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# Absolute algebra

## III–The saturated spectrum

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### Abstract

We investigate the algebraic and topological preliminaries to a geometry in characteristic 1.

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## 1. Introduction

The theory of *characteristic 1 semirings* (*i.e.* semirings with  $1 + 1 = 1$ ) originated in many different contexts : pure algebra (see *e.g.* LaGrassa's PhD thesis [8]), idempotent analysis and the study of  $\mathbf{R}_+^{max}$  ([1, 3]), and Zhu's theory ([12]), itself inspired by considerations of Hopf algebras (see [11]). Its main motivation is now the Riemann Hypothesis, via adeles and the theory of hyperrings (cf. [2, 3, 4], notably §6 from [4]).

For example, it has by now become clear (see [4], Theorem 3.11) that the classification of finite hyperfield extensions of the Krasner hyperring  $K$  is one of the main problems of the theory. If  $H$  denotes an hyperring extension of  $K$ ,  $B_1$  the smallest characteristic one semifield and  $S$  the sign hyperring, then there are canonical mappings  $B_1 \rightarrow S \rightarrow K \rightarrow H$ , whence mappings

$$Spec(H) \rightarrow Spec(K) \rightarrow Spec(S) \rightarrow Spec(B_1) ,$$

thus  $Spec(H)$  “lies over”  $Spec(B_1)$  (see [4], §6, notably diagram (43), where  $B_1$  is denoted by  $\mathbf{B}$ ).

The ultimate goal of our investigations is to provide a proper algebraic geometry in characteristic one. The natural procedure is to construct “affine  $B_1$ -schemes” and endow them with an appropriate topology and a sheaf of semirings ; a suitable glueing procedure will then produce general “ $B_1$ -schemes”. This program is not yet completed ; in this paper, we deal with a natural first step : the extension to  $B_1$ -algebras of the notions of spectrum and Zariski topology, and the fundamental topological properties of these objects. In order to construct a structure sheaf over the spectrum of a  $B_1$ -algebra, Castella's localization procedure ([1]) will probably be useful.

As in our two previous papers, we work in the context of  $B_1$ -algebras, *i.e.* characteristic one semirings. For such an  $A$ , one may define prime ideals by analogy to classical commutative algebra. In order to define the spectrum of a  $B_1$ -algebra  $A$ , two candidates readily suggest themselves : the set  $Spec(A)$  of prime (in a suitable sense) congruences, and the set  $Pr(A)$  of prime ideals ; in contrast to the classical situation, these two approaches are not equivalent. In fact both sets may be equipped with a natural topology of Zariski type (see [10], Theorem 2.4 and Proposition 3.15), but they do not in general correspond bijectively to one another ; nevertheless, the subset  $Pr_s(A) \subseteq Pr(A)$  of *saturated* prime ideals is in natural bijection with the set of *excellent* prime congruences (see below) on  $A$ .

It turns out (§3) that there is another, far less obvious, bijection between  $Pr_s(A)$  and the *maximal spectrum*  $MaxSpec(A) \subseteq Spec(A)$  of  $A$ . This mapping is actually an homeomorphism for the natural (Zariski-type) topologies mentioned above. As a by-product, we find a new point of view on the description of the maximal spectrum of the polynomial algebra  $B_1[x_1, \dots, x_n]$  found in [9] and [12]. The homeomorphism in question is actually functorial in  $A$  (§4).

In §5, we show that the theory of the nilradical and of the root of an ideal carry over, with some precautions, to our setting ; the situation is even better when one restricts oneself to *saturated* ideals. This allows us, in §6, to establish some nice topological properties of

$$MaxSpec(A) \simeq Pr_s(A) ;$$

namely, it is  $T_0$  and quasi-compact (Theorem 6.1), and the open quasi-compact sets constitute a basis stable under finite intersections. Furthermore this space is *sober*, *i.e.* each irreducible closed set has a (necessarily unique) generic point. In other words,  $Pr_s(A)$  satisfies the usual properties of a ring spectrum that are used in algebraic geometry (see *e.g.* the canonical reference [6]):  $Pr_s(A)$  is a *spectral space* in the sense of Hochster([7]).

In the last paragraph, we discuss the particular case of a *monogenic*  $B_1$ -algebra, that is, a quotient of the polynomial algebra  $B_1[x]$  ; in [9], we had listed the smallest finite such algebras.

In a subsequent work I shall investigate how higher concepts and methods of commutative algebra (minimal prime ideals, zero divisors, primary decomposition) carry over to characteristic one semirings.

## 2. Definitions and notation

We shall review some the definitions and notation of our previous two papers ([9], [10]).

$B_1 = \{0, 1\}$  denotes the smallest *characteristic one semifield* ; the operations of addition and multiplication are the obvious ones, with the slight change that

$$1 + 1 = 1 .$$

A  $B_1$ -module  $M$  is a nonempty set equipped with an action

$$B_1 \times M \rightarrow M$$

satisfying the usual axioms (see [9], Definition 2.3); as first seen in [12], Proposition 1 (see also [9], Theorem 2.5),  $B_1$ -modules can be canonically identified with ordered sets having a smallest element (0) and in which any two elements  $a$  and  $b$  have a least upper bound ( $a + b$ ). In particular, one may identify finite  $B_1$ -modules and nonempty finite lattices.

A (commutative)  $B_1$ -algebra is a  $B_1$ -module equipped with an associative multiplication that has a neutral element and satisfies the usual axioms relative to addition (see [9], Definition 4.1). In the sequel, except when otherwise indicated,  $A$  will denote a  $B_1$ -algebra.

An *ideal*  $I$  of  $A$  is by definition a subset containing 0, stable under addition, and having the property that

$$\forall x \in A \ \forall y \in I \ xy \in I ;$$

$I$  is termed prime if  $I \neq A$  and

$$ab \in I \implies a \in I \text{ or } b \in I .$$

By a *congruence* on  $A$ , we mean an equivalence relation on  $A$  compatible with the operations of addition and multiplication. The trivial congruence  $\mathcal{C}_0(A)$  is characterized by the fact that any two elements of  $A$  are equivalent under it ; the congruences are naturally ordered by inclusion, and

$$MaxSpec(A)$$

will denote the set of maximal nontrivial congruences on  $A$ .

For  $\mathcal{R}$  a congruence on  $A$ , we set

$$I(\mathcal{R}) := \{x \in A | x \mathcal{R} 0\} ;$$

it is an ideal of  $A$

A nontrivial congruence  $\mathcal{R}$  is termed *prime* if

$$ab \mathcal{R} 0 \implies a \mathcal{R} 0 \text{ or } b \mathcal{R} 0 ;$$

the set of prime congruences on  $A$  is denoted by  $\text{Spec}(A)$ . It turns out that (see [10], Proposition 2.3)

$$\text{MaxSpec}(A) \subseteq \text{Spec}(A) .$$

For  $J$  an ideal of  $A$ , there is a unique smallest congruence  $\mathcal{R}_J$  such that  $J \subseteq I(\mathcal{R})$  ; it is denoted by  $\mathcal{R}_J$ . Such congruences are termed *excellent* .

