On vehicle following control systems with delays

Woihida Aggoune, Irinel-Constantin Morarescu, Silviu-Iulian Niculescu

Some remarks on vehicle following control systems with delays

Woihida Aggoune[*], Irinel-Constantin Morărescu[†], Silviu-Iulian Niculescu[‡]

Abstract

In this paper, we consider the problem of vehicle following control with delay. To solve the problem of traffic congestion, one of the solutions to be considered consists in organizing the traffic into *platoons*, that is groups of vehicles including a leader and a number of followers "tightly" spaced, all moving in a longitudinal direction. Excepting the stability of individual cars, the problem of avoidance of slinky type effects will be explicitly discussed. Sufficient conditions on the set of control parameters to avoid such a phenomenon will be explicitly derived in a frequency-domain setting.

1 INTRODUCTION

Traffic congestion (irregular flow of traffic) became an important problem in the last decade mainly to the exponential increasing of the transportation around medium- and large-size cities. One of the ideas to help solving this problem was the use of automatic control to replace human drivers and their low-predictable reaction with respect to traffic problems. As an example, human drivers have reaction time between $0.25 - 1.25$ sec of around 30m or more at 60kms/hour (see, for instance, Sipahi and Niculescu [2007] for a complete description of human drivers reactions, and further comments on existing traffic flow models).

[*]Equipe Commande des Systèmes (ECS), ENSEA, 6 Av. du Ponceau, 95014 Cergy-Pontoise Cedex, France. E-mail: aggoune@ensea.fr.

[†]CRAN, Nancy-Université, CNRS, 2, avenue de la forêt de Haye, 54516 Vandoeuvre-lès-Nancy Cedex, France. E-mail: constantin.morarescu@ensem.inpl-nancy.fr Corresponding author.

[‡]Laboratoire des Signaux et Systèmes (L2S), CNRS-Supélec 3 rue Joliot-Curie, 91190 Gif-sur-Yvette, France, E-mail: Silviu.Niculescu@lss.supelec.fr

A way to solve this problem is to organize the traffic into *platoons*, consisting in groups of vehicles including a leader and a number of followers in a longitudinal direction. In this case, the controller of each vehicle of a platoon would use the sensor information to try to reach the speed and acceleration of the preceding vehicle. Another problem to be considered is the so-called *slinky-type effect* (see, e.g. Burnham *et al.* [1974], Ioannou and Chien [1993], Shiekholslam and Desoer [1993] and the references therein). This is a phenomenon of amplification of the spacing errors between subsequent vehicles as vehicle index increases.

In Huang and Ren [1998], a control scheme to solve this multi-objective control problem was proposed. Known as *autonomous intelligent cruise control*, the controller in this scheme has access only to the relative state information of the preceding vehicle. This study is made under the assumptions that the leading vehicle performs a maneuver in finite time before reaching a steady state, and that prior to a maneuver, all the vehicles move at the same steady speed. The stability analysis of the system in closed-loop was performed by using a Lyapunov-Razumikhin approach leading to conservative conditions. The slinky-effect type phenomenon was discussed and some sufficient conditions to avoid slinky effects have been proposed, but without any explicit attempt in computing the whole set of controller's parameters guaranteeing the requested property. To the best of the authors' knowledge, such a problem has not received a definitive answer.

The aim of this paper is to give better answers to the problem mentioned above - construction of explicit control laws guaranteeing simultaneously individual stability and the avoidance of the slinky-type effect phenomenon. We use a frequency-domain method to give necessary and sufficient conditions for the individual stability analysis by computing the explicit delay bounds guaranteeing asymptotic stability. Next, we shall explicitly compute bounds on the controller's gains ensuring the avoidance of the slinky effects.

The remaining paper is organized as follows: In Section 2, the problem formulation is presented. In Section 3, we state and prove our main results concerning the stability of the system and the slinky effect avoidance conditions. In section 4, an illustrative example is presented. Finally, some concluding remarks end the paper.

2 SYSTEM MODEL AND PROBLEM FORMULATION

The general schema of a platoon of n vehicles is represented below, where $x_i(t)$ is the position of the ith vehicle with respect to some reference point O and H_i is the minimum separation distance allowable between the corresponding vehicles.

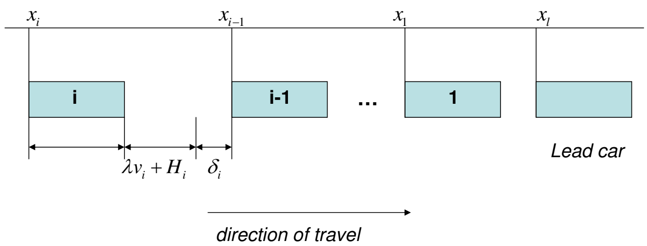


Figure 1: Platoon configuration

The goal is to maintain a distance $\lambda v_i + H_i$ between vehicle i and $i-1$, where λ is a prescribed headway constant and v_i the corresponding velocity (see Huang and Ren [1998]). The spacing error δ_i between the vehicles i and $(i-1)$ is defined as :

$$\delta_i(t) = x_{i-1}(t) - x_i(t) - (\lambda v_i + H_i)$$

in the case of system (1).

2.1 Model of vehicle dynamics

For each vehicle of the platoon, the model is of the form:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \gamma_i(t) \\ \dot{\gamma}_i(t) = -\frac{1}{\eta}\gamma_i(t) + \frac{1}{m\eta}u_i(t-\tau_i) - \frac{1}{m\eta}T_L, \end{cases} \tag{1}$$

where $x_i(t)$, $v_i(t)$ and $\gamma_i(t)$ represent respectively the position, the speed and the acceleration of the ith vehicle. Here, η is the vehicle's engine time-constant, m is the vehicle mass, T_L is the load torque on the engine speed, gear ratio, grade change etc., and it is assumed to be constant. τ_i is the total (corresponding) delay (including fueling and transport, etc.) for the ith vehicle (see Huang and Ren [1997] for more details).

2.2 Control law

In Huang and Ren [1998], the proposed control law is given by:

$$u_i(t) = k'_s\delta_i(t) + k'_v\dot{\delta}_i(t) + T_L, \tag{2}$$

where k'_s and k'_v are design constants. If one applies the control law (2) to the system (1), we shall obtain the following third order delay equation:

$$\begin{aligned} &\frac{d^3}{dt^3}\delta_i(t) = -\alpha\frac{d^2}{dt^2}\delta_i(t) - k_s\delta_i(t-\tau_i) \\ &-(k_v + \lambda k_s)\frac{d}{dt}\delta_i(t-\tau_i) - \lambda k_v\frac{d^2}{dt^2}\delta_i(t-\tau_i) \\ &+k_s\delta_{i-1}(t-\tau_{i-1}) + k_v\frac{d}{dt}\delta_{i-1}(t-\tau_{i-1}), \end{aligned} \tag{3}$$

where k_s and k_v are derived from k'_s and k'_v by an appropriate re-scaling. For the sake of simplicity, the corresponding computations are omitted (see Huang and Ren [1997] and Huang and Ren [1998]).

