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Some remarks on the paper "Rank of mapping tori and companion matrices" by G. Levitt and V. Metaftsis .

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1 Introduction

The present paper considers certain diophantine issues arising from the study of the rank of the so-called *mapping tori groups*. These groups have been studied in a number of papers. For more on this, we referred to the article [5] of Levitt and Metaftsis who in fact pointed out to us the question which we now analyze.

Let $A \in \mathrm{GL}_d(\mathbb{Z})$. The natural action $v \mapsto Av$ induces a semidirect product

$$G = \mathbb{Z}^d \rtimes_A \mathbb{Z} = \langle \mathbb{Z}^d, t \,|\, t\mathbf{v}t^{-1} = A\mathbf{v} \rangle$$

where we identify the generator 1 of $\mathbb{Z}$ with A. Let $\mathrm{OR}(A)$ be the minimum number of vectors whose A-orbits generate $\mathbb{Z}^d$ and let $\mathrm{rank}(G)$ be the minimum number of generators of G. In [5] the Authors shows that

Theorem 1.1 (Levitt-Metaftsis)

$$\mathrm{rank}(G) = 1 + \mathrm{OR}(A) \ .$$

In particular, G can be generated by two elements if and only if A is conjugate in $\mathrm{GL}_d(\mathbb{Z})$ to a companion matrix.

We recall that a *companion matrix* is one of the shape

$$\begin{pmatrix} 0 & & & & * \\ 1 & 0 & & & * \\ & \ddots & \ddots & & * \\ & & 1 & 0 & * \\ & & & 1 & * \end{pmatrix} .$$

See also the end of this section for equivalent properties.

Since the conjugacy problem is decidable in $\mathrm{GL}_d(\mathbb{Z})$ (cf. [4]), one can decide whether G has rank 2 or not.

Motivated by a topological result of J. Souto [6], they then prove:

Theorem 1.2 (Levitt-Metaftsis) *Let $A \in GL_d(\mathbb{Z})$ be of infinite order. Consider the family of finitely generated groups $G_n = \mathbb{Z}^d \rtimes_{A^n} \mathbb{Z}$. Then there exists $n_0 = n_0(A)$ such that* $\mathrm{rank}(G_n) > 2$ *for $n \geq n_0$. In other words, for $n \geq n_0$ the matrix A^n is not conjugate to a companion matrix in* $\mathrm{GL}_d(\mathbb{Z})$.

Their proof is based on the Skolem-Mahler-Lech Theorem on linear recurrence sequence. An alternative approach to this last result follows from a local argument in $\mathbb{Z}_p$ which amount to reduction modulo p, using equations in S-units. This approach actually shows a bit more, upon excluding the matrices all of whose eigenvalues are roots of unity. Namely, under the assumption that some complex eigenvalue of $A \in \mathrm{GL}_d(\mathbb{Z})$ has infinite order, we shall prove the finiteness of the set of $n \in \mathbb{Z}$ such that A^n is conjugate to a companion matrix in $\mathrm{GL}_d(\mathbb{F}_p)$ for all primes p outside a prescribed (but arbitrary) finite set S. In section 2 we prove:

Theorem 1.3 *Let S be a finite set of prime numbers and let $A \in M_d(\mathbb{Z})$. Suppose that A has two nonzero eigenvalues whose ratio is not a root of unity. Then there are only finitely many integers n such that for all primes $p \notin S$ the reduction modulo p of the matrix A^n is conjugate to a companion matrix in* $\mathrm{GL}_d(\mathbb{F}_p)$.

Let us pause for some remarks on this statement.

Remark 1.4
i) Assume $A \in \mathrm{GL}_d(\mathbb{Z})$. If the ratio of any two eigenvalues of A is a root of unity, then in fact all the eigenvalues must be roots of unity, because $\det A = \pm 1$.

ii) We remark that the assumption on the eigenvalues is necessary here, and (if $S \neq \emptyset$) cannot be replaced with the weaker one that $A \in \mathrm{GL}_d(\mathbb{Z})$ has not finite order. This is shown by examples like $S = \{l\}$,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

We notice that A has infinite order, nevertheless each of the powers A^{l^m}, $m \in \mathbb{N}$, is conjugate to a companion matrix over $\mathbb{F}_p$, for all $p \neq l$, and actually over the ring $\mathbb{Z}[1/l]$.

On the other hand, we note that *in the special case $S = \emptyset$, the conclusion of the Theorem holds even if $A \in \mathrm{GL}_d(\mathbb{Z})$ has all eigenvalues roots of unity, but not finite order*, see Proposition 2.1 in section 2.

iii) We also remark that for $d > 1$ there are integral unimodular matrices which are conjugate to a companion matrix over all $\mathbb{F}_p$ (and indeed over all $\mathbb{Z}_p$), but not conjugate to a companion matrix over $\mathbb{Z}$; we shall sketch at the end of section 2 a proof that the matrix

$$\begin{pmatrix} 196 & 3617 \\ 11 & 203 \end{pmatrix}$$

provides indeed such an example. This shows that, even in the case $S = \emptyset$, the finiteness predicted by the Theorem is *a priori* a stronger assertion than the finiteness of the set of integers n such that A^n is conjugate to a companion matrix over $\mathbb{Z}$.

We give other complements to the results of [5]. In section 3 we prove the following effective version of Theorem 1.2:

Theorem 1.5 *Let $A \in \mathrm{GL}_d(\mathbb{Z})$ be a matrix of infinite order. Let Z be the set of positive integers n such that A^n is conjugate in $\mathrm{GL}_d(\mathbb{Z})$ to a companion matrix. Then there exists an effective absolute constant $c > 0$ such that*

$$\max Z \leq c d^6 (\log d)^2 \ .$$

We could ask for a strong version of Theorem 1.2. Let $A \in \mathrm{GL}_d(\mathbb{Z})$ of infinite order. It is true that there exists $n_0 = n_0(A)$ such that $\mathrm{OR}(G_n)$ is "large" for $n \geq n_0$? Levitt and Metaftsis give a simple counterexample to this statement. It is enough to choose integers h, $k > 1$ such that $k + h = d$ and matrices $A_1 \in \mathrm{GL}_k(\mathbb{Z})$, $A_2 \in GL_h(\mathbb{Z})$ with A_1 of infinite order and with A_2 conjugate to a companion matrix of finite order m . The matrix

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

is then of infinite order, but $\mathrm{OR}(A^n) \leq k+1$ for $n \equiv 1 \bmod m$. In section 4 we investigate more closely this problem. We formulate a conjecture which predicts that for a "generic" matrix A we would have $\mathrm{OR}(A^n) = 2$ for infinitely many integers n. We shall relate this conjecture to another one concerning algebraic numbers and we shall give some evidence for it.

