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October 16, 2008

Abstract

We consider branching random walks with binary search trees as underlying trees. We show that the occupation measure of the branching random walk, up to some scaling factors, converges weakly to a deterministic measure. The limit depends on the stable law whose domain of attraction contains the law of the increments. The existence of such stable law is our fundamental hypothesis. As a consequence, using a one-to-one correspondence between binary trees and plane trees, we give a description of the asymptotics of the profile of recursive trees. The main result is also applied to the study of the size of the fragments of some homogeneous fragmentations.

AMS subject classifications. 60F05, 60G50, 68W40, 60J80, 05C05.

Key words. Random binary search tree, branching random walk, occupation measure, fragmentation, recursive tree.

Abbreviated title. Branching random walks on binary search trees.

1 Introduction

The purpose of this paper is to study the asymptotics of the occupation measure of branching random walks (BRWs) when the underlying tree is a binary search tree. This work is motivated by the large number of results on the occupation measure of BRWs on conditioned Galton-Watson trees which limit is ISE (Integrated Super-Brownian Excursion), and also by some applications in term of fragmentation processes and in term of the profile of recursive trees discussed later.

This provisional PDF is the accepted version. The article should be cited as: ESAIM: PS, doi: 10.1051/ps:2008035

Informally, a branching walk is a random object owning two levels of randomness. First an underlying tree is picked up under a certain law and then, on each of its edges, a random variable is attached and is used as the increment of a random walk. Hence, a BRW is a family of dependent random walks indexed by the nodes of the underlying tree (it is formally defined below). Many results on BRWs can be found in the literature.

During the last thirty years, Biggins obtained many different results when the underlying tree is a supercritical Galton Watson tree. For instance, in [3], he studied the convergence of a martingale associated to the BRW.

More recently, Aldous [1] introduced a model of BRW on trees of size n: the underlying tree $\mathcal{T}$ is the family tree of a Galton Watson process conditioned to have n nodes (with offspring distribution having mean 1 and variance σ^2) and the increments are independent, centered, with variance θ^2 and having moments of order $4+\epsilon$. With each node u of $\mathcal{T}$, he associates the variable Y_u which is the sum of the increments associated with the ancestors of u. Aldous stated that the normalized occupation measure converges to a random measure called ISE, that is

$$\frac{1}{n+1}\sum_{u\in\mathcal{T}}\delta_{\frac{\sigma^{1/2}Y_u}{\theta n^{1/4}}} \xrightarrow[n\to\infty]{\mathcal{D}} \mu_{ISE},$$

where δ stands for the Dirac measure, in the space of probability measures. The measure μ_{ISE} owns a representation in term of Gaussian process and appears as the natural limit of many different random phenomena, see e.g. [1, 7, 12, 13, 21] and references therein. In particular, Chassaing and Schaeffer [7] have shown that it is closely related to the limit of the profile of uniform quadrangulations.
By contrast with μ_{ISE}, which is a random measure, we will see that things are very different on a binary search tree, as the limiting occupation measure is deterministic and equal to the stable distribution corresponding to the spatial displacements (exept in the non-centered Gaussian case where variance is different).

The paper is organized as follows. In the continuation of Section 1 we define our model of branching random walk. Then we give our main results on the convergence of the occupation measure in Section 2, according to whether the increments are identically distributed or not. The proof of Theorem 1 is relagated to Section 4. Section 3 provides two applications of Corollary 1 in term of homogeneous fragmentations and in term of the profile of recursive trees.

Binary search trees (BSTs) and finite branching random walks

Let us introduce our model of branching random walk. For this, we begin with a formal description of trees. Let

$$\mathbb{U} := \{\varnothing\} \cup \bigcup_{n\in\{1,2,\cdots\}} \mathbb{N}^n$$

be the set of the finite words on the alphabet $\mathbb{N} := \{0, 1, 2, \cdots\}$ (with $\varnothing$ as the empty word). We write uv the word formed by the concatenation of u and v. A plane tree τ is a finite subset of $\mathbb{U}$ such that

$$\begin{cases} \varnothing \in \mathcal{T}, \\ \forall (u, v) \in \mathbb{U}^2, \quad \text{if } uv \in \mathcal{T} \text{ then } u \in \mathcal{T}, \\ \forall u \in \mathbb{U}\ , \forall d \in \mathbb{N}^* := \{1, 2, \cdots\}, \quad \text{if } ud \in \mathcal{T} \text{ then } u(d-1) \in \mathcal{T}. \end{cases} \tag{1.1}$$

The elements of $\mathcal{T}$ are called *nodes*, and $\varnothing$ is called the *root*. The number of letters in u, denoted by $|u|$, is called the *depth* of u (with $|\varnothing| = 0$). We say that v is an ancestor of u when v is a prefix of u ($\exists w \in \mathbb{U}$; $vw = u$), we denote it by $v \ll u$. We write $\mathcal{P}$ the set of plane trees. Remark that the above definition of a plane tree fits with the usual one: a plane tree is a finite rooted unlabeled loop free connected graph in which the set of children of every vertex is endowed with a total order.

In the present paper we are concerned with complete binary trees. Let $\mathbb{U}^b := \{\varnothing\} \cup \bigcup_{n\geq 1}\{0, 1\}^n$, be the subset of $\mathbb{U}$ of the finite words on the alphabet $\{0, 1\}$. A complete binary tree $\mathcal{T}$ is a plane tree such that every node has zero or two children (in other words $\mathcal{T}$ is a finite subset of $\mathbb{U}^b$ satisfying (1.1) and such that $u1 \in \mathcal{T} \Leftrightarrow u0 \in \mathcal{T}$). We write $\mathcal{B}$ for the set of complete binary trees. A tree $\mathcal{T}$ can be described by giving the set $\partial\mathcal{T}$ of its *leaves* (the nodes with no descendants also called external nodes). The nodes of $\mathcal{T}\backslash\partial\mathcal{T}$ are called *internal* nodes.

As said above a branching random walk is a random object having two levels of randomness: the underlying tree and the increments. We first introduce our model of random underlying tree, the binary search tree (BST).