An ideal  $J$  of  $A$  is termed *saturated* if it is of the form  $I(\mathcal{R})$  for some congruence  $\mathcal{R}$  ; this is the case if and only if  $J = \overline{J}$ , where

$$\overline{J} := I(\mathcal{R}_J) .$$

We shall denote the set of prime ideals of  $A$  by  $\text{Pr}(A)$ , and the set of saturated prime ideals by  $\text{Pr}_s(A)$ .

For  $S \subseteq A$ , let us set

$$W(S) := \{ \mathcal{P} \in \text{Pr}(A) | S \subseteq \mathcal{P} \} ,$$

and

$$V(S) := \{ \mathcal{R} \in \text{Spec}(A) | S \subseteq I(\mathcal{R}) \} .$$

As seen in [10], Theorem 2.4 and Proposition 3.4, the family  $(W(S))_{S \subseteq A}$  is the family of closed sets for a topology on  $\text{Pr}(A)$ , and the family  $(V(S))_{S \subseteq A}$  is the family of closed sets for a topology on  $\text{Spec}(A)$ . We shall always consider  $\text{Spec}(A)$  and  $\text{Pr}(A)$  as equipped with these topologies, and their subsets with the induced topologies.

For  $M$  a commutative monoid, we define the *Deitmar spectrum*  $\text{Spec}_{\mathcal{D}}(M)$  as the set of prime ideals (including  $\emptyset$ ) of  $M$  (in [5], this is denoted by  $\text{Spec } \mathbf{F}_M$ ). We define  $\mathcal{F}(M) = B_1[M]$  as the “monoid algebra of  $M$  over  $B_1$  ”; the functor  $\mathcal{F}$  is adjoint to the forgetful functor from the category of  $B_1$ -algebras to the category of monoids (for the details, see [9], §5). Furthermore, there is an explicit canonical bijection between  $\text{Spec}_{\mathcal{D}}(M)$  and a certain subset of  $\text{Spec}(\mathcal{F}(M))$  (see [10], Theorem 4.2).

For  $S$  a subset of  $A$ , let  $< S >$  denote the intersection of all the ideals of  $A$  containing  $S$  (there is always at least one such ideal :  $A$  itself). It is

clear that  $\langle S \rangle$  is an ideal of  $A$ , and therefore is the smallest ideal of  $A$  containing  $S$ . As in ring theory, one may see that

$$\langle S \rangle = \left\{ \sum_{j=1}^n a_j s_j \mid n \in \mathbf{N}, (a_1, \dots, a_n) \in A^n, (s_1, \dots, s_n) \in S^n \right\}.$$

We shall denote by  $\mathcal{SP}$  the category whose objects are spectra of  $B_1$ -algebras and whose morphisms are the continuous maps between them.

### 3. A new description of maximal congruences

Let  $A$  denote a  $B_1$ -algebra.

For  $\mathcal{P}$  a saturated prime ideal of  $A$ , let us define a relation  $\mathcal{S}_{\mathcal{P}}$  on  $A$  by :

$$x\mathcal{S}_{\mathcal{P}}y \equiv (x \in \mathcal{P} \text{ and } y \in \mathcal{P}) \text{ or } (x \notin \mathcal{P} \text{ and } y \notin \mathcal{P}) .$$

Then  $\mathcal{S}_{\mathcal{P}}$  is a congruence on  $A$  : if  $x\mathcal{S}_{\mathcal{P}}y$  and  $x'\mathcal{S}_{\mathcal{P}}y'$ , then one and only one of the following holds :

- (i)  $x \in \mathcal{P}, y \in \mathcal{P}, x' \in \mathcal{P} \text{ and } y' \in \mathcal{P}$  ,
- (ii)  $x \in \mathcal{P}, y \in \mathcal{P}, x' \notin \mathcal{P} \text{ and } y' \notin \mathcal{P}$  ,
- (iii)  $x \notin \mathcal{P}, y \notin \mathcal{P}, x' \in \mathcal{P} \text{ and } y' \in \mathcal{P}$  ,
- (iv)  $x \notin \mathcal{P}, y \notin \mathcal{P}, x' \notin \mathcal{P} \text{ and } y' \notin \mathcal{P}$  .

In case (i),  $x + x' \in \mathcal{P}$  and  $y + y' \in \mathcal{P}$ , whence  $x + x'\mathcal{S}_{\mathcal{P}}y + y'$  ; in cases (ii) and (iv),  $x + x' \notin \mathcal{P}$  and  $y + y' \notin \mathcal{P}$  (as  $\mathcal{P}$  is saturated), whence  $x + x'\mathcal{S}_{\mathcal{P}}y + y'$ . Case (iii) is symmetrical relatively to case (ii), therefore, in all cases,  $x + x'\mathcal{S}_{\mathcal{P}}y + y'$  :  $\mathcal{S}_{\mathcal{P}}$  is compatible with addition.

In cases (i), (ii) and (iii),  $xx' \in \mathcal{P}$  and  $yy' \in \mathcal{P}$ , whence  $xx'\mathcal{S}_{\mathcal{P}}yy'$  ; in case (iv)  $xx' \notin \mathcal{P}$  and  $yy' \notin \mathcal{P}$  (as  $\mathcal{P}$  is prime), whence also  $xx'\mathcal{S}_{\mathcal{P}}yy'$  :  $\mathcal{S}_{\mathcal{P}}$  is compatible with multiplication, hence is a congruence on  $A$ .

As  $0 \in \mathcal{P}$  and  $1 \notin \mathcal{P}$ ,  $0 \not\mathcal{S}_{\mathcal{P}}1$ , therefore  $\mathcal{S}_{\mathcal{P}}$  is nontrivial ; but each  $x \in A$  is either in  $\mathcal{P}$  (whence  $x\mathcal{S}_{\mathcal{P}}0$ ) or not (whence  $x\mathcal{S}_{\mathcal{P}}1$ ). It follows that

$$\frac{A}{\mathcal{S}_{\mathcal{P}}} = \{\bar{0}, \bar{1}\} \simeq B_1 ;$$

in particular,  $\mathcal{S}_{\mathcal{P}}$  is maximal :  $\mathcal{S}_{\mathcal{P}} \in \text{MaxSpec}(A)$ .

Obviously,  $I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}$ .

