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Information Transmission under Random Emission Capital Constraints

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Abstract

We model the transmission of a message on the complete graph with n vertices and limited resources. The vertices of the graph stand for servers that may broadcast the message at random. Each server has a random emission capital that decreases at each emission. Given an initially informed server, the question is to decide of the number of servers that receive the information before the capital of all the informed servers is exhausted. We establish limit theorems, as $n \rightarrow \infty$, for the proportion of visited vertices before exhaustion and for the total duration. The analysis relies on a description of the transmission procedure as a dynamical selection of successful nodes in a Galton-Watson tree with respect to the success epochs of the coupon collector problem. To the best of our knowledge, this is the first time such a transmission model and such a description are considered. A similar study is possible for expander graphs. Due to the lack of space, we will do it in a separate paper.

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1 Introduction

Dissemination of viruses and transmission of information in computer networks gave rise to many practical as well as theoretical investigations over the two last decades (see [1, 5, 9, 10, 13, 15]).

In this paper, we model the transmission of a message on the complete graph with n vertices and limited resources. Every vertex represents a server, which can be in one of three states: inactive (it did not receive the message yet), active (it has already received it, and is able to transmit it), exhausted (it has already received it, but cannot transmit it anymore because it has exhausted its own capital of emissions). Each server S_i has a random emission capital K_i . The message is initially received from outside by one server, which is then turned from the inactive state to the active one (if it has a positive emission capital) or exhausted (if its emission capital is 0), though the $n - 1$ other servers are inactive. At each integer time, one of the active servers (say S_i) casts the message, it loses one unit of its own emission capital K_i , and it selects the target at random among the n servers. If the target is inactive, it discovers the information, it becomes itself active or exhausted according to its own emission capital. If not, this broadcast is unsuccessful and nothing else happens. When an active server exhausts its emission capital, it enters the exhausted state. The transmission ends at a finite time τ_n , which is at most equal to n plus the sum of all initial capitals.

From a practical point, the graph may be thought as a wireless network, the vertices of which are battery powered sensors with a limited energy capacity. We refer the reader to [1, 5, 9, 10] for applications of graph theory to the performance evaluation of information transmission in wireless networks.

Here we carefully describe the asymptotic behavior of the proportion of visited vertices at the end of the process when n tends to the infinity. The mathematical analysis relies on a twofold structure: a subtree of the Galton-Watson tree, which models the vertices reached by the emission procedure, and the success epochs of the coupon collector problem, which model the successful transmissions. Put it in a probabilistic way, we propose a coupling of the transmission model as a marginal tree of the Galton-Watson tree, obtained by pruning some of the nodes according to the coupon collector problem. Such a coupling provides a direct interpretation of the scenarios when the network ceases broadcasting in a macroscopic time: basically, these scenarios correspond to the extinction event in the Galton-Watson tree. On the survival event, we manage to specify the first-order behavior (in n) of the exhaustion time τ_n and of the proportion of visited nodes. A central limit theorem is established as well for τ_n under suitable square-integrability conditions on the distribution of the capital of a given vertex. We refer to [2, 11] for advanced results concerning the coupon collector problem.

We also refer to [16] and [15] for related transmission models. Machado et al. [16] consider the case where $K_i = 2$ and prove partial transmission results. Obviously, our approach extends this result, as constant capitals are a specific case of random ones. A specific interest of random capitals consists in allowing K_i to be 0 with a non-trivial probability: as we shall see below, a quick stop of the transmission process then occurs with a positive probability, as the extinction event of the Galton-Watson tree. As in [16], Kurtz et al. [15] investigate the case when the K_i 's are constant, but possibly larger than 2, time running continuously. In their model, there is one particle at each vertex of the graph at time 0; one of them is active, the others are inactive. The active particle begins to move as a continuous-time, rate 1, random walk on the graph; as soon as any active particle visits an inactive one, the latter becomes active and starts an independent random walk. Each active particle dies at the instant it reaches a total of L jumps (consecutive or not) without activating any particle. Each active particle starts with L lives and loses one life unit whenever it jumps on a vertex which has already been visited by

the process.

The note is organized as follows. The basic model is presented in Section 2. In Section 3, we provide an alternative construction based on a pruning procedure of the Galton-Watson tree. Main results and sketches of proofs are given in Section 4. Prospects are discussed in Section 5.