In section 5, we partially answer another question posed in [5]. Let $A \in M_d(\mathbb{Z})$ be nonsingular and consider the so-called HNN extension

$$\mathbb{Z}^d *_A = \langle \mathbb{Z}^d, t \,|\, t\mathbf{v}t^{-1} = A\mathbf{v} \rangle \ .$$

By this notation it is meant that $\mathbb{Z}^d *_A$ is generated by $\mathbb{Z}^d$ and t subject to the commutation rule written on the right. We remark that $\mathbb{Z}^d *_A$ is a semidirect product if and only if $A \in \mathrm{GL}_d(\mathbb{Z})$.

Levitt and Metaftsis ask if one can generalize Theorem 1.1 to such groups. We generalize the method of [5], characterizing the rank of $\mathbb{Z}^d *_A$ in terms of the A-orbits. Then we give a positive answer to their question for 2×2 matrices.

Notations Let $\mathcal{O}$ be a ring and let $A \in M_d(\mathcal{O})$ be a $d \times d$ matrix with coefficient in $\mathcal{O}$. We denote by $\mathrm{OR}_{\mathcal{O}}(A)$ the minimum number of A-orbits needed to generate $\mathcal{O}^d$.

Let k be a field and let $A \in M_d(k)$. Then it is well known that $\mathrm{OR}_k(A)$ is equal to the number of invariant factors of A. In particular, $\mathrm{OR}_k(A) = 1$ if and only if A is conjugate in $\mathrm{GL}_d(k)$ to a companion matrix. This is in turn equivalent to require that the eigenspace of A relative to any eigenvalue has dimension 1. It is also equivalent to the fact that the minimal polynomial of A has degree d.

In what follows we shall consider matrices $A \in M_d(\mathbb{Z})$. We simply write $\mathrm{OR}(A)$ for $\mathrm{OR}_{\mathbb{Z}}(A)$.

2 An alternative proof using equations in S-units

In this section we first state and prove a Proposition which shows special cases in which the conclusion of Theorem 1.3 holds, with different assumptions. Then we prove Theorem 1.3 and we give some further remarks.

Proposition 2.1 *Let $A \in M_d(\mathbb{Z})$ and assume that A has at least one eigenvalue counted with multiplicity at least 2. Then, for all integers $n > 1$ and for all primes $p \,|\, n$, the matrix A^n is not conjugate to a companion matrix in $\mathrm{GL}_d(\mathbb{F}_p)$.*

In particular, if all the eigenvalues of A are roots of unity and A has infinite order, then for $n > 1$ there exists a prime p such that the matrix A^n is not conjugate to a companion matrix in $\mathrm{GL}_d(\mathbb{F}_p)$.

Proof. By assumption the characteristic polynomial $f(t) \in \mathbb{Z}[t]$ of A has a multiple factor, and we may write $f(t) = g^2(t)h(t)$ for suitable monic polynomials $g, h \in \mathbb{Z}[t]$, with $\deg g \geq 1$.

Take $n > 1$ and choose p to be any prime divisor of n. If A^n was conjugate to a companion matrix over $\mathbb{F}_p$, its minimal polynomial over $\mathbb{F}_p$ would have degree d, and the same would hold for A^p (because $A^n \in \mathbb{F}_p[A^p]$). However $g(A^p)h(A^p) \equiv g^p(A)h^p(A) = (g^{p-2}(A)h^{p-1}(A))f(A) = 0 \pmod{p}$. But $\deg g(t)h(t) < d$, a contradiction.

Concerning the last assertion of the Proposition, note that if all the eigenvalues of A are roots of unity and A has infinite order, then A has at least one eigenvalue counted with multiplicity at least 2; for otherwise A would be diagonalizable (over $\overline{\mathbb{Q}}$) and hence of finite order. It is now sufficient to choose p as any prime divisor of n and to apply the previous part.

□

Proof of Theorem 1.3. By assumption we may pick nonzero complex eigenvalues λ, ξ of A, such that their ratio is not a root of unity (so in particular $\lambda \neq \xi$). Consider the number field $K = \mathbb{Q}(\lambda, \xi)$ and let x_λ, x_ξ be respective eigenvectors of A in $K^d \setminus \{0\}$.

We let Σ be the set of places of K obtained as the union of the following sets:

(a) the set of archimedean places and the set of places above primes in S;

(b) the set of those places at which either λ or ξ is not a unit;

(c) the set of those places at which the reductions of λ and ξ coincide;

(d) the set of those places at which the reductions of x_λ or x_ξ are not defined or are linearly dependent.

Then Σ is a finite set of places of K.

Suppose now that n is an integer such that A^n is conjugate to a companion matrix over $\mathbb{F}_p$, for every prime $p \notin S$. Fix $v \notin \Sigma$ and denote with a tilde the reduction modulo v.

We contend that $\tilde{\lambda}^n \neq \tilde{\xi}^n$. In fact, assuming the contrary we deduce that $\tilde{A}^n$ has the linearly independent eigenvectors $\tilde{x}_\lambda, \tilde{x}_\xi$ relative to the same eigenvalue $\tilde{\lambda}^n$. But then, in view of the characterization recalled at the end of section 1, $\tilde{A}^n$ cannot be conjugate to a companion matrix over the residue field of v. On the other hand if this residue field

has characteristic p, we have $p \notin S$ (because $v \notin \Sigma$); so A^n is conjugate to a companion matrix over $\mathbb{F}_p$; this is a contradiction which proves the claim.

