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Residual Entanglement and Sudden Death: a Direct Connection

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We explore the results of Coffman *et.al.* [*Phys. Rev. A*, **61**, 052306 (2000)] derived for general tripartite states in a dynamical context. We study a class of physically motivated tripartite systems. We show that whenever entanglement sudden death occurs in one of the partitions residual entanglement will appear. For fourpartite systems however, the appearance of residual entanglement is not conditioned by sudden death of entanglement. We can only say that if sudden death of entanglement occurs in some partition there will certainly be residual entanglement.

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Entanglement, a property at the heart of Quantum Mechanics, has first been brought to scientific debate the intriguing questions posed by Einstein, Podolsky, and Rosen in ref. [1] and since then the matter has always been under investigation. Recently the interest of the physical community in this counterintuitive property has raised even more due to its potential as a resource for information processing and quantum computation [2]. For that purpose having a profound knowledge of entanglement is a must (see, *e.g.*, ref. [3] and references therein), as well as a thorough comprehension of entanglement distribution in composite systems (involving more than two degrees of freedom). In this context, several years ago Coffman *et. al.* [4] studied the entanglement distribution in three qubit systems (ABC), where each one of them can be entangled with the other two. Moreover they proved the existence of what they called residual entanglement, which is not detected by usual two qubits entanglement quantifiers [5, 6]. Their result is valid for pure states in a Hilbert space $2 \otimes 2 \otimes 2$, where they proved that quantum correlation between A and BC will be manifest in one of three forms: *i*) A is entangled with B ; *ii*) A with C ; and *iii*) the entanglement is distributed among ABC , the so-called residual entanglement. To this day this relation is the most general available in the field of quantum information. In spite of its mathematical rigor their relation has not yet been explored in dynamical situations. We know that entanglement distribution is very important for the implementation of quantum communication in general, where the relevance of entanglement distribution is crucial. This work is devoted to the purpose of understanding as deeply as possible relevant dynamical consequences of the relation derived in [4].

As the work of ref. [4] was developed, an apparently disconnected effect about entanglement has been found by Życzkowski *et. al.* [7]. They very recently showed that two parties entanglement can suddenly disappear. Since then this dynamical characteristic of entanglement has been called sudden death of entanglement [7–9] (hereafter ESD) and has been measured [10] using twin photons. A step forward in the solution to this question was given in an example studied by Sainz *et. al.* in ref. [11]. They studied a four qubit system which interacts locally and pairwise and showed the existence of an entanglement invariant. However in such systems entanglement sudden death is also present (noted first in ref. [12]). What happens to the entanglement in a unitary evolution in such a situation? Their result may point to the idea that the amount of quantum correlations present in a closed system should be conserved, however there is nothing to prevent a dynamical redistribution of the initial entanglement. In others words, the initial entanglement might migrate from one partition to others in a way that the initial entanglement be conserved.

The purpose of the present work is to answer the following question: what happens with the entanglement distribution when 3-qubit systems undergo ESD? We show that residual entanglement is intimately related to ESD for a large class of states.

Fourpartite systems are also investigated having the same question in mind but no solid mathematical results to back up our model result about the connection between ESD and appearance of genuine entanglement. In this case environmental effects are taken in to account.

The context of quantum optics the kind of interaction we use and our modeling of reservoir effects has proven very realistic in many situations of physical interest. Since entanglement dynamics is an essential part of the implementation of quantum communication we believe the results presented here may be of use.

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Let us consider a three qubit system A, B, C where entanglement can be found in all partitions. Coffman *et. al* [4] proved that quantum correlation between A and BC will be manifest as follows

$$C_{A(BC)}^2 = C_{AB}^2 + C_{AC}^2 + \tau_{ABC} \quad (1)$$

where τ_{ABC} stands for a tripartite residual entanglement and $C_{i(j)}$ is the concurrence between partitions i and j . Moreover the authors noticed that τ_{ABC} is invariant if one interchanges A and B . From the generality of eq.(1) all entangled physical systems which may be mapped onto a three qubits problem must obey (1).

