force $v_\lambda(\cdot,0)$ to belong to B_α, where B_α is the open ball of $H^1(\mathbb{R})$ with radius $0 < \alpha \ll 1$ and centered at the origin. Therefore the existence and uniqueness of a solution to (5.1) on the time interval $[0,1]$ for small initial data in $H^1(\mathbb{R})$ will ensure the existence and uniqueness of a solution to (1.1) on the time interval $[0,T]$ with $T \sim \lambda^3 \sim \min\{1, \|v_0\|^{-6}_{H^1}\}$ for arbitrary initial data in $H^1(\mathbb{R})$. Moreover, using the conservation of the energy H defined in (1.7), which controls the H^1-norm, will imply the global well-posedness of (1.1) in $H^1(\mathbb{R})$.

Since all the estimates obtained in the precedent sections are still valid for (5.1) with implicit constants independent of $0 < \lambda \leq 1$ and for sake of simplicity, we will continue working with equation (1.1), in the case $\epsilon = 1$, *i.e.*,

$$\partial_t v - b\mathcal{H}\partial_x^2 v - a\partial_x^3 v = cv\partial_x v - d\partial_x(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)), \tag{5.2}$$

instead of equation (5.1). The rest of the proof of Theorem 1.1 follows closely our proof for the Benjamin-Ono equation (see Theorem 1.1 in [23]). For this reason, we will only give a sketch of it.

As a consequence of the well-posedness theory for more regular solutions obtained in Theorem 1.3 of [18], we have the following result.

Proposition 5.1. *For all $v_0 \in H^\infty(\mathbb{R})$, there exists a positive time $T = T(\|v_0\|_{H^2}) \in (0,1]$ and a solution $v \in C([0,T]; H^\infty(\mathbb{R}))$ to equation (5.2). Moreover, T is a non-decreasing function of its argument.*

Note however that at this point, we still do not know wether those solutions are global or not.

The first step is to obtained *a priori* estimates for those smooth solutions. Let $v_0 \in B_\alpha \cap H^\infty(\mathbb{R})$ be given. Here, we recall that $B_\alpha = \{\phi \in H^1(\mathbb{R}) \ : \ \|\phi\|_{H^1} \leq \alpha\}$, where $\alpha > 0$ will be choosen sufficiently small. For any $s \geq 1$, we will also denote

$$N_T^s(v) = \max\left\{\|v\|_{L_T^\infty H_x^s}, \|J_x^s v\|_{L_{x,T}^4}, \|v\|_{L_x^2 L_T^\infty}, \|J_x^s \partial_x v\|_{\widetilde{L_x^\infty L_T^2}}, \|w\|_{X_T^{s,\frac{1}{2},1}}, \right\}. \tag{5.3}$$

Then, estimates (3.7)–(3.11) and (4.1) yield

$$N_T^s(v) \lesssim (1 + \|v_0\|_{H^1}^2)\|v_0\|_{H^s} + q(N_T^1(v))N_T^s(v), \tag{5.4}$$

for any $s \geq 1$, where q is a polynomial with no constant term. By continuity, estimate (5.4) for $s = 1$ ensures that there exist two positive constants α_1 and C_1 such that $N_T^1(v) \leq C_1\alpha$, provided $v_0 \in B_\alpha$ with $0 < \alpha < \alpha_1$. Moreover, using estimate (5.4) again implies that

$$\|v\|_{L_T^\infty H_x^s} \leq N_T^s(v) \lesssim \|v_0\|_{H^s}, \tag{5.5}$$

for any $s \geq 1$, provided $\|v_0\|_{H^1} \leq \alpha < \alpha_1$. Therefore, by using estimate (5.5) for $s = 2$, we can reapply the result of Proposition 5.1 a finite number of time to extend the solution v to the interval $[0, 1]$, as soon as $\|v_0\|_{H^1}$ is small enough. We observe that the scaling argument explained above and the control of the H^1-norm by the Hamiltonian H defined in (1.7), allow to extend our solution v globally in time, so that $v \in C(\mathbb{R}; H^\infty(\mathbb{R}))$.

To prove the uniqueness as well as the continuity of the flow map, we follow our argument in the proof of Theorem 1.1 of [23] and derive a Lipschitz bound on the flow map : $H^s(\mathbb{R}) \to L^\infty([0, 1]; H^s(\mathbb{R}))$ for initial data having the same low frequency part, and where $L^\infty([0, 1]; H^s(\mathbb{R}))$ is considered with the norm N_1^s defined in (5.3). The idea is to apply similar estimates to (3.7)–(3.11) and (4.1) to the difference of two solutions $u := v_1 - v_2$ and to the difference of the gauges $z := w_1 - w_2$. Note that at this point a control on $G := F[v_1] - F[v_2]$ in $L^\infty([0, 1]\times\mathbb{R})$ is needed and can be obtained exactly as in Lemma 4.1 of [23] by splitting G between its low and high frequency parts $G := G_{lo} + G_{hi}$. We use the equation satisfied by G_{lo} and evolving from 0 to control G_{lo} and Bernstein's inequalities to control G_{hi}.

Finally to prove the existence in $H^s(\mathbb{R})$, for initial data and the continuity of the flow map, we fix an initial data $v_0 \in B_\alpha \cap H^s(\mathbb{R})$ and an approximate sequence of initial data $v_0^j = \mathcal{F}_x^{-1}\big(\chi_{|[-j,j]}\mathcal{F}_x v_0\big)$. Then, as explained above, the associated sequence of solutions $\{v^j\}_j$ is a subset of $C([0, 1]; H^\infty(\mathbb{R}))$. Moreover, since it evolves from initial data having the same low frequency part, the Lipschitz bound implies that $\{v^j\}_j$ is a Cauchy sequence in all the norms appearing in N_1^s and therefore converges strongly in those norms to a solution v of (5.2) satisfying $v(\cdot, 0) = v_0$ and (1.8)–(1.10).