The underlying tree model

Random BSTs are fundamental data structures associated with quicksort, one of the most used sorting algorithms. We present a recursive construction of BSTs, or we should say labeled BSTs: $(U_i)_{i\geq 1}$ is a sequence of independent random variables uniformly distributed on $[0, 1]$. With the n first terms $(U_i)_{i=1\cdots n}$ we associate a labeled BST of size n, that is a complete binary tree in which each of the n internal nodes contains one of the values $(U_i)_{i=1\cdots n}$: the root contains U_1, the left subtree is the labeled BST associated with the sequence of the U_i's smaller than U_1 and the right subtree is the labeled BST associated with the sequence of the U_i's larger than U_1. The labeled BST associated with an empty list is reduced to a leaf. See Figure 1 for an example. This construction induces a law on $\mathcal{B}_n := \{\mathcal{T} \in \mathcal{B}, \#\mathcal{T} = 2n + 1\}$, the trees where the labels are deleted (a complete binary tree with n internal nodes always has $n + 1$ external nodes).

This BST model can also be obtained by a sequential construction without the U_i's: $\mathcal{T}_0$ is reduced to a leaf. Assume $\mathcal{T}_n$ is a BST with $n + 1$ leaves, then we construct $\mathcal{T}_{n+1}$ by choosing equally likely one of the $n + 1$ leaves of $\mathcal{T}_n$ and by replacing it by an internal node. The replacement creates two new leaves.

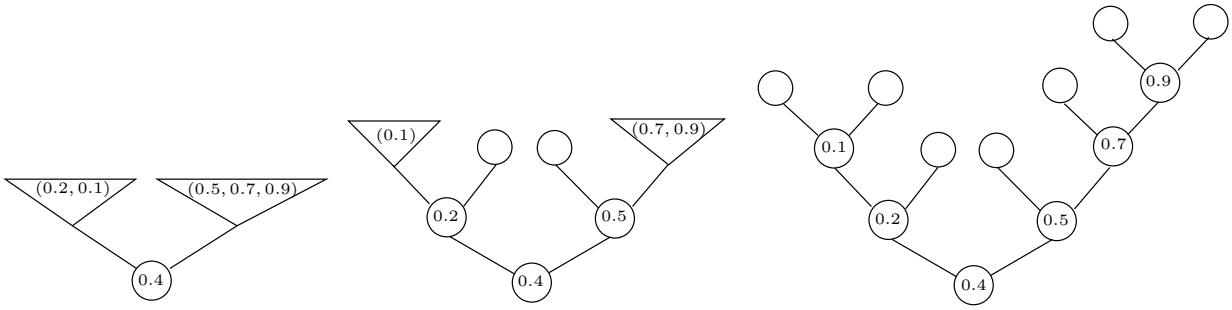

Figure 1 *: First, second and last steps of the construction of the labeled BST associated with* $(0.4, 0.2, 0.5, 0.7, 0.1, 0.6)$*. The associated BST is obtained by deleting the labels on the last picture.*

Many properties of BSTs, like the height or the profile, have been studied in the literature: we refer to Mahmoud [18], Chauvin & al. [8], Drmota [10] and Devroye and Hwang [9] and references therein for an overview of the subject.

The increments model

We endow the elements of $\mathbb{U}^b$ with random variables, that is we define $(c_u)_{u\in\mathbb{U}^b}$ a family of independent random variables on $\mathbb{R}^2$; $c_u := (X_{u0}, X_{u1})$ may be seen as a variable associated with the two edges from u to its children (X_{u0} can be seen to be associated with the edge $(u, u0)$ and X_{u1} with $(u, u1)$); $X_\varnothing := 0$. The X_u will be taken as the increments of our branching random walk. We assume the two families (U_i) and (c_u) to be independent. Moreover, we first assume the marginal distributions of all the c_u (that is the laws of X_{u0} and X_{u1}) to be equal. Notice that the variables X_{u0} and X_{u1} are not necessarily independent.

Following the above description we construct $\mathcal{T}_n$, the BST associated with $(U_i)_{i=1\cdots n}$, and with each $u \in \mathcal{T}_n$ we associate a trajectory of the killed random walk $Y_u = (Y_u(j))_{j\in\{0,\cdots,|u|\}}$ defined by

$$Y_u(0) = 0, \qquad Y_u(j) = \sum_{\substack{v \ll u;\\ |v| \le j}} X_v \quad \forall j \in \{1, \cdots, |u|\}. \tag{1.2}$$

Each $Y_u(j)$ is the sum of the j's increments associated with the j's first edges of the path from the root to u; with the above assumptions on the c_u's, $Y_u(j)$ is a sum of i.i.d. random variables. The finite branching random walk with underlying tree $\mathcal{T}$ is the family of all the trajectories Y_u, that is $(Y_u, u \in \mathcal{T})$. See Figure 2 for an example.

We are interested in the occupation measure $\sum \delta_{Y_u(|u|)}$ of these random walks, suitably rescaled.

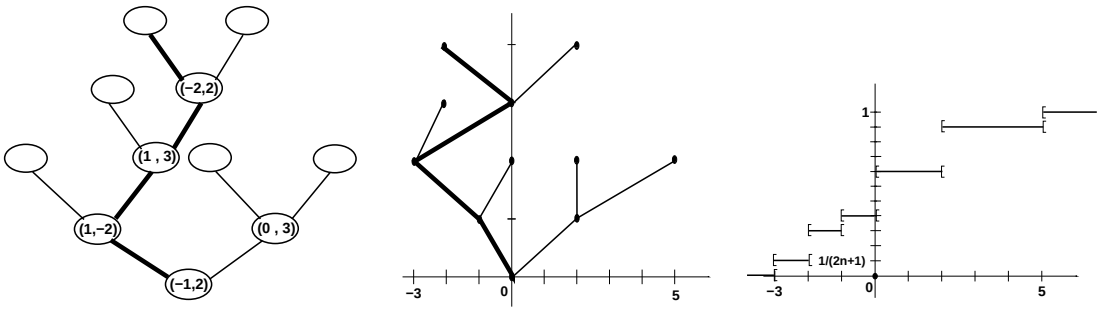

Figure 2 *: BST with the associated (c_u), branching random walk and cumulative function associated with the empirical measure.*

Knowing the height of the node u, the variable $Y_u(|u|)$ is a sum of i.i.d. random variables, and thus the asymptotics behavior of $Y_u(|u|)$, when $|u|$ is large, depends on the stable law, whenever it exists, that contains the distribution of the marginals of the c_u's in its domain of attraction.