2.3 Frequency domain formulation

2.3.1 Individual stability

A basic control requirement for the overall system is the asymptotic stability of the ith vehicle if the preceding, the $(i-1)$th, is at steady-state (i.e. the spacing errors verify: $\delta_{i-1} = \dot{\delta}_{i-1} = 0$). In this case, the system is described by:

$$\begin{aligned} &\frac{d^3}{dt^3}\delta_i(t) = -\alpha\frac{d^2}{dt^2}\delta_i(t) - k_s\delta_i(t-\tau_i) \\ &-(k_v+\lambda k_s)\frac{d}{dt}\delta_i(t-\tau_i) - \lambda k_v\frac{d^2}{dt^2}\delta_i(t-\tau_i). \end{aligned} \tag{4}$$

Taking the Laplace transform, under zero initial conditions, we obtain a third-order transcendental equation of the form

$$\begin{aligned} \Gamma_i(s,\tau_i) &\triangleq \quad s^3 + \alpha s^2 + [\lambda k_v s^2 + (k_v+\lambda k_s)s + k_s]e^{-\tau_i s} \\ &= \quad Q(s) + P(s)e^{-s\tau} = 0. \end{aligned} \tag{5}$$

Assumption 1 *(a)* $P(0) \neq 0$

(b) The polynomials $P(s)$ and $Q(s)$ do not have common zeros

If Assumption 1.(a) is violated, then 0 is a zero of $\Gamma_i(s,\tau_i)$ for any $\tau_i \in \mathbb{R}_+$. Therefore, the system is never asymptotically stable. If assumption 1.(b) is not satisfied, $P(s)$ and $Q(s)$ have a common factor $c(s) \neq$ constant. Simplifying by $c(s)$ we get a system described by (5) which satisfies assumption 1.(b).

The individual vehicle stability is guaranteed if and only if Γ has all its roots in the left half complex plane. This depends on the delay magnitude τ_i.

Then the problem of stability can be formulated as a research of parameters α, λ, k_s and k_v such that this condition is ensured.

2.3.2 Avoiding slinky effect

The second part of the multi-objective problem previously defined consist in controlling the slinky effect. The goal is to find sufficient conditions to guarantee that we avoid such a phenomenon. Considering the system (3) and applying the Laplace transform, one gets

$$G(s) \triangleq \delta_i(s)/\delta_{i-1}(s) = \frac{(k_s + sk_v)e^{-\tau_{i-1}s}}{(k_s + (k_v+\lambda k_s)s + \lambda k_v s^2)e^{-\tau_i s} + \alpha s^2 + s^3}. \tag{6}$$

One has no *slinky-type effect* if:

$$|G(jw)| = |\frac{\delta_i(jw)}{\delta_{i-1}(jw)}| < 1 \tag{7}$$

for any $w > 0$ (see Ioannou and Chien [1993], Shiekholslam and Desoer [1993], Swaroop *et al.* [1994]). Then the problem turns out in finding the set of parameters (k_s, k_v) and the delays τ_i such that the stability of the system (4) is guaranteed and the condition (7) is satisfied.

3 MAIN RESULTS

3.1 Delay stability margin

Before proceeding further, we consider the case without delay. The closed-loop system free of delay is asymptotically stable when the polynomial $\Gamma_i(s,0)$ is Hurwitz. Since $\alpha,\ k_s,\ k_v \in \mathbb{R}_+$, the third-order polynomial:

$$s^3 + (\alpha + \lambda k_v)s^2 + (k_v + \lambda k_s)s + k_s = 0 \tag{8}$$

is Hurwitz if and only if:

$$(\alpha + \lambda k_v)(k_v + \lambda k_s) > k_s, \tag{9}$$

which is equivalent to

$$\lambda k_v^2 + (\alpha + \lambda^2 k_s)k_v + (\alpha\lambda - 1)k_s > 0. \tag{10}$$

Note that a sufficient condition for (10) is:

$$k_v > \frac{1-\alpha\lambda}{\lambda^2}.$$

Denote now by Ω the set of crossing frequencies, that is the set of reals $\omega > 0$, such that $\pm j\omega$ is a solution of the characteristic equation (5). Then the following statement holds.

Proposition 1 *Consider the characteristic equation (5) associated to the system (4). Then:*

- *(a) the crossing frequency set Ω is not empty*
- *(b) the system is asymptotically stable for all delays $\tau_i \in (0, \tau^\star)$ where $\tau^\star$ is defined by:*

$$\tau^\star = \min_{\omega\in\Omega} \Big\{ \tau_k(\omega) \mid \tau_k(\omega) > 0,\, k \in \mathbb{Z} \Big\}, \tag{11}$$

where

$$\tau_k(\omega) = \frac{1}{\omega}((2k+1)\pi + \angle(Q(jw)) - \angle(P(jw))$$

Proof. (a) Straightforward. Assume by contradiction that the delay-independent stability holds. As discussed in Niculescu [2001], a necessary condition for delay-independent stability is the Hurwitz stability of Q, and this is not the case.

(b) Since the system free of delay is asymptotically stable, the conclusion of (a) leads to the existence of a delay margin $\tau^\star$, such that the system is asymptotically stable for all delays $\tau \in [0, \tau^\star)$. Furthermore at $\tau = \tau^\star$ the system becomes unstable if and only if the characteristic equation (5) has at least one root $s = jw$ on the imaginary axis. In other words if there exists $w \in \Omega$ a crossing frequency. Since

$$\frac{P(jw)}{Q(jw)} = -e^{-jw\tau} \tag{12}$$

one can derive the delay values corresponding to each crossing frequency ω as:

$$\tau_k(\omega) = \frac{1}{\omega}\big((2k+1)\pi + \angle(Q(jw)) - \angle(P(jw)\big) \tag{13}$$

Obviously τ^* is the smallest positive value that satisfies the previous relation. ■

The condition *(a)* above simply says that the corresponding system cannot be delay-independent asymptotically stable, and the condition *(b)* above gives an explicit expression of the delay margin $\tau^\star$.

3.2 Stability analysis in controller parameter space (k_v, k_s)

In the sequel, we study the behavior of the system for a fixed delay value τ. More precisely, for a given $\tau = \tau^*$ we search the crossing frequencies ω and the corresponding *crossing points* in the parameter space (k_v, k_s) defined by the control law such that $Q(j\omega, k_v, k_s, \tau^*) + P(j\omega, k_v, k_s, \tau^*)\mathrm{e}^{-j\omega\tau^*} = 0$.