6. Convergence towards the Benjamin-Ono equation in $H^1(\mathbb{R})$ when $\rho = \sqrt{3}\rho_1$

In this section we prove that (1.1) is uniformly in $\varepsilon > 0$ well-posed in $H^1(\mathbb{R})$ whenever the ratio of the density between the two fluids is given by $\rho = \sqrt{3}\rho_1$. This will in a classical way (see for instance [7]) lead to Theorem 1.2. According to Remark 3.1, this condition on this ratio permits to cancel the term $\partial_x P_{+hi}(e^{iF}v^2)$

in (3.4). Note that this term behaves mainly as the nonlinear term of the Benjamin-Ono equation $\partial_x(v^2)$. Therefore we do not know how deal with this term when ε is going to zero. One possibility could be to use the variant of the Bourgain' space introduced in [10, 16] as it was done to study the inviscid limit of the the Benjamin-Ono-Burgers equation in [8]. However, there is another problem here since linear and bilinear estimates involving V_ε are not uniform in ε (see for instance (6.1)-(6.2)). This is due to the fact that the linear terms $\mathcal{H}u_{xx}$ and $\varepsilon\partial_x^3$ compete together as in the Benjamin equation (cf. [2]). This is reflected on the energy H (see (1.7)) by the fact that the $\dot{H}^1$ and $\dot{H}^{1/2}$ components of the quadratic part are of opposite signs.

6.1. **Some linear estimates.** First we establish needed linear estimates on the group $V_\varepsilon(\cdot)$ (see also Lemmas 2.4-2.5).

Lemma 6.1. *For any $0<\varepsilon\leq 1$, any $0<T\leq 1$ and any $\varphi\in L^2(\mathbb{R})$ it holds*

$$\|V_\epsilon(t)\varphi\|_{L^4_T L^\infty_x}\lesssim \epsilon^{-1/6}\|\varphi\|_{L^2} \tag{6.1}$$

and

$$\|V_\epsilon(t)\varphi\|_{L^6_T L^6_x}\lesssim \epsilon^{-1/9}\|\varphi\|_{L^2} \tag{6.2}$$

Moreover, denoting by P_ϵ a smooth space Fourier projector on $\{|\xi|\notin]\frac{1}{4\epsilon},\frac{1}{2\epsilon}[\}$, we have

$$\|V_\epsilon(t)P_\epsilon\varphi\|_{L^4_T L^\infty_x}\lesssim \|\varphi\|_{L^2} \tag{6.3}$$

and

$$\|V_\epsilon(t)P_\epsilon\varphi\|_{L^6_T L^6_x}\lesssim \|\varphi\|_{L^2} \tag{6.4}$$

Proof. By the TT^* argument it suffices to prove that for $0<t\leq 1$,

$$\|V_\epsilon(t)\varphi\|_{L^\infty_x}\lesssim \epsilon^{-1/3}t^{-1/2}\|\varphi\|_{L^2} \tag{6.5}$$

and

$$\|V_\epsilon(t)P_\epsilon\varphi\|_{L^\infty_x}\lesssim t^{-1/2}\|\varphi\|_{L^2} \tag{6.6}$$

By classical arguments, (6.5) will be proven if we show

$$\Big\|\int_{\mathbb{R}} e^{i[x\xi+(\xi|\xi|-\epsilon\xi^3)t]}\,d\xi\Big\|_{L^\infty_x}\lesssim \epsilon^{-1/3}t^{-1/2}\,.$$

Setting $\theta:=\xi\sqrt{t}$ this is equivalent to prove

$$I_\epsilon:=\sup_{|t|\leq 1, X\in\mathbb{R}}\Big|\int_{\mathbb{R}} e^{i[X\theta+|\theta|\theta-\frac{\epsilon}{\sqrt{t}}\theta^3]}\,d\theta\Big|\lesssim \epsilon^{-1/3} \tag{6.7}$$

We set $\Phi(\theta):=X\theta+\theta^2-\frac{\epsilon}{\sqrt{t}}\theta^3$ and notice that for $\theta\neq 0$,

$$\Phi'(\theta):=X+2|\theta|-3\frac{\epsilon}{\sqrt{t}}\theta^2,\ \Phi^{''}(\theta)=2(\operatorname{sgn}\theta-3\frac{\epsilon}{\sqrt{t}}\theta)\ \text{ and } \Phi^{'''}(\theta)=-6\frac{\epsilon}{\sqrt{t}}\,.$$

(6.7) is obvious when restricted on $|\theta|\leq 100$. By symmetry we can assume that $\theta>100$. X,t and ϵ being fixed, there exists $M_{t,\epsilon,X}\geq 100$ such that

$$\forall\theta>M,\quad |\Phi'(\theta)|\geq \max(1+|\theta|,\frac{\epsilon}{\sqrt{t}}\theta^2)\,.$$

Therefore, integration by parts yields that $I_\epsilon\lesssim 1$ in this region.
Now for $\theta<M$, we use Van der Corput lemma and that $|\Phi^{'''}(\theta)|=6\frac{\epsilon}{\sqrt{t}}$ to get