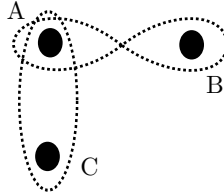


FIG. 1: System ABC is initially in the pure state $|\psi_0\rangle = |AB\rangle|C\rangle$, with AB initially entangled and factored of C . when a unitary interaction between A and C is turned on there will be an entanglement dynamics in the tripartite system.

Initially for the physical situation depicted in fig. 1 we have

$$C_{A(BC)} = C_{AB} = C_0 \quad (2)$$

and

$$C_{AC} = \tau_{ABC} = 0 \quad (3)$$

Let us consider now that the qubits A and C interact. When the interaction is “turned on” A and C will dynamically entangle and, according to monogomy of entanglement [13], A will be less entangled with B . However, C will “see” the state A as

$$\rho_A = \text{tr}_B(|AB\rangle\langle AB|) \quad (4)$$

As during this time evolution the partitions AB and C interact one should expect that the entanglement distribution will be such that C_{AB} will become smaller as a function of time and both C_{AC} and C_{BC} start to grow accordingly. This is an example where eq.(1) must be obeyed all along the dynamics. Therefore it is possible that besides C_{AB} , C_{AC} and C_{BC} there may at some point appear a τ_{ABC} . So, for systems where the partition AC admits interaction among its constituents and shares C_0 with B there are actually two very enlightening dynamical situations: *i*) there is no ESD in any of the partitions and *ii*) there is ESD in at least one of the partitions. In the first case (*i*), with excitation exchange between A and C , one can show [14] that

$$C_{AB}^2 + C_{BC}^2 = C_0^2 \quad (5)$$

Besides this result we also have that

$$C_{B(AC)}^2 = C_{AB}^2 + C_{BC}^2 + \tau_{ABC} \quad (5)$$

must be obeyed so that

$$C_{B(AC)}^2 = C_0^2 + \tau_{ABC} \quad (6)$$

Now, since by hypothesis there is no ESD in any of the partitions and B does not interact with the partition AC , we have $C_{B(AC)} = C_0$ and $\tau_{ABC} = 0$. What happens when one of the partitions undergoes ESD (*ii*)? In this case, given the interaction between A and C the entanglement (C_{AC}) will not disappear suddenly. Therefore ESD can only occur in partitions AB and BC . Let us first consider that during a time interval ESD occurs in partition AB . During this time window, eq.(1) gives

$$C_{A(BC)}^2 = C_{AC}^2 + \tau_{ABC} \quad (7)$$

However, we should remark that C “sees” A as a mixed state and its capacity to entangle with A will depend on how much A is entangled with B and also on the type of interaction. Since initially we have $C_{A(BC)} = C_0$ and knowing that $C_{A(BC)} \geq C_{AC}$ during the whole evolution, in the interval when $C_{AB} = 0$ the residual entanglement τ_{ABC} must be different from zero otherwise eq.(1) will not be satisfied. The same analysis is valid when ESD occurs in the partition BC , from analyzing (5). Last but not least we consider the case in which ESD occurs in both partitions. Then $C_{B(AC)} = C_0 = \tau_{ABC}$.

The above considerations leave no doubt that the appearance of entanglement sudden death [7, 8] in tripartite systems bears very intimate connection with higher order entanglement, *i.e.*, residual entanglement.

A concrete example is the tripartite system studied in ref. [15], consisting of two atoms, only one of which A , say, interacts with the cavity, the other B serves the unique purpose of allowing for an entangled initial state with A . The cavity C interacts resonantly with A according to the usual Jaynes–Cummings model [16], where the interaction is given as

$$H_I = \hbar g(c^\dagger \sigma_-^A + c \sigma_+^A) \quad , \quad (6)$$

where g is a coupling constant, c (c^\dagger) is an operator that annihilates (creates) an excitation in C and $\sigma_-^A = |\downarrow\rangle\langle\uparrow|$ ($\sigma_+^A = |\uparrow\rangle\langle\downarrow|$) analogously for the atoms. Consider the atomic initial state as given by

$$|AB\rangle_\psi = \beta |\uparrow\downarrow\rangle + \alpha |\downarrow\uparrow\rangle \quad (7)$$

and the cavity in vacuum $|C\rangle_0 = |0\rangle$, with $|\beta|^2 + |\alpha|^2 = 1$. For this initial state $|AB\rangle_\psi|C\rangle_0$ the evolved state will be

$$|ABC\rangle_t^{(0)} = [\beta \cos(gt) |\uparrow\downarrow\rangle + \alpha |\downarrow\uparrow\rangle] |0\rangle - i\beta \sin(gt) |\downarrow\downarrow\rangle |1\rangle \quad . \quad (8)$$