According to the continuity of zeros with respect to the delay parameters, the number of roots in the right-half plane (RHP) can change only when some zeros appear and cross the imaginary axis. Thus, it is natural to consider the *frequency crossing set* Ω consisting of all real positive ω such that there exist at least a pair (k_v, k_s) for which

$$H(j\omega, k_v, k_s, \tau^*) \triangleq Q(j\omega) + P(j\omega)\mathrm{e}^{-j\omega\tau} = 0. \tag{14}$$

Remark 1 *Using the conjugate of a complex number we get*

$$H(j\omega, k_v, k_s, \tau) = 0 \Leftrightarrow H(-j\omega, k_v, k_s, \tau) = 0.$$

Therefore, it is natural to consider only positive frequencies, that is $\Omega \subset (0, \infty)$.

Considering that the set Ω and the parameters $\alpha,\ \lambda$ are known we can easily derive all the crossing points in the parameter space (k_v, k_s).

Proposition 2 *For a given $\tau > 0$ and $\omega \in \Omega$ the corresponding crossing point (k_v, k_s) is given by:*

$$k_v = \frac{\omega^2(1-\alpha\lambda)\cos\omega\tau + \omega(\alpha+\lambda\omega^2)\sin\omega\tau}{1+\lambda^2\omega^2} \tag{15}$$

$$k_s = \frac{\omega^2(\lambda\omega^2+\alpha)\cos\omega\tau + \omega^3(\alpha\lambda-1)\sin\omega\tau}{1+\lambda^2\omega^2} \tag{16}$$

Proof. Using the decomposition of the equation (14) into real and imaginary part, straightforward computation lead us to

$$k_v + \lambda k_s = \omega(\omega\cos\omega\tau + \alpha\sin\omega\tau), \tag{17}$$
$$k_s - \lambda k_v\omega^2 = \omega^2(\alpha\cos\omega\tau - \omega\sin\omega\tau) \tag{18}$$

and further we can derive the result stated above. □

To illustrate our purpose, let us consider the case where $\alpha = 5$, $\lambda = 1$ and $\tau = 0.5$, then for each $\omega \in \Omega$ the corresponding crossing points (k_v, k_s) are represented in the following figure.

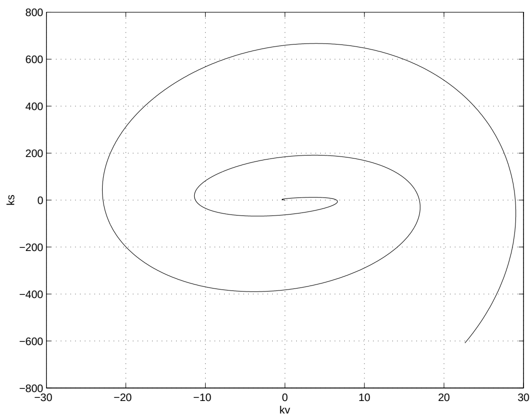


Figure 2: Crossing points

Remark 2 *For all $\omega \in \Omega$ we have $P(j\omega) \neq 0$. Indeed, it is easy to see that if $\omega \in \Omega$, then there exists at least one pair (k_v, k_s) such that $H(j\omega, k, T, \tau) = 0$. Therefore, assuming that $P(j\omega) = 0$ we get also $Q(j\omega) = 0$ which contradicts assumption 1.(b).*

Since we are interested to find the crossing points (k_v, k_s) such that k_v and k_s are *finite* the frequency crossing set Ω is characterized by the following:

Proposition 3 *The frequency crossing set* Ω *consists of a finite number of intervals of finite length.*

Proof. It is obvious from the equations (15) and (16) that the controller parameters k_v and k_s approach infinity when $\omega \to \infty$. Thus, in order to have finite values for k_v and k_s we have to impose an upper limit for the variation of ω. On the other hand, considering $\Omega \subset (0, M]$, it is clear that the inequalities $k_v > 0$ and $k_s > 0$ are simultaneously satisfied for ω into a finite number of intervals included in $(0, M]$.
□

Let us suppose that $\Omega = \bigcup_{\ell=1}^{N} \Omega_\ell$. Then (15) and (16) define a continuous curve. Using the notations introduced in the previous paragraph and the technique developed in Gu *et al.* [2005a] and Morărescu *et al.* [2007], we can easily derive the crossing direction corresponding to this curve.

More exactly, let us denote $\mathcal{T}_\ell$ the curve defined above and consider the following decompositions into real and imaginary parts:

$$\begin{aligned} R_0 + jI_0 &= \frac{j}{s}\frac{\partial H(s,k_v,k_s,\tau)}{\partial s}\bigg|_{s=j\omega} \\ R_1 + jI_1 &= -\frac{1}{s}\frac{\partial H(s,k_v,k_s,\tau)}{\partial k_v}\bigg|_{s=j\omega}, \\ R_2 + jI_2 &= -\frac{1}{s}\frac{\partial H(s,k_v,k_s,\tau)}{\partial k_s}\bigg|_{s=j\omega}. \end{aligned}$$

Then, since $H(s, k_v, k_s, \tau)$ is an analytic function of s, k_v and k_s, the implicit function theorem indicates that the tangent of $\mathcal{T}_\ell$ can be expressed as

$$\begin{pmatrix} \dfrac{\mathrm{d}k_v}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}k_s}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_1 I_0 - R_0 I_1 \\ R_0 I_2 - R_2 I_0 \end{pmatrix}, \tag{19}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{20}$$

It follows that $\mathcal{T}_\ell$ is smooth everywhere except possibly at the points where either (20) is not satisfied, or when

$$\frac{\mathrm{d}k_v}{\mathrm{d}\omega} = \frac{\mathrm{d}k_s}{\mathrm{d}\omega} = 0. \tag{21}$$

From the above discussions, we can conclude with the following:

Proposition 4 *The curve $\mathcal{T}_\ell$ is smooth everywhere except possibly at the point corresponding to $s = j\omega$ such that $s = j\omega$ is a multiple solution of (14).*

Proof. If (21) is satisfied then staightforward computations show us that $R_0 = I_0 = 0$. In other words $s = j\omega$ is a multiple solution of (14).

On the other hand,

$$R_1 I_2 - R_2 I_1 = -\omega(1 + \lambda^2\omega^2) < 0, \, \forall\omega > 0.$$

■

The next paragraph focuses on the characterization of the crossing direction corresponding to each of the curves defined by (15) and (16) (see, for instance, Morărescu [2006] or Morărescu and Niculescu [2007] for similar results for different problems).