$I_\epsilon \lesssim \epsilon^{-1/3}$. This completes the proof of (6.5).
To prove (6.6), we use that $\xi \notin]\frac{1}{4\epsilon}, \frac{1}{2\epsilon}[$ implies $\theta \notin]\frac{\sqrt{t}}{4\epsilon}, \frac{\sqrt{t}}{2\epsilon}[$ and thus $|\Phi''(\theta)| = 2|1 - 3\frac{\epsilon}{\sqrt{t}}\theta| \gtrsim 1$. This yields the result by applying Van der Corput lemma in the region $\theta < M$. □

Lemma 6.2. *For any $0 < \varepsilon \leq 1$, any $0 < T \leq 1$ and any $\varphi \in L^2(\mathbb{R})$ it holds*

$$\|V_\epsilon(t)\varphi\|_{L^4_x L^\infty_T} \lesssim \epsilon^{-1/4}\|\varphi\|_{L^2} \tag{6.8}$$

and for any dyadic integer $N \geq 10/\varepsilon$,

$$\|V_\epsilon(t)P_N\varphi\|_{L^\infty_x L^2_t} \lesssim \varepsilon^{-1/2}\|P_N\varphi\|_{L^2} \tag{6.9}$$

Proof. (6.8) can be proved by using the change of unknown (2.12) exactly as in Lemma 2.6 whereas (6.9) is proven in (2.21).

6.2. **Nonlinear estimates.** According to the linear estimates of Lemmas 2.2-2.3, we have to estimate each terms of the right-hand side member of (3.4) in $X^{1,1/2,1}_\varepsilon$. Recall that the term $\partial_x P_{+hi}(e^{-iF}v^2)$ cancels in this section due to the choice of the ratio between ρ and ρ_1.

Lemma 6.3. *Assume that $\operatorname{supp} v \subset \{|t| \leq T\}$ with $1 < T \leq 1$. Then for any $0 < \epsilon \leq 1$ it holds*

$$\left\|\partial_x P_{+hi}(W\partial_x P_{-hi}v)\right\|_{X^{1,-1/2,1}_\epsilon} \lesssim \|w\|_{X^{1,-1/2,1}}(1 + \|v\|_{L^\infty_t H^1_x})\|v\|_{L^\infty_t H^1_x} \tag{6.10}$$

Proof. We rewrite $\hat{w}$ as $\eta_\epsilon\hat{w} + (1-\eta_\epsilon)\hat{w} := \widehat{w_1} + \widehat{w_2}$ where η_ϵ is a smooth even function with support in $|\xi| \in]\frac{1}{5\epsilon}, \frac{5}{9\epsilon}[$ such that $\eta_\epsilon \equiv 1$ on $]\frac{1}{4\epsilon}, \frac{1}{2\epsilon}[$.
By Sobolev embedding we have

$$J := \left\|\partial_x P_{+hi}(W_2\partial_x P_{-hi}v)\right\|_{X^{1,-1/2,1}_\epsilon} \lesssim T^{1/3}\left\|\partial_x^2 P_{+hi}(W_2\partial_x P_{-hi}v)\right\|_{L^2_T L^2_x}.$$

But for any fixed integer $k \geq 1$, making use of the discrete Young's inequality as in the proof of (3.11), we get

$$\begin{aligned}
\left\|\partial_x^2 P_{2^k}P_{+hi}(W_2\partial_x P_{-hi}v)\right\|_{L^2_t L^2_x} &\lesssim 2^{2k}\sum_{j\geq k}\left\|\partial_x^2 P_{2^k}P_{+hi}(P_{2^j}W_2\partial_x P_{-hi}u)\right\|_{L^2_t L^2_x} \\
&\lesssim 2^{2k}\sum_{j\geq k}2^{-j}\|P_{2^j}w_2\|_{L^4_T L^\infty_x}\|\partial_x u\|_{L^\infty_t L^2_x} \\
&\lesssim \|\partial_x v\|_{L^\infty_T L^2_x}\sum_{j\geq k}2^{-2(j-k)}\|P_{2^j}\partial_x w_2\|_{L^4_t L^\infty_x} \\
&\lesssim \gamma_k\|\partial_x w_2\|_{\widetilde{L^4_t L^\infty_x}}\|\partial_x v\|_{L^\infty_t L^2_x}
\end{aligned}$$

where $(\gamma_k) \in l^2(\mathbb{Z}_+)$. This is acceptable, according to (6.3), since $\widehat{w_2}$ cancels on $]\frac{1}{4\epsilon}, \frac{1}{2\epsilon}[$.
Now to treat the contribution of w_1 we have to use the resonance relation :

$$\sigma - \sigma_1 - \sigma_2 = \xi\xi_2(2 - 3\epsilon\xi_1)$$

Since Supp $\hat{w_1} \subset]\frac{1}{5\epsilon}, \frac{5}{9\epsilon}[$, it follows that $\max(|\sigma|, |\sigma_i|) \gtrsim |\xi\xi_2|$ for this contribution. We set

$$\begin{aligned} I &:= \Big\| \partial_x P_{+hi}(W_2 \partial_x P_{-hi} v) \Big\|_{X_\epsilon^{1,-1/2,1}} \\ &\lesssim \sum_{N,N_1,N_2} \Big\| P_N \partial_x P_{+hi}(P_{N_1} W_2 \partial_x P_{-hi} P_{N_2} v) \Big\|_{X_\epsilon^{1,-1/2,1}} := \sum_{N,N_1,N_2} I_{N,N_1,N_2} \end{aligned}$$