2 Convergence of the occupation measure

2.1 The identically distributed case

Let ν be a distribution on $\mathbb{R}$ with finite mean $m_\nu \in \mathbb{R}^+ \cup \{+\infty\}$ and variance $\sigma_\nu^2 \in \mathbb{R}^+ \cup \{+\infty\}$. We assume that ν belongs to the domain of attraction of a non-degenerate stable law μ with characteristic exponent $\alpha \in (0,2] \setminus \{1\}$ (for sake of simplicity we restrict our study to the case $\alpha \neq 1$), which means that there exist two deterministic sequences (a_n) and (b_n) such that

$$\frac{(\sum_{i=1}^n X_i) - b_n}{a_n} \xrightarrow[n\to\infty]{\mathcal{D}} \mu, \tag{2.3}$$

for any family (X_i) of i.i.d random variables with law ν. The characteristic function of any stable law μ is of the form

$$\phi(t) = \exp\left\{ i\gamma t - c|t|^\alpha \left[1 + i\beta \frac{t}{|t|} \tan\left(\frac{\pi}{2}\alpha\right) \right] \right\},$$

where $\alpha \in (0,2] \setminus \{1\}$, $\gamma \in \mathbb{R}$, $|\beta| \leq 1$ and $c \geq 0$. Moreover
– The sequence (a_n) is *regularly varying of index* $1/\alpha$, that is

$$a_n = L_n \times n^{1/\alpha},$$

where (L_n) is a nonnegative sequence satisfying $L_{\lfloor nt \rfloor}/L_n \xrightarrow[n\to\infty]{} 1$, for every $t > 0$ ((L_n) is slowly varying).
– The sequence (b_n) will be taken as $b_n = nm'_\nu$, where $m'_\nu = m_\nu$ if $\alpha > 1$ and $m'_\nu = 0$ if $\alpha < 1$ (this is possible, see the proof of Theorem 9.34 of [5]).
When b_n is fixed, there exist several choices for $((a_n), \mu)$ in order to get (2.3) but, given ν, all non-degenerate μ have the same characteristic exponent α.
• If $\alpha = 2$, ν is in the domain of attraction of the normal distribution. For $b_n = nm_\nu$, μ can be shown to be centered (using the comparison between $((\sum_{i=1}^n X_i) - b_n)/a_n + ((\sum_{i=1}^n \tilde{X}_i) - b_n)/a_n$ and $((\sum_{i=1}^{2n} X_i) - b_{2n})/a_{2n}$).
– If $\sigma_\nu^2 < +\infty$ we take $a_n = \sqrt{n}$ (that is $L_n \equiv 1$) and thus μ is $\mathcal{N}(0, \sigma_\nu^2)$, according to the central limit theorem.
– If $\sigma_\nu^2 = +\infty$ then $a_n = L_n\sqrt{n}$ for some $L_n \to +\infty$ (as one can check in the proof of [14] Theorem 5.17).
• If $\alpha \in (0, 2) \setminus \{1\}$, the choice of (a_n) determines μ.

From now on, we assume (a_n), (b_n), and μ are fixed.

Let $\mathcal{T}_n$ be a BST. Taking u uniformly at random in $\mathcal{T}_n$ or $\mathcal{T}_n \backslash \partial\mathcal{T}_n$, the value $|u|$ is "close" to $2\log n$ (see (4.9) and [6]). Thus, if we assume that both marginals of the c_u's equal distribution ν, this incites us to set

$$\alpha_n := a_{2\log n} = L_{2\log n} \times (2\log n)^{1/\alpha} \qquad \text{and} \qquad \beta_n := b_{2\log n} = 2m'_\nu \log n$$

as normalizing constants in the internal and global occupation measures of the branching random walks defined by

$$\mu_n = \frac{1}{n} \sum_{u \in \mathcal{T}_n \backslash \partial\mathcal{T}_n} \delta_{\frac{Y_u(|u|) - \beta_n}{\alpha_n}} \quad \text{and} \quad \overline{\mu}_n = \frac{1}{2n+1} \sum_{u \in \mathcal{T}_n} \delta_{\frac{Y_u(|u|) - \beta_n}{\alpha_n}}.$$

Theorem 1 *If the marginals of the c_u's belong to the domain of attraction of μ, then*

if $\alpha \in (0, 2) \setminus \{1\}$ or if $\alpha = 2$ and $\sigma_\nu^2 = +\infty$ we have

$$\mu_n \xrightarrow[n\to\infty]{proba} \mu \qquad \text{and} \qquad \overline{\mu}_n \xrightarrow[n\to\infty]{proba} \mu,$$

and if $\alpha = 2$ and $\sigma_\nu^2 < +\infty$ we have

$$\mu_n \xrightarrow[n\to\infty]{proba} \mathcal{N}(0, \sigma_\nu^2 + m_\nu^2) \qquad \text{and} \qquad \overline{\mu}_n \xrightarrow[n\to\infty]{proba} \mathcal{N}(0, \sigma_\nu^2 + m_\nu^2),$$

in the set of probability measures on $\mathbb{R}$ endowed with the weak topology.

In the following, we denote by $\mu_{\infty,\nu}$ the limit measure. Notice that $\mu_{\infty,\nu}$ is always deterministic and that its characteristic exponent is the same as that of μ. The proof of Theorem 1 is relegated to section 4.

2.2 The non-identically distributed case

We are now going to show that Theorem 1 holds true even if the marginals of the c_u's are not the same. Let $\Lambda^\star$ be a distribution on $\mathbb{R}^2$ and ν_0 and ν_1 its two marginals. We assume that ν_0 and ν_1 are different. We denote by Λ the distribution on $\mathbb{R}^2$ defined by

$$\Lambda := \frac{1}{2}(\Lambda^\star + \Lambda'), \tag{2.4}$$

where Λ' is the image of $\Lambda^\star$ by $(x, y) \mapsto (y, x)$. Let $Y^\star$ and Y be the two branching random walks with underlying tree $\mathcal{T}_n$ and with increments $c_u^\star$ and c_u having distribution $\Lambda^\star$ and Λ. The construction of Y can be seen as follows: for any u in $\mathcal{T}_n$, we toss a fair coin, if it is tail then c_u is distributed as $\Lambda^\star$ and if it is head then c_u is distributed as Λ'.