We will call the direction of the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

Proposition 5 *Assume $\omega \in \Omega_\ell$, k_v, k_s satisfy (15) and (16) respectively, and ω is a simple solution of (14) and $H(j\omega', k_v, k_s, \tau) \neq 0$, $\forall\omega' > 0$, $\omega' \neq \omega$ (i.e. (k_v, k_s) is not an intersection point of two curves or different sections of a single curve).*

Then a pair of solutions of (14) cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_1 I_2 - R_2 I_1 > 0$. The crossing is to the left if the inequality is reversed.

Remark 3 *In the proof of Proposition 4 we have shown that $R_1 I_2 - R_2 I_1$ is always negative. Thus, a system described by (14) may have more than one stability region in controller parameter space (k_v, k_s) if one of the following two items are satisfied:*

- *it has one or more crossing curves with some turning points (the direction of $\mathcal{T}_\ell$ in controller parameter space changes).*
- *it has at least two different crossing curves with opposite direction in (k_v, k_s) - space.*

3.3 Avoiding slinky effects

Now, we treat the second part of the multi-objective problem under consideration. This correspond to the characterization of the conditions guaranteeing that we avoid slinky-effects. We consider the system (3). Applying the

Laplace transform one obtains:

$$G(s) = \frac{\delta_i(s)}{\delta_{i-1}(s)} = \frac{(k_s + sk_v)e^{-\tau_{i-1}s}}{(k_s + (k_v + \lambda k_s)s + \lambda k_v s^2)e^{-\tau_i s} + \alpha s^2 + s^3}. \quad (22)$$

There is no slinky effect if :

$$|G(jw)| < 1, \quad \forall \omega > 0 \quad (23)$$

This condition can be rewritten as:

$$A(w, \tau_i)(w) = w^2 B(w, \tau_i) \geq 0 \quad (24)$$

with

$$\begin{array}{rcl} B(w, \tau_i)(w) & = & w^4 - 2\lambda k_v sin(w\tau_i)w^3 + \\ & & (\lambda^2 k_v^2 + \alpha^2 + 2(\alpha\lambda k_v - k_v - \lambda k_s)cos(w\tau_i))w^2 + \\ & & 2(k_s - \alpha(k_v + \lambda k_s))sin(w\tau_i)w + +\lambda^2 k_s^2 - 2\alpha k_s cos(w\tau_i) \end{array} \quad (25)$$

which should be satisfied for all $w \in \mathbb{R}$.
The objective is to define conditions on the parameters of the controller, in order to satisfy this constraint.
Consider first the case $\tau_i = 0$. Then, we have:

$$\begin{array}{rcl} B(w, 0) & = & w^4 + \left[(\lambda k_v + \alpha)^2 - 2(k_v + \lambda k_s)\right]w^2 \\ & + & \lambda^2 k_s^2 - 2\alpha k_s \end{array} \quad (26)$$

A necessary condition for the positivity of $B(w, 0)$ is

$$\lambda^2 k_s^2 - 2\alpha k_s > 0, \quad (27)$$

which implies that:

$$k_s \in (\frac{2\alpha}{\lambda^2}, +\infty) \quad (28)$$

Under this condition, the positivity of $B(w, 0)$ is guaranteed if:

$$\left[(\lambda k_v + \alpha)^2 - 2(k_v + \lambda k_s)\right]^2 \leq 4(\lambda^2 k_s^2 - 2\alpha k_s). \quad (29)$$

which leads to:

$$\begin{array}{rcl} -2k_s\lambda\sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}} & \leq & (\lambda k_v + \alpha)^2 - 2(k_v + \lambda k_s) \\ & & \leq 2k_s\lambda\sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}} \end{array} \quad (30)$$

In order to complete this analysis, we want to characterize the set of parameters k_v guaranteeing the previous inequality under the constraint (28).
If we consider first the right part of (30), which is equivalent to:

$$\lambda^2 k_v^2 + 2(\lambda\alpha - 1)k_v + \alpha^2 - 2\lambda k_s(1 + \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}) \leq 0$$

we can remark that if

$$k_s > \max\{\frac{2\alpha}{\lambda^2}, \frac{2\alpha\lambda - 1}{2\lambda^3}\} \qquad (31)$$

then there exists at least one positive value k_v, such that the right part of (30) is satisfied. Moreover k_v should satisfy:

$$\max\{0, \frac{1 - \alpha\lambda - \sqrt{\Delta_1}}{\lambda^2}\} \leq k_v \leq \frac{1 - \alpha\lambda + \sqrt{\Delta_1}}{\lambda^2}. \qquad (32)$$

where

$$\Delta_1 = 1 - 2\alpha\lambda + 2\lambda^3 k_s\big(1 + \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}\big).$$

The left inequality in (30) can be rewritten as:

$$\lambda^2 k_v^2 + 2(\lambda\alpha - 1)k_v + \alpha^2 - 2\lambda k_s(1 - \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}) \geq 0$$

This leads to the following condition on k_v :

$$k_v \in (-\infty, \frac{1 - \alpha\lambda - \sqrt{\Delta_2}}{\lambda^2}] \;\cup\; [\frac{1 - \alpha\lambda + \sqrt{\Delta_2}}{\lambda^2}, +\infty). \qquad (33)$$

where

$$\Delta_2 = 1 - 2\alpha\lambda + 2\lambda^3 k_s\big(1 - \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}\big)$$

is assumed to be positive. If $\Delta_2 < 0$, then the left part of (30) will be satisfied for all positive k_v. Finally, using the conditions (32) and (33) function of the sign of Δ_2, it follows that k_v must be chosen in the intersection of the intervals defined by (32) and (33).

Now we analyze the sign of $B(w, \tau_i)$ when $\tau_i \geq 0$. We consider again the expression given in (24) of $B(w, \tau_i)$.
For the terms involving $cos(w\tau_i)$, we have:

$$-2\alpha k_s cos(w\tau_i) \geq -2\alpha k_s$$

and

$$2(\alpha\lambda k_v - k_v - \lambda k_s)cos(w\tau_i) \geq -2|\alpha\lambda k_v - k_v - \lambda k_s|.$$

Concerning the terms involving $sin(w\tau_i)$, since $sin(w\tau_i) \leq w\tau_i$ for $w > 0$ then:

$$-2\lambda k_v sin(w\tau_i)w^3 \geq -2\lambda k_v \tau_i w^4 \geq -2\lambda k_v \tau^\star w^4$$

and

$$\begin{aligned} &2(k_s - \alpha(k_v + \lambda k_s))sin(w\tau_i)w \\ &\quad \geq -2|k_s - \alpha(k_v + \lambda k_s)|\tau_i w^2 \\ &\quad \geq -2|k_s - \alpha(k_v + \lambda k_s)|\tau^\star w^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &B(w,\tau_i) \geq (1 - 2\lambda k_v \tau^\star)w^4 + [\lambda^2 k_v^2 + \alpha^2 \\ &-2|\alpha\lambda k_v - k_v - \lambda k_s| \, -2\tau^\star|k_s - \alpha(k_v + \lambda k_s)|]w^2 \\ &+\lambda^2 k_s^2 - 2\alpha k_s \\ &\geq (1 - 2\lambda k_v \tau^\star)w^4 + [(\lambda k_v - \alpha)^2 - 2k_v - 2\lambda k_s \\ &-2\tau^\star k_s \, -2\tau^\star\alpha(k_v + \lambda k_s)]w^2 + \lambda^2 k_s^2 - 2\alpha k_s \geq 0. \end{aligned}$$