2 A formal definition of the dynamics

Conditionally on the states up to time $t \in \mathbb{N}$, the total emission capital evolves according to

$$S_n(t+1) = S_n(t) + \begin{cases} -1 \\ K(t) - 1 \end{cases} \quad \text{with probability} \quad \begin{cases} N_n(t)/n \\ 1 - N_n(t)/n \end{cases},$$

with $K(t)$ the initial emission capital of the target activated at time $t+1$, and $N_n(t)$ the number of servers which have received the information by time t ; respectively, in the above cases,

$$N_n(t+1) = \begin{cases} N_n(t) \\ N_n(t) + 1 \end{cases}.$$

The variables $(K(t), t \in \mathbb{N})$ are i.i.d., the common distribution being denoted by μ . The transmission process lasts for a duration τ_n which is the first time t when the emission capital is equal to 0.

A natural question consists in determining whether the information will reach all servers, or a proportion of them only. More precisely: Define the full transmission event

$$\mathbf{Trans}_n = \{N_n(\tau_n) = n\},$$

that stands for ‘‘all the servers finally receive the information’’. Then, three regimes are expected:

- (FT; full transmission) This regime is defined by: $\mathbb{P}(\mathbf{Trans}_n) \rightarrow 1$ as $n \rightarrow \infty$.
- (PT; partial transmission) $\mathbb{P}(\mathbf{Trans}_n) \rightarrow 0$ as $n \rightarrow \infty$.
- (ST; successful transmission) $\lim_{n \rightarrow \infty} \mathbb{P}(\mathbf{Trans}_n) \in (0, 1)$.

In all cases, the limits

$$\lim_{n \rightarrow \infty} N_n(\tau_n)/n, \quad \lim_{n \rightarrow \infty} \tau_n/n$$

are to be evaluated.

Fundamental questions of interest are: Under which conditions on the tails of K does one of the above regimes hold? In each of these cases, what is the magnitude of the random fluctuations, and what is their asymptotic law?

3 Labeled Galton-Watson tree

In this section, we give an alternative construction of the information transmission process. We start, for each n , with a Galton-Watson tree with labels, and define a pruning procedure. The pruning uses the labels, and it corresponds in fact to a coupon collector process with an independent Galton-Watson tree. In Proposition 3.1, we will see that the labeled Galton-Watson tree yields an equivalent description of the process defined by the dynamics.

More precisely:

Let $\mathcal{W} = \cup_{k \geq 0} (\mathbb{N}^*)^k$ be the set of all finite words on the alphabet $\{1, 2, \dots\}$. Its elements are of the form $w_1 w_2 \dots w_k, w_i \in \mathbb{N}^*$ when $k \geq 1$, and, for $k = 0$, $(\mathbb{N}^*)^k$ reduces to the empty word \emptyset , that we call the root. We then denote by $|w| = k$ the length of $w = w_1 w_2 \dots w_k \in \mathcal{W}$. (With $|\emptyset| = 0$.) For $w, w' \in \mathcal{W}$, we write $w < w'$ if: $|w| < |w'|$, or $|w| = |w'|$ and $w \leq_{\text{lex}} w'$ in the lexicographic order. We denote by \preceq the usual predecessor relation in \mathcal{W} , that is $w \preceq w'$ if w is a prefix of w' .

Let $(K(w), w \in \mathcal{W})$ be a family of i.i.d. random variables on $\mathbb{N} = \{0, 1, 2, \dots\}$ with common law μ . Assume $\mu(0) < 1$ and $\mu(1) < 1$ for a nontrivial setup. The associated Galton-Watson tree \mathcal{T}^{GW} is the set of $w \in \mathcal{W}$ such that, for all $i = 1, \dots, |w|, w_i \leq K(v)$ with v the predecessor of w of length $i - 1$. (In other words, given a parent w' at the $(i - 1)$ th generation, that is w' is a word of length $i - 1$, the children of w' are the words $w'1, \dots, w'K(w')$, of length i , obtained by concatenation.) In particular, $\emptyset \in \mathcal{T}^{\text{GW}}$. Denote by Z_k^{GW} the size of the k th generation of this tree, $Z_k^{\text{GW}} = \text{card} \{w \in \mathcal{T}^{\text{GW}} : |w| = k\}$, which is given by

$$Z_{k+1}^{\text{GW}} = \sum_{v \in \mathcal{T}^{\text{GW}}, |v|=k} K(v), \quad Z_0^{\text{GW}} = 1.$$