Proposition 1 *Let $\mu_n^\star$, $\overline{\mu}_n^\star$, μ_n and $\overline{\mu}_n$ be the internal and global occupation measures of $Y^\star$ and Y. We have*

$$\mu_n \overset{\mathcal{D}}{=} \mu_n^\star \qquad \text{and} \qquad \overline{\mu}_n \overset{\mathcal{D}}{=} \overline{\mu}_n^\star. \tag{2.5}$$

Proposition 1 is an analogue of Proposition 7 in Marckert [21]. We give here the main arguments of his proof: any binary tree $\mathcal{T}_{n-1}$ marked with the c_u's corresponds, via a composition of transpositions, to a unique marked tree obtained with the $c_u^\star$'s (the transposition around a node u is the map that exchanges the two subtrees rooted in u). Two marked trees are said to be in the same equivalence class if and only if there exists a composition of transpositions that associates one tree with the other one. Each class contains 2^{n-1} trees and one exactly is marked with the $c_u^\star$'s. Moreover each binary tree $\mathcal{T}_{n-1}$ can be marked with the c_u's in 2^{n-1} ways. Since the underlying trees of a same class are equally likely BSTs and since the occupation measure is invariant by transposition, the result (2.5) holds true.

As a consequence of Proposition 1, the two rescaled occupation measures have the same limit (if any) and thus we have the following result on the convergence of the occupation measure of the branching random walk $Y^\star$.

Corollary 1 *If the marginal distributions of Λ, defined in (2.4), belong to the domain of attraction of μ, then*

$$\frac{1}{n}\sum_{u\in\mathcal{T}_n\setminus\partial\mathcal{T}_n} \delta_{\frac{Y_u^\star(|u|)-\beta_n}{\alpha_n}} \xrightarrow[n\to\infty]{proba} \mu_{\infty,\frac{1}{2}\nu_0+\frac{1}{2}\nu_1} \quad \text{and} \quad \frac{1}{2n+1}\sum_{u\in\mathcal{T}_n} \delta_{\frac{Y_u^\star(|u|)-\beta_n}{\alpha_n}} \xrightarrow[n\to\infty]{proba} \mu_{\infty,\frac{1}{2}\nu_0+\frac{1}{2}\nu_1}.$$

in the set of probability measures endowed with the weak topology, where α_n and β_n are given in section 2.

Remark 1 Using Corollary 1 for the case when $c_u^\star = (-1, +1)$ a.s., we get

$$\frac{1}{n}\sum_{u\in\mathcal{T}_n\setminus\partial\mathcal{T}_n} \delta_{\frac{Y_u^\star(|u|)}{\sqrt{2\log n}}} \xrightarrow[n\to+\infty]{proba} \mathcal{N}(0,1). \tag{2.6}$$

Let u_n be a random node uniformly chosen in $\mathcal{T}_n \backslash \partial\mathcal{T}_n$, as a consequence of (2.6) we have

$$\frac{Y^{\star}_{u_n}(|u_n|)}{\sqrt{2\log n}} \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

This particular case of c_u's have been recently studied by Kuba and Panholzer [16]; they show, among other things, the above convergence. They also prove that the rate of convergence is of order $\mathcal{O}(1/\sqrt{\log n})$.

Sections 3.1 and 3.2 handle the study of the behavior of a random process named homogeneous fragmentation and of the profile of another type of random trees named the recursive trees. The results we obtain on both processes are direct consequences of Theorem 1 and Corollary 1.

3 Applications

3.1 Empirical measure of homogeneous fragmentations

Here we are interested in homogeneous fragmentations with no erosion, no loss of mass and where a fragment is always divided, at random, into two parts. The homogeneous fragmentations (and more general fragmentations) have been studied a lot these last years, particularly by Bertoin (see e.g. [2]). Following the steps in Chauvin & al. [8], we give a definition of the fragmentation process $(\mathcal{F}(t))_{t\geq 0}$ of the interval $(0,1)$ we are interested in. This model contains not only the state at time t but also the history of the process. Let $\mathcal{F}(0) := (0,1)$ and $t_1 \sim \mathbf{Exp}(1)$, an exponential random variable, associated with $\mathcal{F}(0)$, seen as the lifetime of the interval. At time t_1, the process $\mathcal{F}$ jumps, the interval $\mathcal{F}(0)$ splits into two parts I_0 and I_1 such that the size of the left fragment is given by a random variable Z having distribution ν_0 on $(0,1)$: $\mathcal{F}(t_1) := (I_0, I_1) = ((0,Z),(Z,1))$. We denote by ν_1 the law of $1-Z$, the length of I_1. After each jump time the fragments of $\mathcal{F}$ behave independently of each other and each fragment splits after an $\mathbf{Exp}(1)$ distributed lifetime into two parts. When I_u splits, we denote by I_{u0} and I_{u1} the left and right fragments and we call them the children of I_u. Conditionally on I_u, the length of the left interval, denoted by $|I_{u0}|$, is $|I_u|Z_u$ where Z_u is an independent copy of Z. Thanks to the lack of memory of the exponential distribution, when n fragments are present, each of them will split first equally likely. Of course the behavior of the process depends on ν_0 and ν_1, the laws of the dependent variables Z_u and $1-Z_u$. We call this fragmentation an homogeneous fragmentation with law ν_0
In this construction, the fragments are naturally indexed by the leaves of a binary tree that encodes the history of the fragmentation. More formally, a continuous binary tree process $(\mathbb{T}_t)_{t\geq 0}$ on $\mathcal{B}$ can be associated with $(\mathcal{F}(t))_{t\geq 0}$. We define $\mathbb{T}_t$ by the set of its leaves

$$\partial\mathbb{T}_t := \{u,\ I_u \in \mathcal{F}(t)\}.$$

Let t_n be the n-th jump time of the processes $(\mathbb{T}_t)$ and $(\mathcal{F}(t))$. Thanks to [8] we have the following results:

$$\frac{t_n}{\log n} \xrightarrow{n\to+\infty} 1 \quad a.s.$$

and, for any $n \geq 1$, $\mathbb{T}_{t_n}$ is a BST of size n.

The size of the fragments in this fragmentation process can be described as an homogeneous fragmentation with the formalism of Bertoin, that is giving the dislocation measure ν_d on $\mathcal{S} := \{(x_1, x_2, \cdots);\ s.t.\ x_1 \geq x_2 \geq \cdots \geq 0, \sum x_i = 1\}$. Here, the measure ν_d is defined by

$$\int_{\mathcal{S}} f(\mathbf{x})\nu_d(d\mathbf{x}) = \int_0^1 f(s \vee (1-s), s \wedge (1-s), 0, \ldots)\nu_0(ds),$$

for every non-negative measurable f.