To quantify the entanglement between A and BC of state (8), we will use concurrence in the form $2\sqrt{\det \rho_A}$ [4], where $\rho_A = \text{tr}_{BC}(|ABC\rangle\langle ABC|_t)$. For the entanglement between A and B and between A and C we use the concurrence which is defined in the refs. [5, 6] as

$$C_\rho = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} \quad (9)$$

where the λ_i 's are the eigenvalues of the $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ organized in decreasing order, σ_y is one of the Pauli matrices and ρ^* is the complex conjugate of ρ . For the state (8) each bipartition will have concurrence

$$\begin{aligned} C_{AB} &= C_0 |\cos gt| \\ C_{AC} &= |\beta|^2 |\sin 2gt| \\ C_{BC} &= C_0 |\sin gt| \end{aligned}$$

with $C_0 = 2|\beta\alpha|$ stands for the initial entanglement of AB and the entanglement between A and BC will be

$$C_{A(BC)} = 2\sqrt{|\beta|^2 \cos^2 gt (|\alpha|^2 + |\beta|^2 \sin^2 gt)} \quad .$$

For this initial state (8) there will be no ESD in AB and AC , since

$$C_{AB}^2 + C_{BC}^2 = C_0^2$$

and according to eq.(5)

$$\tau_{ABC} = 0 \quad ,$$

as discussed above. Otherwise we will also have

$$C_{A(BC)}^2 = C_{AB}^2 + C_{AC}^2 \quad (10)$$

showing explicitly that in this example $\tau_{ABC} = 0$.

Next we consider an initial state which will dynamically be lead to ESD in some partition. This will happen, *e.g.*, if the cavity contains one excitation initially, $|C\rangle_1 = |1\rangle$. For this initial state the evolved state will be

$$|ABC\rangle_t^{(1)} = [\beta \cos(\sqrt{2}gt) |\uparrow\downarrow\rangle + \alpha \cos(gt) |\downarrow\uparrow\rangle] |1\rangle - i[\alpha \sin(gt) |\uparrow\uparrow\rangle |0\rangle + \beta \sin(\sqrt{2}gt) |\downarrow\downarrow\rangle |2\rangle] \quad (11)$$

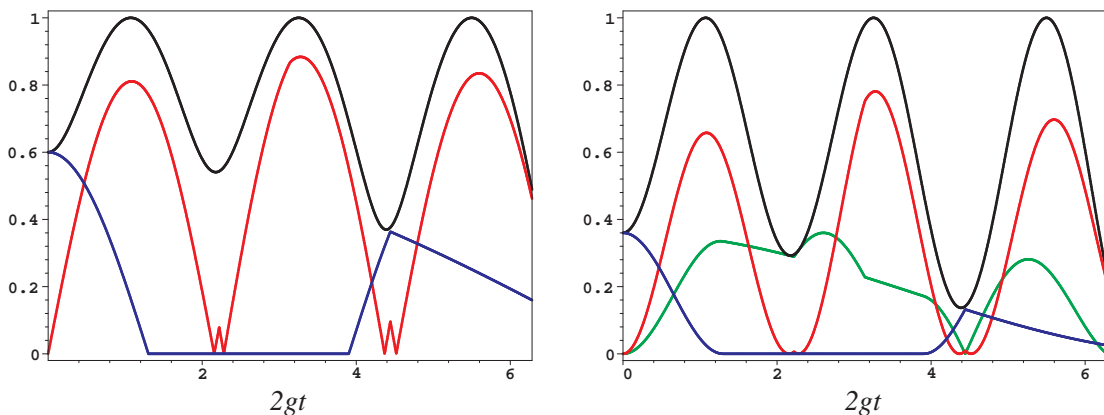


FIG. 2: Graphics for the state (11) with $\beta = 3/\sqrt{10}$ and $\alpha = 1/\sqrt{10}$. **Left Figure (LF)**: Concurrences in AB and AC and between A and BC . The blue, red and black curves are the concurrences C_{AB} , C_{AC} and $C_{A(BC)}$, respectively. **Right Figure (RF)**: Here we show the residual entanglement and the concurrences squared between A and B , A and C , A and BC . The green, blue, red and black curves are the residual entanglement τ_{ABC} and the concurrences squared C_{AB}^2 , C_{AC}^2 and $C_{A(BC)}^2$, respectively.