Furthermore, let  $(x, y) \in A^2$  be such that  $x\mathcal{R}_{\mathcal{P}}y$  ; then there is  $z \in \mathcal{P}$  such that  $x + z = y + z$ . If  $x \in \mathcal{P}$  then  $y + z = x + z \in \mathcal{P}$ , whence  $y \in \mathcal{P}$  (as  $y + (y + z) = y + z$  and  $\mathcal{P}$  is saturated) ; symmetrically,  $y \in \mathcal{P}$  implies  $x \in \mathcal{P}$ , whence the assertions  $(x \in \mathcal{P})$  and  $(y \in \mathcal{P})$  are equivalent, and  $x\mathcal{S}_{\mathcal{P}}y$ . We have shown that

$$\mathcal{R}_{\mathcal{P}} \leq \mathcal{S}_{\mathcal{P}} .$$

We shall denote by  $\alpha_A$  the mapping

$$\begin{aligned} \alpha_A & : \text{Pr}_s(A) \rightarrow \text{MaxSpec}(A) \\ & \mathcal{P} \mapsto \mathcal{S}_{\mathcal{P}} . \end{aligned}$$



Let  $\mathcal{R} \in \text{MaxSpec}(A)$  ; then  $\mathcal{R} \in \text{Spec}(A)$ , whence  $I(\mathcal{R})$  is prime ; by Theorem 3.8 of [10], it is saturated, *i.e.*  $I(\mathcal{R}) \in \text{Pr}_s(A)$ . Let us set

$$\beta_A(\mathcal{R}) := I(\mathcal{R}) .$$

**Theorem 3.1.** *The mappings*

$$\alpha_A : \text{Pr}_s(A) \mapsto \text{MaxSpec}(A)$$

and

$$\beta_A : \text{MaxSpec}(A) \mapsto \text{Pr}_s(A)$$

are bijections, inverse of one another. They are continuous for the topologies on  $\text{Pr}_s(A)$  and  $\text{MaxSpec}(A)$  induced by the topologies on  $\text{Pr}(A)$  and  $\text{Spec}(A)$  mentioned above, whence  $\text{Pr}_s(A)$  and  $\text{MaxSpec}(A)$  are homeomorphic.

*Proof.* Let  $\mathcal{R} \in \text{MaxSpec}(A)$  ; then

$$\alpha_A(\beta_A(\mathcal{R})) = \alpha_A(I(\mathcal{R})) = \mathcal{S}_{I(\mathcal{R})} .$$

Let us assume  $x\mathcal{R}y$  ; then, if  $x \in I(\mathcal{R})$  one has  $x\mathcal{R}0$ , whence  $y\mathcal{R}0$  and  $y \in I(\mathcal{R})$ ; by symmetry,  $y \in I(\mathcal{R})$  implies  $x \in I(\mathcal{R})$ , thus  $(x \in I(\mathcal{R}))$  and  $(y \in I(\mathcal{R}))$  are equivalent, *i.e.*  $x\mathcal{S}_{I(\mathcal{R})}y$ . We have proved that  $\mathcal{R} \leq \mathcal{S}_{I(\mathcal{R})}$ . As  $\mathcal{R}$  is maximal, we have  $\mathcal{R} = \mathcal{S}_{I(\mathcal{R})}$ , whence

$$\alpha_A(\beta_A(\mathcal{R})) = \mathcal{S}_{I(\mathcal{R})} = \mathcal{R} ,$$

and

$$\alpha_A \circ \beta_A = \text{Id}_{\text{MaxSpec}(A)} .$$

Let now  $\mathcal{P} \in \text{Pr}_s(A)$  ; then

$$\begin{aligned} (\beta_A \circ \alpha_A)(\mathcal{P}) &= \beta_A(\alpha_A(\mathcal{P})) \\ &= \beta_A(\mathcal{S}_{\mathcal{P}}) \\ &= I(\mathcal{S}_{\mathcal{P}}) \\ &= \mathcal{P} , \end{aligned}$$

whence

$$\beta_A \circ \alpha_A = \text{Id}_{\text{Pr}_s(A)} ,$$

and the first statement follows.

Let now  $F$  denote a closed subset of  $Pr_s(A)$  ; then  $F = G \cap Pr_s(A)$  for  $G$  a closed subset of  $Pr(A)$  and  $G = W(S) := \{\mathcal{P} \in Pr(A) | S \subseteq \mathcal{P}\}$  for some subset  $S$  of  $A$ . But then, for  $\mathcal{R} \in MaxSpec(A)$ ,  $\mathcal{R} \in \beta_A^{-1}(F)$  if and only if  $\beta_A(\mathcal{R}) \in F$ , *i.e.*  $I(\mathcal{R}) \in G \cap Pr_s(A)$ , that is  $I(\mathcal{R}) \in G$ , or  $S \subseteq I(\mathcal{R})$ , which means  $\mathcal{R} \in V(S)$ . Thus

$$\beta_A^{-1}(F) = V(S) \cap MaxSpec(A)$$

is closed in  $MaxSpec(A)$ . We have shown the continuity of  $\beta_A$ .

Let now  $H \subseteq MaxSpec(A)$  be closed ; then  $H = MaxSpec(A) \cap L$  for some closed subset  $L$  of  $Spec(A)$ , and  $L = V(T)$  for some subset  $T$  of  $A$ . Then a saturated prime ideal  $\mathcal{P}$  of  $A$  belongs to  $\alpha_A^{-1}(H)$  if and only if  $\alpha_A(\mathcal{P}) \in H$ , that is

$$\mathcal{S}_{\mathcal{P}} \in MaxSpec(A) \cap L ,$$

*i.e.*

$$\mathcal{S}_{\mathcal{P}} \in V(T)$$

or  $T \subseteq I(\mathcal{S}_{\mathcal{P}})$ . But  $I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}$  whence  $\mathcal{P}$  belongs to  $\alpha_A^{-1}(H)$  if and only if  $T \subseteq \mathcal{P}$ , that is

$$\alpha_A^{-1}(H) = W(T) \cap Pr_s(A) ,$$

which is closed in  $Pr_s(A)$ . □

Let us consider the special case in which  $A$  is in the image of  $\mathcal{F} : A = \mathcal{F}(M)$ , for  $M$  a commutative monoid. Let  $P$  be a prime ideal of  $M$  ; as seen in [10], Theorem 4.2,  $\tilde{P}$  is a saturated prime ideal in  $A$ , and one obtains in this way a bijection between  $Spec_{\mathcal{D}}(M)$  and  $Pr_s(A)$ . The following is now obvious :

**Theorem 3.2.** *The mapping*

$$\begin{aligned} \psi_M & : Spec_{\mathcal{D}}(M) \rightarrow MaxSpec(\mathcal{F}(M)) \\ P & \mapsto \alpha_{\mathcal{F}(M)}(\tilde{P}) \end{aligned}$$

*is a bijection.*

Two particular cases are of special interest :

1.  $M$  is a group ; then  $Spec_{\mathcal{D}}(M) = \{\emptyset\}$ , whence  $MaxSpec(\mathcal{F}(G))$  has exactly one element.

2.  $M = C_n := \langle x_1, \dots, x_n \rangle$  is the free monoid on  $n$  variables  $x_1, \dots, x_n$ . Then the elements of  $\text{Spec}_{\mathcal{D}}(M)$  are the  $(P_J)_{J \subseteq \{1, \dots, n\}}$ , where

$$P_J := \bigcup_{j \in J} x_j C_n$$

(a fact that was already used in [10], Example 4.3). Then

$$\psi_M(P_J) = \alpha_{\mathcal{F}(M)}(\tilde{P}_J) = \mathcal{S}_{\tilde{P}_J}$$

whence  $x\psi_M(P_J)y$  if and only if either  $(x \in \tilde{P}_J \text{ and } y \in \tilde{P}_J)$  or  $(x \notin \tilde{P}_J \text{ and } y \notin \tilde{P}_J)$ . But we have seen in [9], Theorem 4.5, that

$$\mathcal{F}(M) = B_1[x_1, \dots, x_n]$$

could be identified with the set of finite formal sums of elements of  $M$ . Obviously, an element  $x$  of  $\mathcal{F}(M)$  belongs to  $\tilde{P}_J$  if and only if at least one of its components involves at least one factor  $x_j (j \in J)$ . It is now clear that, using the notation of [9], Definition 4.6 and Theorem 4.7,

$$\psi_M(P_J) = \tilde{J}.$$

We hereby recover the description of  $\text{MaxSpec}(B_1[x_1, \dots, x_n])$  given in [9] (Theorems 4.7, 4.8 and 4.10).