It is well known that the survival event $\text{Surv}^{\text{GW}} = \bigcap_k \{Z_k \geq 1\} = \{\text{card } \mathcal{T}^{\text{GW}} = \infty\}$ has complement probability

$$\sigma^{\text{GW}} = 1 - \mathbb{P}(\text{Surv}^{\text{GW}}) = \begin{cases} = 1 & \text{if } \mathbb{E}K \leq 1, \\ < 1 & \text{if } \mathbb{E}K > 1. \end{cases}$$

On the same probability space, we consider an independent coupon collector process with n images ($n \geq 1$): Let $\Delta_{i,n}, i = 1, \dots, n - 1$, be independent, geometrically distributed r.v.'s on \mathbb{N}^* with parameter $1 - i/n$ (success probability) respectively. The success epochs are

$$T_{1,n} = 0, \quad T_{i,n} = \sum_{j=1}^{i-1} \Delta_{j,n}, \quad i = 2, \dots, n,$$

and the counting function is

$$N_n(t) = \sum_{i=1}^n 1_{\{T_{i,n} \leq t\}}, \quad t = 0, 1, \dots$$

In fact, $N_n(t)$ represents the number of servers having received the information by time t (note that $1 \leq N_n(t) \leq \min\{t + 1, n\}$).

For any fixed integer n , with these two ingredients, we can define the transmission process together with the transmission time length τ_n . Let us start with an informal description. We browse a part of the Galton-Watson tree following the order $<$, and we paint the nodes in \circ or in Δ according to the coupon collector process (success or failure); we only browse nodes which are in stand-by; as soon as a node is painted \circ , its number of children nodes in \mathcal{T}^{GW} is revealed, and these children are put in stand-by. We then move to the next node in stand-by (next for $<$). The procedure runs until there are no more nodes in stand-by.

Here is a precise definition. Recursively for $t = 0, 1, \dots$, we construct $X(t) \in \mathcal{T}^{\text{GW}}$, and disjoint $\mathcal{T}^\circ(t), \mathcal{T}^\boxtimes(t), \mathcal{T}^\Delta(t) \subset \mathcal{T}^{\text{GW}}$ as follows. ($X(t)$ encodes the vertex where the t -th tentative emission takes place, $\mathcal{T}^\circ(t)$ denotes the set of servers already informed by time t , $\mathcal{T}^\boxtimes(t)$ is the set of tentative emissions scheduled but not yet performed at time t , $\mathcal{T}^\Delta(t)$ is set of failed

emissions, i.e. those performed by time t for which the target was already informed.) Start with

$$X(0) = \emptyset, \quad \mathcal{T}^\circ(0) = \{\emptyset\}, \quad \mathcal{T}^{\boxtimes}(0) = \{w \in \mathcal{T}^{\text{GW}} : w \leq K(\emptyset)\}, \quad \mathcal{T}^\Delta(0) = \emptyset,$$

with \emptyset denoting the empty set. With the process $(X(t), \mathcal{T}^\circ(t), \mathcal{T}^{\boxtimes}(t), \mathcal{T}^\Delta(t))$ at time t , its value at the next step $t + 1$ is defined by:

- If $\mathcal{T}^{\boxtimes}(t)$ is nonempty, we let $X(t + 1)$ be its first element,

$$X(t + 1) = \inf\{w \in \mathcal{T}^{\boxtimes}(t)\}, \text{ denoted by } v$$

to ease the notations, and we perform a test:

- If $N_n(t + 1) = N_n(t) + 1$, we define

$$\begin{aligned} \mathcal{T}^\circ(t + 1) &= \mathcal{T}^\circ(t) \cup \{v\}, \\ \mathcal{T}^{\boxtimes}(t + 1) &= (\mathcal{T}^{\boxtimes}(t) \setminus \{v\}) \cup \{v1, v2, \dots, vK(v)\}, \\ \mathcal{T}^\Delta(t + 1) &= \mathcal{T}^\Delta(t). \end{aligned} \tag{1}$$

(Above, the notation vk denotes the word of length $|v| + 1$ obtained by concatenation.)