This conclusion may be reformulated as the assertion that $\eta_n := \lambda^n - \xi^n$ is a Σ-unit for such an integer n; on the other hand both λ^n, ξ^n are Σ-units, due to our choice of Σ (see property (b)). So each relevant integer n provides a solution $(\eta_n/\lambda^n, \xi^n/\lambda^n)$ to the Σ-unit equation $X + Y = 1$ (*i.e.* to be solved in in Σ-units X, Y of K). But this equation has only finitely many solutions, due to a well-known (rather deep) Theorem in the theory of diophantine equations. Hence $(\xi/\lambda)^n$ can take only finitely many values. Since ξ/λ is neither zero nor a root of unity, this proves that n too takes values in a finite set. This proves the Theorem.

□

Remark 2.2
i) We remark that the above equation in Σ-units is of special type, and this allows an entirely elementary treatment in special cases (that is, without relying on the deep result alluded to in the proof).

We also remark that Baker's theory of linear forms in logarithms allows to find effectively the finite set of relevant integers n, provided A and S are given effectively. This shall be implicit in the effective treatment in the next section.

ii) Before going ahead with such effective analysis, we sketch a proof of the assertion made in the introduction that the unimodular integral matrix

$$A := \begin{pmatrix} 196 & 3617 \\ 11 & 203 \end{pmatrix}$$

is conjugate to a companion matrix over all $\mathbb{F}_p$ but not over $\mathbb{Z}$. Actually we shall prove a bit more, namely changing $\mathbb{F}_p$ with $\mathbb{Z}_p$ and moreover proving that *A is conjugate to a companion matrix in $SL_2(\mathbb{Z}_p)$ for each prime p but not over $\mathbb{Z}$*. It is of course possible that there are simpler numerical examples; we have not pursued in the task of finding one.

Proof. To say that A is conjugate to a companion matrix over a ring $\mathcal{O}$ means that there is a vector $z \in \mathcal{O}^2$ such that z, Az form an $\mathcal{O}$-basis of $\mathcal{O}^2$; in turn, this just means that $\det(z, Az)$ belongs to the group $\mathcal{O}^*$ of invertible elements in $\mathcal{O}$. If actually $\det(z, Az) = 1$, then A is conjugate to a companion matrix in $SL_2(\mathcal{O})$.

If we write $z = (t, u)$ with coordinates $t, u \in \mathcal{O}$, this may be rephrased saying that $\det(z, Az) = 11t^2 + 7tu - 3617u^2 \in \mathcal{O}^*$ (resp. $11t^2 + 7tu - 3617u^2 = 1$ in the case of $SL_2(\mathcal{O})$).

Consider then the quadratic form $Q(T, U) = 11T^2 + 7TU - 3617U^2$ of discriminant $\Delta = 7^2 + 4 \cdot 11 \cdot 3617 = 397 \cdot 401$ (where $397, 401$ are primes). We have $44Q(T, U) = (22T + 7U)^2 - \Delta U^2$. It is easily checked that $Q(T, U)$ represents 1 over any $\mathbb{Z}_p$. Indeed, by the usual Hensel's principle we have only to pay attention to the special cases $p = 2, 11, 397, 401$ and prove the solvability of the corresponding congruences (i.e. modulo $8, 11, 397, 401$). For $p = 2$, use $Q(1, 1) \equiv 1 \pmod 8$. In the remaining three cases, use respectively that $Q(-3, -1) \equiv 1 \pmod{11}$ and that 44 is a quadratic residue modulo 397 and modulo 401.

We conclude that A is conjugate to a companion matrix in $SL_2(\mathbb{Z}_p)$ for all primes p.

On the other hand, suppose that $Q(a,b) = \epsilon \in \{1,-1\} = \mathbb{Z}^*$ for some integers $a, b \in \mathbb{Z}$. Then we would have $(22a+7b)^2 - \Delta b^2 = 44\epsilon$. Consider the unit $\omega := (399+\sqrt{\Delta})/2$ of the ring of integers $\mathcal{O}_\Delta$ of $\mathbb{Q}(\sqrt{\Delta})$. Let also $\xi = (22a+7b+b\sqrt{\Delta})/2$ which is again in $\mathcal{O}_\Delta$ since $22a+7b$ and b have the same parity. Then $\xi\xi' = 11\epsilon$, where a dash denotes conjugation in $\mathbb{Q}(\sqrt{\Delta})$.

We could then find an integer m so that $\sqrt{11/\omega} \leq |\xi\omega^m| < \sqrt{11\omega}$. Putting $\rho := \xi\omega^m$ we find $|\rho\rho'| = 11$, whence the above inequalities yield $\sqrt{11/\omega} < |\rho'| \leq \sqrt{11\omega}$. In turn, this gives $|\rho - \rho'| < 2\sqrt{11\omega}$. Finally, $\frac{(\rho-\rho')}{\sqrt{\Delta}}$ is an integer in $\mathbb{Z}$ bounded in absolute value by $2\sqrt{11\omega/\Delta}$, which is < 1. Therefore $\rho = \rho'$, so $\rho \in \mathbb{Z}$. But this is impossible since 11 is not a square, proving the claim.

□

3 An effective bound.

In this section we prove the effective version of Theorem 1.2 announced in the introduction.

Proof of Theorem 1.5. We first remark that, by Proposition 2.1 in section 2, we may assume that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$ and that at least one of these eigenvalues is not a root of unity.