Clearly,

$$\begin{aligned} I_{N,N_1,N_2} = \sum_L \Big[\int_{\mathbb{R}^2} & \frac{\langle \xi \rangle \xi \chi_{\mathbb{R}_+}(\xi) \phi_N(\xi) \psi_L(\xi,\tau)}{\langle \tau - b|\xi|\xi + a\varepsilon\xi^3 \rangle} \Big| \\ & \times \int_{\mathbb{R}^2} \phi_{N_1}(\xi_1) \widehat{W_2}(\xi_1,\tau_1) \xi_2 \phi_{N_2}(\xi_2) \widehat{P_{-hi} v}(\xi_2,\tau_2) d\xi_1\, d\tau_1 \Big|^2 d\xi\, d\tau \Big]^{1/2} . \end{aligned}$$

We separate the contributions of different regions with respect to which σ is dominant. Therefore, to calculate I_{N,N_1,N_2}, we split the integration domain $\mathcal{D}$ in the following disjoint regions

(6.11)

$$\mathcal{A}_{N,N_2} = \big\{ (\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 \mid 2^{-2} N N_2 \le |\sigma| \le 2^2 N N_2 \big\},$$

$$\mathcal{B}_{N,N_2} = \big\{ (\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 \mid |\sigma_1| \ge \frac{1}{6} N N_2 \ , \ |\sigma| \notin [2^{-2} N N_2, 2^2 N N_2] \big\},$$

$$\mathcal{C}_{N,N_2} = \big\{ (\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 \mid |\sigma| \notin [2^{-2} N N_2, 2^2 N N_2], \ |\sigma_1| < \frac{1}{6} N N_2 \ , |\sigma_2| \ge \frac{1}{6} N N_2 \big\},$$

and denote by $I_{N,N_1,N_2}^{\mathcal{A}_{N,N_2}}$, $I_{N,N_1,N_2}^{\mathcal{B}_{N,N_2}}$, $I_{N,N_1,N_2}^{\mathcal{C}_{N,N_2}}$ the restriction of I_{N,N_1,N_2} to each of these regions. Then, it follows that

$$I_{N,N_1,N_2} \le I_{N,N_1,N_2}^{\mathcal{A}_{N,N_2}} + I_{N,N_1,N_2}^{\mathcal{B}_{N,N_2}} + I_{N,N_1,N_2}^{\mathcal{C}_{N,N_2}}$$

and thus

$$I^2 \lesssim \sum_N \big| I_N^{\mathcal{A}} \big|^2 + \sum_N \big| I_N^{\mathcal{B}} \big|^2 + \sum_N \big| I_N^{\mathcal{C}} \big|^2, \tag{6.12}$$

where

$$I_N^{\mathcal{A}} := \sum_{N_1,N_2} I_{N,N_1,N_2}^{\mathcal{A}_{N,N_2}}, \ I_N^{\mathcal{B}} := \sum_{N_1,N_2} I_{N,N_1,N_2}^{\mathcal{B}_{N,N_2}} \text{ and } I_N^{\mathcal{C}} := \sum_{N_1,N_2} I_{N,N_1,N_2}^{\mathcal{C}_{N,N_2}} .$$

Therefore, it suffices to bound $\sum_N \big| I_N^{\mathcal{A}} \big|^2$, $\sum_N \big| I_N^{\mathcal{B}} \big|^2$ and $\sum_N \big| I_N^{\mathcal{C}} \big|^2$.

• Bound on $\sum_N \big| I_N^{\mathcal{B}} \big|^2$. We use that $L^{1+}_{loc} \hookrightarrow \mathcal{B}^{-1/2,1}_{2,loc}$ to get

$$\begin{aligned} I_{2^k}^{\mathcal{B}} &\lesssim T^{1/3} 2^{2k} \sum_{j \ge k} \sum_{q \le j} \Big\| P_{2^k} P_{+hi}(P_{2^j} W_2 \partial_x P_{-hi} P_{2^q} v) \Big\|_{L^2_T L^2_x} \\ &\lesssim T^{1/3} \sum_{j \ge k} \sum_{q \le j} 2^{2(k-j)} 2^{-(k+q)/2} \| P_{2^j} \partial_x w_2 \|_{X^{0,1/2}} \| P_{2^q} \partial_x v \|_{L^\infty_T L^\infty_x} \\ &\lesssim T^{1/3} \sum_{j \ge k} \sum_{q \le j} 2^{\frac{3}{2}(k-j)} \| P_{2^j} \partial_x w_2 \|_{X^{0,1/2}} 2^{(q-j)/2} \| P_{2^q} v \|_{L^\infty_T L^\infty_x} \\ &\lesssim T^{1/3} \gamma_k \| w_2 \|_{X^{1,1/2}} \| v \|_{L^\infty_T H^1_x} \end{aligned}$$

with $\|(\gamma_k)\|_{l^2(\mathbb{Z}_+)} \lesssim 1$.

• Bound on $\sum_N \big| I_N^{\mathcal{A}} \big|^2$. Note that since $|\sigma| \sim N N_2$ in this region, we do not have

to sum on L.

$$\begin{aligned} I_{2^k}^{\mathcal{A}} &\lesssim \sum_{j\geq k}\sum_{q\leq j} 2^{-2(j-k)}2^{-(k+q)/2}\|P_{2^j}\partial_x w_2\|_{L^2_{tx}}\|P_{2^q}\partial_x v\|_{L^\infty_T L^\infty_x} \\ &\lesssim \sum_{j\geq k}\sum_{q\leq j} 2^{\frac{3}{2}(k-j)}\|P_{2^j}\partial_x w_2\|_{L^2_{tx}}2^{(q-j)/2}\|P_{2^q} v\|_{L^\infty_T L^\infty_x} \\ &\lesssim T^{1/2}\gamma_k\|w_2\|_{X^{1,1/2,1}}\|v\|_{L^\infty_T H^1_x} \end{aligned}$$

with $\|(\gamma_k)\|_{l^2(\mathbb{Z}_+)}\lesssim 1$.