The following result will be useful

**Theorem 3.3.** *Any proper saturated ideal of a  $B_1$ -algebra  $A$  is contained in a saturated prime ideal of  $A$ .*

*Proof.* Let  $J$  be a proper saturated ideal of  $A$ ; as  $I(\mathcal{R}_J) = \overline{J} = J \neq A$ ,  $\mathcal{R}_J \neq \mathcal{C}_0(A)$ . By Zorn's Lemma, one has  $\mathcal{R}_J \leq \mathcal{R}$  for some  $\mathcal{R} \in \text{MaxSpec}(A)$ . According to Theorem 2.1,  $\mathcal{R} = \alpha_A(\mathcal{P}) = \mathcal{S}_{\mathcal{P}}$  for a saturated prime ideal  $\mathcal{P}$  of  $A$ , therefore  $\mathcal{R}_J \leq \mathcal{S}_{\mathcal{P}}$  and

$$J = \overline{J} = I(\mathcal{R}_J) \subseteq I(\mathcal{S}_{\mathcal{P}}) = \mathcal{P}.$$

□

#### 4. Functorial properties of spectra

Let  $\varphi : A \rightarrow C$  denote a morphism of  $B_1$ -algebras, and let  $\mathcal{R} \in \text{Spec}(C)$ . We define a binary relation  $\tilde{\varphi}(\mathcal{R})$  on  $A$  by :

$$\forall (a, a') \in A^2 \quad a\tilde{\varphi}(\mathcal{R})a' \equiv \varphi(a)\mathcal{R}\varphi(a') .$$

It is clear that  $\tilde{\varphi}(\mathcal{R})$  is a congruence on  $A$ , and that

$$I(\tilde{\varphi}(\mathcal{R})) = \varphi^{-1}(I(\mathcal{R})) .$$

In particular  $I(\tilde{\varphi}(\mathcal{R}))$  is a prime ideal of  $A$ , hence  $\tilde{\varphi}(\mathcal{R}) \in \text{Spec}(A)$  :  $\tilde{\varphi}$  maps  $\text{Spec}(C)$  into  $\text{Spec}(A)$ . Let  $F := V(S)$  be a closed subset of  $\text{Spec}(A)$ , and let  $\mathcal{R} \in \text{Spec}(C)$  ; then  $\mathcal{R} \in \tilde{\varphi}^{-1}(F)$  if and only if  $\tilde{\varphi}(\mathcal{R}) \in F$ , that is  $S \subseteq I(\tilde{\varphi}(\mathcal{R}))$ , or  $S \subseteq \varphi^{-1}(I(\mathcal{R}))$ , i.e.  $\varphi(S) \subseteq I(\mathcal{R})$ , or  $\mathcal{R} \in V(\varphi(S))$ . Therefore  $\tilde{\varphi}^{-1}(F) = V(\varphi(S))$  is closed in  $\text{Spec}(C)$  :  $\tilde{\varphi}$  is continuous.

Furthermore, for  $\varphi : A \rightarrow C$  and  $\psi : C \rightarrow D$  one has

$$\widetilde{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi} : \text{Spec}(D) \rightarrow \text{Spec}(A) .$$

It follows that the equations  $\mathcal{H}(A) = \text{Spec}(A)$  and  $\mathcal{H}(\varphi) = \tilde{\varphi}$  define a contravariant functor  $\mathcal{H}$  from  $\mathcal{Z}_a$  to  $\mathcal{SP}$ .

Let  $J$  denote an ideal in  $C$ , and let us assume  $a\mathcal{R}_{\varphi^{-1}(J)}a'$  ; then there is an  $x \in \varphi^{-1}(J)$  with  $a + x = a' + x$ . Now  $\varphi(x) \in J$  and

$$\begin{aligned} \varphi(a) + \varphi(x) &= \varphi(a + x) \\ &= \varphi(a' + x) \\ &= \varphi(a') + \varphi(x) , \end{aligned}$$

whence  $\varphi(a)\mathcal{R}_J\varphi(a')$  and  $a\tilde{\varphi}(\mathcal{R}_J)a'$ . We have established

**Proposition 4.1.** *Let  $A$  and  $C$  denote  $B_1$ -algebras,  $\varphi : A \rightarrow C$  a morphism and  $J$  an ideal of  $C$  : then*

$$\mathcal{R}_{\varphi^{-1}(J)} \leq \tilde{\varphi}(\mathcal{R}_J) .$$

**Theorem 4.2.** *Let  $A$  and  $C$  denote two  $B_1$ -algebras, and  $\varphi : A \rightarrow C$  a morphism. Then  $\tilde{\varphi} : \text{Spec}(C) \rightarrow \text{Spec}(A)$  maps  $\text{MaxSpec}(C)$  into  $\text{MaxSpec}(A)$ , and the diagram*

$$\begin{array}{ccc} \text{Pr}_s(C) & \xrightarrow{\varphi^{-1}} & \text{Pr}_s(A) \\ \downarrow \alpha_C & & \downarrow \alpha_A \\ \text{MaxSpec}(C) & \xrightarrow{\tilde{\varphi}} & \text{MaxSpec}(A) \end{array}$$

*commutes.*

*Proof.* Let  $\mathcal{P} \in \text{Pr}_s(C)$ , then, for all  $(a, a') \in A^2$

$$\begin{aligned}
a\tilde{\varphi}(\mathcal{S}_{\mathcal{P}})a' &\iff \varphi(a)\mathcal{S}_{\mathcal{P}}\varphi(a') \\
&\iff (\varphi(a) \in \mathcal{P} \text{ and } \varphi(a') \in \mathcal{P}) \\
\text{or } (\varphi(a) \notin \mathcal{P} \text{ and } \varphi(a') \notin \mathcal{P}) & \\
&\iff (a \in \varphi^{-1}(\mathcal{P}) \text{ and } a' \in \varphi^{-1}(\mathcal{P})) \\
\text{or } (a \notin \varphi^{-1}(\mathcal{P}) \text{ and } a' \notin \varphi^{-1}(\mathcal{P})) & \\
&\iff a\mathcal{S}_{\varphi^{-1}(\mathcal{P})}a' .
\end{aligned}$$

Therefore

$$\begin{aligned}
(\tilde{\varphi} \circ \alpha_C)(\mathcal{P}) &= \tilde{\varphi}(\alpha_C(\mathcal{P})) \\
&= \tilde{\varphi}(\mathcal{S}_{\mathcal{P}}) \\
&= \mathcal{S}_{\varphi^{-1}(\mathcal{P})} \\
&= \alpha_A(\varphi^{-1}(\mathcal{P})) \\
&= (\alpha_A \circ \varphi^{-1})(\mathcal{P})
\end{aligned}$$

whence  $\tilde{\varphi} \circ \alpha_C = \alpha_A \circ \varphi^{-1}$ .