We now recall some arguments from [5]. Define, as in the proof of Proposition 5.3 of op. cit., the linear recurrence sequences $m \mapsto u_m^{(k)}$ $(0 \leq k \leq d-1)$ by

$$A^m = u_m^{(0)} A^0 + \cdots + u_m^{(d-1)} A^{d-1} .$$

These sequences form a basis of the $\overline{\mathbb{Q}}$-vector space V of linear recurrence sequences associated to the characteristic polynomial of A. Let $\Delta_n(A)$ be the determinant of the matrix $(u_{nm}^{(k)})_{0\leq m,k\leq d-1}$. Let $n \in \mathbb{N}$ and $v_0 \in \mathbb{Z}^d$. Then

$$\det(v_0, A^n v_0, \ldots, A^{(d-1)n} v_0) = \Delta_n(A) \det(v_0, Av_0, \ldots, A^{(d-1)} v_0) .$$

Let, as in the statement of the theorem, Z be the set of positive integers n such that A^n is conjugate in $\mathrm{GL}_d(\mathbb{Z})$ to a companion matrix. Thus $n \in Z$ if and only if $\mathbb{Z}^d$ is generated by the A^n-orbit of a vector v_0, which in turn implies that $\mathbb{Z}^d$ is generated by the A-orbit of v_0. Thus, if $n \in Z$ then there exists $v_0 \in \mathbb{Z}^d$ such that

$$|\det(v_0, A^n v_0, \ldots, A^{(d-1)n} v_0)| = |\det(v_0, Av_0, \ldots, A^{(d-1)} v_0)| = 1 .$$

Hence $|\Delta_n(A)| = 1$. On the other hand, let us assume $|\Delta_n(A)| = 1$. We may assume that the set Z is not empty, otherwise the conclusion of the theorem is trivial. Hence there exists $v_0 \in \mathbb{Z}^d$ such that $\det(v_0, Av_0, \ldots, A^{(d-1)} v_0) = \pm 1$. Thus $\det(v_0, A^n v_0, \ldots, A^{(d-1)n} v_0) = \pm 1$ which shows that $n \in Z$.

The previous discussion prove that $n \in Z$ if and only if $|\Delta_n(A)| = 1$. Let $D_n(A)$ be the Vandermonde determinant $D_n(A) = \det(\lambda_{k+1}^{nm})_{0\leq m,k\leq d-1}$. Since the recurrence sequences $m \mapsto \lambda_{k+1}^m$ $(0 \leq k \leq d-1)$ give rise to another basis of V , we see that $D_n(A) =$

$\det(C)\Delta_n(A)$ for some $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$. Thus $n \in Z$ if and only if $|D_n(A)| = |D_1(A)|$ (remark that $\Delta_1(A) = 1$).

Let $n \in Z$. We shall obtain a bound for n from a lower bound for $|D_n(A)|$ and from an upper bound for $|D_1(A)|$. First we recall some definitions. Given two monic polynomials f, $g \in \mathbb{Z}[t]$ we denote by $\mathrm{disc}(f)$ the discriminant of f and by $\mathrm{res}(f,g)$ the resultant of f and g. We let $M(f) \geq 1$ its Mahler's measure, *i.e.* the absolute value of the product of the roots of f lying outside the unit circle. We also denote by $f^{[n]}$ the polynomial whose roots are the n-th powers of the roots of f.

Let $f(t)$ be the characteristic polynomial of A. We factorize f over $\mathbb{Z}$ as $f = f_1 \cdots f_s$. We let $d_j = \deg(f_j)$. Then

$$|D_n(A)|^2 = \prod_{i=1}^{s} |\mathrm{disc}(f_i^{[n]})| \times \prod_{1 \leq i,j \leq s} |\mathrm{res}(f_i^{[n]}, f_j^{[n]})| \geq \prod_{i=1}^{s} |\mathrm{disc}(f_i^{[n]})| \tag{3.1}$$

Observe that $f_j^{[n]}$ has only simple roots (since otherwise $D_n(A) = 0 \neq D_1(A)$) thus of degree d_j. By the main result of [3] (which rests on lower bounds in linear forms in two logarithms), there exists an absolute positive constant c_0 such that

$$|\mathrm{disc}(f_j^{[n]})| \geq M(f_j)^{(d_j-1)(n-c_0 d_j^6 \log d_j \log n)} .$$

We remark that $M(f_j)^{d_j-1} \geq M(f_j)$. This is clear if $d_j \geq 2$. If $d_j = 1$ then $f_j = x \pm 1$ ($A \in \mathrm{GL}(n,\mathbb{Z})$ implies $f(0) = \pm 1$) thus $M(f_j) = 1$, and again $M(f_j)^{d_j-1} = 1 = M(f_j)$. Assume $n > c_0 d^6 \log d \log n$. By the remark above

$$|\mathrm{disc}(f_j^{[n]})| \geq M(f_j)^{n - c_0 d^6 \log d \log n} . \tag{3.2}$$

By (3.1), (3.2) and by the multiplicativity of Mahler's measure we obtain

$$|D_n(A)|^2 \geq M(f)^{n - c_0 d^6 \log d \log n} .$$

By Hadamard's inequality:

$$|D_1(A)| \leq M(f)^{d-1} d^{\rho_1^2 + \cdots + \rho_k^2} \leq M(f)^{d-1} d^{d^2} .$$

Thus $|D_n(A)| = |D_1(A)|$ implies

$$(n - c_0 d^6 \log d \log n - 2d + 2) \log M(f) \leq 2d^2 \log d . \tag{3.3}$$

Since at least one of the eigenvalues of A is not a root of unity, f is not a product of cyclotomic polynomials. By a Theorem of Dobrowolski [2], there exists an absolute positive constant c_1 such that

$$\log M(f) \geq c_1 (\log d)^{-3} .$$

From (3.3) we get

$$n \leq c d^6 (\log d)^2$$

for some absolute positive constant c.

□

4 The rank> 2 problem

Let $d \geq 3$. As recalled in the introduction, Levitt and Metaftsis provide a family of examples of matrices $A \in M_d(\mathbb{Z})$ of infinite order and such that $\mathrm{OR}(A^n) = 2$ for infinitely many n. One can choose

$$A = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}$$

with $a \neq \pm 1$ and $B \in M_{d-1}(\mathbb{Z})$ conjugate to a companion matrix of finite order. Note however that this forces $A \notin \mathrm{GL}_d(\mathbb{Z})$. See remark 4.4, iv) for an example of a matrix $A \in \mathrm{GL}_3(\mathbb{Z})$ such that $\mathrm{OR}(A^n) = 2$ for infinitely many n.