- If $N_n(t + 1) = N_n(t)$, we define

$$\begin{aligned} \mathcal{T}^\circ(t + 1) &= \mathcal{T}^\circ(t), \\ \mathcal{T}^{\boxtimes}(t + 1) &= \mathcal{T}^{\boxtimes}(t) \setminus \{v\}, \\ \mathcal{T}^\Delta(t + 1) &= \mathcal{T}^\Delta(t) \cup \{v\}. \end{aligned} \tag{2}$$

- If $\mathcal{T}^{\boxtimes}(t)$ is empty, we set $\tau_n = t$, and the construction is stopped (as well as the transmission). The set $\mathcal{T}^\circ(t) = \mathcal{T}^\circ(\tau_n) = \mathcal{T}^\circ(\infty)$ is the set of servers finally informed. Note that $\tau_n \leq T_{n,n}$ is a.s. finite.

We observe that for all t , $\mathcal{T}^\circ(t)$ is a tree, as well as $\mathcal{T}^\circ(t) \cup \mathcal{T}^{\boxtimes}(t)$. Moreover, $\mathcal{T}^{\boxtimes}(t) \cup \mathcal{T}^\Delta(t)$ is a cutset of \mathcal{T}^{GW} .

\emptyset

$X(t)$

Figure 1. The Galton-Watson tree is represented up to the 4th generation.

$\mathcal{T}^\circ(t)$, $\mathcal{T}^\boxtimes(t)$ and $\mathcal{T}^\Delta(t)$ and $X(t)$ are represented at time $t = 11$.

We now relate the above construction to the dynamical model for transmission. Consider a new, independent, i.i.d. sequence $(\bar{K}_i, i \in \mathbb{N} \setminus \{0\})$ with law μ , and define, for $i = 1, \dots, n$,

$$K_i = \begin{cases} K(X(T_{i,n})) & \text{if } i \leq \text{card } \mathcal{T}^\circ(\infty), \\ \bar{K}_i & \text{if } i > \text{card } \mathcal{T}^\circ(\infty), \end{cases}$$

and also

$$S_n(t) := \sum_{i=1}^{N_n(t)} K_i - t, \quad t \geq 0. \tag{3}$$

By construction, we have

$$N_n(t) = \text{card } \mathcal{T}^\circ(t), \quad S_n(t) = \text{card } \mathcal{T}^\boxtimes(t). \tag{4}$$

Proposition 3.1 *The sequence $(K_i, i \leq n)$ is independent, identically distributed with law μ , and is independent of $(T_{i,n}, i \leq n)$. Moreover,*

$$\tau_n = \inf\{t \geq 0 : S_n(t) = 0\}.$$

□ The formula for τ_n is easily checked by (4). We now investigate the distribution of the sequence $(K_i, i \leq n)$. Below, we denote by $\mathcal{F}_w = \sigma(K(w'), w' \leq w)$ for $w \in \mathcal{W}$. On the event $A = \{(T_{1,n}, \dots, T_{i,n}) = (k_1, \dots, k_i), X(k_i - 1) = w, \tau_n \geq k_i\}$, $0 = k_1 < \dots < k_i$ and $w \in \mathcal{W}$, $|w| \leq k_i - 1$, $X(T_{i,n})$ coincides with an \mathcal{F}_w -measurable r.v., denoted by χ , which satisfies $w < \chi$ almost-surely. (This follows from the monotonicity of the browsing procedure.)

Similarly, all the variables $K(X(T_{j,n}))$, $1 \leq j \leq i-1$ coincide with \mathcal{F}_w -measurable r.v.'s on A . Clearly, $K(\chi)$ is independent of $\mathcal{F}_w \vee \sigma(T_{1,n}, \dots, T_{n,n})$ and has μ as distribution, since the r.v.'s ($K(w')$, $w' \in \mathcal{W}$) are i.i.d and are independent of the success epochs $(T_{1,n}, \dots, T_{n,n})$. Obviously, the event A belongs to $\mathcal{F}_w \vee \sigma(T_{1,n}, \dots, T_{n,n})$. This proves that, for any bounded and measurable Borel function ϕ ,

$$\mathbb{E}[\phi(K_i) \mathbf{1}_{\{T_{i,n} \leq \tau_n\}} | (T_{j,n}, 1 \leq j \leq n), (K_j, 1 \leq j \leq i-1)] = \mathbf{1}_{\{T_{i,n} \leq \tau_n\}} \int_{\mathbb{N}} \phi d\mu$$

On the event $\{(T_{1,n}, \dots, T_{i,n}) = (k_1, \dots, k_i), \tau_n < k_i\}$, K_i coincides with \bar{K}_i , which is obviously independent of $\sigma((T_{j,n}, 1 \leq j \leq n), (K_j, 1 \leq j \leq i-1))$, so that the above equality also holds with $T_{i,n} \leq \tau_n$ replaced by $T_{i,n} > \tau_n$. \square

The above proposition shows that the process we have constructed here, corresponds to the description of the information transmission process given in Introduction. We then have a graphical construction of the transmission process from a Galton-Watson tree and a coupon collector process. The next one indicates how we can use this coupling to study the transmission process.