We are interested in the empirical measure of the logarithm of the size of the fragments of $\mathcal{F}$ defined by

$$f_t := \sum_{u \in \partial\mathbb{T}(t)} \delta_{-\log|I_u|}.$$

We thus have

$$f_{t_n} = \sum_{u \in \partial\mathbb{T}_{t_n}} \delta_{Y_u(|u|)},$$

where Y is the branching random walk on the binary search tree $\mathbb{T}_{t_n}$, with increments having the following distribution: for any $u \in \mathbb{T}_{t_n}$, X_{u0} is distributed as $-\log(Z_u)$ and X_{u1} is distributed as $-\log(1-Z_u)$, where (Z_u) is a family of i.i.d r.v. having law ν_0. We denote by $\hat{\nu}_0$ and $\hat{\nu}_1$ the two distributions of the increments.

The study of fragmentation processes is concerned with the behavior of the fragment sizes. In [2], Bertoin gives a result on $\sum |I_u|\delta_{-\log|I_u|}$, the empirical measure of the logarithm of the size of the fragments balanced by their own size. He shows, under some assumptions on the dislocation measure of homogeneous fragmentations, that the rescaled measure converges to a Gaussian law. Here we show, in a more general non- gaussian case, that the simple occupation measure f_{t_n}, suitably rescaled, converges weakly to a deterministic measure (as the limiting distribution of Bertoin). The following result is a direct consequence of Corollary 1, used for the external occupation measure.

Theorem 2 *Let $(\mathcal{F}(t))$ be an homogeneous fragmentation with law ν_0. If $\frac{1}{2}\hat{\nu}_0 + \frac{1}{2}\hat{\nu}_1$ belongs to the domain of attraction of μ, then*

$$\frac{1}{n+1} \sum_{I_u \in \mathcal{F}(t_n)} \delta_{\frac{-\log|I_u| - \beta_n}{\alpha_n}} \xrightarrow[n\to+\infty]{proba} \mu_{\infty, \frac{1}{2}\hat{\nu}_0 + \frac{1}{2}\hat{\nu}_1} \tag{3.7}$$

in the set of probability measure on $\mathbb{R}$ endowed with the weak topology, where α_n and β_n are given in section 2.

3.2 Profile of recursive trees

A recursive tree process is a random sequence $(\mathfrak{T}_n, n \geq 1)$ of plane trees, where $\mathfrak{T}_n$ has n nodes, built recursively under the following dynamics: $\mathfrak{T}_1$ is reduced to a simple node; the tree $\mathfrak{T}_{n+1}$ is obtained from $\mathfrak{T}_n$ by adjoining a child to one of the nodes. The choice is uniform over the nodes and the new node is inserted as the rightmost son. Notice that $\mathfrak{T}_2$ is deterministic and that , for $n \geq 3$, recursive trees of size n are not equally likely. Let $\tilde{\mathbb{P}}_n$ be the law on $\mathcal{P}_n := \{\mathfrak{T} \in \mathcal{P}, \#\mathfrak{T} = n\}$ induced by this construction (that is a recursive tree of size n is a random element of $\mathcal{P}_n$ under $\tilde{\mathbb{P}}_n$). Figure 3 shows two recursive trees of size 4 with the possible ways to construct them.

This process of plane trees has been first proposed to study the spread of epidemics (Moon,

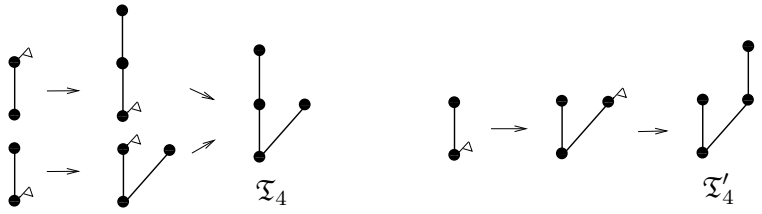

Figure 3 *: Two possible recursive trees of size 4 with the different ways to obtain them (the flags indicate where the nodes are inserted). Since $\mathfrak{T}_4$ is obtained from two different recursive trees of size 3, unlike $\mathfrak{T}'_4$, we have $\tilde{\mathbb{P}}_4(\mathfrak{T}_4) = \frac{1}{3}$ and $\tilde{\mathbb{P}}_4(\mathfrak{T}'_4) = \frac{1}{6}$.*

1974) and many parameters on that trees have been studied: the height,the depth of nodes,the path length,the maximal degree ... (see e.g. Smythe and Mahmoud [20]).

We are interested in $X_n(k)$, the number of nodes in the recursive tree $\mathfrak{T}_n$ at level k. Let $(p_n(k))_k$ be the empirical distribution function of the profile of $\mathfrak{T}_n$ defined by

$$p_n(k) := \sum_{i=1}^{k} \frac{X_n(i)}{n}.$$

We define the rescaled cumulated profile q_n by

$$q_n(\lambda) := p_n(\lfloor \log n + \lambda\sqrt{2\log n} \rfloor) \ , \quad \forall \lambda \in \mathbb{R}. \tag{3.8}$$

We have the following theorem.

Theorem 3

$$(q_n(\lambda))_{\lambda\in\mathbb{R}} \xrightarrow[n\to+\infty]{proba} (\mathbb{P}(N\le\lambda))_{\lambda\in\mathbb{R}}$$

for the Skorohod topology, where N is a random variable with law $\mathcal{N}(0,\frac{1}{2})$.

The proof of Theorem 3 is relegated to section 4.2.

Recently Fuchs, Hwang and Neininger [11] have studied the profile of recursive trees; using the contraction method and the method of moments they proved that $X_n(k)$, normalized by its mean, converges in distribution to a random variable which is the fixed point of an explicit equation (their results are of a different nature from Theorem 3).

4 Proofs

4.1 Proof of Theorem 1

The proof of Theorem 1 we detail below is a consequence of the convergences of the mean and of the variance of the empirical distribution function associated with μ_n (see Lemma 1 and Lemma 2). This approach is different from Janson and Marckert [13]'s one to show the convergence of the occupation measure of a branching random walk on conditioned Galton-Watson (GW) trees to ISE, where the fact that normalized GW trees have a limit (the Aldous continuum random tree) is crucial. There is no similar result for BSTs in the literature.

In this section, we first prove the convergence of μ_n. For the convergence of $\overline{\mu}_n$, the proof runs along the same lines, see Remark 2 for some details.