One could ask if for a "generic" matrix $A \in M_d(\mathbb{Z})$ it is true that $\mathrm{OR}(A^n) = 2$ infinitely often. Let us discuss a bit this problem.

Most of our results are local. Thus it is convenient to introduce the following notations. Let $A \in M_d(\mathbb{Z})$. Given a prime number p we let $\mathrm{OR}_p(A) = \mathrm{OR}_{\mathbb{F}_p}(A \bmod p)$. We define $\mathrm{OR}_{\mathrm{loc}}(A)$ as the maximum of $\mathrm{OR}_p(A)$ for p a prime. We remark that $\mathrm{OR}_{\mathrm{loc}}(A) \leq \mathrm{OR}(A)$.

It seems that it happens only in very special cases that $\mathrm{OR}_{\mathrm{loc}}(A^n)$ is maximal ($= d$) for all large n.

For instance, using Fermat's little Theorem as in the proof of Proposition 2.1, it is easy to prove the following.

Remark 4.1 *Let $A \in M_d(\mathbb{Z})$. Assume that A has only one eigenvalue. Let*

$$\psi(d) = \prod_{q \leq d} q \, .$$

for q running over the prime powers $\leq d$ (we recall that $\log \psi(d) \sim d$ by the Prime Number Theorem). Then $\mathrm{OR}(A^n) = \mathrm{OR}_{\mathrm{loc}}(A^n) = d$ for $n > \psi(d)$.

Proof. Let $f(t) = (t-a)^d$ be the characteristic polynomial of A. Let $n > \psi(d)$. Thus there exists a power q of a prime p, such that $q \,|\, n$ and $q > d$. Then $A^q - a \equiv (A-a)^q \bmod p$. Since $q > d$ we have $A^q \equiv a \bmod p$. Since $q \,|\, n$ we also have $A^n \equiv a^{n/q} \bmod p$. Thus the minimal polynomial of $A^n \bmod p$ is linear, which implies $\mathrm{OR}_p(A^n) = d$. Thus $\mathrm{OR}_{\mathrm{loc}}(A^n) = d$. Since $\mathrm{OR}_{\mathrm{loc}}(A^n) \leq \mathrm{OR}(A^n) \leq d$ we also have $\mathrm{OR}(A^n) = d$.

□

Similarly, the method of the proof of Theorem 1.3 shows:

Remark 4.2 *Let $A \in M_d(\mathbb{Z})$. Assume that A has two nonzero eigenvalues whose ratio is not a root of unity. Let r be the sum of the dimensions of their eigenspaces. Then $\mathrm{OR}(A^n) \geq \mathrm{OR}_{\mathrm{loc}}(A^n) \geq r$ for $n \geq \psi(A)$.*

On the opposite side, we generalize a conjecture of Ailon and Rudnick [1] which would imply that for a "generic" $A \in M_d(\mathbb{Z})$ we had $\mathrm{OR}_{\mathrm{loc}}(A^n) = 2$ infinitely often.

Conjecture 4.3 *Let K be a number field and let $\alpha_1, \ldots, \alpha_d \in \mathcal{O}_K$ be non-zero algebraic integers of K. Let us assume:*

1) $\alpha_1, \ldots, \alpha_d$ do not satisfy non-trivial multiplicative relations of zero degree. (Namely, $\alpha_1^{m_1} \cdots \alpha_d^{m_d} \neq 1$ for integers $m_1, \ldots, m_d$ not all zero but with $m_1 + \ldots + m_d = 0$.)

2) There is no finite places v of K such that three distinct α_j have the same reduction mod v.

Then for infinitely many n there is no finite places v of K such that three distinct α_j^n have the same reduction mod v.

Remark 4.4
i) In the special case $\boldsymbol{\alpha} = (1, a, b)$ with a, $b \in \mathbb{Z}$ multiplicatively independent and such that $\gcd(a-1, b-1) = 1$, conjecture 4.3 reduces to conjecture A of [1]. As for this special case, we have a numerical evidence for it. Moreover, its analogous in function fields should be a consequence of a result of Lang, as in op.cit.

ii) Note that condition 2) is obviously necessary, but not condition 1), as already remarked in [1]. Take for instance a be a non-zero integer, $a \neq \pm 1$ and let, as in op.cit, $\boldsymbol{\alpha} = (1, a, -a)$ which trivially satisfies the conclusion of conjecture 4.3 but not assertion 1). More generally, we are confronted with this curious phenomenon. All the examples of algebraic numbers for which we can *prove* that they satisfy the conclusion of conjecture 4.3, do not satisfy assertion 1).

iii) Here is another examples, which comes from a linear recurrence sequence suggested by C. Ballot. Let $u_n = -1 + F_{n+1}$, where F_n is Fibonacci's sequence. Then u_n satisfies the linear recurrence sequence associated to the polynomial $f(t) = t^3 - 2t^2 + 1 = (t-1)(t^2 - t - 1)$ with roots 1, α and β. Let n be an odd integer not divisible by 3. Then there is no finite places v of $\mathcal{O}_{\mathbb{Q}(\alpha)}$ such that 1, α^n and β^n have the same reduction mod v. Indeed, $f^{[n]}(t) := (t-1)(t-\alpha^n)(t-\beta^n) = (t-1)(t^2 - L_n t + (-1)^n)$, where L_n is Lucas' sequence $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$. Thus for n odd $f^{[n]} \bmod p$ has 1 as triple root if and only if $p = 2$ and L_n is even. In turn, L_n is even if and only if $3 \mid n$.

iv) The above linear recurrence sequence provides an example of a companion matrix $A \in \mathrm{GL}_3(\mathbb{Z})$ such that $\mathrm{OR}(A^n) = 2$ infinitely often. Let

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

be the companion matrix of $f(t) = t^3 - 2t^2 + 1$. Let $v = (0, 1, 0)$ and let $u_n = -1 + F_{n+1}$ as before. Then it easily see that $A^n v = (-u_n, -u_{n-1}, u_{n+1})$. Thus $\mathbb{Z}v \oplus \mathbb{Z}A^n v$ is primitive if and only if $\gcd(u_n, u_{n+1}) = 1$. This proves

$$\gcd(u_n, u_{n+1}) = 1 \Longrightarrow \mathrm{OR}(A^n) \leq 2 \, .$$

An exercise on Fibonacci's number shows that, for n odd not divisible by 3, $\gcd(u_n, u_{n+1}) = 1$. Thus for these integers, $\mathrm{OR}(A^n) \leq 2$. Since A has infinite order, by theorem 1.2 we have $\mathrm{OR}(A^n) = 2$ infinitely often.