Incidentally we have proved that  $\tilde{\varphi}$  maps  $\text{MaxSpec}(C) = \alpha_C(\text{Pr}_s(C))$  into  $\alpha_A(\text{Pr}_s(A)) = \text{MaxSpec}(A)$ , *i.e.* the first assertion.  $\square$

## 5. Nilpotent radicals and prime ideals

The usual theory generalizes without major problem to  $B_1$ -algebras.

**Theorem 5.1.** *In the  $B_1$ -algebra  $A$ , let us define*

$$Nil(A) := \{x \in A \mid (\exists n \geq 1) x^n = 0\} .$$

*Then  $Nil(A)$  is a saturated ideal of  $A$ , and one has*

$$\bigcap_{\mathcal{P} \in Pr(A)} \mathcal{P} = \bigcap_{\mathcal{P} \in Pr_s(A)} \mathcal{P} = Nil(A) .$$

*Proof.* Let  $M := \bigcap_{\mathcal{P} \in Pr(A)} \mathcal{P}$  and  $N = \bigcap_{\mathcal{P} \in Pr_s(A)} \mathcal{P}$ . If  $x \in Nil(A)$  and  $\mathcal{P} \in Pr(A)$ , then, for some  $n \geq 1$ ,  $x^n = 0 \in \mathcal{P}$ , whence (as  $\mathcal{P}$  is prime)  $x \in \mathcal{P} : Nil(A) \subseteq M$ .

As  $Pr_s(A) \subseteq Pr(A)$ , we have  $M \subseteq N$ .

Let now  $x \notin Nil(A)$  ; then

$$(\forall n \in \mathbf{N}) \ x^n \neq 0 .$$

Define

$$\mathcal{E} := \{J \in Id_s(A) \mid (\forall n \geq 0) x^n \notin J\} .$$

This set is nonempty ( $\{0\} \in \mathcal{E}$ ) and inductive for  $\subseteq$ , therefore, by Zorn's Lemma, there exists a maximal element  $\mathcal{P}$  of  $\mathcal{E}$ . As  $1 = x^0 \notin \mathcal{P}$ ,  $\mathcal{P} \neq A$ .

Let us assume  $ab \in \mathcal{P}$ ,  $a \notin \mathcal{P}$  and  $b \notin \mathcal{P}$  ; then  $\overline{\mathcal{P} + Aa}$  and  $\overline{\mathcal{P} + Ab}$  are saturated ideals of  $A$  strictly containing  $\mathcal{P}$ , whence there exists two integers  $m$  and  $n$  with  $x^m \in \overline{\mathcal{P} + Aa}$  and  $x^n \in \overline{\mathcal{P} + Ab}$ . By definition of the closure of an ideal, there are  $u = p_1 + \lambda a \in \mathcal{P} + Aa$  and  $v = p_2 + \mu b \in \mathcal{P} + Ab$  such that  $x^m + u = u$  and  $x^n + v = v$ . Then

$$ub = p_1 b + \lambda(ab) \in \mathcal{P}$$

and

$$x^m b + ub = (x^m + u)b = ub ,$$

whence, as  $\mathcal{P}$  is saturated,  $x^m b \in \mathcal{P}$ .

Then

$$x^m v = x^m p_2 + \mu x^m b \in \mathcal{P} ;$$

as

$$\begin{aligned} x^{m+n} + x^m v &= x^m(x^n + v) \\ &= x^m v, \end{aligned}$$

we obtain  $x^{m+n} \in \mathcal{P}$ , a contradiction.

Therefore  $\mathcal{P}$  is prime and saturated and  $x = x^1 \notin \mathcal{P}$ , whence  $x \notin N$ . We have proved that  $N \subseteq \text{Nil}(A)$ , whence  $M = N = \text{Nil}(A)$ .  $\square$

**Corollary 5.2.**

$$\text{Nil}(A) = \bigcap_{\mathcal{P} \in \text{Pr}(A)} \overline{\mathcal{P}}.$$

*Proof.*

$$\begin{aligned} \text{Nil}(A) &= \bigcap_{\mathcal{P} \in \text{Pr}(A)} \mathcal{P} \text{ (by Theorem 5.1)} \\ &\subseteq \bigcap_{\mathcal{P} \in \text{Pr}(A)} \overline{\mathcal{P}} \\ &\subseteq \bigcap_{\mathcal{P} \in \text{Pr}_s(A)} \overline{\mathcal{P}} \\ &= \bigcap_{\mathcal{P} \in \text{Pr}_s(A)} \mathcal{P} \\ &= \text{Nil}(A) \text{ (also by Theorem 5.1).} \end{aligned}$$

$\square$

**Definition 5.3.** For  $I$  an ideal of  $A$ , we define the *root*  $r(I)$  of  $I$  by

$$r(I) := \{x \in A \mid (\exists n \geq 1) x^n \in I\}.$$

**Lemma 5.4.** (i)  $r(I)$  is an ideal of  $A$ .

(ii)  $\overline{r(I)} \subseteq r(\overline{I})$  ; in particular, if  $I$  is saturated then so is  $r(I)$ .

(iii)  $r(\{0\}) = \text{Nil}(A)$ .

*Proof.* (i) Obviously,  $0 \in r(I)$ .

If  $x \in r(I)$  and  $y \in r(I)$ , then  $x^m \in I$  for some  $m \geq 1$  and  $y^n \in I$  for some  $n \geq 1$ , whence

$$\begin{aligned} (x+y)^{m+n-1} &= \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} x^j y^{m+n-1-j} \\ &= \sum_{j=0}^{m+n-1} x^j y^{m+n-1-j} \\ &\in I, \end{aligned}$$

as  $x^j \in I$  for  $j \geq m$  and  $y^{m+n-1-j} \in I$  for  $j \leq m-1$  (as, then,  $m+n-1-j \geq n$ ). Thus  $x+y \in r(I)$ .

For  $a \in A$ ,  $(ax)^m = a^m x^m \in I$ , whence  $ax \in r(I)$ . Therefore  $r(I)$  is an ideal of  $A$ .

- (ii) Let  $x \in \overline{r(I)}$  then there is  $u \in r(I)$  such that  $x+u = u$ , and there is  $n \geq 1$  such that  $u^n \in I$ . Let us show by induction on  $j \in \{0, \dots, n\}$  that  $u^{n-j}x^j \in \overline{I}$ . This is clear for  $j=0$ . Let then  $j \in \{0, \dots, n-1\}$ , and assume that  $u^{n-j}x^j \in \overline{I}$ ; then

$$\begin{aligned} u^{n-j-1}x^{j+1} + u^{n-j}x^j &= u^{n-j-1}x^j(x+u) \\ &= u^{n-j-1}x^ju \\ &= u^{n-j}x^j, \end{aligned}$$

whence  $u^{n-j-1}x^{j+1} \in \overline{I} = \overline{I}$ . Thus, for  $j=n$ , we obtain

$$x^n = u^{n-n}x^n \in \overline{I},$$

whence  $x \in r(\overline{I})$ .

If now  $I$  is saturated, then

$$\begin{aligned} r(I) &\subseteq \overline{r(I)} \\ &\subseteq r(\overline{I}) \text{ (by the above)} \\ &= r(I), \end{aligned}$$

whence  $r(I) = \overline{r(I)}$  is saturated.