and the concurrences in AB and AC will be

$$C_{AB} = C_0 \max\{0, |\cos(gt) \cos(\sqrt{2}gt)| - |\sin(gt) \sin(\sqrt{2}gt)|\}$$

$$C_{AC} = \left| |\alpha|^2 |\sin(2gt)| - |\beta|^2 |\sin(2\sqrt{2}gt)| \right| .$$

It may be noted that ESD will be in the partition AB , as shown in Figure 2. The entanglement between A and BC is

$$C_{A(BC)} = 2\sqrt{\left(|\alpha|^2 \sin^2(gt) + |\beta|^2 \cos^2(\sqrt{2}gt)\right) \left(|\alpha|^2 \cos^2(gt) + |\beta|^2 \sin^2(\sqrt{2}gt)\right)} \quad (12)$$

and can not be written as (10). It is immediate that $C_{A(BC)}^2 \geq C_{AB}^2 + C_{AC}^2$ and

$$\tau_{ABC} \geq 0 \quad . \quad (13)$$

This inequality reflects the main objective of this work. Observing figure 2, we note that the residual entanglement exists right before of the ESD between A and B . Our interpretation of this result is as follows: the quantum correlations between A and B disappear for a time interval and are distributed throughout the system contributing to the residual entanglement [34].

Residual Entanglement and Sudden Death: a conjecture

We now consider a four partite system, A , B , C and D where initial entanglement C_0 is in the partition AB . We also consider that A interacts locally with C , and B with D as shown in figure 3. More concretely we consider two atoms A and B sharing an entanglement C_0 and the partition C consists of N oscillators initially in vacuum, same for D . The local interaction in the partition AC will be described by the hamiltonian

$$H_{AC} = \frac{\hbar\omega_A}{2}\sigma_z^A + \hbar \sum_{k=1}^N \omega_k c_k^\dagger c_k + \hbar \sum_{k=1}^N g_k (c_k^\dagger \sigma_-^A + c_k \sigma_+^A) \quad (14)$$

where g_k is a coupling constant between the atom and the k -th oscillator of C , c_k (c_k^\dagger) is the operator which annihilates (creates) one excitation in the k -th oscillator of C and $\sigma_-^A = |\downarrow\rangle\langle\uparrow|$ ($\sigma_+^A = |\uparrow\rangle\langle\downarrow|$) in the atom. Similarly for BD .

The fundamental state of the system AC , $|\downarrow\rangle \prod_{k=1}^N |0_k\rangle$, does not evolve in time. However the initial state containing one excitation, $|\uparrow\rangle \prod_{k=1}^N |0_k\rangle$, evolves to the state

$$|\gamma(t)\rangle = \xi(t)|\uparrow\rangle|\tilde{0}\rangle + \chi(t)|\downarrow\rangle|\tilde{1}\rangle \quad (15)$$

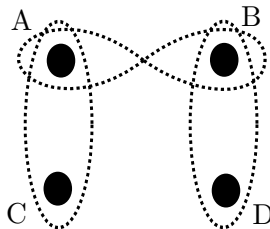


FIG. 3: Initially the partition AB has C_0 of entanglement and is factored from CD . After an initial time a local interaction in the partitions AC and BD begins and there will be an entanglement dynamics in the system.