The conjecture 4.3 immediately implies:

Conjecture 4.5 *Let $A \in M_d(\mathbb{Z})$ be nonsingular with characteristic polynomial f. Let us assume:*

1) The roots of the f do not satisfy non-trivial multiplicative relations of zero degree.

2) For all prime number p the polynomial $f \bmod p$ do not have irreducible factors of multiplicity ≥ 3.

Then $\mathrm{OR}_{\mathrm{loc}}(A^n) \leq 2$ for infinitely many n.

Proof of Conjecture 4.3 $\Rightarrow$ Conjecture 4.5. Let $f(t) = \prod_{j=1}^{d}(t - \alpha_j)$ be the characteristic polynomial of A. Thus the characteristic polynomial $f^{[n]}$ of A^n is $f^{[n]}(t) = \prod_{j=1}^{d}(t - \alpha_j^n)$. Let us assume conjecture 4.3. Then for infinitely many n and for p prime the polynomial $f^{[n]} \bmod p$ has no irreducible factors of multiplicity ≥ 3. Since $\mathrm{OR}_p(A^n)$ is equal to the number of invariant factors of A^n, we deduce that $\mathrm{OR}_p(A^n) \leq 2$. Thus $\mathrm{OR}_{\mathrm{loc}}(A^n) \leq 2$ infinitely often.

□

We remark that if $A \in \mathrm{GL}_d(\mathbb{Z})$, condition 1) of conjecture 4.5 forces the roots of f to be different from roots of unity.

5 HNN extensions

Let G be a finitely generated abelian group and let φ be an injective endomorphism of G. We consider the HNN extension

$$G*_\varphi = \langle G, t \,|\, tgt^{-1} = \varphi(g) \rangle \ .$$

Define $\mathrm{OR}'(\varphi)$ as the last positive integer k such that there exist $g_1, \ldots, g_k \in G$ and $N \in \mathbb{N}$ for which $\mathrm{Im}(\varphi^N)$ is contained in the subgroup generated by the φ-orbits of $g_1, \ldots, g_k$. Remark that $\mathrm{OR}'(\varphi) = \mathrm{OR}(\varphi)$ for $\varphi \in \mathrm{Aut}(G)$. The following Theorem generalizes the first statement of [5], corollary 2.4.

Theorem 5.1 *Let φ be an injective endomorphism of the finitely abelian group G. Let G' be the HNN extension $G*_\varphi$. Then*

$$\mathrm{rank}(G') = \mathrm{OR}'(\varphi) + 1 \ .$$

Before proving this result, we make some simple remarks on HNN extensions. Some of them will be needed in the proof of the theorem.

For $y \in G'$ we denote by F_y the inner automorphism of G' defined by $F_y(x) = yxy^{-1}$. Thus φ is the restriction of F_t.

Remark 5.2
i) Since G is abelian, for g, $g_1 \in G$ we have $F_{gt}(g_1) = g\varphi(g_1)g^{-1} = \varphi(g_1)$.

ii) We note that every $x \in G'$ may be written as $x = t^{-a}gt^b$, with $g \in G$ and a, $b \geq 0$. Even if this representation is not unique, $t^{-a}gt^b \mapsto b - a$ defines a morphism $\chi\colon G' \to \mathbb{Z}$.

iii) Let $G'_+ = \{gt^b \mid g \in G,\ b \geq 0\}$. Then G'_+ is a monoid. More precisely, for $g_1, \ldots, g_r \in G$ and $b_1, \ldots, b_r \geq 0$ we have $g_1 t^{b_1} \cdots g_r t^{b_r} = g t^{b_1 + \cdots + b_r}$ for some $g \in G$.

iv) Let $g \in G$ and let $b \geq 0$. Then $(gt^{-b})^{-1} = \varphi^b(g)^{-1} t^b \in G'_+$.

v) Let $x = t^{-a} g t^b \in G'$ ($g \in G$, a, $b \geq 0$) and let $n \geq a$. Then $F_t^n(x) = t^{n-a} g t^{-n+b} = \varphi^{n-a}(g) t^{b-a}$. If $b \geq a$, we have $F_t^n(x) \in G'_+$. Assume $b < a$. Then, by remark 5.2 iv), $F_t^n(x)^{-1} \in G'_+$.

vi) Let $x_1 = g_1 t^{b_1}$, $x_2 = g_2 t^{b_2} \in G'_+$ ($g_i \in G$, $b_i \geq 0$) and write the euclidean division $b_2 = q b_1 + r$ ($q \geq 0$, $0 \leq r < b_1$). Since $q \geq 0$, by remark 5.2 iii) we have $x_1^q = g'_1 t^{q b_1}$ for some $g'_1 \in G$. Thus

$$x'_2 := x_2 x_1^{-q} = g_2 t^r (g'_1)^{-1} = g_2 \varphi^r (g'_1)^{-1} t^r = g'_2 t^r$$

with $g'_2 \in G$. We have $\langle x_1, x_2 \rangle = \langle x_1, x'_2 \rangle$. By Euclid's algorithm we deduce that there exist $x \in G'_+$ and $g \in G$ such that $\langle x_1, x_2 \rangle = \langle g, x \rangle$. More generally, if $x_1, \ldots, x_{k+1} \in G'_+$, then

$$\langle x_1, \ldots, x_{k+1} \rangle = \langle g_1, \ldots, g_k, g_0 t^b \rangle$$

with $g_i \in G$ and $b \geq 0$.

vii) Assume now that G' can be generated by $k+1$ elements, say $x_1, \ldots, x_{k+1}$. By remark 5.2 v) we can find $n \geq 0$ and $s_i \in \{\pm 1\}$ such that $F_t^n(x_i)^{s_i} \in G'_+$ for $i = 1, \ldots, k+1$. Since F_t is an automorphism, $F_t^n(x_1)^{s_1}, \ldots, F_t^n(x_{k+1})^{s_{k+1}}$ generate again G'. By remark 5.2 vi) there exist $g_0, \ldots, g_k, \in G$ and $b \geq 0$ such that $G' = \langle g_1, \ldots, g_k, g_0 t^b \rangle$. Since $t \in G'$, by remark 5.2 ii) we have $1 = \chi(t) \in b\mathbb{Z}$ which implies $b = 1$.