• Bound on $\sum_N \left|I_N^{\mathcal{C}}\right|^2$.

$$\Big(\sum_N |I_N^{\mathcal{C}}|^2\Big)^{1/2}\lesssim \|\tilde{u}\|_{X^{0,1}}\|\mathcal{F}^{-1}(|\widehat{w_1}|)\|_{L^\infty_{tx}}\lesssim \|w\|_{X^{1,1/2,1}}\|\tilde{v}\|_{X^{0,1}}$$

where $\widehat{\tilde{v}}=\tilde{\eta}_\epsilon\hat{v}$ and $\tilde{\eta}_\epsilon$ is a smooth even function with support in $\{|\xi|\leq\frac{100}{\epsilon}\}$. Here we used that due to the frequency projections together with the frequency localization of w_1, the modulus of the frequencies of v must be less than $10/\varepsilon$. Now from the equation (1.1) satisfied by v and the frequency localization of $\tilde{v}$, we deduce that

$$\|\partial_t V_\epsilon(-t)\tilde{v}\|_{L^2_{tx}}\lesssim\|vv_x\|_{L^2_{tx}}\lesssim\|v\|^2_{L^\infty_t H^1_x}\,.$$

This yields the desired result. □

Lemma 6.4. *Assume that* $\operatorname{supp} v\subset\{|t|\leq T\}$ *with* $1<T\leq 1$. *Then for any* $0<\epsilon\leq 1$ *it holds*

(6.13)

$$\epsilon\Big\|\partial_x P_{+hi}(w\partial_x P_{-hi}v)\Big\|_{X^{1,-1/2,1}_\epsilon}\lesssim\|w\|_{X^{1,-1/2,1}}\Big[\epsilon^{1/9}\|v_x\|_{L^4_{tx}}+(1+\|v\|_{L^\infty_t H^1_x})\|v\|_{L^\infty_t H^1_x}\Big]$$

Proof. We rewrite $\hat{w}$ as $\zeta_\epsilon\hat{w}+(1-\zeta_\epsilon)\hat{w}:=\widehat{w_1}+\widehat{w_2}$ where ζ_ϵ is a smooth even function with support in $|\xi|\in]\frac{1}{2\epsilon},\frac{1}{\epsilon}[$ such that $\zeta_\epsilon\equiv 1$ on $]\frac{5}{9\epsilon},\frac{8}{9\epsilon}[$.
By Sobolev embedding and the frequency localization of w_1 we have

$$J_1:=\epsilon\Big\|\partial_x P_{+hi}(w_1\partial_x P_{-hi}v)\Big\|_{X^{1,-1/2,1}_\epsilon}\lesssim T^{1/3}\Big\|\partial_x P_{+hi}(w_1\partial_x P_{-hi}v)\Big\|_{L^2_tL^2_x}$$

We proceed exactly as in the preceding lemma to get

$$\begin{aligned} \Big\|\partial_x P_{2^k}P_{+hi}(w_1\partial_x P_{-hi}v)\Big\|_{L^2_tL^2_x} &\lesssim 2^{2k}\sum_{j\geq k}\Big\|\partial_x P_{2^k}P_{+hi}(P_{2^j}w_1\partial_x P_{-hi}v)\Big\|_{L^2_tL^2_x} \\ &\lesssim 2^{2k}\sum_{j\geq k}2^{-j}\|P_{2^j}w_1\|_{L^4_tL^\infty_x}\|\partial_x v\|_{L^\infty_tL^2_x} \\ &\lesssim \gamma_k\|\partial_x w_1\|_{\widetilde{L^4_tL^\infty_x}}\|\partial_x v\|_{L^\infty_tL^2_x} \end{aligned}$$

which is acceptable, according to (6.6), since $\widehat{w_1}$ cancels on $]\frac{1}{4\epsilon},\frac{1}{2\epsilon}[$.
Now for the contribution of w_2 we will use that thanks to the frequency localization of w_2 and the resonance relation we have

$$\max(|\sigma|,|\sigma_i|)\gtrsim\epsilon|\xi\xi_1\xi_2|$$

on the space-time Fourier support of this contribution. We decompose this contribution as in the preceding lemma but with respect to ϵNN_1N_2 instead of N_1N_2.

• Bound on $\sum_N \left|I_N^{\mathcal{B}}\right|^2$.

$$\begin{aligned} I_{2^k}^{\mathcal{B}} &\lesssim \varepsilon 2^{2k}\sum_{j\geq k}\sum_{q\leq j}\Big\|P_{2^k}P_{+hi}(P_{2^j}w_2\partial_x P_{-hi}P_{2^q}v)\Big\|_{L_T^{4/3}L_x^2} \\ &\lesssim \sqrt{\varepsilon}\sum_{j\geq k}\sum_{q\leq j}2^{2k}2^{-j}2^{-(k+q+j)/2}\|P_{2^j}\partial_x w_2\|_{X^{0,1/2}}\|P_{2^q}\partial_x v\|_{L_T^4L_x^\infty} \\ &\lesssim \sqrt{\varepsilon}\sum_{j\geq k}\sum_{q\leq j}2^{\frac{3}{2}(k-j)}\|P_{2^j}\partial_x w_2\|_{X^{0,1/2}}\|P_{2^q}D_x^{3/4}v\|_{L_{T,x}^4} \\ &\lesssim \varepsilon^{1/2-1/9}\gamma_k\,\|w_x\|_{X^{1,1/2}}(\varepsilon^{1/9}\|v_x\|_{L_{T,x}^4}) \end{aligned}$$

with $\|\gamma_k\|_{l^2(\mathbb{Z}_+)}\lesssim 1$.