Let us set :

$$\begin{aligned} &C(w,\tau^\star) = (1 - 2\lambda k_v \tau^\star)w^4 + [(\lambda k_v - \alpha)^2 - 2k_v \\ &-2\lambda k_s - 2\tau^\star k_s - 2\tau^\star\alpha(k_v + \lambda k_s)]w^2 + \lambda^2 k_s^2 - 2\alpha k_s \end{aligned}$$

We suppose that :

$$1 - 2\lambda k_v \tau^\star > 0. \tag{34}$$

Then the positivity of $C(w,\tau^\star)$ is ensured if (28) is satisfied and if we have:

$$\begin{aligned} &[(\lambda k_v - \alpha)^2 - 2k_v - 2\lambda k_s - 2\tau^\star k_s \\ &-2\tau^\star\alpha(k_v + \lambda k_s)]^2 \leq 4(1 - 2\lambda k_v \tau^\star)(\lambda^2 k_s^2 - 2\alpha k_s). \end{aligned} \tag{35}$$

This leads to the condition:

$$\begin{aligned} &-2k_s\lambda\sqrt{(1 - \frac{2\alpha}{\lambda^2 k_s})(1 - 2\lambda k_v \tau^\star)} \leq \\ &(\lambda k_v - \alpha)^2 - 2k_v - 2\lambda k_s - 2\tau^\star(k_s + \alpha(k_v + \lambda k_s)) \\ &\leq 2k_s\lambda\sqrt{(1 - \frac{2\alpha}{\lambda^2 k_s})(1 - 2\lambda k_v \tau^\star)} \end{aligned} \tag{36}$$

Now, we search to define the set of parameters k_v which satisfy these inequalities.

If we consider the right part of (36), which can be rewritten as:

$$\begin{aligned} &\lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) \\ &-2\lambda k_s\big(1+\sqrt{(1-\frac{2\alpha}{\lambda^2 k_s})(1-2\lambda k_v\tau^\star)}\big) \le 0, \end{aligned} \tag{37}$$

with k_v under the square root.

Since $1-2\lambda k_v\tau^\star \le 1$ and $1-\dfrac{2\alpha}{\lambda^2 k_s} \le 1$ then

$$\begin{aligned} &\lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) \\ &-2\lambda k_s\big(1+\sqrt{(1-\frac{2\alpha}{\lambda^2 k_s})(1-2\lambda k_v\tau^\star)}\big) \\ \le\ &\lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) \\ &-2\lambda k_s\big(1+(1-2\lambda k_v\tau^\star)(1-\frac{2\alpha}{\lambda^2 k_s})\big) \end{aligned} \tag{38}$$

Thus, if we can find k_v such that:

$$\begin{aligned} &\lambda^2 k_v^2 - 2(1+\alpha\lambda+5\alpha\tau^\star-2\tau^\star\lambda^2 k_s)k_v \\ &+\alpha^2 - 2\tau^\star(1+\alpha\lambda)k_s - 4\lambda k_s + \frac{4\alpha}{\lambda} \le 0 \end{aligned} \tag{39}$$

then the right part of (36), would be satisfied.

A necessary condition to guarantee this previous condition is to have:

$$\begin{aligned} \Delta_{1,\tau^\star} &= \Big(1+\alpha\lambda+5\alpha\tau^\star-2\tau^\star\lambda^2 k_s\Big)^2 \\ &\quad -\lambda^2\Big(\alpha^2-2\tau^\star(1+\alpha\lambda)k_s-4\lambda k_s+\frac{4\alpha}{\lambda}\Big) \\ &\quad \ge 0 \end{aligned} \tag{40}$$

and then under this condition, we choose k_v as follows :

$$\max\{0, \frac{a_1-\sqrt{\Delta_{1,\tau^\star}}}{\lambda^2}\} \le k_v \le \frac{a_1+\sqrt{\Delta_{1,\tau^\star}}}{\lambda^2}. \tag{41}$$

where $a_1 = 1 + \alpha\lambda + 5\alpha\tau^\star - 2\tau^\star\lambda^2 k_s$.
We can remark that (40) can be rewritten as:

$$4{\tau^\star}^2\lambda^4 k_s^2 + 2\lambda^2\big(2\lambda - \tau^\star(1 + 10\alpha\tau^\star + \alpha\lambda)\big)k_s$$
$$+(1 + 5\alpha\tau^\star)^2 + 2\alpha\lambda[5\alpha\tau^\star - 1] \geq 0$$

Note that this last inequality leads to the following condition on k_s:

$$k_s \in \big(-\infty, \xi_1\big] \bigcup [\xi_2, +\infty\big) \tag{42}$$

where
$$\xi_1 = \frac{\lambda^2\big(2\lambda - \tau^\star(1 + 10\alpha\tau^\star + \alpha\lambda)\big) - \sqrt{\overline{\Delta_{1,\tau^\star}}}}{4{\tau^\star}^2\lambda^4}$$
and
$$\xi_2 = \frac{\lambda^2\big(2\lambda - \tau^\star(1 + 10\alpha\tau^\star + \alpha\lambda)\big) + \sqrt{\overline{\Delta_{1,\tau^\star}}}}{4{\tau^\star}^2\lambda^4}.$$

$$\overline{\Delta_{1,\tau^\star}} = \lambda^4\big(2\lambda - \tau^\star(1 + 10\alpha\tau^\star + \alpha\lambda)\big)^2 - 4{\tau^\star}^2\lambda^4[(1 + 5\alpha\tau^\star)^2 + 2\alpha\lambda(5\alpha\tau^\star - 1)]$$

which is supposed to be positive. If it is not the case, then the condition (40) is verified for all $k_s \geq 0$.