Proposition 3.2 *Let $\hat{\tau}_k = \max\{t \geq 0 : |X(t)| = k\}$. Then*

$$\text{card } \mathcal{T}^{\boxtimes}(\hat{\tau}_k) \leq Z_{k+1}^{\text{GW}}. \quad (5)$$

Hence,

$$\text{card} \left((\mathcal{T}^{\circ}(\hat{\tau}_k) \cup \mathcal{T}^{\boxtimes}(\hat{\tau}_k)) \right) \leq \sum_{\ell=0}^{k+1} Z_{\ell}^{\text{GW}}. \quad (6)$$

\square The inequalities directly follow from the construction. \square

Remark 3.3 *We emphasize that the coupling with the Galton-Watson tree could be modified according to specific purposes. For instance, the node $X(t)$ could be chosen as a random vertex among all the vertices belonging to $\mathcal{T}^{\boxtimes}(t)$ and not as the smallest vertex in $\mathcal{T}^{\boxtimes}(t)$. In some sense, this would be a more natural selection procedure as the active broadcasting site in the original model is chosen randomly itself. Such a genealogy would favor sites with a large number of children, so that the resulting global shape of the tree \mathcal{T}° would not be so flat as it is under the coupling we described above. Anyhow, there is no need to consider such a modification of our coupling for the results we prove below: in what follows, we thus make use of the original coupling as its ordered structure is quite comfortable to handle.*

4 Results and sketch of Proofs

Lemma 4.1 *We have the following convergence in law of sequences of processes on the Skorohod space:*

(i) *On $D([0, 1])$,*

$$n^{-1/2} (T_{[nq], n} - n \ln \frac{1}{1-q}) \xrightarrow{\text{law}} B(\sigma_T(q)^2),$$

with B a standard Brownian motion, and

$$\sigma_T(q)^2 = \frac{q}{1-q} + \ln(1-q) > 0.$$

(ii) Denote $N_n(t) = N_n(\lfloor nt \rfloor)$. Then, on $D(\mathbb{R}^+)$,

$$n^{-1/2}(N_n(ns) - n(1 - e^{-s})) \xrightarrow{\text{law}} B(\sigma_N(s)^2),$$

with B a standard Brownian motion, and

$$\sigma_N(s)^2 = e^{-s}(1 - e^{-s}) - se^{-2s} > 0.$$

Both limits are independent increments Gaussian processes with zero mean, and they are martingales.

□ Assertion (i) is a direct application of the invariance principle for triangular array of independent, but not i.i.d., square-integrable r.v.'s. (See Dacunha-Castelle and Duflo [4, Théorème 7.4.28] or Jacod and Shiryaev [8, Chapter VII, Theorem 5.4].) The variance is computed as a Riemann sum,

$$\sigma_T(q)^2 = \lim_n \frac{1}{n} \sum_{i=1}^{qn} \text{Var}(\Delta_{i,n}) = \lim_n \frac{1}{n} \sum_{i=1}^{qn} \frac{i/n}{(1 - i/n)^2} = \int_0^q \frac{y}{(1 - y)^2} dy.$$

Assertion (ii) follows from (i), using that $N_n(\cdot)$ and $T_{[n],n}$ are reciprocal in a generalized sense. With $f(q) = -\ln(1 - q)$, $f^{-1}(s) = 1 - e^{-s}$, we have $\sigma_N(s)^2 = \sigma_T(f^{-1}(s))^2 \times [f' \circ f^{-1}(s)]^{-2}$. (See Billingsley [3, Theorem 17.3]). □

The next lemma shows that when the Galton-Watson tree is infinite, transmission takes place on a macroscopic time level.