• Bound on $\sum_N \left|I_N^{\mathcal{A}}\right|^2$. Note that since $|\sigma|\sim \varepsilon N N_1 N_2$ in this region, we do not have to sum on L.

$$\begin{aligned} I_{2^k}^{\mathcal{A}} &\lesssim \sqrt{\varepsilon}\,2^{2k}\sum_{j\geq k}\sum_{q\leq j}2^{-(k+q+j)/2}\Big\|P_{2^k}P_{+hi}(P_{2^j}w_2\partial_x P_{-hi}P_{2^q}v)\Big\|_{L_{tx}^2} \\ &\lesssim \sqrt{\varepsilon}\sum_{j\geq k}\sum_{q\leq j}2^{\frac{3}{2}(k-j)}\|P_{2^j}\partial_x w_2\|_{L_{tx}^4}\|P_{2^q}D_x^{1/2}v\|_{L_{Tx}^4} \\ &\lesssim \varepsilon^{1/2-1/9}\gamma_k\,\|w_x\|_{X^{1,1/2}}(\varepsilon^{1/9}\|v_x\|_{L_{tx}^4}) \end{aligned}$$

with $\|\gamma_k\|_{l^2(\mathbb{Z}_+)}\lesssim 1$.

• Bound on $\sum_N \left|I_N^{\mathcal{C}}\right|^2$. We separate two subregions.

a) $\xi_1\leq \frac{50}{\epsilon}$ Then we have $|\xi_2|\leq \frac{50}{\epsilon}$. We write

$$\Big(\sum_N |I_N^{\mathcal{C}}|^2\Big)^{1/2}\lesssim \|\tilde{v}\|_{X^{0,1}}\|\mathcal{F}^{-1}(|\widehat{w_2}|)\|_{L_{tx}^\infty}\lesssim \|w_2\|_{X^{1,1/2,1}}\|\tilde{v}\|_{X^{0,1}}$$

where $\widehat{\tilde{v}}=\tilde{\eta}_\epsilon\hat{v}$ and $\tilde{\eta}_\epsilon$ is a smooth even function with support in $\{|\xi|\leq \frac{100}{\epsilon}\}$ This is acceptable since, as in the preceding lemma due to the frequency localization of v,

$$\|\tilde{v}\|_{X^{0,1}}\lesssim (1+\|v\|_{L_t^\infty H_x^1})\|v\|_{L_t^\infty H_x^1}\ .$$

b) $\xi_1>\frac{50}{\epsilon}$ Then we write

$$\begin{aligned} I_{2^k}^{\mathcal{C}} &\lesssim \sum_{j\geq k}\epsilon\Big\|\mathcal{F}^{-1}\Big(|\xi^2\xi_2||\hat{v}|\,|\widehat{P_{2^j}w_2}|\Big)\Big\|_{L_t^{4/3}L_x^2} \\ &\lesssim \|v\|_{X^{-1,1}}\sum_{j\geq k}\|\mathcal{F}^{-1}(|\widehat{P_{2^j}\partial_x w_2}|)\|_{L_t^4L_x^\infty} \\ &\lesssim \gamma_k\|v\|_{X^{-1,1}}\|w\|_{X^{1,1/2}} \end{aligned}$$

where we used (6.3) and the frequency localization of w_2 in this subregion. This is acceptable, since in view of (1.1) it is not too hard to check that

$$\|v\|_{X^{-1,1}}\lesssim (1+\|v\|_{L_t^\infty H_x^1})\|v\|_{L_t^\infty H_x^1}\ .$$

□

Now the contribution of the term $\epsilon\partial_x P_{+hi}(WP_-(vv_x))$ is easy to estimate as follows

$$\begin{aligned} \epsilon\Big\|\partial_x P_{+hi}(WP_-(vv_x))\Big\|_{X^{1,-1/2,1}} &\lesssim \epsilon\Big\|\partial_x^2 P_{+hi}(WP_-(vv_x))\Big\|_{L_{x,T}^2} \\ &\lesssim \epsilon^{2/3}(\epsilon^{1/9}\|w_x\|_{L_{x,T}^6})(\epsilon^{1/9}\|v\|_{L_{x,T}^6})(\epsilon^{1/9}\|v_x\|_{L_{x,T}^6}) \end{aligned} \tag{6.14}$$

which is acceptable thanks to (6.2). Moreover, the contributions of each term of the left-hand side of (4.17) is controlled by $\|v\|^2_{L^\infty_t H^1_x}$ since, by the frequency projections, only low frequencies of v, v^2 and $e^{iF}v$ are involved. It thus remains to control the term $\varepsilon\|\partial_x P_{+hi}\big(e^{iF}v^3\big)\|_{X^{1,-\frac{1}{2},1}}$. This is the aim of the following lemma.

Lemma 6.5.

$$\begin{aligned}\varepsilon\|\partial_x P_{+hi}\big(e^{iF}v^3\big)\|_{X^{1,-\frac{1}{2},1}} \quad &\lesssim \quad \|v\|^3_{L^\infty_T H^1} + \varepsilon\|v\|^4_{L^\infty_t H^1_x} \\ &\quad +\varepsilon\Big[\|v\|^3_{L^\infty_T H^1}(1+\|v\|_{L^\infty_T H^1}) + \|v_{xx}\|_{\widetilde{L^\infty_x L^2_t}}\|v\|^2_{L^4_x L^\infty_t}\Big]\end{aligned} \tag{6.15}$$