Proof of Theorem 5.1. We first show that $\mathrm{rank}(G') \leq \mathrm{OR}'(\varphi) + 1$. Let $k = \mathrm{OR}'(\varphi)$. Thus there exist $g_1, \ldots, g_k \in G$ and $N \in \mathbb{N}$ for which $\mathrm{Im}(\varphi^N)$ is contained in the subgroup generated by the φ-orbits of $g_1, \ldots, g_k$. Let $g \in G$. Then $t^N g t^{-N} = \varphi^N(g) \in \langle g_1, \ldots, g_k, t \rangle$. Thus $g \in \langle g_1, \ldots, g_k, t \rangle$ and $G' = \langle g_1, \ldots, g_k, t \rangle$.

We now show that $\mathrm{OR}'(\varphi) + 1 \leq \mathrm{rank}(G')$. Let $\mathrm{rank}(G') = k+1$. By remark 5.2 vii) there exist $g_0, \ldots, g_k \in G$ such that $G' = \langle g_1, \ldots, g_k, g_0 t \rangle$. Let $g \in G$. Then there exist $i_1, \ldots, i_l \in \{1, \ldots, k\}$, $\lambda_1, \ldots, \lambda_l \in \mathbb{Z}$ and $\mu_1, \ldots, \mu_l \in \mathbb{Z}$ such that

$$g = (g_0 t)^{\lambda_1} g_{i_1}^{\mu_1} \cdots (g_0 t)^{\lambda_l} g_{i_l}^{\mu_l} = F^{\rho_1}(g_{i_1})^{\mu_1} \cdots F^{\rho_l}(g_{i_l})^{\mu_l} (g_0 t)^{\rho_l}$$

where $F = F_{g_0 t}$ is the inner automorphism $x \mapsto (g_0 t) x (g_0 t)^{-1}$ and where $\rho_i = \lambda_1 + \cdots + \lambda_i$ ($i = 1, \ldots, l$). Let $N_g \geq 0$ such that $m_i := \rho_i + N_g \geq 0$ for $i = 1, \ldots, l$. Then, by remark 5.2 i),

$$\begin{aligned} \varphi^{N_g}(g) = F^{N_g}(g) &= F^{m_1}(g_{i_1})^{\mu_1} \cdots F^{m_k}(g_{i_k})^{\mu_k} (g_0 t)^{\rho_k} \\ &= \varphi^{m_1}(g_{i_1})^{\mu_1} \cdots \varphi^{m_k}(g_{i_k})^{\mu_k} (g_0 t)^{\rho_k} \ . \end{aligned}$$

By remark 5.2 ii) we have $0 = \chi(\varphi^{N_g}(g)) = \chi((g_0 t)^{\rho_k}) = \rho_k$. Thus $\varphi^{N_g}(g)$ is in the subgroup generated by the φ-orbits of $g_1, \ldots, g_k$. It is now enough to choose $N = \max_g N_g$ for g running over a finite system of generators of G.

□

From now on we fix $G = \mathbb{Z}^d$. We translate the assertion $\mathrm{OR}'(\varphi) = 1$ in term of local conditions. Given a prime p we denote by $\overline{\varphi}\colon \mathbb{F}_p^d \to \mathbb{F}_p^d$ the reduction mod p of φ. For $v \in \mathbb{Z}^d$ we let Λ_v be the subgroup generated by $v, \varphi(v), \ldots, \varphi^{d-1}(v)$ and we denote by $\overline{\Lambda_v}$ its reduction mod p.

Let K be a field and let ψ be an endomorphism of a d-dimensional K-vector space V. We recall that $\dim \mathrm{Im}(\psi^j) = \dim \mathrm{Im}(\psi^d)$ for $j \geq d$.

Theorem 5.3 *Let $\varphi\colon \mathbb{Z}^d \to \mathbb{Z}^d$ be an injective morphism. Then $\mathrm{OR}'(\varphi) = 1$ if and only if there exists a vector $v \in \mathbb{Z}^d$ such that for all prime p*

$$\dim \mathrm{Im}(\overline{\varphi}^d) = \dim \overline{\varphi}^d(\overline{\Lambda}_v)\ . \tag{5.4}$$

Remark. Condition (5.4) is obviously satisfied if $p \nmid \det(\Lambda_v)$. If $p \mid \det(\Lambda_v)$ then $\dim(\overline{\Lambda}_v) \leq d-1$ and condtition (5.4) is equivalent to $\dim \mathrm{Im}(\overline{\varphi}^d) = \dim \overline{\varphi}^{d-1}(\overline{\Lambda}_v)$.

Proof. Assume first $\mathrm{OR}'(\varphi) = 1$. Then by definition, there exist a vector $v \in \mathbb{Z}^d$ and $N \in \mathbb{N}$ such that $\mathrm{Im}(\varphi^N) \subseteq \Lambda_v$. Then $\mathrm{Im}(\varphi^{N+d}) \subseteq \varphi^d(\Lambda_v)$. Let p be a prime. By the remark preceding the theorem,

$$\dim \mathrm{Im}(\overline{\varphi}^d) = \dim \mathrm{Im}(\overline{\varphi}^{N+d}) \leq \dim \overline{\varphi}^d(\overline{\Lambda}_v)\ .$$