Lemma 4.2 *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n \geq n\varepsilon, \text{Surv}^{\text{GW}}) = \mathbb{P}(\text{Surv}^{\text{GW}}) = 1 - \sigma^{\text{GW}}.$$

□ The claim being trivial for $\sigma^{\text{GW}} = 1$, we just need to consider the case when $\mathbb{E}K > 1$. Letting here $k = \ln^2 n$, we estimate

$$\mathbb{P}(\tau_n \leq n\varepsilon, \text{Surv}^{\text{GW}}) \leq \mathbb{P}(N_n(k) \leq k) + \mathbb{P}(k < \tau_n \leq n\varepsilon, \text{Surv}^{\text{GW}}), \quad (7)$$

using that $\{N_n(k) = k + 1, \text{Surv}^{\text{GW}}\} \subset \{\tau_n > k, \text{Surv}^{\text{GW}}\}$ which implies that

$$\mathbb{P}\{N_n(k) \leq k, \text{Surv}^{\text{GW}}\} \geq \mathbb{P}\{\tau_n \leq k, \text{Surv}^{\text{GW}}\}.$$

We start with

$$\begin{aligned} \mathbb{P}(N_n(k) \leq k) &= 1 - (1 - 1/n) \times \dots \times (1 - k/n) \\ &\leq 1 - (1 - k/n)^k \\ &\sim k^2/n \quad \text{as } k^2/n \rightarrow 0. \end{aligned} \quad (8)$$

Fix $\varepsilon > 0$ with $(1 - 2\varepsilon)\mathbb{E}K > 1$. It remains to prove the convergence

$$\mathbb{P}(k < \tau_n \leq n\varepsilon, \text{Surv}^{\text{GW}}) = \sum_{i=k}^{n\varepsilon} \mathbb{P}(\tau_n = i, \text{Surv}^{\text{GW}}) \leq \sum_{i=k}^{n\varepsilon} \mathbb{P}(S_n(i) \leq 0) \rightarrow 0, \quad (9)$$

where the inequality holds since $\text{card } \mathcal{T}^\boxtimes(i) = S_n(i)$ holds provided $i \leq \tau_n$. We start to show that

$$\mathbb{P}(N_n(i) < (1 - 2\varepsilon)i) \leq \exp(-C_\varepsilon i), \quad \forall i \leq n\varepsilon.$$

Indeed, using the short notation $T_{s,n}$ for $T_{[s],n}$, the above probability is equal to

$$\mathbb{P}(T_{(1-2\varepsilon)i,n} > i) \leq \mathbb{P}(\bar{T}_{(1-2\varepsilon)i}^\varepsilon > i)$$

with $\bar{T}_{(1-2\varepsilon)i}^\varepsilon$ a sum of a number $(1-2\varepsilon)n$ of i.i.d. geometric r.v.'s with parameter $1-\varepsilon$; now, the desired estimate follows from Chernov's bound. Next, we note that, for $z \in (0, 1)$, $i \leq n\varepsilon$ and $G(z) = \mathbb{E}z^K$,

$$\begin{aligned} \mathbb{P}(S_n(i) \leq 0, N_n(i) > (1-2\varepsilon)i) &\leq \mathbb{E}[z^{S_n(i)}; N_n(i) > (1-2\varepsilon)i] \\ &\leq z^{-i} \mathbb{E}[z^{R((1-2\varepsilon)i)}; N_n(i) > (1-2\varepsilon)i] \\ &\leq z^{-i} G(z)^{(1-2\varepsilon)i}. \end{aligned}$$

Since $(1-2\varepsilon)\mathbb{E}K > 1$, we have $r := z^{-1}G(z)^{(1-2\varepsilon)} < 1$ by picking $z < 1$ close enough to 1. Thus, the left-hand side of (9) is bounded by

$$\mathbb{P}(k < \tau_n \leq n\varepsilon) \leq \sum_{i=k}^{n\varepsilon} [r^i + \exp(-C_\varepsilon i)] \leq 2(1-r_1)^{-1}r_1^k,$$

with $r_1 = \max\{r, \exp(-C_\varepsilon)\} < 1$. Collecting the above estimates in (7) and taking $k = \ln^2 n$, we conclude that $\mathbb{P}(\tau_n \leq n\varepsilon) = \mathcal{O}(n^{-a})$ for all $a \in (0, 1)$. \square