We consider now the left part of (36), which can be rewritten as :

$$\begin{aligned} 0 \leq & \lambda^2 k_v^2 - 2(1 + \alpha\lambda + \alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s + \alpha\lambda k_s) \\ & -2\lambda k_s\Big(1 - \sqrt{(1 - \frac{2\alpha}{\lambda^2 k_s})(1 - 2\lambda k_v\tau^\star)}\Big). \end{aligned} \tag{43}$$

Proceeding as above, we have:

$$\begin{aligned} & \lambda^2 k_v^2 - 2(1 + \alpha\lambda + \alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s + \alpha\lambda k_s) \\ & -2\lambda k_s\Big(1 - (1 - 2\lambda k_v\tau^\star)(1 - \frac{2\alpha}{\lambda^2 k_s})\Big) \\ \leq & \lambda^2 k_v^2 - 2(1 + \alpha\lambda + \alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s + \alpha\lambda k_s) \\ & -2\lambda k_s\Big(1 - \sqrt{(1 - \frac{2\alpha}{\lambda^2 k_s})(1 - 2\lambda k_v\tau^\star)}\Big) \end{aligned} \tag{44}$$

If there exists k_v such that:

$$
\begin{aligned}
0 \le\ & \lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star \\
& +2\tau^\star\lambda^2 k_s(1-\frac{2\alpha}{\lambda^2 k_s}))k_v \\
& +\alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) - 2\lambda k_s\big(1-(1-\frac{2\alpha}{\lambda^2 k_s})\big)
\end{aligned}
\tag{45}
$$

then the left part of (36), will be verified.
This inequality can be simplified as :

$$
\begin{aligned}
0 \le\ & \lambda^2 k_v^2 - 2(1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s)k_v \\
& +\alpha^2 - 2\tau^\star(1+\alpha\lambda)k_s - \frac{4\alpha}{\lambda}
\end{aligned}
\tag{46}
$$

This is satisfied for all k_v such that :

$$
\begin{aligned}
k_v \in & \Big(-\infty, \frac{1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s-\sqrt{\Delta_{2,\tau^\star}}}{\lambda^2}] \\
& \bigcup[\frac{1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s+\sqrt{\Delta_{2,\tau^\star}}}{\lambda^2}, +\infty\Big).
\end{aligned}
\tag{47}
$$

where

$$
\begin{aligned}
\Delta_{2,\tau^\star} =\ & \Big(1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s\Big)^2 \\
& -\lambda^2\Big(\alpha^2-2\tau^\star(1+\alpha\lambda)k_s-\frac{4\alpha}{\lambda}\Big)
\end{aligned}
\tag{48}
$$

is supposed to be positive.
If this quantity is negative, then the inequality (45) and by consequence (43), would be satisfied for all $k_v \ge 0$.
The positivity of $\Delta_{2,\tau^\star}$ can be rewritten as:

$$
\begin{aligned}
& 4{\tau^\star}^2\lambda^4 k_s^2 + 6\lambda^2\tau^\star[1+\alpha-2\alpha\tau^\star]k_s \\
& +(1-3\alpha\tau^\star)^2 + 6\alpha\lambda(1-\alpha\tau^\star) \ge 0
\end{aligned}
$$

which leads to the condition on k_s given by:

$$k_s \in \Big(-\infty, \frac{3\lambda^2\tau^\star(2\alpha\tau^\star - 1 - \alpha) - \sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}] \bigcup [\frac{3\lambda^2\tau^\star(2\alpha\tau^\star - 1 - \alpha) + \sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}, +\infty\Big). \tag{49}$$

if $\overline{\Delta_{2,\tau^\star}}$ defined by :

$$\overline{\Delta_{2,\tau^\star}} = 9\lambda^4{\tau^\star}^2[1+\alpha-2\alpha\tau^\star]^2 - 4\lambda^4{\tau^\star}^2[(1-3\alpha\tau^\star)^2 + 6\alpha\lambda(1-\alpha\tau^\star)] \tag{50}$$

is positive.
It is clear that if $\overline{\Delta_{2,\tau^\star}}$ is negative, then the positivity of $\Delta_{2,\tau^\star}$ would be satisfied for all $k_s \geq 0$.
Now the hypothesis of negativity of $\Delta_{2,\tau^\star}$, which would imply that the left part of (36) is satisfied for all k_v positive, turns out to write that :

$$4{\tau^\star}^2\lambda^4 k_s^2 + 6\lambda^2\tau^\star[1+\alpha-2\alpha\tau^\star]k_s + (1-3\alpha\tau^\star)^2 + 6\alpha\lambda(1-\alpha\tau^\star) \leq 0$$

which is satisfied for

$$\max\{0, \frac{3\lambda^2\tau^\star(2\alpha\tau^\star - 1 - \alpha) - \sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}\} \leq k_s \leq \frac{3\lambda^2\tau^\star(2\alpha\tau^\star - 1 - \alpha) + \sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}. \tag{51}$$

where $\overline{\Delta_{2,\tau^\star}}$ is assumed to be positive.

In conclusion, the determination of the parameters k_v and k_s guaranteeing that (36) is satisfied, can be summarized for the right part of (36), by the choice of k_v in the interval defined by (41) under the necessary condition that $\Delta_{1,\tau^\star}$ is positive. And for the left part of (36), we can choose any $k_v > 0$ or k_v in the interval defined by (47), according to the sign of $\Delta_{2,\tau^\star}$.
We can note that $\Delta_{1,\tau^\star}$ and $\Delta_{2,\tau^\star}$ are function of k_s. Their sign are conditioned by the sign of $\overline{\Delta_{1,\tau^\star}}$ and $\overline{\Delta_{2,\tau^\star}}$.
In the following section, we illustrate our results with some examples.

4 Simulation results

We consider a platoon of 4 following vehicles. We suppose that initially these vehicles travel at the the steady-state velocity of $v_0 = 20m/s$. The following figure correspond to the velocity and acceleration profile of the lead vehicle.

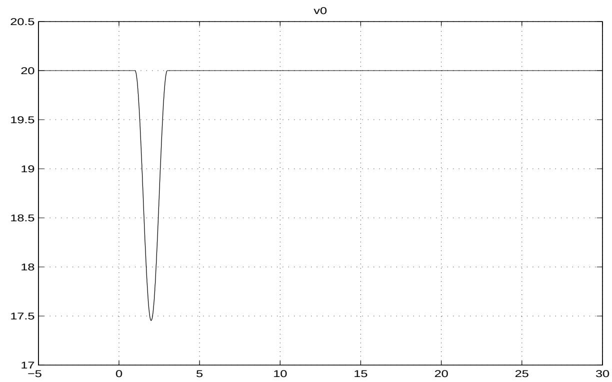


Figure 3: Velocity profile of the lead vehicle

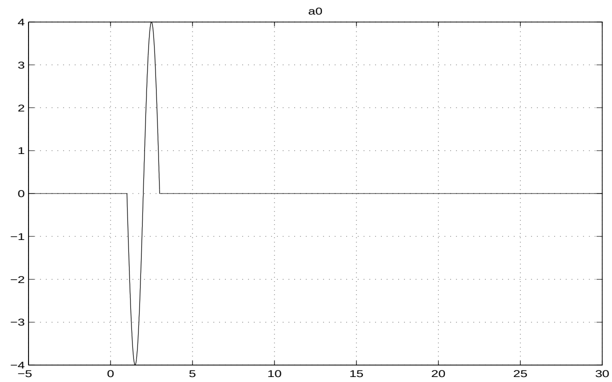


Figure 4: Acceleration profile of the lead vehicle

We assume that the safety distance is characterized by $\lambda = 1$ and $H_i = 2m$ with $\alpha = 5$. We choose the controller parameters $k_s = 19$ and $k_v = 0.12$. Then by Proposition 1, we obtain the *optimal delay margin* equal to $\tau^* = 0.215$. The system (4) is then asymptotically stable for all delays $\tau < 0.215$.