- (iii) That assertion is obvious.

□



**Proposition 5.5.** *For each saturated ideal  $I$  of the  $B_1$ -algebra  $A$ , one has*

$$r(I) = \bigcap_{\mathcal{P} \in \text{Pr}_s(A); I \subseteq \mathcal{P}} \mathcal{P} .$$

**Remark 5.6.** For  $I = \{0\}$ , this is part of Theorem 5.1.

*Proof.* Let  $x \in r(I)$ , and let  $\mathcal{P} \in \text{Pr}_s(A)$  with  $I \subseteq \mathcal{P}$ ; then, for some  $n \geq 1$   $x^n \in I$ , whence  $x^n \in \mathcal{P}$  and  $x \in \mathcal{P}$ :

$$r(I) \subseteq \bigcap_{\mathcal{P} \in \text{Pr}_s(A); I \subseteq \mathcal{P}} \mathcal{P} .$$

Let now  $y \in A$ ,  $y \notin r(I)$ , and denote by  $\pi$  the canonical projection

$$\pi : A \twoheadrightarrow A_0 := \frac{A}{\mathcal{R}_I} .$$

As  $I$  is saturated, one has

$$\forall n \geq 1 \ y^n \notin \bar{I} ,$$

whence

$$\forall n \geq 1 \ y^n \notin \mathcal{R}_I 0 ,$$

or

$$\forall n \geq 1 \ \pi(y)^n = \pi(y^n) \neq \bar{0} .$$

Therefore  $\pi(y) \notin \text{Nil}(A_0)$ , whence, according to Theorem 5.1, there exists a saturated prime ideal  $\mathcal{P}_0$  of  $A_0$  such that  $\pi(y) \notin \mathcal{P}_0$ . But then  $\mathcal{P} := \pi^{-1}(\mathcal{P}_0)$  is a saturated prime ideal of  $A$  containing  $I$  with  $y \notin \mathcal{P}$ , whence

$$y \notin \bigcap_{\mathcal{P} \in \text{Pr}_s(A); I \subseteq \mathcal{P}} \mathcal{P} .$$

□

## 6. Topology of spectra

We can now establish the basic topological properties of the spectra  $Pr_s(A)$  (analogous, in our setting, to Corollary 1.1.8 and Proposition 1.1.10(ii) of [6]).

**Theorem 6.1.**  *$Pr_s(A)$  and  $MaxSpec(A)$  are  $T_0$  and quasi-compact.*

*Proof.* According to Theorem 3.1,  $Pr_s(A)$  and  $MaxSpec(A)$  are homeomorphic, therefore it is enough to establish the result for  $Pr_s(A)$ .

Let  $\mathcal{P}$  and  $\mathcal{Q}$  denote two different points of  $Pr_s(A)$  ; then either  $\mathcal{P} \not\subseteq \mathcal{Q}$  or  $\mathcal{Q} \not\subseteq \mathcal{P}$ . Let us for instance assume that  $\mathcal{P} \not\subseteq \mathcal{Q}$  ; then  $\mathcal{Q} \notin W(\mathcal{P})$  ; set

$$O := Pr_s(A) \cap (Pr(A) \setminus W(\mathcal{P})) .$$

Then  $O$  is an open set in  $Pr_s(A)$ ,  $\mathcal{Q} \in O$  and, obviously,  $\mathcal{P} \notin O$ . Therefore  $Pr_s(A)$  is  $T_0$ .

Let  $(U_i)_{i \in I}$  denote an open cover of  $Pr_s(A)$  :

$$Pr_s(A) = \bigcup_{i \in I} U_i ;$$

each  $Pr_s(A) \setminus U_i$  is closed, whence  $Pr_s(A) \setminus U_i = Pr_s(A) \cap W(S_i)$  for some subset  $S_i$  of  $A$ . Therefore  $Pr_s(A) \cap (\bigcap_{i \in I} W(S_i)) = \emptyset$ , *i.e.*  $Pr_s(A) \cap W(\bigcup_{i \in I} S_i) = \emptyset$ . Therefore  $Pr_s(A) \cap W(\overline{\langle \bigcup_{i \in I} S_i \rangle}) = \emptyset$ , whence, according to Theorem 3.3,  $\overline{\langle \bigcup_{i \in I} S_i \rangle} = A$ . Let  $J = \langle \bigcup_{i \in I} S_i \rangle$  ; then  $1 \in \overline{J}$ , hence there is  $x \in J$  such that  $1 + x = x$ . Furthermore, there exist  $n \in \mathbf{N}$ ,  $(i_1, \dots, i_n) \in I^n$ ,  $x_{i_k} \in S_{i_k}$  and  $(a_1, \dots, a_n) \in A^n$  such that  $x = a_1 x_{i_1} + \dots + a_n x_{i_n}$ . But then

$$1 + a_1 x_{i_1} + \dots + a_n x_{i_n} = a_1 x_{i_1} + \dots + a_n x_{i_n}$$

whence

$$1 \in \overline{\langle \{x_{i_1}, \dots, x_{i_n}\} \rangle} \subseteq \overline{\bigcup_{j=1}^n S_{i_j}}$$

and

$$\overline{\bigcup_{j=1}^n S_{i_j}} = A .$$

It follows that

$$Pr_s(A) \cap W\left(\bigcup_{j=1}^n S_{i_j}\right) = \emptyset ,$$

that is

$$Pr_s(A) \cap \bigcap_{j=1}^n W(S_{i_j}) = \emptyset ,$$

or

$$Pr_s(A) = \bigcup_{j=1}^n U_{i_j} :$$

$Pr_s(A)$  is quasi-compact. □

For  $f \in A$ , let

$$\begin{aligned} D(f) &:= Pr_s(A) \setminus (Pr_s(A) \cap W(\{f\})) \\ &= \{\mathcal{P} \in Pr_s(A) \mid f \notin \mathcal{P}\}. \end{aligned}$$

- Proposition 6.2.** 1. *Each  $D(f)$  ( $f \in A$ ) is open and quasi-compact in  $Pr_s(A)$  (see [6], Proposition 1.1.10 (ii)).*
2. *The family  $(D(f))_{f \in A}$  is an open basis for  $Pr_s(A)$  (see [6], Proposition 1.1.10(i)); in particular, the open quasi-compact sets constitute an open basis.*
3. *A subset  $O$  of  $Pr_s(A)$  is open and quasi-compact if and only if it is of the form  $Pr_s(A) \cap W(I)$  for  $I$  an ideal of finite type in  $A$ .*
4. *The family of open quasi-compact subsets of  $Pr_s(A)$  is stable under finite intersections.*
5. *Each irreducible closed set in  $Pr_s(A)$  has a unique generic point (see [6], Corollary 1.1.14(ii)).*

*Proof.* 1. The openness of  $D(f)$  is obvious.