where $\xi(t)$ and $\chi(t)$ are functions to be determined which depend on N . We define the collective states

$$|\tilde{0}\rangle = \prod_{k=1}^N |0_k\rangle \quad (16)$$

$$|\tilde{1}\rangle = (1/\chi(t)) \sum_{k=1}^N \lambda_k(t) |1_k\rangle \quad (17)$$

with $|\chi(t)|^2 = \sum_{k=1}^N |\lambda_k(t)|^2$ and $|\xi(t)|^2 + |\chi(t)|^2 = 1$. When $N = 1$, in the resonant limit, we have in AC and BD the so called Double Jaynes–Cummings [11, 12]. The explicit forms for $\xi(t)$ and $\chi(t)$ are

$$\xi(t) = \cos(gt) \quad (18)$$

$$\chi(t) = -i \sin(gt) \quad (19)$$

Otherwise, when $N \rightarrow \infty$ the subsystem C is a reservoir in vacuum and A will decay exponentially as studied in references [8, 10, 17]. In this case we have

$$\xi(t) \rightarrow e^{-\gamma t/2} \quad (20)$$

$$\chi(t) \rightarrow \sqrt{1 - e^{-\gamma t}} \quad (21)$$

where γ is a damping constant.

Now let us consider the atoms prepared, as before, in $|AB\rangle_\psi$ and the $2N$ oscillators in vacuum. This initial state dynamically evolves to

$$|ABDC\rangle_t^{(\psi)} = \beta |\gamma(t)\rangle_{AC} |\tilde{0}\rangle_{BD} + \alpha |\tilde{0}\rangle_{AC} |\gamma(t)\rangle_{BD} \quad (22)$$

The concurrencies of each pair are given by

$$C_{AB} = C_0 |\xi(t)|^2 \quad (23)$$

$$C_{AC} = 2|\beta|^2 |\xi(t)\chi(t)| \quad (24)$$

$$C_{AD} = C_0 |\xi(t)\chi(t)| \quad (25)$$

$$C_{BC} = C_0 |\xi(t)\chi(t)| \quad (26)$$

$$C_{BD} = 2|\alpha|^2 |\xi(t)\chi(t)| \quad (27)$$

$$C_{CD} = C_0 |\chi(t)|^2 \quad (28)$$

From eqs.(23 – 28) we may check that there will be no sudden death in any of the partitions of $ABCD$. The concurrences between A and the rest of the system is given by

$$C_{A(BCD)} = 2|\beta\xi(t)| \sqrt{|\alpha|^2 + |\beta|^2 |\chi(t)|^2} \quad (29)$$

which can be rewritten as

$$C_{A(BCD)} = \sqrt{C_{AB}^2 + C_{AC}^2 + C_{AD}^2} \quad (30)$$

i.e., for the atoms initially prepared in the state $|AB\rangle_\psi$ and the $2N$ oscillators in their vacuum state, the entanglement that A shares with the rest of the system is completely distributed in the partitions AB , AC , and AD . Therefore there will be no residual entanglement.

A qualitatively different situation arises if one considers the initial state

$$|AB\rangle_\phi = \beta|\uparrow\uparrow\rangle + \alpha|\downarrow\downarrow\rangle \quad (31)$$

for the atoms and the $2N$ oscillators in vacuum. This initial state evolves to the state

$$|ABDC\rangle_t^{(\phi)} = \beta|\gamma(t)\rangle_{AC}|\gamma(t)\rangle_{BD} + \alpha|\downarrow\tilde{0}\rangle_{AC}|\downarrow\tilde{0}\rangle_{BD} \quad (32)$$

It is well known that this initial condition, for $N = 1$ [11, 12] and $N \rightarrow \infty$ [8, 10, 17], presents ESD in some partition when $|\beta| > 2|\alpha|$. We focus our attention on the entanglement that A shares with the rest of the system. The entanglement between A and any other subsystem and that of A with BCD are given by

$$C_{AB} = 2|\beta\xi(t)^2| \max\{0, |\alpha| - |\beta\chi(t)^2|\} \quad (33)$$

$$C_{AD} = 2|\beta\xi(t)\chi(t)| \max\{0, |\alpha| - |\beta\xi(t)\chi(t)|\} \quad (34)$$

$$C_{AC} = 2|\beta^2\xi(t)\chi(t)| \quad (35)$$

$$C_{A(BCD)} = 2|\beta\xi(t)|\sqrt{|\beta\chi(t)|^2 + |\alpha|^2} \quad (36)$$

It becomes apparent that the entanglement in partitions AB and AD may disappear suddenly. In the partition AB there will be ESD for times such that

$$|\chi(t)|^2 \geq \left|\frac{\alpha}{\beta}\right| \quad (37)$$

In AD , ESD will occur at times such that

$$|\chi(t)|\sqrt{1 - |\chi(t)|^2} \geq \left|\frac{\alpha}{\beta}\right| \quad (38)$$