Lemma 1 *Under the assumptions of Theorem 1, we have $\forall x\in\mathbb{R}$*

$$\mathbb{E}(\mu_n((-\infty,x]) \xrightarrow[n\to\infty]{} \mu_\infty((-\infty,x]).$$

Proof:

Let u_n be chosen equally likely among the nodes of $\mathcal{T}_n\backslash\partial\mathcal{T}_n$. We get the equalities

$$\begin{aligned}\mathbb{E}(\mu_n((-\infty,x]) &= \mathbb{E}\left(\frac{1}{n}\times\sum_{u\in\mathcal{T}_n\backslash\partial\mathcal{T}_n}\delta_{\frac{Y_u(|u|)-\beta_n}{\alpha_n}}((-\infty,x])\right)\\ &= \mathbb{E}\left(\mathbb{1}_{(-\infty,x]}(\frac{Y_{u_n}(|u_n|)-\beta_n}{\alpha_n})\right).\end{aligned}$$

By (1.2), this is equal to $C_n := \mathbb{P}\left((\sum_{v\ll u_n}X_v)-\beta_n)/\alpha_n\le x\right)$.
The sum contains $|u_n|$ terms. In terms of quicksort the random variable $|u_n|$ is the cost of a successful search in a labeled binary search tree of size n. Louchard [17] shows that

$$\frac{|u_n| - 2\log(n)}{\sqrt{2\log(n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{4.9}$$

Let $H_{\varepsilon,n}$ be the event $\{||u_n| - 2\log(n)| \leq (2\log(n))^{\frac{1}{2}+\varepsilon}\}$, with $\varepsilon \in]0, \frac{1}{2}[$. Since $\mathbb{P}(H_{\varepsilon,n})$ goes to 1, by (4.9), C_n and $\mathbb{P}\big(((\sum_{v \ll u_n} X_v) - \beta_n)/\alpha_n \leq x \big| H_{\varepsilon,n}\big)$ have the same limit (if any).

The sum $(\sum_{v \ll u_n} X_v) - \beta_n$ has the following representation

$$\sum_{i=1}^{d_n}(X_i - m'_\nu) + \sum_{i=1}^{|u_n|-d_n} (\tilde{X}_i - m'_\nu) + m'_\nu(|u_n| - 2\log n), \tag{4.10}$$

where $d_n = 2\log n - (2\log n)^{\frac{1}{2}+\varepsilon}$ and (X_i) and $(\tilde{X}_i)$ are two independent families of i.i.d. random variables with common law ν. Recall that m'_ν is the mean of ν if it is finite, and zero otherwise. To end the proof of Lemma 1 we study the asymptotics of the three terms in (4.10).

• The term $\sum_{i=1}^{d_n}(X_i - m'_\nu)/\alpha_n$ converges in distribution to the stable law μ (recall that $\mu = \mathcal{N}(0, \sigma_\nu^2)$ when $\alpha = 2$ and $\sigma^2 < +\infty$).

• It is known (see Theorem 16.14 of [14]) that (2.3) implies the convergence of the process $\left(\sum_{i=1}^{\lfloor e_n t \rfloor} (\tilde{X}_i - m'_\nu) \,/\, (f_n t)\ ,\ t \geq 0\right)$ in the Skorohod space, to a stable Levy process of index α, if $f_n \sim L_n \times e_n^{1/\alpha}$ with $e_n \to +\infty$ and (L_n) is slowly varying. Thus

$$\left(\sum_{i=1}^{\lfloor c_n t \rfloor} (\tilde{X}_i - m'_\nu) \,/\, \alpha_n, t \geq 0\right) \xrightarrow[n\to\infty]{\mathcal{D}} 0,$$

where $c_n := 2(2\log n)^{\frac{1}{2}+\varepsilon}$. Since, if $X_n \to X$ for the Skhohorod topology then $\max(X_n) \to \max(X)$ in distribution, we get

$$\max_{1 \leq k \leq c_n} \sum_{i=1}^{k} (\tilde{X}_i - m'_\nu) \,/\, \alpha_n \xrightarrow[n\to\infty]{\mathcal{D}} 0$$

On $H_{\varepsilon,n}$, $|u_n| - d_n$ is smaller than c_n. Hence, we have shown that $\sum_{i=1}^{|u_n|-d_n}(\tilde{X}_i - m'_\nu)/\alpha_n$ goes to zero in probability.

• For the third term in (4.10) normalized by α_n, when $\sigma_\nu^2 < +\infty$ we have $\alpha_n = \sqrt{2\log n}$ and $m'_\nu = m_\nu$. Thus, by (4.9)

$$m'_\nu \frac{|u_n| - 2\log n}{\alpha_n} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, m_\nu^2).$$

When $\alpha \in (0,2) \setminus \{1\}$ or when $\alpha = 2$ and $\sigma_\nu^2 = +\infty$ (recall that in this case $L_n \to +\infty$), we have $\alpha_n/\sqrt{2\log n} \to +\infty$, and then, by (4.9)

$$m'_\nu \frac{|u_n| - 2\log n}{\alpha_n} \xrightarrow[n\to\infty]{proba} 0.$$

Now, since the extremal terms in (4.10) are independent and the middle one vanishes in probability, the result follows by using, for example, Theorem 25.4 of [4] (that is, if $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{proba} 0$ then $X_n + Y_n \xrightarrow{\mathcal{D}} X$). □

In (4.10), the two terms that contribute to the limit are the first and the third one (corresponding to a centered version of the law, and to a corrected term due to the mean of the increments). When $\sigma_\nu^2 < +\infty$, these two terms have the same order. When $\sigma_\nu^2 = +\infty$ the first one dominates.

Lemma 2 *Under the assumptions of Theorem 1, we have* $\forall x \in \mathbb{R}$

$$\mathbb{V}(\mu_n((-\infty, x])) \xrightarrow[n\to\infty]{} 0.$$

Proof: Let u_n and v_n be two nodes uniformly and independently chosen on $\mathcal{T}_n \backslash \partial\mathcal{T}_n$. We get for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}(\mu_n((-\infty, x]))^2 &= \mathbb{E}\left(\mathbb{1}_{(-\infty,x]}\left(\frac{Y_{u_n}(|u_n|) - \beta_n}{\alpha_n}\right) \mathbb{1}_{(-\infty,x]}\left(\frac{Y_{v_n}(|v_n|) - \beta_n}{\alpha_n}\right)\right) \\
&= \mathbb{P}\left(\frac{Y_{u_n}(|u_n|) - \beta_n}{\alpha_n} \le x, \frac{Y_{v_n}(|v_n|) - \beta_n}{\alpha_n} \le x\right). \qquad (4.11)
\end{aligned}$$

Let $u_n \wedge v_n$ be the deepest common ancestor of u_n and v_n. To compute the limit in (4.11) we investigate the dependence between $Y_{u_n}(|u_n|)$ and $Y_{v_n}(|v_n|)$. This dependence comes from the values of the ancestors of $u_n \wedge v_n$ which contribute both to $Y_{u_n}(|u_n|)$ and to $Y_{v_n}(|v_n|)$ and can also come from the values of the children of $u_n \wedge v_n$.