Let us assume  $D(f) = \bigcup_{i \in I} U_i$ , where the  $U_i$ 's are open sets in  $D(f)$ . Each  $U_i$  can be written as

$$U_i = D(f) \cap V_i ,$$

for  $V_i$  an open set in  $Pr_s(A)$ , i.e.  $Pr_s(A) \setminus V_i = W(S_i)$  for  $S_i$  a subset of  $A$ . Then

$$D(f) \subseteq \bigcup_{i \in I} V_i = Pr_s(A) \setminus \left( \bigcap_{i \in I} W(S_i) \right) ,$$

whence

$$Pr_s(A) \cap W\left(\bigcup_{i \in I} S_i\right) \subseteq W(\{f\}) ,$$

that is, setting

$$S := \bigcup_{i \in I} S_i ,$$

$$f \in \bigcap_{\mathcal{P} \in W(S) \cap Pr_s(A)} \mathcal{P} = \bigcap_{\mathcal{P} \in Pr_s(A); S \subseteq \mathcal{P}} \mathcal{P} .$$

Therefore, by Proposition 5.5,  $f \in r(\overline{\langle S \rangle})$  : there is  $n \geq 1$  such that  $f^n \in \overline{\langle S \rangle}$ . Thus, there is  $g \in \langle S \rangle$  such that  $f^n + g = g$  ; one has  $g = \sum_{j=1}^m a_j s_j$  for  $a_j \in A$ ,  $s_j \in S$  ; for each  $j \in \{1, \dots, m\}$ ,  $s_j \in S_{i_j}$  for some  $i_j \in I$ . Let  $S_0 = \{s_1, \dots, s_m\}$  ; then  $g \in \langle \bigcup_{j=1}^m S_{i_j} \rangle$ , whence  $f^n \in \overline{\langle \bigcup_{j=1}^m S_{i_j} \rangle}$ , and reading the above argument in reverse order with  $S$  replaced by  $\bigcup_{j=1}^m S_{i_j}$  yields that

$$D(f) = \bigcup_{j=1}^m U_{i_j} ,$$

whence the quasi-compactness of  $D(f)$ .

2. Let  $U$  be an open set in  $Pr_s(A)$ , and  $\mathcal{P} \in U$ . We have  $Pr_s(A) \setminus U = Pr_s(A) \cap W(S)$  for some subset  $S$  of  $A$ . As  $\mathcal{P} \notin W(S)$ ,  $S \not\subseteq \mathcal{P}$ , whence there is an  $s \in S$  with  $s \notin \mathcal{P}$ . It is now clear that  $\mathcal{P} \in D(s)$  and

$$D(s) \subseteq Pr_s(A) \setminus W(S) = U .$$

3. Let  $O \subseteq Pr_s(A)$  be open and quasi-compact ; according to (2), one may write  $O = \bigcup_{j \in J} D(f_j)$  with  $f_j \in A$ . But then, there is a finite subset  $J_0$  of  $J$  such that  $O = \bigcup_{j \in J_0} D(f_j)$ . Now

$$\begin{aligned} Pr_s(A) \setminus O &= \bigcap_{j \in J_0} D(f_j) \\ &= Pr_s(A) \cap W(\langle f_j | j \in J_0 \rangle) \end{aligned}$$

is of the required type.

Conversely, if  $Pr_s(A) \setminus O = Pr_s(A) \cap W(I)$  with  $I = \langle g_1, \dots, g_n \rangle$ , it is clear that  $O = \bigcup_{i=1}^n D(g_i)$ ; as a finite union of quasi-compact subspaces of  $Pr_s(A)$ ,  $O$  is therefore quasi-compact.

4. Let  $O_1, \dots, O_n$  denote quasi-compact open subsets of  $Pr_s(A)$  ; then, according to (iii), we may write

$$Pr_s(A) \setminus O_j = Pr_s(A) \cap W(I_j)$$

for some finitely generated ideal  $I_j$  of  $A$ . Thus

$$\begin{aligned}
Pr_s(A) \setminus (O_1 \cap \dots \cap O_m) &= \bigcup_{j=1}^m (Pr_s(A) \setminus O_j) \\
&= \bigcup_{j=1}^m (Pr_s(A) \cap W(I_j)) \\
&= Pr_s(A) \cap \bigcup_{j=1}^m W(I_j) \\
&= Pr_s(A) \cap W\left(\prod_{j=1}^m I_j\right) \\
&= Pr_s(A) \cap W(I_1 \dots I_m) ,
\end{aligned}$$

whence, according to (iii),  $O_1 \cap \dots \cap O_m$  is quasi-compact, as  $I_1 \dots I_m$  is finitely generated.

5. Let  $F$  denote an irreducible closed set in  $Pr_s(A)$  ; then  $F = Pr_s(A) \cap \overline{W(S)}$  for  $S$  a subset of  $A$ . We have seen above that, setting  $I := \overline{\langle S \rangle}$ , one has  $F = Pr_s(A) \cap W(I)$ . As  $F$  is not empty,  $I \neq A$ . Let us assume  $ab \in I$  ; then, for each  $\mathcal{P} \in F$ , one has  $ab \in I \subseteq \mathcal{P}$ , whence  $a \in \mathcal{P}$  or  $b \in \mathcal{P}$ , *i.e.*  $\mathcal{P} \in F \cap W(\{a\})$  or  $\mathcal{P} \in F \cap W(\{b\})$  :

$$F = (F \cap W(\{a\})) \cup (F \cap W(\{b\})) .$$

As  $F$  is irreducible, it follows that either  $F = F \cap W(\{a\})$  or  $F = F \cap W(\{b\})$ . In the first case we get  $F \subseteq W(\{a\})$ , *i.e.*

$$a \in \bigcap_{\mathcal{P} \in Pr_s(A); I \subseteq \mathcal{P}} \mathcal{P} = I \text{ (Proposition 5.5) ;}$$

similarly, in the second case,  $b \in I$  :  $I$  is prime. But then

$$\begin{aligned}
\overline{\{I\}} &= Pr_s(A) \cap W(I) \\
&= F
\end{aligned}$$

and  $I$  is a generic point for  $F$ .

It is unique as, in a  $T_0$ -space, an (irreducible) closed set admits **at most one** generic point (see [6], (0.2.1.3)).

□

**Corollary 6.3.**  $Pr_s(A)$  and  $MaxSpec(A)$  are spectral spaces in the sense of Hochster ([7], p. 43).

**Theorem 6.4.** (cf. [6], Corollary 1.1.14) Let  $F = Pr_s(A) \cap W(S)$  be a nonempty closed set in  $Pr_s(A)$  ; then  $F$  is homeomorphic to  $Pr_s(B)$ , where  $B := \frac{A}{\mathcal{R}_I}$  with  $I := \overline{\langle S \rangle}$ .

*Proof.* As seen above, one has  $F = Pr_s(A) \cap W(I)$ , whence, as  $F \neq \emptyset$ ,  $I \neq A$ . Let  $A_0 := \frac{A}{\mathcal{R}_I}$ , and let  $\pi : A \rightarrow A_0$  denote the canonical projection.

Let us now define

$$\begin{aligned} \psi &: Pr_s(A_0) \rightarrow F \\ \mathcal{Q} &\mapsto \pi^{-1}(\mathcal{Q}) . \end{aligned}$$

Then  $\psi$  is well-defined (as  $\pi^{-1}(\mathcal{Q})$  is a saturated prime ideal of  $A$  that contains  $I$ ), and injective (as, for each  $\mathcal{Q} \in Pr_s(A_0)$ ,  $\pi(\psi(\mathcal{Q})) = \mathcal{Q}$ ).