where we used the fact that $|\xi(t)| = \sqrt{1 - |\chi(t)|^2}$. When we solve the inequality in (38) we find

$$\frac{1}{2} - \sqrt{\frac{1}{4} - \left|\frac{\alpha}{\beta}\right|^2} < |\chi(t)|^2 < \frac{1}{2} + \sqrt{\frac{1}{4} - \left|\frac{\alpha}{\beta}\right|^2} \quad (39)$$

which imposes the condition $|\beta| > 2|\alpha|$. So when we observe the inequalities (37) and (39) it is easy to see that when

$$\left|\frac{\alpha}{\beta}\right| \leq |\chi(t)|^2 \leq \frac{1}{2} + \sqrt{\frac{1}{4} - \left|\frac{\alpha}{\beta}\right|^2} \quad (40)$$

with $|\beta| > 2|\alpha|$, $C_{AB} = 0$ and $C_{AD} = 0$ at the same time. This will always be the case for the initial state (31) with $|\beta| > 2|\alpha|$, as shown in figure 4.

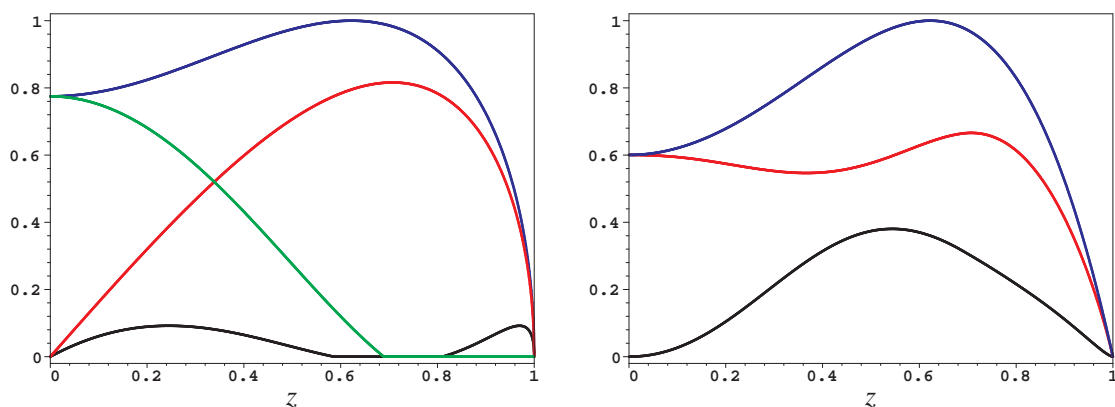


FIG. 4: Concurrences as a function of $z = |\chi(t)|$, with $\beta \approx 0.905$ and $\alpha \approx 0.429$ ($|\beta| > 2|\alpha|$) in the state (32). **LF**: $C_{A(BCD)}$ in blue, C_{AC} in red, C_{AD} in black, and C_{AB} in green. **RF**: $C_{A(BCD)}^2$ in blue, $C_{AB}^2 + C_{AC}^2 + C_{AD}^2$ in red, and E_{ABCD} in black.

The entanglement between A and BCD may be rewritten as

$$C_{A(BCD)}^2 = C_{AC}^2 + C_0^2 |\xi(t)|^2 \quad , \quad (41)$$

which is valid during the whole evolution. The relationship above shows that the entanglement shared by A with the rest of the system is divided in two parts. i) one which is in AC due to the local interactions between AC and ii) one which is spread over the rest of the system. We may now define the positive semidefinite quantity [18] E_{ABCD} which represents the entanglement between A and BCD which cannot be accounted for by the entanglement of A with B , C and D separately, *i.e.*, $E_{ABCD} = C_{A(BCD)}^2 - [C_{AB}^2 + C_{AC}^2 + C_{AD}^2]$, which for our case gives