We arrive to the same conclusion by using the Matlab package DDE-

BIFTOOL (bifurcation analysis of delay differential equations), (see Engelborghs *et al.* [2001], Engelborghs *et al.* [2002]) to represent the rightmost roots of the characteristic equation. Indeed, if we choose the limit value of the delay $\tau = 0.215$ then we can observe that rightmost roots of the characteristic equation are on the imaginary axis. When we choose a delay larger, the system is unstable since there exists roots in the right half plane.

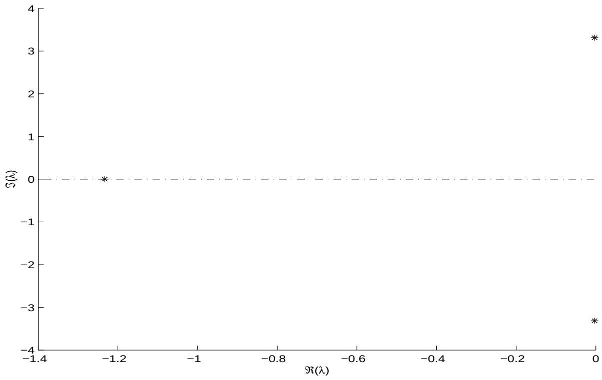


Figure 5: Rightmost roots of the characteristic equation for $\tau = 0.215$

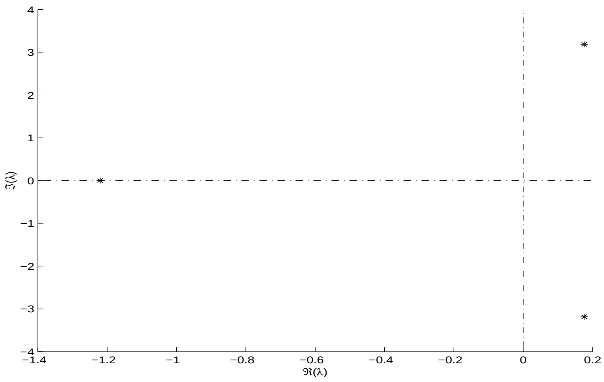
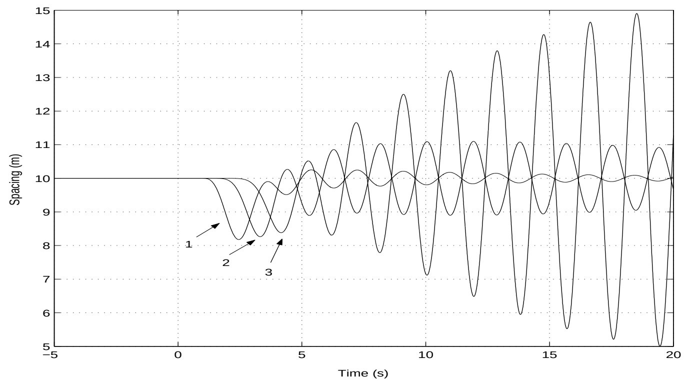
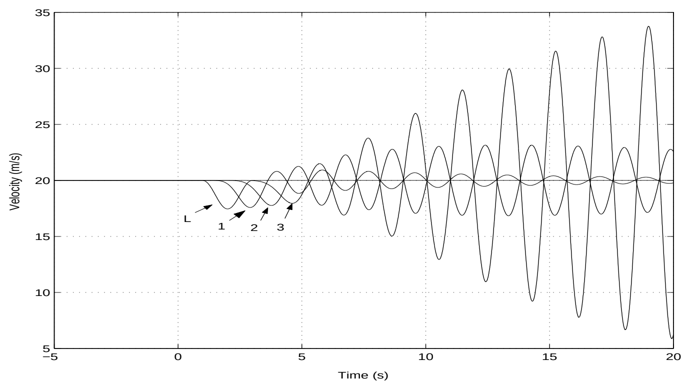


Figure 6: Rightmost roots of the characteristic equation for $\tau = 0.25$

Now, if we consider the second part of the multi-objective problem, we can remark that the conditions to avoid slinky-effect We can also note that

in order to have no slinky effects we just have to restrict this bound to $\tau = 0.0504$.
Then, if we choose a delay $\tau = 0.2$, we can observe the phenomenon of slinky effect. This is what we can observe in the following figures.





If we choose a delay $\tau = 0.05$, then we can remark that there is no slinky effect.

Thus, in order to guarantee the individual stability of vehicles of the platoon and to avoid the slinky effect phenomenon, it suffices to choose the delay $\tau \leq \min(0.215, 0.0504) = 0.0504$.

5 CONCLUSIONS

In this paper, we have considered the problem of vehicle following control system. For a given controller structure, we have developed conditions

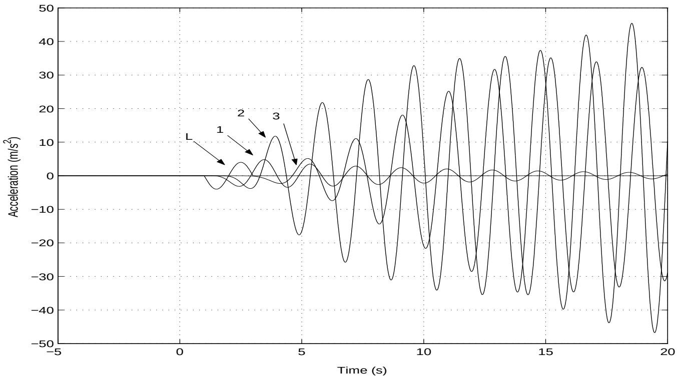
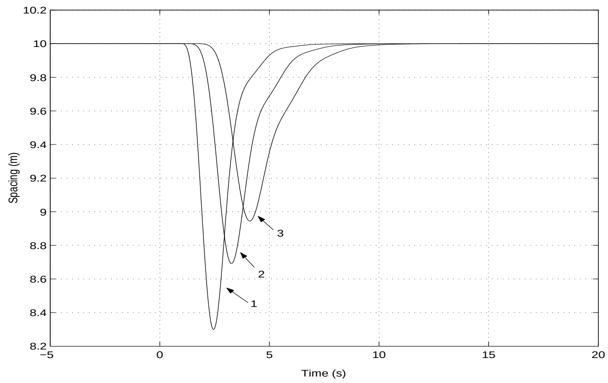
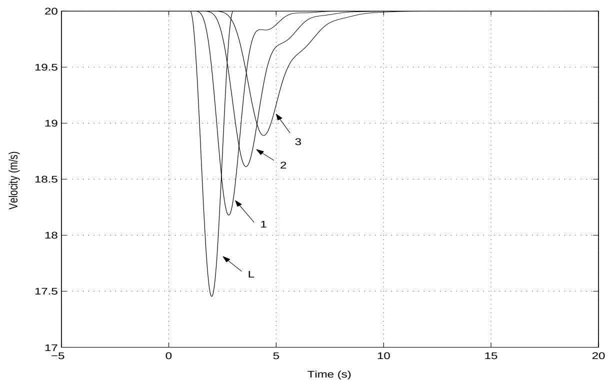