Let Δ_{u_n,v_n} be the path distance between u_n and v_n in $\mathcal{T}_n$. Mahmoud and Neininger [19] have shown

$$\frac{\Delta_{u_n,v_n} - 4\log(n)}{\sqrt{4\log(n)}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1). \qquad (4.12)$$

Using the relation $\Delta_{u_n,v_n} = |u_n| + |v_n| - 2|u_n \wedge v_n|$, (4.12) and (4.9) one gets, for any $\varepsilon > 0$,

$$\frac{|u_n \wedge v_n|}{(2\log(n))^{\frac{1}{2}+\varepsilon}} \xrightarrow[n\to\infty]{proba} 0. \tag{4.13}$$

In order to get the limit in (4.11) we write

$$Y_{u_n}(|u_n|) \stackrel{\mathcal{D}}{=} \tilde{X}_1 + \sum_{i=1}^{|u_n\wedge v_n|} X_i + \sum_{i=2}^{|u_n|-|u_n\wedge v_n|} \tilde{X}_i \tag{4.14}$$

and

$$Y_{v_n}(|v_n|) \stackrel{\mathcal{D}}{=} \hat{X}_1 + \sum_{i=1}^{|u_n\wedge v_n|} X_i + \sum_{i=2}^{|v_n|-|u_n\wedge v_n|} \hat{X}_i, \tag{4.15}$$

where the random variables $(X_i)_{i\geq 1}$, $(\tilde{X}_i)_{i\geq 2}$ and $(\hat{X}_i)_{i\geq 2}$ are all i.i.d. with law ν; $\tilde{X}_1$ and $\hat{X}_1$ are the values attached to the children of $u_n \wedge v_n$ and may be dependent. Using (4.13) we prove, with the same arguments as those in the proof of Lemma 1, that the common sum of (4.14) and (4.15), that is the sum of the X_i, normalized by α_n, goes to zero in probability. Moreover we have $\frac{\tilde{X}_1}{\alpha_n} \xrightarrow[n\to+\infty]{proba} 0$ and $\frac{\tilde{X}_1}{\alpha_n} \xrightarrow[n\to+\infty]{proba} 0$. Whence, using the independence between the remaining sums of (4.14) and (4.15), that is the sum on the $\tilde{X}_i$ and the sum on the $\hat{X}_i$, we get that the quantity (4.11) has the same limit (if any) as

$$\mathbb{P}\left((-\beta_n + \sum_{i=2}^{|u_n|-|u_n\wedge v_n|} \tilde{X}_i)/\alpha_n \leq x\right) \mathbb{P}\left((-\beta_n + \sum_{i=2}^{|v_n|-|u_n\wedge v_n|} \hat{X}_i)/\alpha_n \leq x\right).$$

Using the same methods as in Lemma 1, we finally find that $\mathbb{E}\left[(\mu_n(]-\infty, x]))^2\right]$ has the same limit as $[\mathbb{E}(\mu_n(]-\infty, x]))]^2$. □

Proof of Theorem 1:
Thanks to Lemma 1 and Lemma 2, for any $x \in \mathbb{R}$, the sequence $(F_n(x))$ converges in probability to $F_{\infty,\nu}(x)$, where F_n and $F_{\infty,\nu}$ are respectively the distribution functions of μ_n and of $\mu_{\infty,\nu}$ ($F_{\infty,\nu}$ and $\mu_{\infty,\nu}$ are deterministic).
A well known result of analysis is the following: if a sequence of bounded increasing functions, having limits zero at $-\infty$ and one at $+\infty$, converges pointwise to a continuous function then the convergence is uniform. Thus, since the function $F_{\infty,\nu}$ is continuous, we deduce the uniform convergence of F_n to $F_{\infty,\nu}$ in probability. This entails the weak convergence of μ_n to $\mu_{\infty,\nu}$ in probability. □

Remark 2 We proved Theorem 1 for the internal occupation measure μ_n. To prove the result for $\overline{\mu}_n$ it suffices to find similar results to (4.9) and (4.13) for the external nodes. Brown and Shubert [6] have shown that the asymptotics (4.9) is true for u_n uniform on $\partial\mathcal{T}_n$ and we obtain (4.13) for the leaves using (4.12), a proof runs as follows: Let C_I and C_E be the sets

$$C_I := \{(u,v) \in (\mathcal{T}_n \backslash \partial\mathcal{T}_n)^2 \text{ such that } |\Delta_{u,v} - 4\log n| > (4\log n)^{\frac{1}{2}+\varepsilon}\}.$$

and

$$C_E := \{(u,v) \in (\partial\mathcal{T}_n)^2 \text{ such that } |\Delta_{u,v} - 4\log n| > (4\log n)^{\frac{1}{2}+\varepsilon} + 2\}$$

Using the projection from C_E to C_I, which returns the parent of each leaf, we get $\#C_E \leq 4\times\#C_I$. Thus $\#C_E/n^2$ goes to zero in probability and, using the relation $\Delta_{u,v} = |u| + |v| - 2|(u \wedge v)|$ we get that (4.13) holds for u_n and v_n uniformly and independently chosen on $\partial\mathcal{T}_n$.

Moreover, the main result is restricted here to $\alpha \neq 1$, but the symetric case for $\alpha = 1$ also works, since we can take $b_n \equiv 0$ for the convergence (2.3).