$$E_{ABCD} = C_0^2 |\xi(t)|^2 - [C_{AB}^2 + C_{AD}^2] \quad . \quad (42)$$

Fig. 4 illustrates the entanglement distribution in the case of the initial condition (32). Note (on the LF) that ESD only occurs in the partitions AB and AD . When $0.584 \lesssim z \lesssim 0.812$, $C_{AD} = 0$. However when $0.689 \lesssim z$ we will have $C_{AB} = 0$. In this situation when $0.689 \lesssim z \lesssim 0.812$ we will have $C_{AB} = C_{AD} = 0$ at the same time. The **RF** illustrates the behavior of $C_{A(BCD)}^2$. It shows a smooth behavior when it is increasing or decreasing. It represents all the entanglement between A and BCD including the one coming from the unitary interaction. The curve in red $C_{AB}^2 + C_{AC}^2 + C_{AD}^2$ initially decreases due to the entanglement decrease followed by ESD in AB and AD . Right after that it increases since the entanglement provided by the interaction in the partition AC becomes quantitatively significant. E_{ABCD} (curve in black) presents a maximum before the other graphs. This is due to the fact that the entanglement between AB and between AD are decreasing and the AC entanglement is not yet qualitatively significant. After the ESD in AB and AD the entanglement due to the AC dynamics grows, so that the E_{ABCD} curves starts to decrease.

Interestingly enough the E_{ABCD} entanglement will be present during the whole evolution for any value of β and α in state (32). This means that for the initial state $|AB\rangle_\phi$ there will always be an entanglement between A and BCD which cannot be accounted for by the entanglement of A with B , C , D separately. This is not true for the initial state $|AB\rangle_\psi$ where we have $E_{ABCD} = 0$ and all the entanglement content between the partitions A and BCD may be accounted for by two partite concurrences. When $|\beta| > 2|\alpha|$ in the initial state (31) there will be a time interval Δt , defined by eq. (40), during which $C_{AB} = 0$ and $C_{AD} = 0$, as discussed above. In this situation we have

$$E_{ABCD} = C_0^2 |\xi(t)|^2 \quad (43)$$

which represents the entanglement distributed in the whole system that cannot be accounted for by C_{AB} , C_{AC} , and C_{AD} .

Discussion and Conclusion

The interaction presented here where excitations are exchanged between the atoms and the field, simulate quantum circuits in Quantum Optics [19]. This interaction provides for the possibility of exchange information [20] and also transfer of entanglement in systems like those represented here [21]. For example in eq. (14) in the limit when $N \rightarrow \infty$ with system C initially in the vacuum state under the well know Born–Markov approximation [22] simulates the vacuum fluctuations responsible for the atomic exponential decay. In spite of its broad range of applicability, this type of interaction does not cover all phenomena in Quantum Optics. A phase coupling between A and the N oscillators is also a useful kind of environment without excitation exchange. When $N \rightarrow \infty$ this dynamics leads to the disappearance of coherence [23] and may also induce ESD. In such situation one should expect that the entanglement distribution be very similar to the genuine entanglement in tripartite systems. This phase interaction between atoms and fields (when $N \rightarrow \infty$) is similar to the one modeled in ref. [24] where ESD is observed when two entangled atoms are subjected to a classical noisy environment simulated by a stochastic classical field. This results in phase damping of the collective and individual atomic states.

Recent studies show the existence of ESD in systems qubits–qutrits ($2 \otimes 3$) [25, 26] and sudden death of nonlocality in three qubit systems ($2 \otimes 2 \otimes 2$) [26–28] have also been investigated. In the first case, as also discussed here, if the phase reservoir interacts either locally or globally with the qubit and qutrit, then one should expect residual entanglement in the subsystem of interest. The second case, where a sudden disappearance of nonlocality is observed, requires, however a more careful analysis since there exists entangled states which do not violate Bell inequalities [29–32] even when they are tripartite with residual entanglement [33]. In other words the investigation between entanglement and nonlocality is a very promising, open area of research.

As for the present work we have shown that tripartite systems subjected to a local interaction will exhibit a very close connection between ESD and residual entanglement, based on eq. (1).

A four partite system has also been investigated and the same phenomenon is observed, Next a natural conjecture is in order: is ESD a general mechanics through which entanglement flows from partitions involving two qubits to larger ones? This is our belief based on the fact that it can be rigorously demonstrated for three qubits and several examples involving more qubits point in the same direction. Proving this conjecture remain an open intriguing challenge.