Figure 7: Control responses of 4 following vehicles with time delay 0.2 s





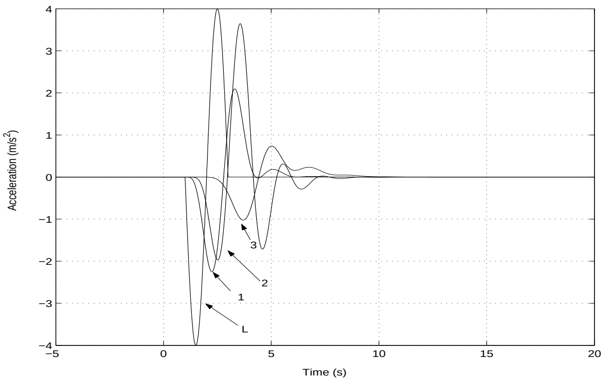


Figure 8: Control responses of 4 following vehicles with time delay 0.05 s

guaranteeing the individual stability of each vehicle of the platoon, and the derived conditions depend on the size of the delay. Moreover, we considered the problem of slinky-effect phenomenon, and we proposed sufficient conditions to avoid it. We have given an explicit characterization of some sets of controller parameters which solve the problem.

References

Burnham, G.O., Seo, J. and Bekey, G.A. (1974). Identification of human drivers models in car following, *IEEE Transactions on Automatic Control*, vol. **19**, pp. 911–915.

Cooke, K. L. and van den Driessche, P. (1986). On zeroes of some transcendental equations. in *Funkcialaj Ekvacioj*, vol. **29**, pp. 77-90.

Engelborghs, K., Luzyanina, T. and Samaey, G. (2001). *DDE-BIFTOOL v. 2.00: a Matlab package for bifurcation analysis of delay differential equations*, Technical Report TW-330, Departement of Computer Science, K.U.Leuven, Belgium.

Engelborghs, K., Luzyanina, T. and Roose, R. (2002) Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL, *ACM Trans. Math. Softw.*, vol. **28**, pp. 1–21.

El'sgol'ts, L.E. and Norkin, S.B. (1973). *Introduction to the theory and applications of differential equations with deviating arguments* (Academic Press: New York).

Gu, K., Kharitonov, V.L. and Chen, J. (2003). *Stability of time-delay systems* (Birkhauser: Boston).

Gu, K., Niculescu, S.-I. and Chen, J. (2005). “On stability crossing curves for general systems with two delays,” in *J. Math. Anal. Appl.*, vol. 311, pp. 231-253.

Hale, J. K. and Verduyn Lunel, S. M. (1993). *Introduction to Functional Differential Equations* (Applied Math. Sciences, **99**, Springer-Verlag, New York).

Huang, S. and Ren, W. (1997). Design of vehicle following control systems with actuator delays, in *International Journal of Systems Science*, vol. **28**, pp. 145–151.

Huang, S. and Ren, W. (1998). Autonomous Intelligent Cruise Control with Actuator Delays, in *Journal of Intelligent and Robotic Systems*, vol. **23**, pp. 27–43.

Huang, S. and Ren, W. (1999). Automatic vehicle following with integrated throttle and brake control, in *International Journal of Control*, vol. **72**, pp. 75–83.

Ioannou, P.A. and Chien, C.C. (1993). Autonomous intelligent cruise control, in *IEEE Transactions on Vehicular Technology*, vol. **18**, pp. 657–672.

Michiels, W. (2002). *Stability and stabilization of time-delay systems* (Ph.D. Thesis, Katholieke Universiteit Leuven, Belgium, May 2002).

Morărescu, C.I. (2006). *Qualitative analysis of distributed delay systems: Methodology and algorithms* (Ph.D. thesis, University of Bucharest/Université de Technologie de Compiègne, September 2006).

Morărescu, C.I. and Niculescu, S.-I. (2007). Stability crossing curves of SISO systems controlled by delayed output feedback. *Dynamics of Continuous, Discrete and Impulsive Systems* (to be published 2007).

Morărescu, C.-I., Niculescu, S.-I. and Gu, K. (2007). Stability Crossing Curves of Shifted Gamma-Distribution Delay Systems. *SIAM Journal on Applied Dynamical Systems* (to be published 2007)

Niculescu, S.-I. (2001). *Delay effects on stability. A robust control approach* (Springer-Verlag: Heidelberg, LNCIS, vol. **269**).

Pasino, K. (1995). A mixture of intelligent and conventional control methods may be the best way to implement autonomous systems, *IEEE Spectrum*, vol. **32**, pp. 55–62.

Sipahi, R. (2005). *Cluster Treatment of Characteristic Roots, CTCR, A Unique Methodology for the Complete Stability Robustness Analysis of Linear Time Invariant Multiple Time Delay Systems Against Delay Uncertainties* (Ph.D. Thesis, University of Connecticut, Mechanical Engineering Department, August 2005).

Sipahi, R. and Niculescu, S.-I. (2007). Deterministic time-delayed traffic flow models: A survey. in (F.M. ATAY, ED.), Springer-Verlag (to appear).

Shiekholslam, S. and Desoer, C.A. (1993). Longitudinal control of a platoon of vehicles with no communication of lead vehicle information: a system study, in *IEEE Transactions on Vehicular Technology*, vol. **42**, pp. 546–554.

Swaroop, D., Hedrick, J.K., Chien, C.C. and Ioannou, P.A. (1994). A comparison of spacing and headway control laws for automatically controlled vehicles, *Vehicle System Dynamics*, vol. **23**, pp. 597–625.

Varaiya, P. (1993). Smart cars on smart roads: problems of control, in *IEEE Transactions on Automatic Control*, vol. **38**, pp. 195–207.