4.2 Proof of Theorem 3

Let us see that Theorem 3 is a consequence of Corollary 1. Following the formalism of Marckert [21], we define a map Φ, named rotation correspondence, from the set $\mathcal{P}$ of plane trees to the set $\mathcal{B}$ of complete binary trees (this is referred to as the natural correspondence in Knuth [15]). With a plane tree $\mathfrak{T}_n$ is associated a binary tree $\mathcal{T}_{n-1}$ such that $\Phi(\mathfrak{T}_n) = \mathcal{T}_{n-1}$. It means that to each node a of $\mathfrak{T}_n$ corresponds an internal node of $\mathcal{T}_{n-1}$, denoted by $\Phi(a)$, verifying the following three points:

(a) If a is the leftmost son of the root of $\mathfrak{T}_n$ then $\Phi(a) = \varnothing$.
(b) $(\{a,b\} \subset \mathfrak{T}_n, b = a0) \Leftrightarrow (\{\Phi(a), \Phi(b)\} \subset \mathcal{T}_{n-1}, \Phi(b) = \Phi(a)0)$.
(c) $(\{a,b\} \subset \mathfrak{T}_n, a = dc, b = d(c+1)) \Leftrightarrow (\{\Phi(a), \Phi(b)\} \subset \mathcal{T}_{n-1}, \Phi(b) = \Phi(a)1)$.

$\mathcal{T}_{n-1}$ is obtained by adding the leaves to the internal nodes. **(b)** means that a relation *father-leftmost son* in $\mathfrak{T}_n$ corresponds to a relation *father-left son* in $\mathcal{T}_{n-1}$. **(c)** means that a relation between a node and its first brother on its right in $\mathfrak{T}_n$ corresponds to a relation *father-right son* in $\mathcal{T}_{n-1}$. Figure 4 illustrates why the correspondence is called a rotation. To get $\mathcal{T}_{n-1}$ from $\mathfrak{T}_n$, remove the root, keep the edges father-leftmost son and add an edge between each node and its first brother on its right (if any). Make a $\pi/4$ rotation. This gives the internal nodes of $\mathcal{T}_{n-1}$. Add the leaves to obtain a complete binary tree.

Let $\Psi := \Phi^{-1}$. The replacement of a leaf by an internal node in a complete binary tree $\mathcal{T}$ results in the addition of a node as a rightmost son in $\Psi(\mathcal{T})$. Thus, since the insertion in both processes is uniform, the random process of complete binary trees $(\Phi(\mathfrak{T}_n), n \geq 1)$ is a BST process: this implies that both processes $(\Phi(\mathfrak{T}_n), n \geq 1)$ and $(\mathcal{T}_n, n \geq 0)$ have the same law.

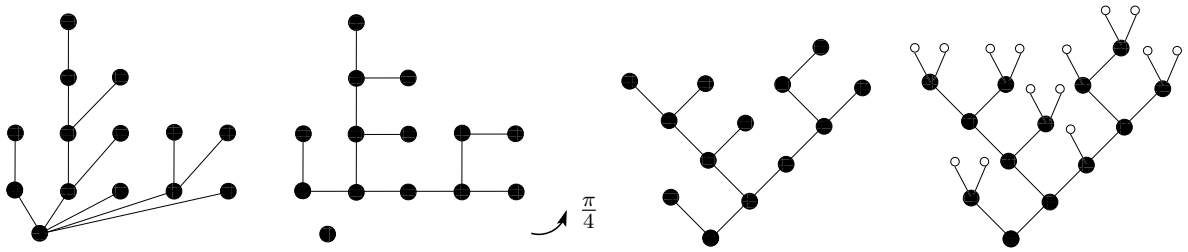

Figure 4 *: The three steps of the map ϕ.*

Hence the random variables on $\mathcal{P}_{n+1}$ can be seen as random variables on $\mathcal{B}_n$ through the map Ψ.

From now on we denote by $\mathfrak{T}_n$ a recursive tree and by $\mathcal{T}_{n-1}$ the BST $\Phi(\mathfrak{T}_n)$. For any $u = d_1 d_2 \cdots d_k \in \mathcal{T}_{n-1}$, we denote by $L(u)$ the left-depth of u, that is $L(u) = \#\{i \,|\, d_i = 0\}$ ($L(u)$ is the number of left-steps from the root to u). By the rotation correspondence the height of a node a in $\mathfrak{T}_n - \{\varnothing\}$ minus 1 is equal to the left-depth of $\Phi(a)$ in $\mathcal{T}_{n-1}$. In order to count the left-depth of each internal node, we construct a branching random walk $Y^\star$ with underlying tree $\mathcal{T}_{n-1}$ and where the law of the increments is given by

$$\mathbb{P}\big(c_u^\star = (1,0)\big) = 1, \quad \forall u \in \mathbb{U}^b.$$

With these increments, $Y_u^\star(|u|)$ is equal to $L(u)$ and is also equal to $|\Psi(u)| - 1$. Thus we have

$$\begin{aligned} X_n(k) &= \#\{a \in \mathfrak{T}_n, |a| = k\} \\ &= \#\{u \in \mathcal{T}_{n-1}, L(u) = k-1\} \\ &= \#\{u \in \mathcal{T}_{n-1}, Y_u^\star(|u|) = k-1\}. \end{aligned}$$

The occupation measure associated with $Y^\star$ is

$$\mu_{n-1}^\star = \frac{1}{n-1} \sum_{u \in \mathcal{T}_{n-1} \setminus \partial\mathcal{T}_{n-1}} \delta_{\frac{Y_u^\star(|u|) - \log(n-1)}{\sqrt{2\log(n-1)}}}.$$

We thus have

$$p_n(k) = \frac{n-1}{n} \mu_{n-1}^\star \left((-\infty, \frac{k-1-\log(n-1)}{\sqrt{2\log(n-1)}}] \right).$$

Using Corollary 1 for the distribution Λ given by

$$\mathbb{P}(c_u = (1,0)) = \mathbb{P}(c_u = (0,1)) = \frac{1}{2}, \quad \forall u \in \mathbb{U}^b,$$

we get

$$\mu^{\star}_{n-1} \xrightarrow[n\to+\infty]{proba} \mathcal{N}(0, \frac{1}{2})$$

in the set of probability measures for the weak topology, which ends the proof of Theorem 3. □

Conclusion and perspectives

Two important tools of the proof of Theorem 1 are the properties of the stable laws and some asymptotics properties of the BSTs. If we consider the branching random walk with another type of underlying trees that still verify (4.9) and (4.13) then an analog of Theorem 1 can be proved. It seems to be the case of the recursive trees studied in section 3.2.

Acknowledgment

Jean Francois Marckert is at the origin of this work, I would like to thank him for all his helpful remarks. I also thank one referee for his very efficacious comments.

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