Proof. . First, proceeding as in the proof of Proposition 4.4, it is not too hard to see that

$$\varepsilon\|\partial_x P_{+hi}\big(e^{iF}v^3\big)\|_{X^{1,-\frac{1}{2},1}} \lesssim \|v\|^3_{L^\infty_t H^1_x} + \varepsilon\|v\|^4_{L^\infty_t H^1_x} + \|v\|^4_{L^\infty_t H^1_x} + \varepsilon\|\partial^2_x P_{HI}(v^3)\|_{L^2_t L^2_x} .$$

It thus remains to bound the last term of the right-hand side of the above inequality. We notice that for $k \geq 4$ we may bound its P_{2^k}-projection by

$$\|\partial^2_x P_{2^k} v^3\|_{L^2_{tx}} \lesssim \Big\|\partial^2_x P_{2^k}\Big(\sum_{j\geq k-4} P_{2^j} v S_{2^j} v S_{2^j} v\Big)\Big\|_{L^2_{tx}} .$$

Hence, proceeding as in Proposition 4.4, it is easy to check that for $k \geq 4$,

$$\|\partial^2_x P_{2^k} v^3\|_{L^2_t L^2_x} \lesssim \gamma_k \|v_{xx}\|_{L^\infty_x L^2_t}\|v\|^2_{L^4_x L^\infty_t}$$

where $\|(\gamma_k)\|_{l^2(\mathbb{Z}_+)} \lesssim 1$. This completes the proof of the lemma. □

Finally to close the estimates we have to control some norms of v in terms of w.

Lemma 6.6. *Let $0 < T \leq 1$, $0 \leq \theta \leq 1$, $0 < \epsilon \leq 1$ and v be a solution to* (1.1) *in the time interval $[0, T]$. Then, it holds that*

$$\varepsilon^{1/9}\|\partial_x v\|_{L^6_{x,t}} \lesssim \|v_0\|_{H^1} + \big(1 + \|v\|^2_{L^\infty_T H^1_x}\big)\|w\|_{X^{s,1/2,1}} + \varepsilon^{1/9}\|v\|_{L^\infty_T H^1_x}\|\partial_x v\|_{L^6_{x,T}}, \tag{6.16}$$

$$\varepsilon^{1/4}\|v\|_{L^4_x L^\infty_t} \lesssim \|v_0\|_{H^1} + \big(1+\|v\|^2_{L^\infty_T H^1_x}\big)\|w\|_{X^{s,1/2,1}} + \varepsilon^{1/4}\|v\|_{L^\infty_T H^1_x}\|v\|_{L^4_x L^\infty_T}, \tag{6.17}$$

and

$$\begin{aligned}\varepsilon^{1/2}\|\partial^2_x v\|_{\widetilde{L^\infty_x L^2_T}} \lesssim &\|v_0\|_{H^1} + \big(1 + \|v\|^2_{L^\infty_T H^1_x}\big)\|w\|_{X^{s,1/2,1}} \\ &+ \|v\|_{L^\infty_T H^1_x}\Big(\|v\|_{L^\infty_T H^1_x} + \varepsilon^{1/2}\|\partial^2_x v\|_{\widetilde{L^\infty_x L^2_T}}\Big).\end{aligned} \tag{6.18}$$

Proof. (6.16) and (6.17) can be proven exactly as in Proposition 3.2 with (6.1)-(6.8) in hand. (6.18) can be proven as (3.11) in Proposition 3.2 with (6.9) in hand, using that by Bernstein inequality, for any $v \in L^2_{T,x}$ and $\varepsilon > 0$ it holds

$$\|P_{\leq 100/\varepsilon} v\|_{L^\infty_x L^2_T} \lesssim \varepsilon^{-1/2}\|v\|_{L^2_{Tx}} .$$

Gathering Lemmas 6.3-6.6 we obtain that (1.1) is uniformly well-posed in $H^1(\mathbb{R})$, i.e.

Proposition 6.7. *For any $R > 0$ there exists a positive time $T = T(R)$ and a positive real number $C(R)$ such that for all $0 < \varepsilon \leq 1$ and any initial data $v_0 \in H^1(\mathbb{R})$, with $\|v_0^j\|_{H^1} \leq C(R)$, it holds*

$$\|S_\varepsilon(t)v_0^j\|_{L^\infty_T H^1_x} \leq C(R) . \tag{6.19}$$

Moreover for any couple of initial data $(v_0^1, v_0^2) \in H^1(\mathbb{R})^2$ *with* $\|v_0^j\|_{H^1} \leq C(R)$ *and* $P_1(v_0^1) = P_1(v_0^2)$ *it holds*

$$\|S_\varepsilon(t)v_0^1 - S_\varepsilon(t)v_0^2\|_{L_T^\infty H_x^1} \leq C(R)\|v_0^1 - v_0^2\|_{H^1} \tag{6.20}$$

With this proposition in hand, Theorem 1.2 follows by general arguments developed for instance in [7]. We fix an initial data $v_0 \in H^1(\mathbb{R})$ and $\alpha > 0$ and we would like to prove that for $\varepsilon > 0$ small enough,

$$\|S_\varepsilon(t)v - S(t)v\|_{L_T^\infty H_x^1} \leq \alpha$$

where $T = T(\|v_0\|_{H^1})$. The result for any fixed $T > 0$ follows by iterating the argument and using the continuity of the flow-map for the Benjamin-Ono equation. First, thanks to (6.20) there exists $r_\alpha > 1$ such that for all $\varepsilon \in]0, 1]$ it holds

$$\|S_\varepsilon(t)v_0 - S_\varepsilon(t)P_{\leq r_\alpha}v_0\|_{L_T^\infty H_x^1} \leq \alpha/3 \text{ and } \|S(t)v_0 - S(t)P_{\leq r_\alpha}v_0\|_{L_T^\infty H_x^1} \leq \alpha/3$$