Let  $\mathcal{P} \in F$  ; then  $\pi(\mathcal{P})$  is an ideal of  $A_0$ . Let us assume  $\pi(v) \in \overline{\pi(\mathcal{P})}$  ; then

$$\pi(v) + \pi(a) = \pi(a)$$

for some  $a \in \mathcal{P}$ , that is

$$\pi(a + v) = \pi(a) .$$

But then

$$a + v + i = a + i$$

for some  $i \in I$ , whence

$$v + (a + i) = a + i$$

As  $a + i \in \mathcal{P}$  and  $\mathcal{P}$  is saturated, it follows that  $v \in \mathcal{P}$  :  $\pi(\mathcal{P})$  is saturated.

Furthermore , if  $\pi(1) \in \pi(\mathcal{P})$ , one has  $\pi(1) + \pi(v) = \pi(v)$  for some  $v \in \mathcal{P}$ , whence there is  $w \in I$  such that  $1 + v + w = v + w$ , whence  $1 + v + w \in \mathcal{P}$  and (as  $\mathcal{P}$  is saturated)  $1 \in \mathcal{P}$  and  $\mathcal{P} = A$ , a contradiction. Therefore  $\pi(\mathcal{P}) \neq A_0$ .

Let us assume  $\pi(x)\pi(y) \in \pi(\mathcal{P})$  : then  $xy + i = q + i$  for some  $i \in I$ , whence

$$(x + i)(y + i) = xy + xi + iy + i^2 \in \mathcal{P} ,$$

and  $x + i \in \mathcal{P}$  or  $y + i \in \mathcal{P}$  ; as  $\mathcal{P}$  is saturated, it follows that  $x \in \mathcal{P}$  or  $y \in \mathcal{P}$ , whence  $\pi(x) \in \pi(\mathcal{P})$  or  $\pi(y) \in \pi(\mathcal{P})$  :  $\pi(\mathcal{P})$  is prime.

As  $\mathcal{P}$  is saturated, one sees in the same way that  $\psi(\pi(\mathcal{P})) = \pi^{-1}(\pi(\mathcal{P})) = \mathcal{P}$ , whence  $\psi$  is surjective.

Let  $G := F \cap W(S_0)$  be closed in  $F$  ; then  $\mathcal{P} \in \psi^{-1}(G)$  if and only if  $\psi(\mathcal{P}) \in F \cap W(S_0)$ , that is  $S \subseteq \pi^{-1}(\mathcal{P})$  and  $S_0 \subseteq \pi^{-1}(\mathcal{P})$ , *i.e.*  $\pi(S \cup S_0) \subseteq \mathcal{P}$  :

$$\psi^{-1}(G) = Pr_s(A_0) \cap W(\pi(S \cup S_0))$$

is closed in  $F$ , and  $\psi$  is continuous.

Let now  $H := Pr_s(A_0) \cap W(\bar{G})$  be closed in  $Pr_s(A_0)$ , and let  $\mathcal{Q} \in Pr_s(A_0)$  ; as  $\pi$  is surjective,  $\bar{G} \subseteq \mathcal{Q}$  if and only if  $\pi^{-1}(\bar{G}) \subseteq \pi^{-1}(\mathcal{Q}) = \psi(\mathcal{Q})$ , and it follows that

$$\psi(H) = F \cap W(\pi^{-1}(\bar{G}))$$

is closed in  $F$ . Therefore  $\psi$  is a homeomorphism. □

## 7. Remarks on the one-generator case

Let us now consider the case of a nontrivial monogenic  $B_1$ -algebra containing strictly  $B_1$ , *i.e.*  $A = \frac{B_1[x]}{\sim}$  is a quotient of the free algebra  $B_1[x]$  with  $x \approx 0$ ,  $x \approx 1$ . Denote by  $\alpha$  the image of  $x$  in  $A$ ; then  $\alpha \notin \{0, 1\}$ , and  $\alpha$  generates  $A$  as a  $B_1$ -algebra.

Let us suppose that, for some  $(u, v) \in A^2$ ,  $\alpha u = 1 + \alpha v$ ; then  $\alpha$  is not nilpotent, as from  $\alpha^n = 0$  would follow

$$0 = \alpha^n v = \alpha^{n-1}(\alpha v) = \alpha^{n-1}(1 + \alpha u) = \alpha^{n-1} + \alpha^n u = \alpha^{n-1} ,$$

whence  $\alpha^{n-1} = 0$  and, by induction on  $n$ ,  $1 = \alpha^0 = 0$ , a contradiction.

Therefore three cases may appear

- (i)  $\alpha$  is nilpotent.
- (ii)  $\alpha$  is not nilpotent and there does not exist  $(u, v) \in A^2$  such that  $\alpha u = 1 + \alpha v$ .
- (iii) ( $\alpha$  is not nilpotent) and there exists  $(u, v) \in A^2$  such that  $\alpha u = 1 + \alpha v$ .

In case (i), any prime ideal of  $A$  must contain  $\alpha$ , hence contain  $\alpha A$ ; the ideal  $\alpha A$  is, according to the above remark, saturated, and is not contained in a strictly bigger saturated ideal other than  $A$  itself (in both cases, as any element of  $A$  not in  $\alpha A$  is of the shape  $1 + \alpha x$ ). Therefore  $Pr_s(A) = \{\alpha A\}$ , whence  $Nil(A) = \alpha A$ . In this case we see that

$$\frac{A}{\mathcal{R}_{Nil(A)}} \simeq B_1 .$$

In cases (ii) and (iii), no power of  $\alpha$  belongs to  $Nil(A)$ ; as  $Nil(A)$  is saturated, it follows that  $Nil(A) = \{0\}$ . In fact,  $A$  is integral, whence  $\{0\} \in Pr_s(A)$ . If  $\mathcal{P} \in Pr_s(A)$  and  $\mathcal{P} \neq \{0\}$ , then  $\mathcal{P}$  contains some power of  $\alpha$ , hence contains  $\alpha$ , hence contains  $\alpha A$ . As above we see that  $\mathcal{P} = \alpha A$ ; but, in case (iii),  $\alpha A$  is not saturated. In case (ii) it is easy to see that  $\alpha A$  is prime and saturated. Therefore

1. In case (ii),  $Pr_s(A) = \{\{0\}, \alpha A\}$ ;  $\{0\}$  is a generic point, that is

$$\overline{\{\{0\}\}} = Pr_s(A) ,$$

and  $\alpha A$  a “closed point” ( $\{\alpha A\}$  is closed);



2. In case (iii),  $Pr_s(A) = \{\{0\}\}$ .

One may remark that  $B_1[x]$  itself falls into case (ii).

In [9], pp. 75–79, we have enumerated (up to isomorphism) monogenic  $B_1$ -algebras of cardinality  $\leq 5$ . It is easy to see where these algebras fall in the above classification ; we keep the numbering used in [9]. Let then  $3 \leq |A| \leq 5$ . We have the following repartition

Case (i) : (6),(8),(12),(15),(18),(24)

Case (ii): (7),(10),(11),(16),(19),(25),(26)

Case (iii): (5),(9),(13),(14),(17),(20),(21),(22),(23),(27),(28)

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