Assume now that there exists $v \in \mathbb{Z}^d$ such that (5.4) holds for every prime p. Let p be a prime. Since $\overline{\varphi}^d(\overline{\Lambda}_v) \subseteq \mathrm{Im}(\overline{\varphi}^d)$ and since these $\mathbb{F}_p$-vector spaces have the same dimension,

$$\mathrm{Im}(\overline{\varphi}^d) = \overline{\varphi}^d(\overline{\Lambda}_v) \subseteq \overline{\Lambda}_v\ .$$

Let b be the product of the primes dividing $\det(\Lambda_v)$. By Bezout's identity we easily see that

$$\mathrm{Im}(\varphi^d) \subseteq \Lambda_v + b\mathbb{Z}^d\ .$$

By induction we deduce

$$\mathrm{Im}(\varphi^{dN}) \subseteq \Lambda_v + b^N\mathbb{Z}^d$$

for $N \in \mathbb{N}$. We chose for N a natural number such that $\det(\Lambda_v)$ divides b^N. Then

$$\mathrm{Im}(\varphi^{dN}) \subseteq \Lambda_v + b^N\mathbb{Z}^d \subseteq \Lambda_v + \det(\Lambda_v)\mathbb{Z}^d \subseteq \Lambda_v\ .$$

□

We consider the following even special case: $G = \mathbb{Z}^2$, $\varphi \in M_2(\mathbb{Z})$ non-singular. In this case, the assertion (5.4) is equivalent to the following two statements:

1) $p \mid \det(\Lambda_v) \Rightarrow p \mid \det(\varphi)$.

2) $\varphi(v) \equiv 0 \bmod p \Rightarrow p \mid \mathrm{tr}(\varphi)$.

Indeed, assume that p satisfies (5.4). Let $p \mid \det(\Lambda_v)$. Then $\dim \mathrm{Im}(\overline{\varphi}^2) = \dim \overline{\varphi}^2(\overline{\Lambda}_v) < 2$. Thus $\overline{\varphi}$ is not injective and $p \mid \det(\varphi)$. Moreover, if $\varphi(v) \equiv 0 \bmod p$ then $\overline{\varphi}(\overline{\Lambda}_v) = 0$, thus $\dim \mathrm{Im}(\overline{\varphi}^2) = \dim \overline{\varphi}^2(\overline{\Lambda}_v) = 0$ and $\overline{\varphi}$ is nilpotent mod p which in turn implies $p \mid \mathrm{tr}(\varphi)$.

Conversely, let p be a prime satisfying 1) and 2). We have already remarked that (5.4) is trivially satisfied if $p \nmid \det(\Lambda_v)$. Assume that $p \mid \det(\Lambda_v)$. By 1), $p \mid \det(\varphi)$. Thus

$\dim \mathrm{Im}(\overline{\varphi}^2) \leq 1$. Assume first $\varphi(v) \not\equiv 0 \bmod p$. Since $p \,|\, \det(\Lambda_v)$, we must have $\varphi(v) \equiv \lambda v \bmod p$ with $\lambda \not\equiv 0 \bmod p$. Hence $\overline{\Lambda}_v =< v >_{\mathbb{F}_p}$ and $\overline{\varphi}^2(\overline{\Lambda}_v) =< \lambda^2 v >_{\mathbb{F}_p}$. Thus $\dim \overline{\varphi}^2(\overline{\Lambda}_v) = 1 \geq \dim \mathrm{Im}(\overline{\varphi}^2)$. If $\varphi(v) \equiv 0 \bmod p$, then, by 2), φ is nilpotent mod p, and again $\dim \overline{\varphi}^2(\overline{\Lambda}_v) = \dim \mathrm{Im}(\overline{\varphi}^2) = 0$.

We write $v = (x, y)$. Then $\det(\Lambda_v) = Q(x, y)$ with $Q(X, Y)$ a quadratic form. Thus the existence of a vector $v \in \mathbb{Z}^2$ which satisfies conditions 1) and 2) above translate into the following statements on Q. There exist x, $y \in \mathbb{Z}$ such that for p prime we have:

$$p \,|\, Q(x, y) \Longrightarrow p \,|\, \det(\varphi)$$
$$\varphi(x, y) \equiv 0 \bmod p \Longrightarrow p \,|\, \mathrm{tr}(\varphi)\ .$$

This last requirement amounts to certain finitely many congruence conditions, depending explicitly only on φ.

It is now a well-known matter to decide about the existence of x, $y \in \mathbb{Z}$ satisfying these congruences and moreover such that $Q(x, y)$ is composed only of primes dividing $\det(\varphi)$ (and one can also calculate such x, y if there exist any). Thus:

Corollary 5.4 *Let $\varphi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be an injective morphisme. Then one can compute* $\mathrm{rank}(\mathbb{Z}^2 *_\varphi)$.

We have not made any particular effort to generalize this statement to higher dimension.

We finally remark that the analogous of Theorem 1.2 still holds in HNN extension and it is indeed an easy corollary of Theorem 1.3.

Corollary 5.5 *Let $A \in M_d(\mathbb{Z})$ be a nonsingular matrix of infinite order. Suppose that A has two eigenvalues whose ratio is not a root of unity. Consider the family of HNN extension $G_n = \mathbb{Z}^d *_{A^n}$. Then there exists $n_0 = n_0(A)$ such that* $\mathrm{rank}(G_n) > 2$ *for $n \geq n_0$.*

Proof. Let S be the set of primes dividing the discriminant of A. By Theorem 1.3 there exists $n_0 = n_0(A)$ such that for all $n \geq n_0$ the matrix A^n is not conjugate in $\mathrm{GL}(\mathbb{F}_p)$ to a companion matrix for all $p \notin S$. Let $n \geq n_0$ and assume $\mathrm{OR}'(A^n) = 1$. Let $p \notin S$. Then, by the choice of S, the matrix A is in $\mathrm{GL}_d(\mathbb{F}_p)$. Thus $\mathrm{OR}(A^n) = \mathrm{OR}'(A^n) = 1$ and A^n is conjugate to a companion matrix in $\mathrm{GL}(\mathbb{F}_p)$, contradiction. Thus for $n \geq n_0$ we have $\mathrm{OR}'(A^n) > 1$ and, by Proposition 5.1, $\mathrm{rank}(G_n) > 2$.

□

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