For $\mathbb{E}K > 1$, define $\theta \in (0, \infty)$ by

$$\frac{1 - e^{-\theta}}{\theta} = \frac{1}{\mathbb{E}K}. \quad (10)$$

Extend this definition by $\theta = 0$ if $\mathbb{E}K \leq 1$, $\theta = \infty$ if $\mathbb{E}K = \infty$. The function $\mathbb{E}K \mapsto \theta$ is increasing from $[0, \infty]$ to $[0, \infty]$. Let also

$$p = 1 - e^{-\theta} \in [0, 1],$$

and note from (10), that when $\mathbb{E}K \in (1, \infty)$, $p = 1 - e^{-\theta} \in (0, 1)$ is the unique solution of

$$p \mathbb{E}K = -\ln(1-p), \quad (11)$$

Theorem 4.3 *Let $\mathbb{E}K \in (0, \infty]$.*

(i) *As $n \rightarrow \infty$,*

$$\tau_n/n \longrightarrow \theta \mathbf{1}_{\text{Surv}^{\text{GW}}}$$

in probability, with θ defined by (10), though

$$N_n(\tau_n)/n \longrightarrow p \mathbf{1}_{\text{Surv}^{\text{GW}}}$$

in probability, with $p = 1 - e^{-\theta}$.

(ii) *If $\mathbb{E}K \leq 1$,*

$$\lim_{n \rightarrow \infty} \tau_n = \lim_{n \rightarrow \infty} N_n(\tau_n) = \sum_{k \geq 0} Z_k^{\text{GW}}$$

in probability.

□ (i) Assume first $\mathbb{E}K < \infty$. Then, we can apply the law of large numbers to the process

$$R(m) = \sum_{i=1}^m K_i, \quad (12)$$

to show that \mathbb{P} -a.s., $R(nq)/n \rightarrow q\mathbb{E}K$ uniformly on $[0, 1]$. Observe that

$$S_n(t) = R(N_n(t)) - t. \quad (13)$$

In addition to Lemma 4.1, this shows that, in probability,

$$S_n(ns)/n \rightarrow (1 - e^{-s})\mathbb{E}K - s, \quad \text{uniformly on compacts of } \mathbb{R}_+ \quad (14)$$

as $n \rightarrow \infty$. Now, we write

$$\begin{aligned} \mathbb{P}(|\tau_n - n\theta \mathbf{1}_{\text{Surv}^{\text{GW}}}| > n\delta) &= \mathbb{P}(|\tau_n - n\theta| > n\delta, \text{Surv}^{\text{GW}}) + \mathbb{P}(\tau_n > n\delta, (\text{Surv}^{\text{GW}})^c) \\ &\leq \mathbb{P}(\tau_n < n\varepsilon, \text{Surv}^{\text{GW}}) + \mathbb{P}(|\tau_n - n\theta| > n\delta, \text{Surv}^{\text{GW}}, \tau_n \geq n\varepsilon) \\ &\quad + \mathbb{P}(\tau_n > n\delta, (\text{Surv}^{\text{GW}})^c) \end{aligned}$$

where the first term of the right-hand side vanishes from lemma 4.2. The last term vanishes because τ_n is smaller than the extinction time of the Galton-Watson process, which is a.s. finite. Since τ_n is the first time such that $S_n(\tau_n) = 0$ (except if $\tau_n = T_{n,n}$, an event of small probability), the second term vanishes too by (14). This ends the proof for finite $\mathbb{E}K$. In the opposite case, for a truncation level $L > 0$ we consider $K^{(L)}(w) = \min\{K(w), L\}$, we can apply the above proof, and we obtain $\lim_n \tau_n^{(L)}/n = \theta^{(L)}$ in obvious notations. Since $\tau_n^{(L)} \leq \tau_n$ and $\lim_{L \rightarrow \infty} \theta^{(L)} = \theta = \infty$, the first claim is proved by letting $L \rightarrow \infty$. For the second one, note that $N_n^{(L)}(\tau_n^{(L)}) \leq N_n(\tau_n)$, $\text{Surv}^{\text{GW}(L)} \nearrow \text{Surv}^{\text{GW}}$ as L is increased, and the proof is clear.

(ii) follows directly from (8). □

Theorem 4.4 *Assume $\mathbb{E}K > 1, \mathbb{E}K^2 < \infty$ and $\mathbb{P}(K \geq 2) > 0$. Let σ_K^2 denote the variance of K . As $n \rightarrow \infty$, we have conditionnally on Surv^{GW} ,*

$$n^{-1/2}(\tau_n - n\theta) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma_\tau^2),$$

with $\sigma_\tau^2 = [(1 - p)\mathbb{E}K - 1]^{-2}[p\sigma_K^2 + (1 - p)^2(\mathbb{E}K)^2\sigma_N^2]$.