On the other hand, denoting by w_{r_α} the gauge transform of $S(t)P_{r_\alpha}v_0$ and noticing that the Benjamin-Ono equation (1.5) can be rewritten as

$$\begin{aligned}&\partial_t v - b\mathcal{H}\partial_x^2 v - a\epsilon\partial_x^3 v - cv\partial_x v + d\epsilon\partial_x(v\mathcal{H}\partial_x v - \mathcal{H}(v\partial_x v))\\ &= d\epsilon\partial_x(v\mathcal{H}\partial_x v - \mathcal{H}(v\partial_x v)) - a\epsilon\partial_x^3 v\ ,\end{aligned}$$

we can proceed exactly as in the obtention of the Lipschitz bound (6.20) to get

$$\|S_\varepsilon(t)P_{\leq r_\alpha}v_0 - S(t)P_{\leq r_\alpha}v_0\|_{L_T^\infty H_x^1} \lesssim C(R)T\varepsilon\Big(\|w_{r_\alpha}\|_{L_T^\infty H^3} + p\Big(N_T^3(S(t)P_{\leq r_\alpha}v_0\Big)\Big)$$

where p is a polynomial function and the N_T^s-norm is defined in (5.3). This yields the result by taking $\varepsilon > 0$ small enough.

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References

1. L. Abdelouhab, J. L. Bona, M. Felland, and J-C. Saut, *Nonlocal models for nonlinear, dispersive waves,* Phys. D, **40** (1989), no. 3, 360–392.
2. J. Albert, J. L. Bona and J.-M. Restrepo *Solitary-wave solutions of the Benjamin equation,* SIAM J. Appl. Math., **59** (1999), 2139–2161.
3. J. L. Bona, D. Lannes and J.-C. Saut, *Asymptotic models for internal waves,* J. Math. Pures Appl., **89** (2008), 538-566.
4. N. Burq, F. Planchon, *On well-posedness for the Benjamin-Ono equation,* Math. Annal., **440** (2008), 497–542.
5. W. Craig, P. Guyenne and H. Kalisch, *Hamiltonian long wave expansions for free surfaces and interfaces,* Comm. Pure Appl. Math., **58** (2005), 1587–1641.
6. J. Ginibre, *Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain),* Astérisque, **237** (1996), 163–187.
7. Z. Guo and B. Wang, *Global well-posedness and inviscid limit for the Korteweg-de Vries-Burgers equation,* J. Diff. Eq., **246** (2009), 3864Ð3901.
8. Z. Guo, L. Peng, B. Wang and Y. Wang *Uniform well-posedness and inviscid limit for the Benjamin-Ono-Burgers equation* , Advances Math., **228** (2011), 647-677.
9. A. D. Ionescu, C. E. Kenig, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces,* J. Amer. Math. Soc., **20** (2007), 753–798.

10. A. D. Ionescu, C. E. Kenig and D. Tataru, *Global well-posedness of the KP-I initial value problem in the energy space,* Invent. Math., **173** (2008), 265–304.
11. R. Iorio, *On the Cauchy problem for the Benjamin-Ono equation,* Comm. Part. Diff. Eq., **11** (1986), 1031–1081.
12. C. E. Kenig, K. D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations,* Math. Res. Lett., **10** (2003), 879–895.
13. C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations,* Ind. Univ. Math. J., **40** (1991), 33–69.
14. C. E. Kenig, G. Ponce and L. Vega, *Well-posedness for the initial value problem of the Korteweg-de Vries equation,* J. Amer. Math. Soc., **4** (1991), 323–347.
15. C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle,* Comm. Pure Appl. Math., **46** (1993), 527–620.
16. H. Koch and D.Tataru, *A priori bounds for the 1D cubic NLS in negative Sobolev spaces,* Int. Math. Res. Not., (2007), no. 16.
17. H. Koch, N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$,* Int. Math. Res. Not. **26** (2003), 1449–1464.
18. F. Linares, D. Pilod and G. Ponce, *Well-posedness for a higher-order Benjamin-Ono equation,* J. Diff. Eq., **250** (2011), 450–475.
19. F. Linares and G. Ponce, "Introduction to nonlinear dispersive equations," Universitext. Springer, New York, 2009.
20. L. Molinet, *Global well-posedness in the energy space for the Benjamin-Ono equation on the circle,* Math. Ann., **337** (2007), 353–383.
21. L. Molinet, *Global well-posedness in L^2 for the periodic Benjamin-Ono equation,* Amer. J. Math., **130** (2008), 635–683.
22. L. Molinet, *A note on the inviscid limit of the Benjamin-Ono-Burgers equation in the energy space,* preprint (2011).
23. L. Molinet and D. Pilod, *The Cauchy problem for the Benjamin-Ono equation in L^2 revisited,* to appear in Anal. PDE, (2011).
24. L. Molinet, J-C. Saut and N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations,* SIAM J. Math. Anal., **33** (2001), no. 4, 982–988.
25. D. Pilod, *On the Cauchy problem for higher-order nonlinear dispersive equations,* J. Diff. Eq., **245** (2008), 2055-2077.
26. G. Ponce, *On the global well-posedness of the Benjamin-Ono equation,* Diff. Int. Eq. **4** (1991), 527–542.
27. T. Tao, *Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$,* J. Hyp. Diff. Eq., **1** (2004), 27–49.
28. T. Tao, *Multilinear weighted convolution of L^2-functions, and applications to nonlinear dispersive equations,* Amer. J. Math., **123** (2001), 839–908.
29. D. Tataru, *On global existence and scattering for the wave maps equation,* Amer. J. Math., **123** (2001), 37–77.