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- [1] A. Einstein, E. Podolsky, and N. Rosen, *Phys. Review*, **47**, 777 (1935).
- [2] M. Nielsen, and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2000).
- [3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.*, **81**, 865 (2009).
- [4] V. Coffman, J. Kundu, and W. K. Wootters, *Phys. Rev. A*, **61**, 052306 (2000).
- [5] S. Hill e W. K. Wootters, *Phys. Rev. Lett.*, **78**, 5022 (1997).
- [6] W. K. Wootters, *Phys. Rev. Lett.*, **80**, 2245(1998).
- [7] K. Zyczkowski, P. Horodecki, M. Horodecki, and R. Horodecki, *Phys. Rev. A*, **65**, 012101 (2001).
- [8] T. Yu, and J. H. Eberly, *Phys. Rev. Lett.*, **93**, 140404 (2004).
- [9] T. Yu, and J. H. Eberly, *Science*, **323**, 598-601 (2009).
- [10] M. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. Walborn, P. Ribeiro, and L. Davidovich, *Science*, **316**, 579 (2007).
- [11] I. Sainz, and G. Björk, *Phys. Rev. A*, **76**, 042313 (2007).
- [12] M. Yönaç, T. Yu, and J. H. Eberly, *J. Phys. B: At. Mol. Phys.*, **39**, S621 (2006).
- [13] D. Yang, *Phys. Lett. A*, **360**, 249 (2006); M. P. Seevinck, *Quantum Inf. Process.*, **9**, 273 (2010).
- [14] S. Chan, M. Reid, and Z. Ficek, *J. Phys. B: At. Mol. Phys.*, **43**, 215505 (2010).
- [15] Zhi-Jian Li, Jun-Qi Li, Yan-Hong Jin and Yi-Hang Nie, *J. Phys. B: At. Mol. Phys.*, **40**, (2007) 3401–3411.
- [16] E. Jaynes, and F. Cummings, *Proc. IEEE*, **51**, 89 (1963).
- [17] C. E. López, G. Romero, F. Lastra, E. Solano, and J. C. Retamal, *Phys. Rev. Lett.*, **101**, 080503 (2008).
- [18] T. J. Osborne and F. Verstraete, *Phys. Rev. Lett.*, **96**, 220503 (2006).
- [19] S. Haroche and J. M. Raimond, *Exploring the Quantum: Atoms, Cavities and Photons* (Oxford University Press, New York, USA, 2006).
- [20] X. Maÿtre, E. Hagle, G. Nogues, C. Wunderlich, P. Goy, M. Brune, J. M. Raimond, and S. Haroche, *Phys. Rev. Lett.* , **79**, 769 (1997).
- [21] D. Cavalcanti, J. G. Oliveira Jr., J. G. Peixoto de Faria, M. O. Terra Cunha, and M. França Santos, *Phys. Rev. A* , **74**, 042328 (2006).
- [22] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer-Verlag, Berlin, Germany, 2000).
- [23] H. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag, Berlin, Germany, 1993).
- [24] T. Yu and J. H. Eberly, *Opt. Commun.*, **264**, 393 (2006).
- [25] K. Ann and G. Jaeger, *Phys. Lett. A*, **372**, 570 (2008).
- [26] K. Ann and G. Jaeger, *Found. Phys.*, **39**, 790 (2009).
- [27] G. Jaeger and K. Ann, *Phys. Lett. A*, **372**, 2212 (2008).
- [28] K. Ann and G. Jaeger, *Phys. Lett. A*, **372**, 6853 (2008).
- [29] R. F. Werner, *Phys. Rev. A* , **40**, 4277 (1989).
- [30] J. Barrett, *Phys. Rev. A* , **65**, 042302 (2002).
- [31] A. Acin, N. Gisin, and B. Toner, *Phys. Rev. A* , **73**, 062105 (2006).
- [32] D. Cavalcanti, M. L. Almeida, V. Scarani, and A. Acin, *Nature Commun.*, **2**, 184 (2011).
- [33] G. Toth and A. Acin, *Phys. Rev. A* , **76**, 030306(R) (2006).
- [34] The same reasoning follows if we considers as initial state $|AB\rangle_\phi = \beta|\uparrow\uparrow\rangle + \alpha|\downarrow\downarrow\rangle$.