□ By the invariance principle, $n^{-1/2}(R(nq) - nq\mathbb{E}K) \xrightarrow{\text{law}} \bar{B}(q\sigma_K^2)$, with \bar{B} a Brownian motion. By independence of $(K_i)_i$ and $(\Delta_{i,n})_i$, the vector

$$n^{-1/2} \begin{pmatrix} R(nq) - nq\mathbb{E}K \\ N_n(ns) - n(1 - e^{-s}) \end{pmatrix} \xrightarrow{\text{law}} \begin{pmatrix} \bar{B}(q\sigma_K^2) \\ B(\sigma_N(s)^2) \end{pmatrix}$$

as $n \rightarrow \infty$, where B and \bar{B} are independent. Inserting this in the relation $R(N_n(\tau_n)) = \tau_n$, which holds with probability going to 1, we obtain the result. □

The above Theorem extends results in [15, 16]. We do not describe random fluctuations any further, but we turn to illustrate the regimes of successful/full transmission.

Theorem 4.5 *If there exist $c > 0$ and $\alpha \in (0, 1)$ such that $\liminf_{n \rightarrow +\infty} [n^\alpha \mathbb{P}(K \geq n)] \geq c$, then, $\mathbb{P}(\text{Trans}_n) \equiv \mathbb{P}(N_n(\tau_n) = n) \rightarrow \mathbb{P}(\text{Surv}^{\text{GW}})$ as $n \rightarrow +\infty$.*

□ Clearly, $\mathbb{E}(K) = +\infty$. By Theorem 4.3, for any $\varepsilon > 0$, $\mathbb{P}(N_n(\tau_n)/n \geq 1 - \varepsilon, \text{Surv}^{\text{GW}}) \rightarrow \mathbb{P}(\text{Surv}^{\text{GW}})$ as $n \rightarrow +\infty$. Thus, for any $\beta \in (1, 1/\alpha)$,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \mathbb{P}(N_n(\tau_n) = n, \text{Surv}^{\text{GW}}) &\geq \liminf_{n \rightarrow +\infty} \mathbb{P}\left(\sum_{i=1}^{\lfloor n/2 \rfloor} K_i \geq T_{n,n}, \text{Surv}^{\text{GW}}\right). \\ &\geq \liminf_{n \rightarrow +\infty} \mathbb{P}\left(\sum_{i=1}^{\lfloor n/2 \rfloor} K_i \geq n^\beta, T_{n,n} \leq n^\beta, \text{Surv}^{\text{GW}}\right). \end{aligned}$$

By Markov inequality, $\mathbb{P}(T_{n,n} > n^\beta) \rightarrow 0$ as $n \rightarrow +\infty$. Moreover, $\mathbb{P}(K_1 \geq n^\beta) \geq (c/2)n^{-\alpha\beta}$ for n large enough, so that, for n large,

$$\mathbb{P}\left(\bigcap_{i=1}^{\lfloor n/2 \rfloor} (K_i < n^\beta)\right) \leq \left(1 - \frac{c}{2n^{\alpha\beta}}\right)^{\lfloor n/2 \rfloor} \sim \exp\left(-\frac{c}{4}n^{1-\alpha\beta}\right).$$

We deduce that $\liminf_{n \rightarrow +\infty} \mathbb{P}(N_n(\tau_n) = n, \text{Surv}^{\text{GW}}) = \mathbb{P}(\text{Surv}^{\text{GW}})$.

Finally, on $(\text{Surv}^{\text{GW}})^c$, $N_n(\tau_n)/n \rightarrow 0$ in probability, so that $\mathbb{P}(N_n(\tau_n) = n, (\text{Surv}^{\text{GW}})^c) \rightarrow 0$. □

5 Conclusion and further research aspects

We have proposed and analysed a new probabilistic model for the information transmission on complete graphs under capital emission constraints. The description of the limit behavior for the proportion of vertices visited at the end of the process relies on a coupling with the Galton-Watson tree: Theorem 4.3 specifies the (PT) regime and Theorem 4.5 gives typical conditions for (FT) and (CT) regimes according to the survival probability of the Galton-Watson tree. Several interesting subsequent questions are to be investigated in further works: (i) What is the typical number of needed emissions before a given node of the graph is reached? (ii) What is the scaling limit of the tree $\mathcal{T}^\circ(\tau_n)$? We also plan to conduct experimentations on real computer networks and to extend this analysis to expander graphs [7]. Finally, another prospect is to investigate the case when the information is transmitted in a dynamical environment, that is when the emission capitals K_i ($1 \leq i \leq n$) depend on some time-dependent environment process.

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